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# Utility maximization with proportional transaction costs under model uncertainty

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## Abstract

We consider a discrete time financial market with proportional transaction costs under model uncertainty, and study a semi-static utility maximization for the case of exponential utility preference. The randomization techniques recently developed in [12] allow us to transform the original problem into a frictionless market framework, however, with the extra probability uncertainty on an enlarged space. Using the one-period duality result in [3], together with measurable selection arguments and minimax theorem, we are able to prove all together the existence of the optimal strategy, convex duality theorem as well as the auxiliary dynamic programming principle in our context with transaction costs. As an application of the duality representation, some important features of utility indifference prices are investigated in the robust setting.

**Key words.** Utility maximization, transaction costs, model uncertainty, randomization method, convex duality, utility indifference pricing.

**MSC (2010).** Primary: 60G40, 60G05; Secondary: 49M29.

## 1 Introduction

The optimal investment via utility maximization has always been one of the fundamental problems in quantitative finance. In particular, the optimal semi-static portfolio among risky assets and liquid options and the associated utility indifference pricing of unhedgeable illiquid contingent claims have attracted a lot of research interests recently. In the classical dominated market model, the so-called utility maximization with random endowments was extensively investigated, see among [29], [19], [21], [20], [7] and [28]. In particular, the duality approach has been proposed and developed as a powerful tool to deal with general incomplete market models. Without knowing the specific underlying model structures, the convex duality relationship enables one to obtain the existence of the primal optimizer by solving the corresponding dual optimization problem first. Typically, the dual problem is formulated on the set of equivalent (local) martingale measures (EMM), whose existence is ensured by some appropriate no arbitrage assumptions. Depending on the domain of the utility function, different technics will be involved in order to close the strong duality gap. For utilities defined on the positive real line, to handle the random payoffs and to establish the bipolar relationship, the appropriate closure of the dual set of EMM plays the key role, see [19] and [21] for instance. On the other hand, for

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utilities defined on the whole real line, a subset of EMM with finite general entropy is usually chosen to define the dual problem while the appropriate definition of working portfolios turns out to be critical to guarantee and relate the primal and dual optimizers, see [20], [7] and [28] and the references therein.

Because of the growing complexity of real financial markets, the aforementioned optimization problems have been actively extended mainly in two directions. The first fruitful extension incorporates the practical trading frictions, namely transaction costs, into decision making and the resulting wealth process. As transaction costs will generically break the (local) martingale property of the self-financing wealth process under EMM, the dual pricing kernel is not expected to be the same as in the frictionless counterpart. Instead, the no arbitrage condition is closely related to the existence of a pair of dual elements named the consistent price system (CPS). Briefly speaking, the first component of CPS is a process evolving inside the bid-ask spread, while the second component is an equivalent probability measure under which the first component becomes a martingale. However, similar to the case in the frictionless model, for utility maximization with random endowments, the set of CPS can only serve as the first step to formulate the naive dual problem. More efforts are demanded to deal with the random payoffs from options, see some related work in [8], [31] and [25].

The second compelling extension in the literature is to take into account the model uncertainty, for instance the volatility uncertainty, by starting with a set of possibly mutual singular probability measures that can describe the investor's all believes of the market. In the discrete time framework, the no-arbitrage condition and the fundamental theorem in robust finance have been essentially studied in [1, 13, 17, 16], etc. for frictionless markets, and in [5, 14, 15] for market with transaction costs. Analogous to the dominated case, the pricing-hedging duality can usually be obtained by studying the superhedging problem under some appropriate no-arbitrage conditions. The non-dominated robust utility maximization in the discrete time frictionless was first examined by [27], where the dynamic programming principle plays the major role to derive the existence of the optimal primal strategy without passing to the dual problem, see some further extensions in [26, 10, 11]. However, whether the convex duality holds remained open in these pioneer work of utility maximization. Recently, [3] established the duality representation for the exponential utility preference in the frictionless model under some restrictive no arbitrage conditions, which motivates us to reconsider the validity of duality theorem in this paper with proportional transaction costs under weaker market conditions using some distinctive arguments. We also note a recent paper [4], in which the authors proved a robust utility maximization duality using medial limits and a functional version of Choquet's capacitability theorem.

The main objective of this paper is therefore to study the existence of the optimal strategy, the convex duality theorem and the auxiliary dynamic programming principle for a semi-static utility maximization problem with transaction costs in a discrete time framework. To be precise, we envision an investor who chooses the optimal semi-static portfolio in stocks and liquid options with an extra random endowment for the case of exponential utility preference and meanwhile each trading incurs proportional transaction fees. The core idea of our analysis is to reduce the complexity of transaction costs significantly by employing the randomization method as in [12]. Consequently, the unpleasant impacts caused by trading fees can be hidden in an enlarged space of probability uncertainty and some technics in the literature of robust hedging and utility maximization in frictionless models can be modified and adopted.

Our contributions are two-fold. Firstly, we apply the randomization technique to transform the initial problem into a utility maximization problem in a fictitious market without frictions, which is formally similar to the problem studied in [3]. Nevertheless, the admissible strategy in our fictitious problem is not allowed to be adapted to the fictitious underlying process, which does not fit into the standard framework. Moreover, the quasi-sure version of no-arbitrage

condition of the second order we adopt under the framework of [14] corresponds to the robust no-arbitrage condition in the fictitious market, which is weaker than the pointwise robust no-arbitrage condition assumed in [3]. Contrary to the dominated counterpart, the existence of the dual optimizer is not granted because of the probability uncertainty. To close the strong duality gap in our paper, the dynamic programming principle and the minimax theorem will be the remedy to relate the primal and dual problems. In particular, due to our weaker no-arbitrage condition, we will suggest a modified dynamic programming argument comparing to [3] to establish our main result. In fact, our modified arguments can also be applied to the framework of [3] to improve their results. Secondly, to manifest the value of the duality representation, we also investigate an application to utility indifference pricing. Several fundamental properties of indifference prices including the asymptotic convergence of indifference prices to the superhedging price and some continuity results with respect to random endowments are confirmed in the robust setting with transaction costs.

The rest of the paper is organized as follows. In Section 2, we introduce the market model with transaction costs, and show how to reformulate the robust utility maximization problem on a frictionless market on an enlarged space using the randomization method. In Section 3, we restrict to the case of the exponential utility preference. A convex duality theorem and the existence of the optimal trading strategy are first obtained in the presence of both model uncertainty and transaction costs. As an application, several properties of the utility indifference prices are concluded. Section 4 mainly provides the proof of the duality result using a dynamic programming argument.

**Notation.** Given a measurable space  $(\Omega, \mathcal{F})$ , we denote by  $\mathfrak{B}(\Omega, \mathcal{F})$  the set of all probability measures on  $(\Omega, \mathcal{F})$ . For a topological space  $\Omega$ ,  $\mathcal{B}(\Omega)$  denotes its Borel  $\sigma$ -field with the abbreviate notation  $\mathfrak{B}(\Omega) := \mathfrak{B}(\Omega, \mathcal{B}(\Omega))$ . For a Polish space  $\Omega$ , a subset  $A \subseteq \Omega$  is called analytic if it is the image of a Borel subset of another Polish space under a Borel measurable mapping. A function  $f : \Omega \rightarrow \overline{\mathbb{R}} := [-\infty, \infty]$  is upper semianalytic if  $\{\omega \in \Omega : f(\omega) > a\}$  is analytic for all  $a \in \mathbb{R}$ . Given a probability measure  $\mathbb{P} \in \mathfrak{B}(\Omega)$  and a measurable function  $f : \Omega \rightarrow \overline{\mathbb{R}}$ , we define the expectation

$$\mathbb{E}^{\mathbb{P}}[f] := \mathbb{E}^{\mathbb{P}}[f^+] - \mathbb{E}^{\mathbb{P}}[f^-], \quad \text{with the convention } \infty - \infty = -\infty.$$

For a family  $\mathcal{P} \subseteq \mathfrak{B}(\Omega)$  of probability measures, a subset  $A \subset \Omega$  is called  $\mathcal{P}$ -polar if  $A \subset A'$  for some universally measurable set  $A'$  satisfying  $\mathbb{P}[A'] = 0$  for all  $\mathbb{P} \in \mathcal{P}$ , and a property is said to hold  $\mathcal{P}$ -quasi surely or  $\mathcal{P}$ -q.s if it holds true outside a  $\mathcal{P}$ -polar set. For  $\mathbb{Q} \in \mathfrak{B}(\Omega)$ , we write  $\mathbb{Q} \lll \mathcal{P}$  if there exists  $\mathbb{P}' \in \mathcal{P}$  such that  $\mathbb{Q} \ll \mathbb{P}'$ . Given a sigma algebra  $\mathcal{G}$ , we denote by  $L^0(\mathcal{G})$  the collection of  $\mathbb{R}^d$ -valued random variable that are  $\mathcal{G}$ -measurable,  $d$  being given by the context.

## 2 Market model and Problem Formulation

Let us first introduce a financial market with proportional transaction costs in a multivariate setting under model uncertainty. A utility maximization problem is formulated afterwards and we then reformulate the problem further in a fictionless market setting on an enlarged space. Although the reformulation technique can be used for a more general framework, we will stay essentially in the context of Bouchard and Nutz [13, 14].

### 2.1 Market model and preliminaries

**A product space with a set of probability measures** Let  $\Omega_0 := \{\omega_0\}$  be a singleton and  $\Omega_1$  be a Polish space. For each  $t = 1, \dots, T$ , we denote by  $\Omega_t := \Omega_1^t$  the  $t$ -fold Cartesian

product of  $\Omega_1$  and let  $\mathcal{F}_t^0 := \mathcal{B}(\Omega_t)$  and  $\mathcal{F}_t$  its universal completion. In particular,  $\mathcal{F}_0$  is trivial. We define the filtered measurable space  $(\Omega, \mathcal{F})$  by

$$\Omega := \Omega_T, \mathcal{F} := \mathcal{F}_T, \mathbb{F} := (\mathcal{F}_t)_{0 \leq t \leq T} \text{ and } \mathbb{F}^0 := (\mathcal{F}_t^0)_{0 \leq t \leq T}.$$

We then introduce a set  $\mathcal{P}$  of probability measures on  $(\Omega, \mathcal{F})$  by

$$\mathcal{P} := \{\mathbb{P} := \mathbb{P}_0 \otimes \mathbb{P}_1 \otimes \cdots \otimes \mathbb{P}_{T-1} : \mathbb{P}_t(\cdot) \in \mathcal{P}_t(\cdot) \text{ for } t \leq T-1\}. \quad (2.1)$$

In above,  $\mathbb{P}_t : \Omega_t \mapsto$  is a probability kernel so that the probability measure  $\mathbb{P}$  is defined by Fubini's theorem in the sense that

$$\mathbb{P}(A) := \int_{\Omega_1} \cdots \int_{\Omega_1} \mathbf{1}_A(\omega_1, \omega_2, \dots, \omega_T) \mathbb{P}_{T-1}(\omega_1, \dots, \omega_{T-1}; d\omega_T) \cdots \mathbb{P}_0(d\omega_1),$$

and  $\mathcal{P}_t(\omega)$  is a non-empty convex set in  $\mathfrak{B}(\Omega_1)$ , which represents the set of all possible models for the  $(t+1)$ -th period, given the state  $\omega \in \Omega_t$  at time  $t = 0, 1, \dots, T-1$ . As in the literature, we assume that, for each  $t$ ,

$$\text{graph}(\mathcal{P}_t) := \{(\omega, \mathbb{P}) : \omega \in \Omega_t, \mathbb{P} \in \mathcal{P}_t(\omega)\} \subseteq \Omega_t \times \mathcal{P}(\Omega_1) \text{ is analytic.} \quad (2.2)$$

This ensures in particular that  $\mathcal{P}$  in (2.1) is nonempty.

**A financial market with proportional transaction cost** The financial market with proportional transaction cost is formulated in terms of random cones. Let  $d \geq 2$ , for every  $t \in \{0, 1, \dots, T\}$ ,  $K_t : \Omega \rightarrow 2^{\mathbb{R}^d}$  is a  $\mathcal{F}_t^0$ -measurable random set in the sense that  $\{\omega \in \Omega : K_t(\omega) \cap O \neq \emptyset\} \in \mathcal{F}_t^0$  for every closed (open) set  $O \subset \mathbb{R}^d$ . Here, for each  $\omega \in \Omega$ ,  $K_t(\omega)$  is a closed convex cone containing  $\mathbb{R}_+^d$ , called the solvency cone at time  $t$ . It represents the collection of positions, labelled in units of different  $d$  financial assets, that can be turned into non-negative ones (component by component) by performing immediately exchanges between the assets. We denote by  $K_t^* \subset \mathbb{R}_+^d$  its (nonnegative) dual cone:

$$K_t^*(\omega) := \{y \in \mathbb{R}^d : x \cdot y \geq 0 \text{ for all } x \in K_t(\omega)\}, \quad (2.3)$$

where  $x \cdot y := \sum_{i=1}^d x^i y^i$  is the inner product on  $\mathbb{R}^d$ . For later use, let us also introduce

$$K_t^{*,0}(\omega) := \{y = (y^1, \dots, y^d) \in K_t^*(\omega), y^d = 1\}.$$

As in [14], we assume the following conditions throughout the paper:

**Assumption 2.1.**  $K_t^* \cap \partial \mathbb{R}_+^d = \{0\}$  and  $\text{int} K_t^*(\omega) \neq \emptyset$  for every  $\omega \in \Omega$  and  $t \leq T$ .

It follows from the above assumption and [14, Lemma A.1] that  $K_t^*$ ,  $K_t^{*,0}$  and  $\text{int} K_t^*$  are all  $\mathcal{F}_t^0$ -measurable. Moreover, there is a  $\mathbb{F}^0$ -adapted process  $S$  satisfying

$$S_t(\omega) \in K_t^{*,0}(\omega) \cap \text{int} K_t^*(\omega) \text{ for every } \omega \in \Omega, t \leq T. \quad (2.4)$$

We also assume that transaction costs are bounded and uniformly strictly positive. This is formulated in terms of  $S$  above.

**Assumption 2.2.** There is some constant  $c > 1$  such that

$$c^{-1} S_t^i(\omega) \leq y^i \leq c S_t^i(\omega), \text{ for every } i \leq d-1 \text{ and } y \in K_t^{*,0}(\omega).$$

Finally, we define the collection of admissible strategies as follows.

**Definition 2.3.** We say that an  $\mathbb{F}$ -adapted process  $\eta = (\eta_t)_{0 \leq t \leq T}$  is an admissible trading strategy if

$$\eta_t \in -K_t \quad \mathcal{P}\text{-q.s. for all } t \leq T.$$

We denote by  $\mathcal{A}$  the collection of all admissible strategies.

The constraint  $\eta_t \in -K_t$  means that  $0 - \eta_t \in K_t$ , i.e., starting at  $t$  with 0, one can perform immediate transfers to reach the position  $\eta_t$ . Then, given  $\eta \in \mathcal{A}$ , the corresponding wealth process associated to a zero initial endowment at time 0 is  $(\sum_{s=0}^t \eta_s)_{t \leq T}$ . We can refer to [12, 14] for concrete examples. See also the monograph [23].

## 2.2 A utility maximization problem and its reformulation

Let  $U : \mathbb{R} \rightarrow \mathbb{R} \cup \{-\infty\}$  be a non-decreasing concave utility function. We are interested in the following robust utility maximization problem with random endowments:

$$V(\xi) := \sup_{\eta \in \mathcal{A}_0} \inf_{\mathbb{P} \in \mathcal{P}} \mathbb{E}^{\mathbb{P}} \left[ U \left( \left( \xi + \sum_{t=0}^T \eta_t \right)^d \right) \right], \quad (2.5)$$

where  $\mathcal{A}_0$  denotes the collection of all  $\eta \in \mathcal{A}$  such that  $(\xi + \sum_{t=0}^T \eta_t)^i = 0$  for  $i = 1, \dots, d-1$ . Namely, we take the  $d$ -th asset as numéraire and require the liquidation of the other assets at the final time  $T$ . The mixture of model uncertainty, transaction costs and random endowments can bring a lot of new mathematical challenges. Our paramount remedy to reduce the complexity is to reformulate it on a fictitious market without transaction cost. In particular, this allows us to use some well known results and techniques in the existing literature.

**A frictionless market on the enlarged space** Given the constant  $c > 1$  in Assumption 2.2, we define  $\Lambda_1 := [c^{-1}, c]^{d-1}$  and  $\Lambda := (\Lambda_1)^{T+1}$ , and then introduce the canonical process  $\Theta_t(\theta) := \theta_t$ ,  $\forall \theta = (\theta_t)_{t \leq T} \in \Lambda$ , as well as the  $\sigma$ -fields  $\mathcal{F}_t^\Lambda := \sigma(\Theta_s, s \leq t)$ ,  $t \leq T$ . We next introduce an enlarged space  $\bar{\Omega} := \Omega \times \Lambda$ , an enlarged  $\sigma$ -field  $\bar{\mathcal{F}} := \mathcal{F} \otimes \mathcal{F}_T^\Lambda$ , together with three filtrations  $\bar{\mathbb{F}}^0 = (\bar{\mathcal{F}}_t^0)_{0 \leq t \leq T}$ ,  $\tilde{\mathbb{F}} = (\tilde{\mathcal{F}}_t)_{0 \leq t \leq T}$  and  $\bar{\mathbb{F}} = (\bar{\mathcal{F}}_t)_{0 \leq t \leq T}$  in which  $\bar{\mathcal{F}}_t^0 := \mathcal{F}_t^0 \otimes \mathcal{F}_t^\Lambda$ ,  $\tilde{\mathcal{F}}_t := \mathcal{F}_t \otimes \{\emptyset, \Lambda\}$  and  $\bar{\mathcal{F}}_t := \mathcal{F}_t \otimes \mathcal{F}_t^\Lambda$  for  $t \leq T$ .

Next, let us introduce our randomized market model with the fictitious underlying stock  $X = (X_t)_{0 \leq t \leq T}$  defined by

$$X_t(\bar{\omega}) := \Pi_{K_t^{*,0}(\bar{\omega})}[S_t(\omega)\theta_t], \quad \text{for all } \bar{\omega} = (\omega, \theta) \in \bar{\Omega}, \quad t \leq T, \quad (2.6)$$

where  $S_t(\omega)\theta_t := (S_t^1(\omega)\theta_t^1, \dots, S_t^{d-1}(\omega)\theta_t^{d-1}, S_t^d(\omega))$ , and  $\Pi_{K_t^{*,0}(\bar{\omega})}[y]$  stands for the projection of  $y \in \mathbb{R}^d$  on the convex closed set  $K_t^{*,0}(\bar{\omega})$ . It is worth noting that  $S_t \in K_t^{*,0}$  for  $t \leq T$  and that  $X$  is  $\bar{\mathbb{F}}^0$ -adapted by Lemma 2.6 of [12].

We then define two sets of admissible strategies:

$$\mathcal{H} := \{\text{All } \tilde{\mathbb{F}}\text{-predictable processes}\} \quad \text{and} \quad \bar{\mathcal{H}} := \{\text{All } \bar{\mathbb{F}}\text{-predictable processes}\}.$$

Recall that  $\tilde{\mathcal{F}}_t := \mathcal{F}_t \otimes \{\emptyset, \Lambda\}$ , and hence a  $\tilde{\mathbb{F}}$ -predictable process can be identified to be a  $\mathbb{F}$ -predictable process. Consequently, the set  $\mathcal{H}$  of admissible strategies coincides with the set of admissible strategies in the sense of Definition 2.3. Given a strategy  $H \in \bar{\mathcal{H}}$ , the resulting wealth process is given by  $(H \circ X)_t := \sum_{s=1}^t H_s \cdot (X_s - X_{s-1})$ ,  $t \leq T$ .

Finally, let us introduce some sets of probability measures on the enlarged space  $(\bar{\Omega}, \bar{\mathcal{F}})$ . Let

$$\bar{\mathcal{P}} := \{\bar{\mathbb{P}} \in \mathfrak{B}(\bar{\Omega}, \bar{\mathcal{F}}) \text{ such that } \bar{\mathbb{P}}|_{\Omega} \in \mathcal{P}\}.$$

We next introduce a subset  $\overline{\mathcal{P}}_{\text{int}} \subset \overline{\mathcal{P}}$  as follows. Recall that  $\overline{\Omega}$  has a product structure as  $\Omega$ . Concretely, let  $\overline{\Omega}_0 := \Omega_0 \times \Lambda_1$  and  $\overline{\Omega}_t := \overline{\Omega}_0 \times (\Omega_1 \times \Lambda_1)^t$ ,  $\overline{\Omega} := \Omega \times \Lambda = \overline{\Omega}_T$ .

- For  $t = 0, 1, \dots, T-1$  and  $\bar{\omega} = (\omega, \theta) \in \overline{\Omega}_t$ , we define  $\overline{\mathcal{P}}(t, \bar{\omega}) := \{\overline{\mathbb{P}} \in \mathfrak{B}(\Omega_1 \times \Lambda_1) : \overline{\mathbb{P}}|_{\Omega_1} \in \mathcal{P}_t(\omega)\}$ , and

$$\overline{\mathcal{P}}_{\text{int}}(t, \bar{\omega}) := \{\overline{\mathbb{P}} \in \overline{\mathcal{P}}(t, \bar{\omega}) : \delta_{\bar{\omega}} \otimes \overline{\mathbb{P}}[X_{t+1} \in \text{int}K_{t+1}^*] = 1\}, \quad (2.7)$$

where  $\delta_{\bar{\omega}} \otimes \overline{\mathbb{P}}$  is a probability measure on  $\overline{\Omega}_{t+1} = \overline{\Omega}_t \times (\Omega_1 \times \Lambda_1)$  and  $X_{t+1}$  (defined in (2.6)) is considered as a random variable defined on  $\overline{\Omega}_{t+1}$ .

- Let  $\overline{\mathcal{P}}_{\text{int}, \emptyset}$  be the collection of all probability measures  $\overline{\mathbb{P}}$  on  $\overline{\Omega}_0$  such that  $\overline{\mathbb{P}}[X_0 \in \text{int}K_0^*] = 1$ . We define

$$\overline{\mathcal{P}}_{\text{int}} := \{\overline{\mathbb{P}}_{\emptyset} \otimes \overline{\mathbb{P}}_0 \otimes \dots \otimes \overline{\mathbb{P}}_{T-1} : \overline{\mathbb{P}}_{\emptyset} \in \overline{\mathcal{P}}_{\text{int}, \emptyset} \text{ and } \overline{\mathbb{P}}_t(\cdot) \in \overline{\mathcal{P}}_{\text{int}}(t, \cdot) \text{ for } t \leq T-1\},$$

where  $\overline{\mathbb{P}}_t(\cdot)$  is a universally measurable selector of  $\overline{\mathcal{P}}_{\text{int}}(t, \cdot)$ .

**Remark 2.4.** Assume that the analyticity condition (2.2) for  $\text{graph}(\mathcal{P}_t)$  holds, Lemma 2.13 of [12] asserts that

$$[[\overline{\mathcal{P}}_{\text{int}}(t)]] := \{(\bar{\omega}, \overline{\mathbb{P}}) : \bar{\omega} \in \overline{\Omega}_t, \overline{\mathbb{P}} \in \overline{\mathcal{P}}_{\text{int}}(t, \bar{\omega})\} \text{ is also analytic,}$$

which in particular ensures that  $\overline{\mathcal{P}}_{\text{int}}$  is nonempty.

**Reformulation on the enlarged space** We now reformulate the utility maximization (2.5) on the enlarged space  $\overline{\Omega}$ , with underlying stock  $X$ . Let us set

$$g(\bar{\omega}) := \xi(\omega) \cdot X_T(\bar{\omega}), \text{ for all } \bar{\omega} = (\omega, \theta) \in \overline{\Omega},$$

as the contingent claim.

**Proposition 2.5.** Suppose that Assumptions 2.1 and 2.2 hold, then

$$V(\xi) = \sup_{H \in \mathcal{H}} \inf_{\overline{\mathbb{P}} \in \overline{\mathcal{P}}} \mathbb{E}^{\overline{\mathbb{P}}} \left[ U \left( g + (H \circ X)_T \right) \right] = \sup_{H \in \mathcal{H}} \inf_{\overline{\mathbb{P}} \in \overline{\mathcal{P}}_{\text{int}}} \mathbb{E}^{\overline{\mathbb{P}}} \left[ U \left( g + (H \circ X)_T \right) \right].$$

*Proof.* To simplify the notation, we write  $\Delta X_t := X_t - X_{t-1}$ . We shall follow closely the arguments in Proposition 3.3 of [12].

*Step 1:* Fix  $\eta \in \mathcal{A}_0$  and define the  $\tilde{\mathbb{F}}$ -predictable process  $H$  by  $H_t := \sum_{s=1}^t \Delta H_s$  with  $\Delta H_t := \eta_{t-1}$  for  $t = 1, \dots, T$ . By rearranging all terms, we have

$$\left( \xi + \sum_{t=0}^T \eta_t \right)^d = \left( \xi + \sum_{t=0}^T \eta_t \right) \cdot X_T = \sum_{t=1}^T H_t \cdot \Delta X_t + \sum_{t=0}^T \eta_t \cdot X_t + g \leq \sum_{t=1}^T H_t \cdot \Delta X_t + g,$$

where the last inequality follows by the fact that  $\eta_t \in -K_t$  and hence  $\eta_t \cdot X_t \leq 0$ . As  $U$  is non-decreasing, it follows that

$$\inf_{\overline{\mathbb{P}} \in \overline{\mathcal{P}}} \mathbb{E}^{\overline{\mathbb{P}}} \left[ U \left( \left( \xi + \sum_{t=0}^T \eta_t \right)^d \right) \right] \leq \inf_{\overline{\mathbb{P}} \in \overline{\mathcal{P}}} \mathbb{E}^{\overline{\mathbb{P}}} \left[ U \left( g + (H \circ X)_T \right) \right],$$

which yields that

$$V(\xi) \leq \sup_{H \in \mathcal{H}} \inf_{\overline{\mathbb{P}} \in \overline{\mathcal{P}}} \mathbb{E}^{\overline{\mathbb{P}}} \left[ U \left( g + (H \circ X)_T \right) \right].$$

The same arguments imply that the same inequality holds if one replaces  $\overline{\mathcal{P}}$  by  $\overline{\mathcal{P}}_{\text{int}}$ .

*Step 2:* To prove the reverse inequality, we fix  $H \in \mathcal{H}$ . Define  $\eta = (\eta_t)_{0 \leq t \leq T}$  by  $\eta_t^i := \Delta H_{t+1}^i$ ,  $t \leq T-1$  and  $\eta_T^i := -\xi^i - \sum_{s=0}^{T-1} \eta_s^i$  for  $i \leq d-1$ , and

$$\eta_t^d(\omega) := \inf_{\theta \in \Lambda} m_t^d(\omega, \theta) \text{ with } m_t^d(\bar{\omega}) := - \sum_{i=1}^{d-1} \eta_t^i(\omega) X_t^i(\bar{\omega}), \quad t \leq T. \quad (2.8)$$

for all  $\bar{\omega} = (\omega, \theta) \in \overline{\Omega}$ . Notice that  $m_t^d(\omega, \theta)$  is bounded and continuous in  $\theta$ , then  $\eta_t^d$  is  $\mathcal{F}_t$ -measurable, by the Measurable Maximum Theorem (see e.g. Theorem 18.19 of [2]). From the construction, we know  $\eta \in \mathcal{A}_0$ . Furthermore, using the fact that each  $X_t$  depends on  $\theta$  only through  $\theta_t$  and the definition of  $\eta^d$ , we have

$$\begin{aligned} \inf_{\theta \in \Lambda} ((H \circ X)_T + g)(\cdot, \theta) &= \inf_{\theta \in \Lambda} \left( \left( \sum_{t=0}^T \eta_t + \xi \right) \cdot X_T - \sum_{t=0}^{T-1} \eta_t \cdot X_t \right)(\cdot, \theta) \\ &= \inf_{\theta \in \Lambda} \left\{ \left( \sum_{t=0}^T \eta_t + \xi \right) \cdot X_T \right\}(\cdot, \theta) - \sum_{t=0}^{T-1} \sup_{\theta \in \Lambda} \{ \eta_t \cdot X_t \}(\cdot, \theta). \\ &= \inf_{\theta \in \Lambda} \left\{ \left( \sum_{t=0}^T \eta_t + \xi \right) \cdot X_T \right\}(\cdot, \theta) = \left( \xi + \sum_{t=0}^T \eta_t \right)^d. \end{aligned} \quad (2.9)$$

Take a countable dense subset  $\Lambda^0$  of  $\Lambda$ , as  $\mathbb{P} \times \delta_\theta \in \overline{\mathcal{P}}$  for all  $\mathbb{P} \in \mathcal{P}$ , it follows that

$$\begin{aligned} \inf_{\mathbb{P} \in \overline{\mathcal{P}}} \mathbb{E}^{\mathbb{P}} \left[ U(g + (H \circ X)_T) \right] &\leq \inf_{\mathbb{P} \in \mathcal{P}} \mathbb{E}^{\mathbb{P}} \left[ \inf_{\theta \in \Lambda^0} U \left( (H \circ X)_T(\cdot, \theta) + g(\cdot, \theta) \right) \right] \\ &= \inf_{\mathbb{P} \in \mathcal{P}} \mathbb{E}^{\mathbb{P}} \left[ U \left( \inf_{\theta \in \Lambda^0} [(H \circ X)_T(\cdot, \theta) + g(\cdot, \theta)] \right) \right] \\ &= \inf_{\mathbb{P} \in \mathcal{P}} \mathbb{E}^{\mathbb{P}} \left[ U \left( \left( \xi + \sum_{t=0}^T \eta_t \right)^d \right) \right]. \end{aligned} \quad (2.10)$$

This leads to

$$\sup_{H \in \mathcal{H}} \inf_{\mathbb{P} \in \overline{\mathcal{P}}} \mathbb{E}^{\mathbb{P}} \left[ U(g + (H \circ X)_T) \right] \leq V(\xi),$$

and hence we have the equality.

*Step 3:* For the case with  $\overline{\mathcal{P}}_{\text{int}}$  in place of  $\overline{\mathcal{P}}$ , it is enough to notice as in *step 2* that

$$\inf_{\theta \in \Lambda_{\text{int}}^0(\cdot)} [(H \circ X)_T(\cdot, \theta) + g(\cdot, \theta)] = \left( \xi + \sum_{t=0}^T \eta_t \right)^d,$$

for some countable dense subsets  $\Lambda_{\text{int}}^0(\cdot)$  of  $\Lambda_{\text{int}}(\cdot)$ , where  $\Lambda_{\text{int}}(\omega)$  is defined as the collection of  $\theta \in \Lambda$  such that  $S_t(\omega)\theta_t \in \text{int}K_t^*(\omega)$ .

Next, for each  $\theta \in \Lambda$ , we associate the probability kernels

$$q_s^\theta : \omega \in \Omega \mapsto q_s^\theta(\cdot | \omega) := \delta_{\theta_s} \mathbf{1}_{A_s^\theta(\omega)} + \delta_{\mathbf{1}} \mathbf{1}_{(A_s^\theta(\omega))^c} \in \mathfrak{B}(\Lambda_1), \quad s \leq T, \quad (2.11)$$

where  $\mathbf{1}$  is the vector of  $\mathbb{R}^d$  with all entries equal to 1,  $A_s^\theta(\omega) := \emptyset$  for  $s \neq t$  and  $A_t^\theta(\omega) := \{S_t(\omega)\theta_t \in \text{int}K_t^*(\omega)\}$ . It follows that  $\mathbb{P} \otimes (q_0^\theta \otimes q_1^\theta \otimes \cdots \otimes q_T^\theta) \in \overline{\mathcal{P}}_{\text{int}}$  for every  $\mathbb{P} \in \mathcal{P}$ . Then it suffices to argue as in *Step 2* above to obtain that

$$\begin{aligned} \inf_{\mathbb{P} \in \overline{\mathcal{P}}_{\text{int}}} \mathbb{E}^{\mathbb{P}} \left[ U(g + (H \circ X)_T) \right] &\leq \inf_{\mathbb{P} \in \mathcal{P}} \mathbb{E}^{\mathbb{P}} \left[ U \left( \inf_{\theta \in \Lambda_{\text{int}}^0(\cdot)} \left( (H \circ X)_T(\cdot, \theta) + g(\cdot, \theta) \right) \right) \right] \\ &= \inf_{\mathbb{P} \in \mathcal{P}} \mathbb{E}^{\mathbb{P}} \left[ U \left( \left( \xi + \sum_{t=0}^T \eta_t \right)^d \right) \right], \end{aligned}$$



and we hence conclude as in *Step 2*.  $\square$

### 2.3 The robust no-arbitrage condition of Bouchard and Nutz

To conclude, we will discuss the no-arbitrage condition on  $\Omega$  and its link to that on the enlarged space  $\overline{\Omega}$ .

**Definition 2.6.** (i) We say the robust no-arbitrage condition of second kind  $\text{NA2}(\mathcal{P})$  on  $\Omega$  holds true if for all  $t \leq T - 1$  and all  $\xi \in L^0(\mathcal{F}_t)$ ,

$$\xi \in K_{t+1} \text{ } \mathcal{P}\text{-q.s.} \implies \xi \in K_t \text{ } \mathcal{P}\text{-q.s.}$$

(ii) Let  $(\mathbb{Q}, Z)$  be a couple where  $\mathbb{Q} \in \mathfrak{B}(\Omega)$  and  $Z = (Z_t)_{t=0, \dots, T}$  an adapted process,  $(\mathbb{Q}, Z)$  is called a strict consistent price system (SCPS) if  $\mathbb{Q} \lll \mathcal{P}$ ,  $Z_t \in \text{int}K_t^*$   $\mathbb{Q}$ -a.s. for all  $t = 0, \dots, T$  and  $Z$  is a  $\mathbb{Q}$ -martingale.

We denote by  $\mathcal{S}$  the collection of all SCPS, and also denote the subset

$$\mathcal{S}_0 := \{(\mathbb{Q}, Z) \in \mathcal{S} \text{ such that } Z^d \equiv 1\}. \quad (2.12)$$

**Remark 2.7.** As stated in the fundamental theorem of asset pricing proved in [14] (see also [5, 6]), the no-arbitrage condition  $\text{NA2}(\mathcal{P})$  is equivalent to: for all  $t \leq T - 1$ ,  $\mathbb{P} \in \mathcal{P}$  and  $\mathcal{F}_t$ -random variable  $Y$  taking value in  $\text{int}K_t^*$ , there exists a SCPS  $(\mathbb{Q}, Z)$  such that  $\mathbb{P} \ll \mathbb{Q}$ ,  $\mathbb{P} = \mathbb{Q}$  on  $\mathcal{F}_t$  and  $Y = Z_t$   $\mathbb{P}$ -a.s..

On the enlarged space  $\overline{\Omega}$ , we also follow [13] to introduce a notion of the robust no-arbitrage condition.

**Definition 2.8.** We say that the robust no-arbitrage condition  $\text{NA}(\overline{\mathcal{P}}_{\text{int}})$  on  $\overline{\Omega}$  holds true if, for every  $H \in \overline{\mathcal{H}}$ ,

$$(H \circ X)_T \geq 0, \overline{\mathcal{P}}_{\text{int}}\text{-q.s.} \implies (H \circ X)_T = 0, \overline{\mathcal{P}}_{\text{int}}\text{-q.s.}$$

**Remark 2.9.** The fundamental theorem of asset pricing in [13] proves that the condition  $\text{NA}(\overline{\mathcal{P}}_{\text{int}})$  (resp.  $\text{NA}(\overline{\mathcal{P}})$ ) is equivalent to : for all  $\overline{\mathbb{P}} \in \overline{\mathcal{P}}_{\text{int}}$  (resp.  $\overline{\mathcal{P}}$ ), there exists  $\overline{\mathbb{Q}} \in \mathfrak{B}(\overline{\Omega})$  such that  $\overline{\mathbb{P}} \ll \overline{\mathbb{Q}} \lll \overline{\mathcal{P}}_{\text{int}}$  (resp.  $\overline{\mathcal{P}}$ ) and  $X$  is a  $(\overline{\mathbb{Q}}, \overline{\mathbb{F}})$ -martingale.

Hereafter, we denote by  $\overline{\mathcal{Q}}_0$  the collection of measures  $\overline{\mathbb{Q}} \in \mathfrak{B}(\overline{\Omega})$  such that  $\overline{\mathbb{Q}} \lll \overline{\mathcal{P}}_{\text{int}}$  and  $X$  is a  $(\overline{\mathbb{F}}, \overline{\mathbb{Q}})$ -martingale. The above two no-arbitrage conditions on  $\Omega$  and on  $\overline{\Omega}$  are related by Proposition 2.16 of [12], that we recall as below.

**Proposition 2.10.** The condition  $\text{NA2}(\mathcal{P})$  on  $\Omega$  is equivalent to the condition  $\text{NA}(\overline{\mathcal{P}}_{\text{int}})$  on  $\overline{\Omega}$ .

## 3 Exponential utility maximization

In this section, we will restrict ourselves to the case of the exponential utility function, i.e.,

$$U(x) := -\exp(-\gamma x), \text{ for some constant } \gamma > 0,$$

and provide a detailed study on the corresponding utility maximization problem.

We will in fact study a more general exponential utility maximization problem, where some liquid options are available to construct static strategies. For  $e \in \mathbb{N} \cup \{0\}$ , there are a finite class of  $\mathcal{F}_T^0$ -measurable random vectors  $\zeta_i : \Omega \rightarrow \mathbb{R}^d$ ,  $i = 1, \dots, e$ , where each  $\zeta_i$  represents the

payoff of some option  $i$  labeled in units of different risky assets. Let  $\xi : \Omega \rightarrow \mathbb{R}^d$  represent the payoff of the random endowment, then our maximization problem is given by.

$$V(\xi, \gamma) := \sup_{(\ell, \eta) \in \mathcal{A}_e} \inf_{\mathbb{P} \in \mathcal{P}} \mathbb{E}^{\mathbb{P}} \left[ U \left( \left( \xi + \sum_{i=1}^e (\ell_i \zeta_i - |\ell_i| c_i 1_d) + \sum_{t=0}^T \eta_t \right)^d \right) \right]. \quad (3.13)$$

where  $1_d$  is the vector with all components equal to 0 but the last one that is equal to 1 and  $\mathcal{A}_e$  denotes the collection of all  $(\ell, \eta) \in \mathbb{R}^e \times \mathcal{A}$  such that  $\xi + \sum_{i=1}^e (\ell_i \zeta_i - |\ell_i| c_i 1_d) + \sum_{t=0}^T \eta_t = 0$  for  $i = 1, \dots, d-1$ . In above, we write  $\gamma$  in  $V(\xi, \gamma)$  to emphasize the dependence of value in parameter  $\gamma$  in the utility function  $U$ . Also, each static option  $\zeta_i$  has price 0, but the static trading induces the proportional transaction cost with rate  $c_i > 0$ .

### 3.1 The convex duality result

In the robust setting without transaction costs, the exponential utility maximization problem has been studied by Bartl [3], where essentially a convex duality result has been established. Here, we apply and generalize their results in our context to obtain a convex duality theorem.

Let us introduce a robust version of the relative entropy associated to a probability measure  $\mathbb{Q}$  as

$$\mathcal{E}(\mathbb{Q}, \mathcal{P}) := \inf_{\mathbb{P} \in \mathcal{P}} \mathcal{E}(\mathbb{Q}, \mathbb{P}), \quad \text{where } \mathcal{E}(\mathbb{Q}, \mathbb{P}) := \begin{cases} \mathbb{E}^{\mathbb{P}} \left[ \frac{d\mathbb{Q}}{d\mathbb{P}} \log \frac{d\mathbb{Q}}{d\mathbb{P}} \right], & \text{if } \mathbb{Q} \ll \mathbb{P}, \\ +\infty, & \text{otherwise.} \end{cases} \quad (3.14)$$

Notice that  $\mathcal{S}_0$  is a subset of the collection of SCPS  $(\mathbb{Q}, Z)$  defined in (2.12), we then define

$$\mathcal{S}_e^* := \left\{ (\mathbb{Q}, Z) \in \mathcal{S}_0 : \mathbb{E}^{\mathbb{Q}}[(\xi \cdot Z_T)_-] + \mathcal{E}(\mathbb{Q}, \mathcal{P}) < +\infty \text{ and } \mathbb{E}^{\mathbb{Q}}[(\zeta_i \cdot Z_T)] \in [-c_i, c_i], i = 1, \dots, e \right\}.$$

**Theorem 3.1.** *Let  $\xi$  and  $(\zeta_i)_{i \leq e} : \Omega \rightarrow \mathbb{R}^d$  be Borel measurable and assume that NA2( $\mathcal{P}$ ) holds. Assume either that  $e = 0$ , or that  $e \geq 1$  and for all  $\ell \in \mathbb{R}^e$  and  $\eta \in \mathcal{A}$ ,*

$$\sum_{i=1}^e (\ell_i \zeta_i - |\ell_i| c_i 1_d) + \sum_{t=0}^T \eta_t \in K_T \text{ } \mathcal{P}\text{-q.s.} \implies \ell = 0. \quad (3.15)$$

Then, we have

$$V(\xi, \gamma) = -\exp \left( - \inf_{(\mathbb{Q}, Z) \in \mathcal{S}_e^*} \left\{ \mathbb{E}^{\mathbb{Q}}[\gamma \xi \cdot Z_T] + \mathcal{E}(\mathbb{Q}, \mathcal{P}) \right\} \right), \quad (3.16)$$

Moreover, the infimum over  $(\ell, \eta) \in \mathcal{A}_e$  is attained by an optimal strategy  $(\hat{\ell}, \hat{\eta})$ .

**Remark 3.2.** Notice that up to taking logarithm on both sides and replacing  $\gamma \xi$  by  $-\xi$ , the equality (3.16) is equivalent to

$$\begin{aligned} & \inf_{(\ell, \eta) \in \mathcal{A}_e} \sup_{\mathbb{P} \in \mathcal{P}} \log \mathbb{E}^{\mathbb{P}} \left[ \exp \left( \left( \xi - \sum_{i=1}^e (\ell_i \zeta_i - |\ell_i| c_i 1_d) - \sum_{t=0}^T \eta_t \right)^d \right) \right] \\ &= \sup_{(\mathbb{Q}, Z) \in \mathcal{S}_e^*} \left\{ \mathbb{E}^{\mathbb{Q}}[\xi \cdot Z_T] - \mathcal{E}(\mathbb{Q}, \mathcal{P}) \right\}. \end{aligned} \quad (3.17)$$

### 3.2 Properties of utility indifference prices

It is well known that the superhedging price is too high in practice. As an alternative way, the utility-based indifference price has been a highly active area of research, in which the investor's risk aversion is inherently incorporated. This section presents an application of the convex duality relationship (3.17) for the exponential utility maximization and provides some interesting features of indifference prices in the presence of both proportional transaction costs and model uncertainty. Generally speaking, the indifference pricing in our setting can be generated by semi-static trading strategies on risky assets and liquid options.

In the robust framework, similar to Theorem 2.4 of [3] in the frictionless model, the duality representation (3.17) can help us to derive that the asymptotic indifference prices converge to the superhedging price as the risk aversion  $\gamma \rightarrow \infty$  regardless of the transaction costs. To see this, let us first recall the superhedging price defined by

$$\begin{aligned}\pi(\xi) &:= \inf \left\{ y + \sum_{i=1}^e c_i |\ell_i| : y1_d + \sum_{i=1}^e \ell_i \zeta_i + \sum_{t=0}^T \eta_t - \xi \in K_T, \mathcal{P} - q.s., (\ell, \eta) \in \mathcal{A}_e \right\} \\ &= \sup_{(\mathbb{Q}, Z) \in \mathcal{S}_e} \mathbb{E}^{\mathbb{Q}}[\xi \cdot Z_T],\end{aligned}$$

where the finite entropy constraint is not enforced in the set

$$\mathcal{S}_e := \left\{ (\mathbb{Q}, Z) \in \mathcal{S}_0 : \mathbb{E}^{\mathbb{Q}}[(\zeta_i \cdot Z_T)] \in [-c_i, c_i], i = 1, \dots, e \right\},$$

and the equality follows from Theorem 3.1 of [12].

The indifference price, on the other hand, is denoted by  $\pi_\gamma(\xi)$  which satisfies the equation that

$$V(0, \gamma) = V_{\pi_\gamma(\xi)}(\xi; \gamma)$$

where  $V_x(B; \gamma)$  is defined in (3.13) with the initial wealth  $x \in \mathbb{R}$ , the option payoff  $B$  and the risk aversion coefficient  $\gamma > 0$ . Equivalently, we can rewrite it as

$$\begin{aligned}\pi_\gamma(\xi) &= \inf \left\{ p \in \mathbb{R} : \sup_{\mathbb{P} \in \mathcal{P}} \frac{1}{\gamma} \log \mathbb{E}^{\mathbb{P}} \left[ \exp \left( \gamma \left( \xi - p1_d - \sum_{i=1}^e (\ell_i \zeta_i - |\ell_i| c_i 1_d) - \sum_{t=0}^T \eta_t \right)^d \right) \right] \right. \\ &\quad \left. \leq \sup_{\mathbb{P} \in \mathcal{P}} \frac{1}{\gamma} \log \mathbb{E}^{\mathbb{P}} \left[ \exp \left( \gamma \left( - \sum_{i=1}^e (\ell_i \zeta_i - |\ell_i| c_i 1_d) - \sum_{t=0}^T \eta_t \right)^d \right) \right], (\ell, \eta) \in \mathcal{A}_e \right\} \\ &= \sup_{(\mathbb{Q}, Z) \in \mathcal{S}_e^*} \left\{ \mathbb{E}^{\mathbb{Q}}[\xi \cdot Z_T] - \frac{1}{\gamma} \mathcal{E}(\mathbb{Q}, \mathcal{P}) \right\} - \sup_{(\mathbb{Q}, Z) \in \mathcal{S}_e^*} \left\{ -\frac{1}{\gamma} \mathcal{E}(\mathbb{Q}, \mathcal{P}) \right\},\end{aligned}\tag{3.18}$$

where the first equality follows by the definition of indifference price and the second equality is a direct consequence of the duality representation (3.17).

The formula (3.18) yields directly the next few properties of the utility indifference price.

**Lemma 3.3.** *The following basic properties hold:*

- (i)  $\pi_\gamma(\xi)$  dose not depend on the initial wealth  $x_0$ .
- (ii)  $\pi_\gamma(\xi)$  is increasing in  $\gamma$  (monotonicity in  $\gamma$ ).
- (iii)  $\pi_\gamma(\beta \xi) = \beta \pi_{\beta \gamma}(\xi)$  for any  $\beta \in (0, 1]$  (volume scaling).
- (iv)  $\pi_\gamma(\xi + c) = c + \pi_\gamma(\xi)$  for  $c \in \mathbb{R}$  (translation invariance).
- (v)  $\pi_\gamma(\alpha \xi_1 + (1 - \alpha) \xi_2) \leq \alpha \pi_\gamma(\xi_1) + (1 - \alpha) \pi_\gamma(\xi_2)$  (convexity).

(vi)  $\pi_\gamma(\xi_1) \leq \pi_\gamma(\xi_2)$  if  $\xi_1 \leq \xi_2$  (monotonicity).

The next result shows the risk-averse asymptotics on the utility indifference prices. Similar results can also be found in [18, 11, 3].

**Proposition 3.4.** *In the robust setting of Theorem 3.1 with proportional transaction costs, we have*

$$\pi(\xi) = \lim_{\gamma \rightarrow \infty} \pi_\gamma(\xi). \quad (3.19)$$

Because the proof of the above result needs the assistance of the proof of the main duality result and some new notations can not be avoided, we postpone it in Section 4.4.

**Remark 3.5.** *Observing the scaling property in item (iii) of Lemma 3.3, the limit (3.19) can be rewritten as  $\lim_{\beta \rightarrow \infty} \frac{1}{\beta} \pi_\gamma(\beta \xi) = \pi(\xi)$ , in which the term  $\frac{1}{\beta} \pi_\gamma(\beta \xi)$  can be understood as the price per unit for a given amount volume  $\beta$  of the contingent claim  $\xi$ .*

Furthermore, with the increasing risk aversion, the convex duality result (3.17) also yields that the optimal hedging strategies under the exponential utility preference converge to the superhedging counterpart in the following sense.

**Proposition 3.6.** *We have that*

$$\lim_{\gamma \rightarrow \infty} \sup_{\mathbb{P} \in \mathcal{P}} \mathbb{E}^{\mathbb{P}} \left[ \left( \pi(\xi) 1_d + \sum_{i=1}^e (\ell_i^{*,\gamma} \zeta_i - |\ell_i^{*,\gamma}| c_i 1_d) + \sum_{t=0}^T \eta_t^{*,\gamma} - \xi \right)^- \right] = 0,$$

where  $(\ell^{*,\gamma}, \eta^{*,\gamma})$  is an optimal semi-static strategy to the problem (3.17) under the risk aversion level  $\gamma$ .

*Proof.* Let us set  $\Gamma_\gamma := \pi(\xi) 1_d + \sum_{i=1}^e (\ell_i^{*,\gamma} \zeta_i - |\ell_i^{*,\gamma}| c_i 1_d) + \sum_{t=0}^T \eta_t^{*,\gamma} - \xi$  and it follows by (3.17) that

$$\sup_{\mathbb{P} \in \mathcal{P}} \log \mathbb{E}^{\mathbb{P}} [e^{-\gamma \Gamma_\gamma}] = \sup_{(\mathbb{Q}, Z) \in \mathcal{S}_e^*} \{ \gamma \mathbb{E}^{\mathbb{Q}} [\xi \cdot Z_T] - \gamma \pi(\xi) - \mathcal{E}(\mathbb{Q}, \mathcal{P}) \}. \quad (3.20)$$

If  $\pi(\xi) = +\infty$ , it is clear that  $\sup_{\mathbb{P} \in \mathcal{P}} \log \mathbb{E}^{\mathbb{P}} [e^{-\gamma \Gamma_\gamma}] = -\infty$ . Otherwise, if  $\pi(\xi) < +\infty$ , it follows by item (ii) of Lemma 3.3 that  $\pi_\gamma(\xi)$  is increasing in  $\gamma$  and moreover  $\pi_\gamma(\xi) \leq \pi(\xi)$ . Therefore, it yields that  $\sup_{\mathbb{P} \in \mathcal{P}} \log \mathbb{E}^{\mathbb{P}} [e^{-\gamma \Gamma_\gamma}] \leq 0$  and hence  $\mathbb{E}^{\mathbb{P}} [e^{-\gamma \Gamma_\gamma}] \leq 1$  uniformly for all  $\mathbb{P} \in \mathcal{P}$ . By Jensen's inequality, we have

$$\sup_{\mathbb{P} \in \mathcal{P}} \mathbb{E}^{\mathbb{P}} [\Gamma_\gamma^-] \leq \frac{1}{\gamma} \sup_{\mathbb{P} \in \mathcal{P}} \log \mathbb{E}^{\mathbb{P}} [e^{\gamma \Gamma_\gamma^-}] \leq \frac{1}{\gamma} \sup_{\mathbb{P} \in \mathcal{P}} \log(1 + \mathbb{E}^{\mathbb{P}} [e^{-\gamma \Gamma_\gamma}]),$$

which completes the proof.  $\square$

**Remark 3.7.** *Similar results have been obtained in Corollary 5.1 and Theorem 5.2 of [20] in the classical dominated frictionless market model. Thanks to the convex duality (3.17), this paper makes nontrivial extension of the asymptotic convergence on risk aversion level to the setting with both proportional transaction costs and model uncertainty.*

Again, based on the convex duality representation obtained in the enlarged space, the continuity property and Fatou property of the indifference prices can be shown in the following sense.

**Proposition 3.8.** (i) If  $(\xi_n)_{n \in \mathbb{N}}$  is a sequence of option payoffs such that

$$\sup_{(\mathbb{Q}, Z) \in \mathcal{S}_e^*} \mathbb{E}^\mathbb{Q}[(\xi_n - \xi) \cdot Z_T] \rightarrow 0 \quad \text{and} \quad \inf_{(\mathbb{Q}, Z) \in \mathcal{S}_e^*} \mathbb{E}^\mathbb{Q}[(\xi_n - \xi) \cdot Z_T] \rightarrow 0. \quad (3.21)$$

then  $\pi_\gamma(\xi_n) \rightarrow \pi_\gamma(\xi)$  for any  $\gamma > 0$ .

(ii) For  $\xi_n \geq 0$ , we have

$$\pi_\gamma(\liminf_n \xi_n) \leq \liminf_n \pi_\gamma(\xi_n). \quad (3.22)$$

(iii) If  $(\xi_n)_{n \in \mathbb{N}}$  is a sequence of option payoffs such that  $\xi_n \nearrow \xi$ ,  $\mathbb{P}$ -a.s., then  $\pi_\gamma(\xi_n) \nearrow \pi_\gamma(\xi)$ .

*Proof.* (i) Recall that  $\pi_\gamma(\xi) = \sup_{(\mathbb{Q}, Z) \in \mathcal{S}_e^*} \left\{ \mathbb{E}^\mathbb{Q}[\xi \cdot Z_T] - \frac{1}{\gamma} \mathcal{E}(\mathbb{Q}, \mathcal{P}) \right\} - \sup_{(\mathbb{Q}, Z) \in \mathcal{S}_e^*} \left\{ -\frac{1}{\gamma} \mathcal{E}(\mathbb{Q}, \mathcal{P}) \right\}$  in (3.18), we can obtain that

$$\begin{aligned} |\pi_\gamma(\xi_n) - \pi_\gamma(\xi)| &= \left| \sup_{(\mathbb{Q}, Z) \in \mathcal{S}_e^*} \left\{ \mathbb{E}^\mathbb{Q}[\xi_n \cdot Z_T] - \frac{1}{\gamma} \mathcal{E}(\mathbb{Q}, \mathcal{P}) \right\} - \sup_{(\mathbb{Q}, Z) \in \mathcal{S}_e^*} \left\{ \mathbb{E}^\mathbb{Q}[\xi \cdot Z_T] - \frac{1}{\gamma} \mathcal{E}(\mathbb{Q}, \mathcal{P}) \right\} \right| \\ &\leq \sup_{(\mathbb{Q}, Z) \in \mathcal{S}_e^*} |\mathbb{E}^\mathbb{Q}[(\xi_n - \xi) \cdot Z_T]|. \end{aligned}$$

The continuity  $\pi_\gamma(\xi_n) \rightarrow \pi_\gamma(\xi)$  follows directly by (3.21).

(ii) The Fatou property can be derived by observing that

$$\begin{aligned} \pi_\gamma(\liminf_n \xi_n) &= \sup_{(\mathbb{Q}, Z) \in \mathcal{S}_e^*} \left\{ \mathbb{E}^\mathbb{Q}[\liminf_n \xi_n \cdot Z_T] - \frac{1}{\gamma} \mathcal{E}(\mathbb{Q}, \mathcal{P}) \right\} - \sup_{(\mathbb{Q}, Z) \in \mathcal{S}_e^*} \left\{ -\frac{1}{\gamma} \mathcal{E}(\mathbb{Q}, \mathcal{P}) \right\} \\ &\leq \sup_{(\mathbb{Q}, Z) \in \mathcal{S}_e^*} \left\{ \liminf_n \mathbb{E}^\mathbb{Q}[\xi_n \cdot Z_T] - \frac{1}{\gamma} \mathcal{E}(\mathbb{Q}, \mathcal{P}) \right\} - \sup_{(\mathbb{Q}, Z) \in \mathcal{S}_e^*} \left\{ -\frac{1}{\gamma} \mathcal{E}(\mathbb{Q}, \mathcal{P}) \right\} \\ &\leq \liminf_n \left( \sup_{(\mathbb{Q}, Z) \in \mathcal{S}_e^*} \left\{ \mathbb{E}^\mathbb{Q}[\xi_n \cdot Z_T] - \frac{1}{\gamma} \mathcal{E}(\mathbb{Q}, \mathcal{P}) \right\} - \sup_{(\mathbb{Q}, Z) \in \mathcal{S}_e^*} \left\{ -\frac{1}{\gamma} \mathcal{E}(\mathbb{Q}, \mathcal{P}) \right\} \right) \\ &= \liminf_n \pi_\gamma(\xi_n). \end{aligned}$$

(iii) By the Fatou property from part (ii) and item (vi) of Lemma 3.3, we have

$$\pi_\gamma(\xi) \geq \liminf_n \pi_\gamma(\xi_n) \geq \pi_\gamma(\xi),$$

which completes the proof.  $\square$

## 4 Proof of main results

This section provides the technical arguments that how we can establish the convex duality (3.17) in the model by first working in the fictitious frictionless market model on an enlarged space. All three results, namely the convex duality theorem, the dynamic programming principle and the existence of the optimal portfolio will be confirmed. The way we translate the randomized transaction costs by the extra model uncertainty in both primal and dual problems plays the crucial role to develop some key equivalences.

#### 4.1 Reformulation of the dual problem

As a first step to reduce the complexity of the proof, the standard dual problem based on CPS in the model with transaction costs will be reformulated on the enlarged dual space. For a random variable  $\varphi : \bar{\Omega} \rightarrow \mathbb{R}_+$ , we define

$$\bar{\mathcal{Q}}^* := \{\bar{\mathbb{Q}} \in \bar{\mathcal{Q}}_0 : \mathbb{E}^{\bar{\mathbb{Q}}}[(\xi \cdot X_T)_-] + \mathcal{E}(\bar{\mathbb{Q}}, \bar{\mathcal{P}}_{\text{int}}) < \infty\} \text{ and } \bar{\mathcal{Q}}_\varphi^* := \{\bar{\mathbb{Q}} \in \bar{\mathcal{Q}}^* : \mathbb{E}^{\bar{\mathbb{Q}}}[\varphi] < \infty\},$$

where  $\mathcal{E}(\bar{\mathbb{Q}}, \bar{\mathcal{P}}_{\text{int}})$  is defined exactly as  $\mathcal{E}(\mathbb{Q}, \mathcal{P})$  in (3.14). We also define

$$\bar{\mathcal{Q}}_\varphi^*(0, \theta_0) := \{\bar{\mathbb{Q}} \in \bar{\mathcal{Q}}_\varphi^* : \bar{\mathbb{Q}}[\Theta_0 = \theta_0] = 1\}.$$

**Lemma 4.1.** *For any universally measurable random vector  $\xi : \Omega \rightarrow \mathbb{R}^d$ , one has*

$$\sup_{(\mathbb{Q}, Z) \in \mathcal{S}_0^*} \{\mathbb{E}^{\mathbb{Q}}[\xi \cdot Z_T] - \mathcal{E}(\mathbb{Q}, \mathcal{P})\} = \sup_{\bar{\mathbb{Q}} \in \bar{\mathcal{Q}}^*} \{\mathbb{E}^{\bar{\mathbb{Q}}}[\xi \cdot X_T] - \mathcal{E}(\bar{\mathbb{Q}}, \bar{\mathcal{P}}_{\text{int}})\}.$$

*Proof.* First, for a given  $(\mathbb{Q}, Z) \in \mathcal{S}_0^*$ , we associate the probability kernel:

$$q^Z : \omega \in \Omega \mapsto q^Z(\cdot | \omega) := \delta_{(Z/S)(\omega)} \in \mathfrak{B}(\Lambda),$$

and define  $\bar{\mathbb{Q}} := \mathbb{Q} \otimes q^Z$ . The construction implies that  $\mathbb{E}^{\mathbb{Q}}[\xi \cdot Z_T] = \mathbb{E}^{\bar{\mathbb{Q}}}[\xi \cdot X_T]$  and that  $\bar{\mathbb{Q}} \in \bar{\mathcal{Q}}^*$ . Moreover, given  $\mathbb{P} \in \mathcal{P}$ , one can similarly define  $\bar{\mathbb{P}} := \mathbb{P} \otimes q^Z \in \bar{\mathcal{P}}_{\text{int}}$  so that  $\bar{\mathbb{Q}} \ll \bar{\mathbb{P}}$  and  $d\mathbb{Q}/d\mathbb{P} = d\bar{\mathbb{Q}}/d\bar{\mathbb{P}}$ ,  $\bar{\mathbb{P}}$ -a.s. This implies that  $\mathcal{E}(\mathbb{Q}, \mathcal{P}) \geq \mathcal{E}(\bar{\mathbb{Q}}, \bar{\mathcal{P}}_{\text{int}})$ . Therefore,

$$\sup_{(\mathbb{Q}, Z) \in \mathcal{S}_0^*} \{\mathbb{E}^{\mathbb{Q}}[\xi \cdot Z_T] - \mathcal{E}(\mathbb{Q}, \mathcal{P})\} \leq \sup_{\bar{\mathbb{Q}} \in \bar{\mathcal{Q}}^*} \{\mathbb{E}^{\bar{\mathbb{Q}}}[\xi \cdot X_T] - \mathcal{E}(\bar{\mathbb{Q}}, \bar{\mathcal{P}}_{\text{int}})\}.$$

Conversely, let us fix  $\bar{\mathbb{Q}} \in \bar{\mathcal{Q}}^*$ , and define  $\mathbb{Q} := \bar{\mathbb{Q}}|_\Omega$  and  $Z_t := \mathbb{E}^{\bar{\mathbb{Q}}}[X_t | \mathcal{F}_t]$  for  $t \leq T$ . As  $\bar{\mathbb{Q}} \ll \bar{\mathbb{P}}$  for some  $\bar{\mathbb{P}} \in \bar{\mathcal{P}}_{\text{int}}$ , then  $\mathbb{Q} \ll \mathbb{P} := \bar{\mathbb{P}}|_\Omega \in \mathcal{P}$ . Moreover, the fact that  $X$  is a  $(\bar{\mathbb{P}}, \bar{\mathbb{Q}})$ -martingale implies that  $Z$  is  $(\mathbb{P}, \mathbb{Q})$ -martingale. Then,  $(\mathbb{Q}, Z) \in \mathcal{S}_0^*$  and  $\mathbb{E}^{\mathbb{Q}}[\xi \cdot Z_T] = \mathbb{E}^{\bar{\mathbb{Q}}}[\xi \cdot X_T]$ . Now as  $d\mathbb{Q}/d\mathbb{P} = \mathbb{E}^{\bar{\mathbb{P}}}[d\bar{\mathbb{Q}}/d\bar{\mathbb{P}} | \mathcal{F}_T]$  and  $x \mapsto x \log(x)$  is convex on  $\mathbb{R}_+$ , we have  $\mathcal{E}(\mathbb{Q}, \mathcal{P}) \leq \mathcal{E}(\bar{\mathbb{Q}}, \bar{\mathbb{P}})$  by Jensen's inequality. It follows that

$$\sup_{(\mathbb{Q}, Z) \in \mathcal{S}_0^*} \{\mathbb{E}^{\mathbb{Q}}[\xi \cdot Z_T] - \mathcal{E}(\mathbb{Q}, \mathcal{P})\} \geq \sup_{\bar{\mathbb{Q}} \in \bar{\mathcal{Q}}^*} \{\mathbb{E}^{\bar{\mathbb{Q}}}[\xi \cdot X_T] - \mathcal{E}(\bar{\mathbb{Q}}, \bar{\mathcal{P}}_{\text{int}})\},$$

and we hence conclude the proof.  $\square$

#### 4.2 Proof of Theorem 3.1 (Case $e = 0$ )

In view of Lemma 4.1 and Proposition 2.5, one can first establish the duality result of the utility maximization problem on the enlarged space  $\bar{\Omega}$ , in order to prove Theorem 3.1.

**Proposition 4.2.** *Let  $g := \xi \cdot X_T$  and  $\text{NA}(\bar{\mathcal{P}}_{\text{int}})$  hold true. Then for any universally measurable random variable  $\varphi : \bar{\Omega} \rightarrow \mathbb{R}_+$ , one has*

$$\begin{aligned} \bar{V} &:= \inf_{H \in \mathcal{H}} \sup_{\bar{\mathbb{P}} \in \bar{\mathcal{P}}_{\text{int}}} \log \mathbb{E}^{\bar{\mathbb{P}}}[\exp(g + (H \circ X)_T)] = \sup_{\bar{\mathbb{Q}} \in \bar{\mathcal{Q}}^*} \{\mathbb{E}^{\bar{\mathbb{Q}}}[g] - \mathcal{E}(\bar{\mathbb{Q}}, \bar{\mathcal{P}}_{\text{int}})\} \quad (4.23) \\ &= \sup_{\bar{\mathbb{Q}} \in \bar{\mathcal{Q}}_\varphi^*} \{\mathbb{E}^{\bar{\mathbb{Q}}}[g] - \mathcal{E}(\bar{\mathbb{Q}}, \bar{\mathcal{P}}_{\text{int}})\}. \end{aligned}$$

Moreover, the infimum of the problem  $\bar{V}$  is attained by some optimal trading strategy  $\hat{H} \in \mathcal{H}$ .

**Remark 4.3.** *The above duality result is similar to that in [3], but differs substantially with theirs in the following two points:*

(i) *In our current work, we have relaxed the strong one-period no-arbitrage condition for all  $\omega_t \in \Omega_t$  assumed in [3]. Indeed, the strong no-arbitrage condition is needed in [3] because their duality and dynamic programming are mixed with each other. More precisely, with the notations in [3, Section 4], they need the relation “ $\mathcal{E}_t(\omega, x) = \mathcal{D}_t(\omega) + x$ ” to hold for all  $t$  and  $\omega \in \Omega_t$  to guarantee the measurability of  $\mathcal{E}_t$  through  $\mathcal{D}_t$  (see in particular their equation (21) and their Proof of Lemma 4.6).*

(ii) *It is worth noting that the reformulations in Proposition 2.5 on the enlarged space do not exactly correspond to standard quasi-sure utility maximization problem. Indeed, we still restrict the class of strategies to  $\tilde{\mathbb{F}}$ -predictable processes, as opposed to  $\bar{\mathbb{F}}$ -predictable processes. The fact that the formulation with these two different filtrations are equivalent will be proved by using a minimax argument.*

**Proof of Theorem 3.1 (case  $e = 0$ )** First, using Lemma 4.1 and Proposition 2.5, the duality (3.17) can be deduced immediately from (4.23) in Proposition 4.2. Moreover, given the optimal trading strategy  $\hat{H} \in \mathcal{H}$  in Proposition 4.2, we can construct  $\hat{\eta}$  by (2.8) and show its optimality by almost the same arguments as in Step 2 of Proposition 2.5 (ii).  $\square$

In the rest of Section 4.2, we will provide the proof of Proposition 4.2 in several steps.

**The weak duality** As in the classical results, one can easily obtain a weak duality result.

**Lemma 4.4.** *For any universally measurable function  $g : \bar{\Omega} \rightarrow \mathbb{R} \cup \{\infty\}$ , one has*

$$\inf_{H \in \mathcal{H}} \sup_{\bar{\mathbb{P}} \in \bar{\mathcal{P}}_{\text{int}}} \log \mathbb{E}^{\bar{\mathbb{P}}} [\exp (g + (H \circ X)_T)] \geq \sup_{\bar{\mathbb{Q}} \in \bar{\mathcal{Q}}^*} \{ \mathbb{E}^{\bar{\mathbb{Q}}} [g] - \mathcal{E}(\bar{\mathbb{Q}}, \bar{\mathcal{P}}_{\text{int}}) \}.$$

*Proof.* Using the result in the [3, Proof of Theorem 4.1 - dynamic programming principle], one knows that for any  $H \in \mathcal{H}$ ,  $\bar{\mathbb{P}} \in \bar{\mathcal{P}}_{\text{int}}$  and  $\bar{\mathbb{Q}} \in \bar{\mathcal{Q}}^*$ , one has

$$\log \mathbb{E}^{\bar{\mathbb{P}}} [\exp (g + (H \circ X)_T)] \geq \mathbb{E}^{\bar{\mathbb{Q}}} [g] - \mathcal{E}(\bar{\mathbb{Q}}, \bar{\mathbb{P}}).$$

(Note that  $\mathcal{E}(\bar{\mathbb{Q}}, \bar{\mathbb{P}}) = \infty$  if  $\bar{\mathbb{Q}}$  is not dominated by  $\bar{\mathbb{P}}$ .) Therefore it is enough to take supremum over  $\bar{\mathbb{Q}}$  (and  $\bar{\mathbb{P}}$ ) and then take infimum over  $H \in \mathcal{H}$  to obtain the two weak duality results in the claim.  $\square$

We can next turn to (and for the duality, it suffices to) prove that

$$\inf_{H \in \mathcal{H}} \sup_{\bar{\mathbb{P}} \in \bar{\mathcal{P}}_{\text{int}}} \log \mathbb{E}^{\bar{\mathbb{P}}} [\exp (g + (H \circ X)_T)] \leq \sup_{\bar{\mathbb{Q}} \in \bar{\mathcal{Q}}_\varphi^*} \{ \mathbb{E}^{\bar{\mathbb{Q}}} [g] - \mathcal{E}(\bar{\mathbb{Q}}, \bar{\mathcal{P}}_{\text{int}}) \}, \quad (4.24)$$

for any universally measurable random variable  $\varphi : \bar{\Omega} \rightarrow [0, \infty)$ .

**The one-period case  $T = 1$**  Let us first consider the one-period case  $T = 1$ . Define

$$\Lambda_{\text{int}}(0, \omega_0) := \{ \theta_0 \in \Lambda_1 : S_0(\omega_0) \theta_0 \in \text{int} K_0^* \},$$

and for each  $\theta_0 \in \Lambda_{\text{int}}(0, \omega_0)$ ,

$$\bar{\mathcal{P}}_{\text{int}}^\delta(0, \theta_0) := \{ \bar{\mathbb{P}} \in \bar{\mathcal{P}}_{\text{int}} : \bar{\mathbb{P}}[\Theta_0 = \theta_0] = 1 \}.$$

Define  $\text{NA}(\bar{\mathcal{P}}_{\text{int}}^\delta(0, \theta_0))$  as  $\text{NA}(\bar{\mathcal{P}}_{\text{int}})$  in Definition 2.8 with  $\bar{\mathcal{P}}_{\text{int}}^\delta(0, \theta_0)$  in place of  $\bar{\mathcal{P}}_{\text{int}}$ . Then,  $\text{NA}(\bar{\mathcal{P}}_{\text{int}})$  implies that  $\text{NA}(\bar{\mathcal{P}}_{\text{int}}^\delta(0, \theta_0))$  holds for every  $\theta_0 \in \Lambda_{\text{int}}(0, \omega_0)$ .

**Lemma 4.5.** *Let  $T = 1$ , and  $g_1 : \overline{\Omega} \rightarrow \mathbb{R} \cup \{\infty\}$  be upper semi-analytic and also  $(\omega, \theta_0, \theta_1) \in \Omega \times \Lambda_1 \times \Lambda_1 \rightarrow g_1(\omega, \theta_0, \theta_1)$  depend only on  $(\omega, \theta_1)$ . Assume that  $\text{NA}(\overline{\mathcal{P}}_{\text{int}})$  holds. Then, for  $g = g_1$ , the inequality (4.24) holds for any random variable  $\varphi : \overline{\Omega} \rightarrow [0, \infty)$  and both terms are not equal to  $-\infty$ . Moreover, there exists an optimal solution  $\hat{H} \in \mathcal{H}$  for the infimum problem at the left hand side. In consequence, Proposition 4.2 holds true for the case  $T = 1$ .*

*Proof. Step 1:* Although the context is slightly different, we can still follow the same arguments line by line in step (b) of the proof of Theorem 3.1 and Lemma 3.2 of [3] to obtain the existence of the optimal strategy  $\hat{H}$  (see also the proof of Theorem 2.2 of [27]), where the key argument is to show that  $h \mapsto \sup_{\overline{\mathbb{P}} \in \overline{\mathcal{P}}_{\text{int}}} \log \mathbb{E}^{\overline{\mathbb{P}}}[\exp(g + h(X_1 - X_0))]$  is lower-semicontinuous.

*Step 2:* We then turn to prove the duality. First, notice that  $\mathcal{H} = \mathbb{R}^d$  when  $T = 1$ , and that  $(g_1, X_1)(\omega, \theta_0, \theta_1)$  is independent of  $\theta_0$ . Then, for all  $\theta_0 \in \Lambda_{\text{int}}(0, \omega_0)$ ,

$$\{\overline{\mathbb{P}} \circ (g_1, X_1)^{-1} : \overline{\mathbb{P}} \in \overline{\mathcal{P}}_{\text{int}}(0, \theta_0)\} = \{\overline{\mathbb{P}} \circ (g_1, X_1)^{-1} : \overline{\mathbb{P}} \in \overline{\mathcal{P}}_{\text{int}}(0, \mathbf{1})\},$$

where  $\mathbf{1}$  represents the vector of  $\mathbb{R}^d$  with all entries equal to 1. Thanks to the standard concatenation argument, it is clear that

$$\overline{V} = \inf_{h_1 \in \mathbb{R}^d} \sup_{\theta_0 \in \Lambda_{\text{int}}(0, \omega_0)} \sup_{\overline{\mathbb{P}} \ll \overline{\mathcal{P}}_{\text{int}}^\delta(0, \theta_0)} \log \mathbb{E}^{\overline{\mathbb{P}}}[\exp(g_1 + h_1 \cdot X_1 - h_1 \cdot S_0 \theta_0)].$$

Define the function

$$\alpha(h_1, \theta_0) := \sup_{\overline{\mathbb{P}} \ll \overline{\mathcal{P}}_{\text{int}}^\delta(0, \theta_0)} \log \mathbb{E}^{\overline{\mathbb{P}}}[\exp(g_1 + h_1 \cdot X_1 - h_1 \cdot S_0 \theta_0)].$$

Notice that

$$\frac{1}{2} \log \mathbb{E}^{\overline{\mathbb{P}}}[\exp(Y_1)] + \frac{1}{2} \log \mathbb{E}^{\overline{\mathbb{P}}}[\exp(Y_2)] \leq \log \mathbb{E}^{\overline{\mathbb{P}}}[\exp((Y_1 + Y_2)/2)],$$

it follows that  $h_1 \mapsto \alpha(h_1, \theta_0)$  is convex. Moreover, the map

$$\theta_0 \mapsto \alpha(h_1, \theta_0) = \sup_{\overline{\mathbb{P}} \ll \overline{\mathcal{P}}_{\text{int}}^\delta(0, \mathbf{1})} \log \mathbb{E}^{\overline{\mathbb{P}}}[\exp(g_1 + h_1 \cdot X_1 - h_1 \cdot S_0 \theta_0)] \text{ is linear.}$$

This allows us to use minimax theorem to deduce that

$$\begin{aligned} \overline{V} &= \inf_{h_1 \in \mathbb{R}^d} \sup_{\theta_0 \in \Lambda_{\text{int}}(0, \omega_0)} \alpha(h_1, \theta_0) = \inf_{h_1 \in \mathbb{R}^d} \sup_{\theta_0 \in \Lambda(0, \omega_0)} \alpha(h_1, \theta_0) = \sup_{\theta_0 \in \Lambda(0, \omega_0)} \inf_{h_1 \in \mathbb{R}^d} \alpha(h_1, \theta_0) \\ &= \sup_{\theta_0 \in \Lambda_{\text{int}}(0, \omega_0)} \inf_{h_1 \in \mathbb{R}^d} \sup_{\overline{\mathbb{P}} \ll \overline{\mathcal{P}}_{\text{int}}^\delta(0, \theta_0)} \log \mathbb{E}^{\overline{\mathbb{P}}}[\exp(g_1 + h_1 \cdot X_1 - h_1 \cdot S_0 \theta_0)]. \end{aligned}$$

In above,  $\Lambda(0, \omega_0)$  denotes the closure of  $\Lambda_{\text{int}}(0, \omega_0)$ , and we can replace  $\Lambda_{\text{int}}(0, \omega_0)$  by  $\Lambda(0, \omega_0)$  since  $\theta_0 \mapsto \alpha(h_1, \theta_0)$  is linear, and  $\theta_0 \mapsto \inf_{h_1 \in \mathbb{R}^d} \alpha(h_1, \theta_0)$  is lower semicontinuous. Using the one period duality result in [3, Theorem 3.1], we obtain

$$\overline{V} = \sup_{\theta_0 \in \Lambda_{\text{int}}(0, \omega_0)} \sup_{\overline{\mathbb{Q}} \in \overline{\mathcal{Q}}_\varphi^*(0, \theta_0)} \left\{ \mathbb{E}^{\overline{\mathbb{Q}}}[g_1] - \mathcal{E}(\overline{\mathbb{Q}}, \overline{\mathcal{P}}_{\text{int}}^\delta(0, \theta_0)) \right\}.$$

*Step 3:* To conclude the proof, it is enough to prove that

$$\sup_{\theta_0 \in \Lambda_{\text{int}}(0, \omega_0)} \sup_{\overline{\mathbb{Q}} \in \overline{\mathcal{Q}}_\varphi^*(0, \theta_0)} \left\{ \mathbb{E}^{\overline{\mathbb{Q}}}[g_1] - \mathcal{E}(\overline{\mathbb{Q}}, \overline{\mathcal{P}}_0^{\text{int}, \delta}(\theta_0)) \right\} \geq \sup_{\overline{\mathbb{Q}} \in \overline{\mathcal{Q}}_\varphi^*} \left\{ \mathbb{E}^{\overline{\mathbb{Q}}}[g_1] - \mathcal{E}(\overline{\mathbb{Q}}, \overline{\mathcal{P}}_{\text{int}}) \right\}, \quad (4.25)$$



as the reverse inequality is trivial by the fact that  $\overline{\mathcal{Q}}_\varphi^*(0, \theta_0) \subset \overline{\mathcal{Q}}_\varphi^*$  and that  $\mathcal{E}(\overline{\mathcal{Q}}, \overline{\mathcal{P}}_{\text{int}}^\delta(0, \theta_0)) = \mathcal{E}(\overline{\mathcal{Q}}, \overline{\mathcal{P}}_{\text{int}})$  for all  $\overline{\mathcal{Q}} \in \overline{\mathcal{Q}}_\varphi^*(0, \theta_0)$ . Let  $\overline{\mathcal{Q}} \in \overline{\mathcal{Q}}_\varphi^*$  and denote by  $(\overline{\mathcal{Q}}_{\theta_0})_{\theta_0 \in \Lambda_{\text{int}}(0, \omega_0)}$  a family of r.c.p.d. of  $\overline{\mathcal{Q}}$  knowing  $\theta_0$ , then by [3, Lemma 4.4], we have

$$\begin{aligned} \mathbb{E}^{\overline{\mathcal{Q}}}[g_1] - \mathcal{E}(\overline{\mathcal{Q}}, \overline{\mathcal{P}}_{\text{int}}) &= \mathbb{E}^{\overline{\mathcal{Q}}}[\mathbb{E}^{\overline{\mathcal{Q}}_{\theta_0}}[g_1] - \mathcal{E}(\overline{\mathcal{Q}}_{\theta_0}, \overline{\mathcal{P}}_{\text{int}}^\delta(0, \theta_0))] - \mathcal{E}(\overline{\mathcal{Q}} \circ \theta_0^{-1}, \overline{\mathcal{P}}_{\text{int}}) \\ &\leq \sup_{\theta_0 \in \Lambda_{\text{int}}(0, \omega_0)} \sup_{\overline{\mathcal{Q}} \in \overline{\mathcal{Q}}_\varphi^*(0, \theta_0)} \left\{ \mathbb{E}^{\overline{\mathcal{Q}}}[g_1] - \mathcal{E}(\overline{\mathcal{Q}}, \overline{\mathcal{P}}_0^{\text{int}, \delta}(\theta_0)) \right\}. \end{aligned}$$

Taking the supremum over  $\overline{\mathcal{Q}}$  in  $\overline{\mathcal{Q}}_\varphi^*$ , we verify (4.25).  $\square$

**The multi-period case: measurable selection of the dynamic strategy** Let us extend the above definitions of  $\Lambda_{\text{int}}(0, \omega_0)$ ,  $\overline{\mathcal{P}}_{\text{int}}^\delta(0, \theta_0)$  and  $\overline{\mathcal{Q}}_\varphi^*(0, \theta_0)$  to an arbitrary initial time  $t$  and initial path  $\bar{\omega}^t$ . For  $t \geq 1$  and  $\bar{\omega} = \bar{\omega}^t = (\omega^t, \theta^t) \in \overline{\Omega}_t$ , let us first recall the definition of  $\Lambda_{\text{int}}(t, \omega^t)$  :

$$\Lambda_{\text{int}}(t, \omega^t) := \{\theta_t \in \Lambda_1 : S_t(\omega^t)\theta_t \in \text{int}K_t^*(\omega^t)\} \subset \Lambda_1.$$

Note that  $\overline{\mathcal{P}}_{\text{int}}(t, \bar{\omega}) \subset \mathfrak{B}(\Omega_1 \times \Lambda_1)$  is defined in (2.7). We introduce

$$\overline{\mathcal{P}}_{\text{int}}^\delta(t, \bar{\omega}) := \{\delta_{\bar{\omega}^t} \otimes \overline{\mathbb{P}}_{t+1} : \overline{\mathbb{P}}_{t+1} \in \overline{\mathcal{P}}_{\text{int}}(t, \bar{\omega})\},$$

and

$$\tilde{\mathcal{P}}_{\text{int}}^\delta(t, \omega) := \{(\delta_{\omega^t} \times \mu(d\theta^t)) \otimes \overline{\mathbb{P}}_{t+1} : \overline{\mathbb{P}}_{t+1} \in \overline{\mathcal{P}}_{\text{int}}(t, \bar{\omega}), \mu \in \mathfrak{B}(\Lambda_{\text{int}}(0, \omega_0) \times \cdots \times \Lambda_{\text{int}}(t, \omega^t))\},$$

where the latter consists in a version of  $\overline{\mathcal{P}}_{\text{int}}^\delta(t, \bar{\omega})$  in which  $\theta^t$  is not fixed anymore.

**Remark 4.6.** (i) For a fixed  $\omega \in \Omega_t$ , let us define  $\text{NA}(\tilde{\mathcal{P}}_{\text{int}}^\delta(t, \omega))$  by

$$h(X_t) \cdot (X_{t+1} - X_t) \geq 0, \tilde{\mathcal{P}}_{\text{int}}^\delta(t, \omega)\text{-q.s.} \implies h(X_t) \cdot (X_{t+1} - X_t) = 0, \tilde{\mathcal{P}}_{\text{int}}^\delta(t, \omega)\text{-q.s.},$$

for every universally measurable function  $h : \mathbb{R}^d \rightarrow \mathbb{R}^d$ . By applying Proposition 2.10 with  $\mathcal{P}(t, \omega)$  in place of  $\mathcal{P}$ , one obtains that  $\text{NA}2(t, \omega)$  defined in (4.26) is equivalent to  $\text{NA}(\tilde{\mathcal{P}}_{\text{int}}^\delta(t, \omega))$ .

(ii) We recall that for each  $t \leq T$  and  $\omega \in \Omega_t$ , the condition  $\text{NA}2(t, \omega)$  is satisfied if

$$\zeta \in K_{t+1}(\omega, \cdot), \mathcal{P}_t(\omega)\text{-q.s.} \implies \zeta \in K_t(\omega), \text{ for all } \zeta \in \mathbb{R}^d. \quad (4.26)$$

Then by [14, Lemma 3.6], the set  $N_t := \{\omega : \text{NA}2(t, \omega) \text{ fails}\}$  is universally measurable. Moreover,  $N_t$  is a  $\mathcal{P}$ -polar set if  $\text{NA}2(\mathcal{P})$  holds.

(iii) It follows from (i) and (ii) that  $\text{NA}2(t, \omega)$  or equivalently  $\tilde{\mathcal{P}}_{\text{int}}^\delta(t, \omega)$  holds for all  $\omega$  outside a  $\mathcal{P}$ -polar set  $N$ , whenever  $\text{NA}2(\mathcal{P})$  holds. The later is equivalent to  $\text{NA}(\overline{\mathcal{P}}_{\text{int}})$  by Proposition 2.10. Therefore, if  $\text{NA}(\overline{\mathcal{P}}_{\text{int}})$  holds, there exists a  $\overline{\mathcal{P}}_{\text{int}}$ -polar  $\overline{N} := N \times \Lambda$ , such that for all  $\bar{\omega} = (\omega, \theta) \notin \overline{N}$ ,  $\text{NA}(\tilde{\mathcal{P}}_{\text{int}}^\delta(t, \omega))$  holds.

(iv) Finally, for a fixed  $\bar{\omega} \in \overline{\Omega}_t$ , we define  $\text{NA}(\overline{\mathcal{P}}_{\text{int}}^\delta(t, \bar{\omega}))$  by

$$h \cdot (X_{t+1} - X_t) \geq 0, \overline{\mathcal{P}}_{\text{int}}^\delta(t, \bar{\omega})\text{-q.s.} \implies h \cdot (X_{t+1} - X_t) = 0, \overline{\mathcal{P}}_{\text{int}}^\delta(t, \bar{\omega})\text{-q.s.},$$

for every  $h \in \mathbb{R}^d$ . Then,  $\text{NA}(\tilde{\mathcal{P}}_{\text{int}}^\delta(t, \omega))$  implies  $\text{NA}(\overline{\mathcal{P}}_{\text{int}}^\delta(t, \omega, \theta))$  for all  $\theta \in \Lambda$  (see also Remark 3.9 of [12]).

Let us fix a functional  $g_{t+1} := \bar{\Omega}_{t+1} \rightarrow \mathbb{R} \cup \{\infty\}$  which is upper semi-analytic and such that  $g_{t+1}(\omega^{t+1}, \theta_0, \dots, \theta_{t+1})$  depends only on  $(\omega^{t+1}, \theta_{t+1})$ . Then for any universally measurable random variable  $Y_{t+1} : \bar{\Omega}_{t+1} \rightarrow \mathbb{R}_+$ , we introduce

$$\bar{\mathcal{Q}}_{Y_{t+1}}^*(t, \bar{\omega}) := \left\{ \bar{\mathbb{Q}} \in \mathfrak{B}(\bar{\Omega}_{t+1}) : \begin{array}{l} \bar{\mathbb{Q}} \lll \bar{\mathcal{P}}_{\text{int}}^\delta(t, \bar{\omega}), \mathbb{E}^{\bar{\mathbb{Q}}}[X_{t+1} - X_t] = 0, \mathbb{E}^{\bar{\mathbb{Q}}}[Y_{t+1}] < \infty, \\ \mathbb{E}^{\bar{\mathbb{Q}}}[g_{t+1}^- + |X_{t+1} - X_t|] + \mathcal{E}(\bar{\mathbb{Q}}, \bar{\mathcal{P}}_{\text{int}}^\delta(t, \bar{\omega})) < \infty \end{array} \right\},$$

and by setting  $Y_{t+1} \equiv 0$ , we define

$$g_t(\bar{\omega}^t) := \sup_{\bar{\mathbb{Q}} \in \bar{\mathcal{Q}}_0^*(t, \bar{\omega}^t)} \left\{ \mathbb{E}^{\bar{\mathbb{Q}}}[g_{t+1}] - \mathcal{E}(\bar{\mathbb{Q}}, \bar{\mathcal{P}}_{\text{int}}^\delta(t, \bar{\omega}^t)) \right\}, \quad \text{for all } \bar{\omega}^t \in \bar{\Omega}_t. \quad (4.27)$$

**Remark 4.7.** Let  $\bar{\omega} = (\omega, \theta)$  and  $\bar{\omega}' = (\omega', \theta')$  be such that  $\omega^t = (\omega')^t$  and  $\theta_t = \theta'_t$ . Then, it follows from the definition of  $\bar{\mathcal{P}}_{\text{int}}^\delta(t, \bar{\omega})$  and  $\bar{\mathcal{Q}}_{Y_{t+1}}^*(t, \bar{\omega})$  that

$$\left\{ \bar{\mathbb{Q}} \circ (g_{t+1}, X_t, X_{t+1})^{-1} : \bar{\mathbb{Q}} \in \bar{\mathcal{Q}}_{Y_{t+1}}^*(t, \bar{\omega}) \right\} = \left\{ \bar{\mathbb{Q}} \circ (g_{t+1}, X_t, X_{t+1})^{-1} : \bar{\mathbb{Q}} \in \bar{\mathcal{Q}}_{Y_{t+1}}^*(t, \bar{\omega}') \right\}.$$

Since  $g_{t+1}(\omega^{t+1}, \theta_0, \dots, \theta_{t+1})$  is assumed to be independent of  $(\theta_0, \dots, \theta_t)$ , then it is clear that  $g_t(\bar{\omega}^t)$  depends only on  $(\omega^t, \theta_t)$  for  $\bar{\omega}^t = (\omega^t, \theta_0, \dots, \theta_t)$ .

The above remark allows us to define

$$g'_t(\omega^t, h) := \sup_{\theta_t \in \Lambda_{\text{int}}(t, \omega^t)} \left\{ g_t(\omega^t, \theta_t) + h \cdot S_t(\omega) \theta_t \right\}, \quad \forall (\omega^t, h) \in \Omega_t \times \mathbb{R}^d. \quad (4.28)$$

**Remark 4.8.** From Remark 4.6,  $\text{NA}(\bar{\mathcal{P}}_{\text{int}})$  implies that  $\text{NA}(\bar{\mathcal{P}}_{\text{int}}^\delta(t, \bar{\omega}))$  holds for  $\bar{\mathbb{P}}$ -a.e.  $\bar{\omega} \in \bar{\Omega}$  under any  $\bar{\mathbb{P}} \in \bar{\mathcal{P}}_{\text{int}}$ . We can in fact apply Theorem 3.1 of [3] to obtain that

$$g_t(\bar{\omega}) = \sup_{\bar{\mathbb{Q}} \in \bar{\mathcal{Q}}_{Y_{t+1}}^*(t, \bar{\omega})} \left\{ \mathbb{E}^{\bar{\mathbb{Q}}}[g_{t+1}] - \mathcal{E}(\bar{\mathbb{Q}}, \bar{\mathcal{P}}_{\text{int}}^\delta(t, \bar{\omega})) \right\}, \quad \bar{\mathcal{P}}_{\text{int}}\text{-q.s.},$$

for all universally measurable random variables  $Y_{t+1} : \bar{\Omega}_{t+1} \rightarrow \mathbb{R}_+$ .

**Lemma 4.9.** For every  $t$ , the graph set

$$\left[ \left[ \bar{\mathcal{Q}}_0^*(t) \right] \right] := \left\{ (\bar{\omega}, \bar{\mathbb{Q}}) : \bar{\omega} \in \bar{\Omega}_t, \bar{\mathbb{Q}} \in \bar{\mathcal{Q}}_0^*(t, \bar{\omega}) \right\} \text{ is analytic.}$$

*Proof.* We follow the arguments in Lemma 4.5 of [3] and Lemma 4.8 of [13]. First notice that  $g_{t+1} \wedge 0 + |X_{t+1} - X_t|$  is upper semi-analytic, an application of Proposition 7.46 of [9] shows that  $(\bar{\omega}, \bar{\mathbb{Q}}) \mapsto \mathbb{E}^{\bar{\mathbb{Q}}}[g_{t+1} \wedge 0 - |X_{t+1} - X_t|]$  is upper semi-analytic. Then because  $\left[ \left[ \bar{\mathcal{P}}_{\text{int}}^\delta(t) \right] \right]$  is analytic and the relative entropy is Borel measurable by Lemma 4.2 of [3], Proposition 7.47 of [9] implies that  $(\bar{\omega}, \bar{\mathbb{Q}}) \mapsto -\mathcal{E}(\bar{\mathbb{Q}}, \bar{\mathcal{P}}_{\text{int}}^\delta(t, \bar{\omega}))$  is upper semi-analytic. It follows that

$$A := \left\{ (\bar{\omega}, \bar{\mathbb{Q}}) : \mathbb{E}^{\bar{\mathbb{Q}}}[g_{t+1} \wedge 0 - |X_{t+1} - X_t|] - \mathcal{E}(\bar{\mathbb{Q}}, \bar{\mathcal{P}}_{\text{int}}^\delta(t, \bar{\omega})) > -\infty \right\}$$

is an analytic set. By Lemma 4.8 of [13], we know

$$B(\bar{\omega}) := \{(\bar{\mathbb{Q}}, \bar{\mathbb{P}}) \in \mathfrak{B}(\bar{\Omega}_{t+1}) \times \mathfrak{B}(\bar{\Omega}_{t+1}) : \bar{\mathbb{P}} \in \bar{\mathcal{P}}_{\text{int}}^\delta(t, \bar{\omega}), \bar{\mathbb{Q}} \ll \bar{\mathbb{P}}\}$$

has an analytic graph. Notice that the set

$$C := \{(\bar{\omega}, \bar{\mathbb{Q}}) : \bar{\mathbb{Q}} \lll \bar{\mathcal{P}}_{\text{int}}^\delta(t, \bar{\omega})\}$$

is the image of  $\text{graph}(B)$  under canonical projection  $\overline{\Omega}_t \times \mathfrak{B}(\overline{\Omega}_{t+1}) \times \mathfrak{B}(\overline{\Omega}_{t+1}) \mapsto \overline{\Omega}_t \times \mathfrak{B}(\overline{\Omega}_{t+1})$  and thus analytic. Finally, it is shown that

$$\left[ \left[ \overline{\mathcal{Q}}_0^*(t) \right] \right] = A \cap C$$

is analytic.  $\square$

**Lemma 4.10.** *Assume that  $\text{NA}(\overline{\mathcal{P}}_{\text{int}})$  holds true. Then both  $g_t$  and  $g'_t$  are upper semi-analytic, and there is a universally measurable map  $h_{t+1} : \Omega_t \times \mathbb{R}^d \rightarrow \mathbb{R}^d$  together with a  $\mathcal{P}$ -polar set  $N$  such that, for every  $(\omega, h_t) \in N^c \times \mathbb{R}^d$ , one has*

$$g'_t(\omega^t, h_t) = \sup_{\theta_t \in \Lambda_{\text{int}}(t, \omega^t)} \sup_{\overline{\mathbb{P}} \in \overline{\mathcal{P}}_{\text{int}}^\delta(t, \bar{\omega})} \log \mathbb{E}^{\overline{\mathbb{P}}} \left[ \exp(g_{t+1} + h_{t+1}(\omega^t, h_t)(X_{t+1} - X_t) + h_t X_t) \right] > -\infty.$$

*Proof.* The proof follows the track of measurable selection arguments as in Lemma 3.7 of [27] with some modifications for our setting. Let us denote, for all  $\omega^t \in \Omega_t$  and  $h_t \in \mathbb{R}^d$ ,

$$V'_t(\omega^t, h_t) := \inf_{h_{t+1} \in \mathbb{R}^d} \sup_{\theta_t \in \Lambda_{\text{int}}(t, \omega^t)} \sup_{\overline{\mathbb{P}} \in \overline{\mathcal{P}}_{\text{int}}^\delta(t, \bar{\omega})} \log \mathbb{E}^{\overline{\mathbb{P}}} \left[ \exp(g_{t+1} + h_{t+1}(X_{t+1} - X_t) + h_t X_t) \right].$$

By Remark 4.7, we can employ the same minimax theorem argument as in Lemma 4.5 above and obtain that

$$V'_t(\omega^t, h_t) = g'_t(\omega^t, h_t) > -\infty, \quad \text{if } \text{NA}(\tilde{\mathcal{P}}_{\text{int}}^\delta(t, \omega)) \text{ holds true.}$$

In view of (iii) in Remark 4.6, this holds true outside a  $\mathcal{P}$ -polar set  $N$ .

Further, as  $g_{t+1}$  is assumed to be upper semi-analytic,  $\left[ \left[ \overline{\mathcal{Q}}_0^*(t) \right] \right]$  is analytic by Lemma 4.9,  $\bar{\omega} \mapsto \mathcal{E}(\overline{\mathcal{Q}}, \overline{\mathcal{P}}_{\text{int}}^\delta(t, \bar{\omega}))$  is lower semi-analytic by Lemma 4.2 of [3] and [9, Proposition 7.47]. It thus follows from a measurable selection argument (see e.g. [9, Propositions 7.26, 7.47, 7.48]) that the maps  $\bar{\omega}^t \mapsto g_t(\bar{\omega}^t)$  and  $(\omega^t, h_t) \mapsto g'_t(\omega^t, h_t)$  are both upper semi-analytic. Thus for fixed  $h_t$ , by [9, Proposition 7.47] and as the graph of  $\text{int}K_t^*$  is Borel,  $\omega^t \mapsto g'_t(\omega^t, h_t)$  is upper semi-analytic. Further, as  $h_t \mapsto g'_t(\omega^t, h_t)$  is upper semi-continuous and Carathéodory functions are jointly measurable ([13, Lemma 4.12]), we have  $(\omega^t, h_t) \mapsto g'_t(\omega^t, h_t)$  is in  $\mathcal{F}_t \otimes \mathcal{B}(\mathbb{R}^d)$ .

Next, we claim that the function

$$\phi(\omega^t, h_t, h) := \sup_{\theta_t \in \Lambda_{\text{int}}(t, \omega^t)} \sup_{\overline{\mathbb{P}} \in \overline{\mathcal{P}}_{\text{int}}^\delta(t, \bar{\omega})} \log \mathbb{E}^{\overline{\mathbb{P}}} \left[ \exp(g_{t+1} + h(X_{t+1} - X_t) + h_t X_t) \right].$$

is  $\mathcal{F}_t \otimes \mathcal{B}(\mathbb{R}^d) \otimes \mathcal{B}(\mathbb{R}^d)$ -measurable. To see this, we first fix  $h$  and  $h_t$ . Then from the same argument as above, as  $\left[ \left[ \overline{\mathcal{P}}_{\text{int}}^\delta(t) \right] \right]$  is analytic and by [9, Propositions 7.26, 7.47, 7.48], we have that

$$(\omega^t, \theta_t) \mapsto \sup_{\overline{\mathbb{P}} \in \overline{\mathcal{P}}_{\text{int}}^\delta(t, \bar{\omega})} \log \mathbb{E}^{\overline{\mathbb{P}}} \left[ \exp(g_{t+1} + h(X_{t+1} - X_t) + h_t X_t) \right]$$

is upper semi-analytic. Now as the graph of  $\text{int}K_t^*$  is Borel, by [9, Proposition 7.47] we have that  $\omega^t \mapsto \phi(\omega^t, h_t, h)$  is upper semi-analytic. On the other hand, for fixed  $\omega^t$ , it follows by an application of Fatou's lemma (as [3, Lemma 4.6]) that  $(h, h_t) \mapsto \phi(\omega^t, h_t, h)$  is lower semi-continuous. By [13, Lemma 4.12], we have that  $\phi$  is indeed  $\mathcal{F}_t \otimes \mathcal{B}(\mathbb{R}^d) \otimes \mathcal{B}(\mathbb{R}^d)$ -measurable.

Let us consider the random set

$$\Phi(\omega^t, h_t) := \{h \in \mathbb{R}^d : \phi(\omega^t, h_t, h) = g'_t(\omega^t, h_t)\}.$$

By the previous arguments, we have that  $\text{graph}(\Phi)$  is in  $\mathcal{F}_t \otimes \mathcal{B}(\mathbb{R}^d) \otimes \mathcal{B}(\mathbb{R}^d)$  and thus in  $\mathcal{U}(\Omega_t \times \mathbb{R}^d) \otimes \mathcal{B}(\mathbb{R}^d)$ . Thus  $\Phi$  admits an  $\mathcal{U}(\Omega_t \times \mathbb{R}^d)$ -measurable selector  $h_{t+1}$  on the universally measurable set  $\Phi(\omega^t, h_t) \neq \emptyset$ ; cf. the corollary and scholium of [24, Theorem 5.5]. Moreover, Lemma 4.5 and Remark 4.6 imply that  $\Phi(\omega^t, h_t) \neq \emptyset$  holds true outside a  $\mathcal{P}$ -polar set  $N$ , it yields that  $h_{t+1}$  solves the infimum  $\mathcal{P}$ -q.s.  $\square$

**The multi-period case: the final proof** We provide a last technical lemma and then finish the proof of Proposition 4.2. Recall that  $g_{t+1} := \bar{\Omega}_{t+1} \rightarrow \mathbb{R} \cup \{\infty\}$  is a given upper semi-analytic functional, such that  $g_{t+1}(\omega^{t+1}, \theta_0, \dots, \theta_{t+1})$  depends only on  $(\omega^{t+1}, \theta_{t+1})$ , and  $g_t$  is defined in (4.27). Given a universally measurable random variable  $Y_t : \bar{\Omega}_t \rightarrow \mathbb{R}_+$ , we define

$$\bar{\mathcal{Q}}_{Y_t, t}^* := \{\bar{\mathbb{Q}} \in \bar{\mathcal{Q}}_0|_{\bar{\Omega}_t} : \mathbb{E}^{\bar{\mathbb{Q}}} [g_t^-] + \mathcal{E}(\bar{\mathbb{Q}}, \bar{\mathcal{P}}_{\text{int}}|_{\bar{\Omega}_t}) < +\infty, \mathbb{E}^{\bar{\mathbb{Q}}} [Y_t] < +\infty\}.$$

**Lemma 4.11.** *Let  $t+1 \leq T$ , then for any universally measurable random variable  $Y_{t+1} : \bar{\Omega}_{t+1} \rightarrow \mathbb{R}_+$  and  $\varepsilon > 0$ , there is a universally measurable random variable  $Y_t^\varepsilon : \bar{\Omega}_t \rightarrow \mathbb{R}_+$  such that*

$$\sup_{\bar{\mathbb{Q}} \in \bar{\mathcal{Q}}_{Y_t^\varepsilon, t}^*} \left\{ \mathbb{E}^{\bar{\mathbb{Q}}} [g_t] - \mathcal{E}(\bar{\mathbb{Q}}, \bar{\mathcal{P}}_{\text{int}}|_{\bar{\Omega}_t}) \right\} \leq \sup_{\bar{\mathbb{Q}} \in \bar{\mathcal{Q}}_{Y_{t+1}, t+1}^*} \left\{ \mathbb{E}^{\bar{\mathbb{Q}}} [g_{t+1}] - \mathcal{E}(\bar{\mathbb{Q}}, \bar{\mathcal{P}}_{\text{int}}|_{\bar{\Omega}_{t+1}}) \right\} + \varepsilon. \quad (4.29)$$

*Proof.* In view of Proposition A.1, we can assume w.l.o.g. that  $Y_{t+1} \equiv 0$ . Then Lemma 4.9 and a measurable selection argument (see e.g. Proposition 7.50 of [9]) guarantees that there exists a universally measurable kernel  $\bar{\mathbb{Q}}_t^\varepsilon(\cdot) : \bar{\Omega}_t \rightarrow \mathfrak{B}(\bar{\Omega}_1)$  such that  $\delta_{\bar{\omega}} \otimes \bar{\mathbb{Q}}_t^\varepsilon(\bar{\omega}) \in \bar{\mathcal{Q}}_0^*(t, \bar{\omega})$  for all  $\bar{\omega} \in \bar{\Omega}_t$ , and

$$g_t(\bar{\omega}) \leq \mathbb{E}^{\delta_{\bar{\omega}} \otimes \bar{\mathbb{Q}}_t^\varepsilon(\bar{\omega})} [g_{t+1}] - \mathcal{E}(\bar{\mathbb{Q}}_t^\varepsilon(\bar{\omega}), \bar{\mathcal{P}}_{\text{int}}(t, \bar{\omega})) + \varepsilon.$$

Let us then define  $Y_t^\varepsilon$  by

$$Y_t^\varepsilon(\cdot) := \mathbb{E}^{\delta_{\cdot} \otimes \bar{\mathbb{Q}}_t^\varepsilon(\cdot)} [g_{t+1}^- + |X_{t+1} - X_t|] + \mathcal{E}(\bar{\mathbb{Q}}_t^\varepsilon(\cdot), \bar{\mathcal{P}}_{\text{int}}(t, \cdot)).$$

By the definition of  $\bar{\mathcal{Q}}_0^*(t, \cdot)$  and [3, Lemma 4.2],  $Y_t^\varepsilon$  is  $\mathbb{R}_+$ -valued and universally measurable.

Then for any  $\bar{\mathbb{Q}} \in \bar{\mathcal{Q}}_{Y_t^\varepsilon, t}^*$ , one has

$$\begin{aligned} & \mathbb{E}^{\bar{\mathbb{Q}} \otimes \bar{\mathbb{Q}}_t^\varepsilon(\cdot)} [g_{t+1}^- + |X_{t+1} - X_t|] + \mathcal{E}(\bar{\mathbb{Q}} \otimes \bar{\mathbb{Q}}_t^\varepsilon(\cdot), \bar{\mathcal{P}}_{\text{int}}|_{\bar{\Omega}_{t+1}}) \\ &= \mathbb{E}^{\bar{\mathbb{Q}}} \left[ \mathbb{E}^{\bar{\mathbb{Q}}_t^\varepsilon(\cdot)} [g_{t+1}^- + |X_{t+1} - X_t|] \right] + \mathcal{E}(\bar{\mathbb{Q}}, \bar{\mathcal{P}}_{\text{int}}|_{\bar{\Omega}_t}) + \mathbb{E}^{\bar{\mathbb{Q}}} \left[ \mathcal{E}(\bar{\mathbb{Q}}_t^\varepsilon(\cdot), \bar{\mathcal{P}}_{\text{int}}(t, \cdot)) \right] \\ &\leq \mathcal{E}(\bar{\mathbb{Q}}, \bar{\mathcal{P}}_{\text{int}}|_{\bar{\Omega}_t}) + \mathbb{E}^{\bar{\mathbb{Q}}} [Y_t^\varepsilon] < +\infty. \end{aligned}$$

Further,  $\bar{\mathbb{Q}} \otimes \bar{\mathbb{Q}}_t^\varepsilon(\cdot)$  is a martingale measure on  $\bar{\Omega}_{t+1}$  by the martingale property of  $\bar{\mathbb{Q}}$  and  $\bar{\mathbb{Q}}_t^\varepsilon(\cdot)$ . Finally, because  $\bar{\mathbb{Q}} \ll \bar{\mathcal{P}}_{\text{int}}|_{\bar{\Omega}_t}$  and  $\bar{\mathbb{Q}}_t^\varepsilon(\cdot) \ll \bar{\mathcal{P}}_{\text{int}}(t, \cdot)$ , it follows that  $\bar{\mathbb{Q}} \otimes \bar{\mathbb{Q}}_t^\varepsilon(\cdot) \ll \bar{\mathcal{P}}_{\text{int}}|_{\bar{\Omega}_{t+1}}$ . This implies that  $\bar{\mathbb{Q}} \otimes \bar{\mathbb{Q}}_t^\varepsilon(\cdot) \in \bar{\mathcal{Q}}_{0, t+1}^*$ .

Thus for any  $\bar{\mathbb{Q}} \in \bar{\mathcal{Q}}_{Y_t^\varepsilon, t}^*$ , one has

$$\begin{aligned} \mathbb{E}^{\bar{\mathbb{Q}}} [g_t] - \mathcal{E}(\bar{\mathbb{Q}}, \bar{\mathcal{P}}_{\text{int}}|_{\bar{\Omega}_t}) &\leq \mathbb{E}^{\bar{\mathbb{Q}}} \left[ \mathbb{E}^{\bar{\mathbb{Q}}_t^\varepsilon(\cdot)} [g_{t+1}] - \mathcal{E}(\bar{\mathbb{Q}}_t^\varepsilon(\cdot), \bar{\mathcal{P}}_{\text{int}}(t, \cdot)) + \varepsilon \right] - \mathcal{E}(\bar{\mathbb{Q}}, \bar{\mathcal{P}}_{\text{int}}|_{\bar{\Omega}_t}) \\ &= \mathbb{E}^{\bar{\mathbb{Q}} \otimes \bar{\mathbb{Q}}_t^\varepsilon(\cdot)} [g_{t+1}] - \mathcal{E}(\bar{\mathbb{Q}} \otimes \bar{\mathbb{Q}}_t^\varepsilon(\cdot), \bar{\mathcal{P}}_{\text{int}}|_{\bar{\Omega}_{t+1}}) + \varepsilon \\ &\leq \sup_{\bar{\mathbb{Q}} \in \bar{\mathcal{Q}}_{0, t+1}^*} \left\{ \mathbb{E}^{\bar{\mathbb{Q}}} [g_{t+1}] - \mathcal{E}(\bar{\mathbb{Q}}, \bar{\mathcal{P}}_{\text{int}}|_{\bar{\Omega}_{t+1}}) \right\} + \varepsilon. \end{aligned}$$

We hence conclude the proof as  $\bar{\mathbb{Q}} \in \bar{\mathcal{Q}}_{Y_t^\varepsilon, t}^*$  is arbitrary.  $\square$

**Proof of Proposition 4.2.** We will use an induction argument. First, Proposition 4.2 in case  $T = 1$  is already proved in Proposition 4.5. Next, assume that Proposition 4.2 holds true for the case  $T = t$ , we then consider the case  $T = t+1$ .

In the case  $T = t+1$ , let us denote  $g_{t+1} := g := \xi \cdot X_{t+1}$ . It is clear that  $g_{t+1}$  is a Borel random variable and  $g_{t+1}(\omega^{t+1}, \theta_0, \dots, \theta_{t+1})$  depends only on  $(\omega^{t+1}, \theta_{t+1})$ . Let  $g_t$  be defined by (4.27). Since Proposition 4.2 is assumed to hold true for the case  $T = t$ , it follows that

there is  $\hat{H} = (\hat{H}_1, \dots, \hat{H}_t) \in \mathcal{H}_t$  such that, for any universally measurable random variable  $Y_t : \bar{\Omega}_t \rightarrow \mathbb{R}_+$ , one has

$$\sup_{\bar{\mathbb{P}} \in \bar{\mathcal{P}}_{\text{int}}} \log \mathbb{E}^{\bar{\mathbb{P}}} [\exp(g_t + (\hat{H} \circ X)_t)] = \sup_{\bar{\mathbb{Q}} \in \bar{\mathcal{Q}}_{Y_t, t}^*} \left\{ \mathbb{E}^{\bar{\mathbb{Q}}} [g_t] - \mathcal{E}(\bar{\mathbb{Q}}, \bar{\mathcal{P}}_{\text{int}} |_{\bar{\Omega}_t}) \right\}. \quad (4.30)$$

Then with the function  $h_{t+1}$  defined in Lemma 4.10, we define

$$\hat{H}_{t+1}(\omega^t) := h_{t+1}(\omega^t, \hat{H}_t(\omega^{t-1})). \quad (4.31)$$

Further, for any  $\bar{\mathbb{P}} \in \bar{\mathcal{P}}_{\text{int}}$ , one has the representation  $\bar{\mathbb{P}} = \bar{\mathbb{P}}_0 \otimes \dots \otimes \bar{\mathbb{P}}_t$ , where  $\bar{\mathbb{P}}_s(\cdot)$  is measurable kernel in  $\bar{\mathcal{P}}_{\text{int}}^\delta(s, \cdot)$ . It follows by direct computation that

$$\begin{aligned} \mathbb{E}^{\bar{\mathbb{P}}} \left[ \exp \left( g_{t+1} + (\hat{H} \circ X)_{t+1} \right) \right] &= \mathbb{E}^{\bar{\mathbb{P}}_0 \otimes \dots \otimes \bar{\mathbb{P}}_{t-1}} \left[ \exp \left( \log \mathbb{E}^{\bar{\mathbb{P}}_t} \left[ \exp \left( g_{t+1} + (\hat{H} \circ X)_{t+1} \right) \right] \right) \right] \\ &\leq \mathbb{E}^{\bar{\mathbb{P}}} \left[ \exp \left( \sup_{\bar{\mathbb{P}}' \in \bar{\mathcal{P}}_{\text{int}}^\delta(t, \cdot)} \log \mathbb{E}^{\bar{\mathbb{P}}'} \left[ \exp \left( g_{t+1} + (\hat{H} \circ X)_{t+1} \right) \right] \right) \right] \\ &\leq \sup_{\bar{\mathbb{P}} \in \bar{\mathcal{P}}_{\text{int}}} \mathbb{E}^{\bar{\mathbb{P}}} \left[ \exp \left( g'_t(\cdot, \hat{H}_t) + (\hat{H} \circ X)_{t-1} - \hat{H}_t X_{t-1} \right) \right], \end{aligned}$$

where the last inequality follows by the definition of  $\hat{H}_{t+1}$  in (4.31) and Lemma 4.10. Taking the supremum over  $\bar{\mathbb{P}} \in \bar{\mathcal{P}}_{\text{int}}$ , it follows from the definition of  $g'_t$  in (4.28) together with a dynamic programming argument that

$$\begin{aligned} \sup_{\bar{\mathbb{P}} \in \bar{\mathcal{P}}_{\text{int}}} \mathbb{E}^{\bar{\mathbb{P}}} \left[ \exp \left( g_{t+1} + (\hat{H} \circ X)_{t+1} \right) \right] &\leq \sup_{\bar{\mathbb{P}} \in \bar{\mathcal{P}}_{\text{int}}} \mathbb{E}^{\bar{\mathbb{P}}} \left[ \exp \left( g_t + \hat{H}_t X_t + (\hat{H} \circ X)_{t-1} - \hat{H}_t X_{t-1} \right) \right] \\ &= \sup_{\bar{\mathbb{P}} \in \bar{\mathcal{P}}_{\text{int}}} \mathbb{E}^{\bar{\mathbb{P}}} \left[ \exp \left( g_t + (\hat{H} \circ X)_t \right) \right]. \end{aligned} \quad (4.32)$$

Then for any universally measurable random variable  $\varphi : \bar{\Omega} \rightarrow \mathbb{R}_+$ , we set  $Y_{t+1} := \varphi$  and use sequentially Lemma 4.11, (4.30), (4.32), to obtain

$$\begin{aligned} \sup_{\bar{\mathbb{Q}} \in \bar{\mathcal{Q}}_\varphi^*} \left\{ \mathbb{E}^{\bar{\mathbb{Q}}} [g_{t+1}] - \mathcal{E}(\bar{\mathbb{Q}}, \bar{\mathcal{P}}_{\text{int}} |_{\bar{\Omega}_{t+1}}) \right\} &\geq \sup_{\bar{\mathbb{Q}} \in \bar{\mathcal{Q}}_{Y_t, t}^*} \left\{ \mathbb{E}^{\bar{\mathbb{Q}}} [g_t] - \mathcal{E}(\bar{\mathbb{Q}}, \bar{\mathcal{P}}_{\text{int}} |_{\bar{\Omega}_t}) \right\} \\ &= \sup_{\bar{\mathbb{P}} \in \bar{\mathcal{P}}_{\text{int}}} \log \mathbb{E}^{\bar{\mathbb{P}}} [\exp(g_t + (\hat{H} \circ X)_t)] \\ &\geq \sup_{\bar{\mathbb{P}} \in \bar{\mathcal{P}}_{\text{int}}} \log \mathbb{E}^{\bar{\mathbb{P}}} \left[ \exp \left( g_{t+1} + (\hat{H} \circ X)_{t+1} \right) \right] \\ &\geq \inf_{H \in \mathcal{H}} \sup_{\bar{\mathbb{P}} \in \bar{\mathcal{P}}_{\text{int}}} \log \mathbb{E}^{\bar{\mathbb{P}}} [\exp(g_{t+1} + (H \circ X)_{t+1})]. \end{aligned}$$

Because the reverse inequality is the weak duality in Lemma 4.4, we obtain the equality everywhere in the above formula, which is the duality result (4.23) for the case  $T = t + 1$ . In particular,  $(\hat{H}_1, \dots, \hat{H}_t, \hat{H}_{t+1})$  is the optimal trading strategy for the case  $T = t + 1$ .  $\square$

### 4.3 Proof of Theorem 3.1 (Case $e \geq 1$ )

In this section, we are interested in the utility maximization problem with semi-static strategy. To take into account of the transaction costs caused by trading the static options  $(\zeta_i, i = 1, \dots, e)$ , we work in the framework of [12] and introduce a further enlarged space by

$$\hat{\Lambda} := \prod_{i=1}^e [-c_i, c_i], \quad \hat{\Omega} := \bar{\Omega} \times \hat{\Lambda}, \quad \hat{\mathcal{F}}_t := \bar{\mathcal{F}}_t \otimes \mathcal{B}(\hat{\Lambda}), \quad \hat{\mathcal{P}}_{\text{int}} := \left\{ \hat{\mathbb{P}} \in \mathfrak{B}(\hat{\Omega}) : \hat{\mathbb{P}}|_{\bar{\Omega}} \in \bar{\mathcal{P}}_{\text{int}} \right\},$$

and define

$$\hat{f}_i : \hat{\Omega} \rightarrow \mathbb{R}, \quad \hat{f}_i(\hat{\omega}) = \zeta_i(\omega) \cdot X_T(\bar{\omega}) - \hat{\theta}_i \text{ for all } \hat{\omega} = (\bar{\omega}, \hat{\theta}) = (\omega, \theta, \hat{\theta}) \in \hat{\Omega}.$$

The process  $(X_t)_{0 \leq t \leq T}$  and the random variable  $g := \xi \cdot X_T$  defined on  $\bar{\Omega}$  can be naturally extended on  $\hat{\Omega}$ .

We can then consider the exponential utility maximization problem on  $\hat{\Omega}$ :

$$\inf_{(H, \ell) \in \mathcal{H} \times \mathbb{R}^e} \sup_{\hat{\mathbb{P}} \in \hat{\mathcal{P}}_{\text{int}}} \log \mathbb{E}^{\hat{\mathbb{P}}} \left[ \exp \left( g + \sum_{i=1}^e \ell_i \hat{f}_i + (H \circ X)_T \right) \right].$$

Let us also introduce

$$\hat{\mathcal{Q}}_e^* := \left\{ \hat{\mathbb{Q}} \in \mathfrak{B}(\hat{\Omega}) : \begin{array}{l} \hat{\mathbb{Q}} \lll \hat{\mathcal{P}}_{\text{int}}, \text{ } X \text{ is } (\hat{\mathbb{F}}, \hat{\mathbb{Q}})\text{-martingale, } \mathbb{E}^{\hat{\mathbb{Q}}}[\hat{f}_i] = 0, \text{ } i = 1, \dots, e, \\ \mathbb{E}^{\hat{\mathbb{Q}}}[(\xi \cdot X_T)_-] + \mathcal{E}(\hat{\mathbb{Q}}, \hat{\mathcal{P}}_{\text{int}}) < +\infty \end{array} \right\},$$

and

$$\hat{\mathcal{Q}}_{e, \varphi}^* := \{\hat{\mathbb{Q}} \in \hat{\mathcal{Q}}_e^* : \mathbb{E}^{\hat{\mathbb{Q}}}[\varphi] < +\infty\}, \text{ for all } \varphi : \hat{\Omega} \rightarrow \mathbb{R}_+.$$

It is easy to employ similar arguments for Lemma 4.1 and Proposition 2.5 to obtain

$$\begin{aligned} & \inf_{(\ell, \eta) \in \mathcal{A}_e} \sup_{\mathbb{P} \in \mathcal{P}} \log \mathbb{E}^{\mathbb{P}} \left[ \exp \left( \left( \xi - \sum_{i=1}^e (\ell_i \zeta_i - |\ell_i| c_i 1_d) - \sum_{t=0}^T \eta_t \right)^d \right) \right] \\ &= \inf_{(H, \ell) \in \mathcal{H} \times \mathbb{R}^e} \sup_{\hat{\mathbb{P}} \in \hat{\mathcal{P}}_{\text{int}}} \log \mathbb{E}^{\hat{\mathbb{P}}} \left[ \exp \left( g + \sum_{i=1}^e \ell_i \hat{f}_i + (H \circ X)_T \right) \right], \end{aligned}$$

and

$$\sup_{(\mathbb{Q}, Z) \in \mathcal{S}_e^*} \{\mathbb{E}^{\mathbb{Q}}[\xi \cdot Z_T] - \mathcal{E}(\mathbb{Q}, \mathcal{P})\} = \sup_{\hat{\mathbb{Q}} \in \hat{\mathcal{Q}}_e^*} \{\mathbb{E}^{\hat{\mathbb{Q}}}[g] - \mathcal{E}(\hat{\mathbb{Q}}, \hat{\mathcal{P}}_{\text{int}})\},$$

with  $g := \xi \cdot X_T$ . Hence, to conclude the proof of Theorem 3.1 (case  $e \geq 1$ ), it is sufficient to prove that, for any universally measurable  $\varphi : \hat{\Omega} \rightarrow \mathbb{R}_+$ , one has

$$\inf_{(H, \ell) \in \mathcal{H} \times \mathbb{R}^e} \sup_{\hat{\mathbb{P}} \in \hat{\mathcal{P}}_{\text{int}}} \log \mathbb{E}^{\hat{\mathbb{P}}} \left[ \exp \left( g + \sum_{i=1}^e \ell_i \hat{f}_i + (H \circ X)_T \right) \right] = \sup_{\hat{\mathbb{Q}} \in \hat{\mathcal{Q}}_{e, \varphi}^*} \{\mathbb{E}^{\hat{\mathbb{Q}}}[g] - \mathcal{E}(\hat{\mathbb{Q}}, \hat{\mathcal{P}}_{\text{int}})\}. \quad (4.33)$$

Let us first provide a useful lemma.

**Lemma 4.12.** *Let  $g : \bar{\Omega} \rightarrow \mathbb{R}$  be upper semi-analytic, and assume that NA2( $\mathcal{P}$ ) holds. Assume either that  $e = 0$ , or that  $e \geq 1$  and for all  $\ell \in \mathbb{R}^e$  and  $\eta \in \mathcal{A}$ , (3.15) holds. Then, for all  $\varphi : \hat{\Omega} \rightarrow \mathbb{R}_+$ , one has*

$$\inf \left\{ y \in \mathbb{R} : y + \sum_{i=1}^e \ell_i \hat{f}_i + (H \circ X)_T \geq g, \hat{\mathcal{P}}_{\text{int}}\text{-q.s.}, \ell \in \mathbb{R}^e, H \in \mathcal{H} \right\} = \sup_{\bar{\mathbb{Q}} \in \bar{\mathcal{Q}}_{e, \varphi}^*} \mathbb{E}^{\bar{\mathbb{Q}}}[g]. \quad (4.34)$$

*Proof.* By Proposition 2.10, NA2( $\mathcal{P}$ ) implies NA( $\bar{\mathcal{P}}_{\text{int}}$ ). For the case  $e = 0$ , as observed by [3, Lemma 3.5], Lemma 3.3 of [13] has indeed proved the following stronger version of fundamental lemma with  $T = 1$ :

$$0 \in \text{ri}\{ \mathbb{E}^{\bar{\mathbb{Q}}}[\Delta X], \bar{\mathbb{Q}} \in \bar{\mathcal{Q}}_{\varphi}^* \}. \quad (4.35)$$

Using (4.35), we can proceed as [13, Lemma 3.5, 3.6, Theorem 4.1] to prove (4.34) in the case without options ( $e = 0$ ).

For the case  $e \geq 1$ , we can argue by induction. Suppose the super-replication theorem with  $e - 1$  options holds with  $g = \hat{f}_e$ :

$$\begin{aligned}\hat{\pi}_{e-1}(g) &:= \inf \left\{ y : y + \sum_{i=1}^{e-1} \ell_i \hat{f}_i + (H \circ X)_T \geq g, \hat{\mathcal{P}}_{\text{int-q.s.}}, \ell \in \mathbb{R}^{e-1}, H \in \mathcal{H} \right\} \\ &= \sup_{\hat{\mathbb{Q}} \in \hat{\mathcal{Q}}_{e-1, \varphi}^*} \mathbb{E}^{\hat{\mathbb{Q}}}[g],\end{aligned}\tag{4.36}$$

and we shall pass to  $e$ . By the no arbitrage condition (3.15), there is no  $H \in \mathcal{H}$ ,  $\ell_1, \dots, \ell_{e-1}$  and  $\ell_e \in \{-1, 1\}$  such that  $\sum_{i=1}^{e-1} \ell_i \hat{f}_i + (H \circ X)_T \geq -\ell_e \hat{f}_e$ ,  $\hat{\mathcal{P}}_{\text{int-q.s.}}$ . It follows that  $\hat{\pi}_{e-1}(\hat{f}_e), \hat{\pi}_{e-1}(-\hat{f}_e) > 0$ , which, by [13, Lemma 3.12] and (4.36), implies that there is  $\hat{\mathbb{Q}}_-, \hat{\mathbb{Q}}_+ \in \hat{\mathcal{Q}}_{e-1, \varphi}^*$  such that

$$-\hat{\pi}_{e-1}(-\hat{f}_e) < \mathbb{E}^{\hat{\mathbb{Q}}_-}[\hat{f}_e] < 0 < \mathbb{E}^{\hat{\mathbb{Q}}_+}[\hat{f}_e] < \hat{\pi}_{e-1}(\hat{f}_e).\tag{4.37}$$

In particular, we have

$$0 \in \text{ri}\{ \mathbb{E}^{\hat{\mathbb{Q}}}[\hat{f}_e], \hat{\mathbb{Q}} \in \hat{\mathcal{Q}}_{e-1, \varphi}^* \}.\tag{4.38}$$

Then we can argue line by line as [12, Proof of Theorem 3.1(case  $e \geq 1$ )] to prove that

$$\text{there exists a sequence } (\hat{\mathbb{Q}}_n)_{n \geq 1} \subset \hat{\mathcal{Q}}_{e, \varphi}^* \text{ s.t. } \mathbb{E}^{\hat{\mathbb{Q}}_n}[g] \rightarrow \hat{\pi}_e(g),$$

which shows that

$$\sup_{\hat{\mathbb{Q}} \in \hat{\mathcal{Q}}_{e, \varphi}^*} \mathbb{E}^{\hat{\mathbb{Q}}}[g] \geq \hat{\pi}_e(g).$$

To conclude, it is enough to notice that the reverse inequality is the classical weak duality which can be easily obtained from [13, Lemmas A.1 and A.2].  $\square$

**Proof of Theorem 3.1 (case  $e \geq 1$ ).** Notice that (4.33) has been proved for the case  $e = 0$  in Section 4.2, although the formulations are slightly different. The proof is still based on the induction as in the proof of [3, Theorem 2.2]. Let us assume that (4.33) holds for  $e - 1 \geq 0$  and then prove it for the case of  $e$ . Define

$$J : \hat{\mathcal{Q}}_{e-1, \varphi}^* \times \mathbb{R} \rightarrow \mathbb{R}, \quad (\hat{\mathbb{Q}}, \beta) \mapsto \mathbb{E}^{\hat{\mathbb{Q}}}[g] + \beta \mathbb{E}^{\hat{\mathbb{Q}}}[\hat{f}_e] - H(\hat{\mathbb{Q}}, \hat{\mathcal{P}}_{\text{int}}).$$

Clearly,  $J$  is concave in the first argument and convex in the second argument. By (4.38),  $J$  satisfies the compactness-type condition (14) in [3], thus we can apply the minimax theorem. Using the induction hypothesis and the same arguments as in [3], we have

$$\begin{aligned}& \inf_{(H, \ell) \in \mathcal{H} \times \mathbb{R}^e} \sup_{\hat{\mathbb{P}} \in \hat{\mathcal{P}}_{\text{int}}} \log \mathbb{E}^{\hat{\mathbb{P}}} \left[ \exp \left( g + \sum_{i=1}^e \ell_i \hat{f}_i + (H \circ X)_T \right) \right] \\ &= \inf_{\beta \in \mathbb{R}} \min_{(H, \ell) \in \mathcal{H} \times \mathbb{R}^{e-1}} \sup_{\hat{\mathbb{P}} \in \hat{\mathcal{P}}_{\text{int}}} \mathbb{E}^{\hat{\mathbb{P}}} \left[ \exp \left( g + \sum_{i=1}^{e-1} \ell_i \hat{f}_i + \beta \hat{f}_e + (H \circ X)_T \right) \right] \\ &= \inf_{\beta \in \mathbb{R}} \sup_{\hat{\mathbb{Q}} \in \hat{\mathcal{Q}}_{e-1, \varphi}^*} J(\hat{\mathbb{Q}}, \beta) \\ &= \sup_{\hat{\mathbb{Q}} \in \hat{\mathcal{Q}}_{e-1, \varphi}^*} \inf_{\beta \in \mathbb{R}} J(\hat{\mathbb{Q}}, \beta) = \sup_{\hat{\mathbb{Q}} \in \hat{\mathcal{Q}}_{e, \varphi}^*} (\mathbb{E}^{\hat{\mathbb{Q}}}[g] - H(\hat{\mathbb{Q}}, \hat{\mathcal{P}}_{\text{int}})).\end{aligned}\tag{4.39}$$

The duality holds as a consequence. Moreover, from (15) of [3],  $\forall c < \inf_{\beta \in \mathbb{R}} \sup_{\hat{\mathbb{Q}} \in \hat{\mathcal{Q}}_{e-1, \varphi}^*} J(\hat{\mathbb{Q}}, \beta)$ ,  $\exists n$ , s.t. for all  $\beta$  satisfying  $|\beta| > n$ ,  $\sup_{\hat{\mathbb{Q}} \in \hat{\mathcal{Q}}_{e-1, \varphi}^*} J(\hat{\mathbb{Q}}, \beta) > c$ . Thus (4.39) can be rewritten as

$$\inf_{|\beta| \leq n} \sup_{\hat{\mathbb{Q}} \in \hat{\mathcal{Q}}_{e-1, \varphi}^*} J(\hat{\mathbb{Q}}, \beta).$$

Now the lower-continuity of  $\beta \mapsto \sup_{\widehat{\mathbb{Q}} \in \widehat{\mathcal{Q}}_{e,\varphi}^*} J(\widehat{\mathbb{Q}}, \beta)$  implies the existence of some  $\hat{\beta}$  such that

$$\sup_{\widehat{\mathbb{Q}} \in \widehat{\mathcal{Q}}_{e,\varphi}^*} J(\widehat{\mathbb{Q}}, \hat{\beta}) = \inf_{\beta \in \mathbb{R}} \sup_{\widehat{\mathbb{Q}} \in \widehat{\mathcal{Q}}_{e,\varphi}^*} J(\widehat{\mathbb{Q}}, \beta).$$

Combining  $\hat{\beta}$  with the optimal strategy with  $e-1$  options  $(\hat{H}, \hat{\ell}^*)$ , we deduce the existence of an optimal strategy for  $e$  options, namely  $(\hat{H}, \hat{\ell}) := (\hat{H}, (\hat{\ell}^*, \hat{\beta}))$ . Using the construction (2.8), one can obtain  $(\hat{\eta}, \hat{\ell})$  explicitly attaining the infimum in (3.17) from  $(\hat{H}, \hat{\ell})$  which is constructed already in previous steps.  $\square$

#### 4.4 Proof of Proposition 3.4

Using the expression in (3.18), one has

$$\lim_{\gamma \rightarrow \infty} \pi_\gamma(\xi) = \lim_{\gamma \rightarrow \infty} \sup_{(\mathbb{Q}, Z) \in \mathcal{S}_e^*} \left\{ \mathbb{E}^\mathbb{Q}[\xi \cdot Z_T] - \frac{1}{\gamma} \mathcal{E}(\mathbb{Q}, \mathcal{P}) \right\},$$

where the r.h.s. is increasing in  $\gamma$ . Replacing the limit by supremum, and then interchanging the order of two supremums, we have

$$\lim_{\gamma \rightarrow \infty} \pi_\gamma(\xi) = \sup_{(\mathbb{Q}, Z) \in \mathcal{S}_e^*} \sup_{\gamma} \left\{ \mathbb{E}^\mathbb{Q}[\xi \cdot Z_T] - \frac{1}{\gamma} \mathcal{E}(\mathbb{Q}, \mathcal{P}) \right\} = \sup_{(\mathbb{Q}, Z) \in \mathcal{S}_e^*} \mathbb{E}^\mathbb{Q}[\xi \cdot Z_T].$$

By similar arguments as in Section 3.2 of [12], we can reformulate the problem at the r.h.s. on the enlarged space  $\widehat{\Omega}$  and then use Lemma 4.12 to obtain that

$$\sup_{(\mathbb{Q}, Z) \in \mathcal{S}_e^*} \mathbb{E}^\mathbb{Q}[\xi \cdot Z_T] = \sup_{\widehat{\mathbb{Q}} \in \widehat{\mathcal{Q}}_e^*} \mathbb{E}^{\widehat{\mathbb{Q}}}[\xi \cdot X_T] = \pi(\xi).$$

This concludes the proof.  $\square$

## A Appendix

The main objective of this section is to prove Proposition A.1 below, which consists an important technical step in the proof of Lemma 4.11. We shall stay in the enlarged space  $\overline{\Omega}$  context as introduced in Section 2.2.

**Proposition A.1.** *Let  $\bar{g} : \overline{\Omega} \rightarrow \mathbb{R}$  be upper semi-analytic,  $\varphi : \overline{\Omega} \rightarrow \mathbb{R}_+$  a universally measurable random variable, and  $\bar{\mathbb{P}} \in \overline{\mathcal{P}}_{\text{int}}$  a fixed probability measure. Then for any  $\overline{\mathbb{Q}}^* \in \overline{\mathcal{Q}}^*$ , one has*

$$\mathbb{E}^{\overline{\mathbb{Q}}^*}[\bar{g}] - \mathcal{E}(\overline{\mathbb{Q}}^*, \bar{\mathbb{P}}) \leq \sup_{\overline{\mathbb{Q}} \in \overline{\mathcal{Q}}_\varphi^*} \mathbb{E}^{\overline{\mathbb{Q}}}[\bar{g}] - \mathcal{E}(\overline{\mathbb{Q}}, \bar{\mathbb{P}}). \quad (\text{A.40})$$

To prove Proposition A.1, we will fixe a martingale measure  $\overline{\mathbb{Q}}^*$  such that  $\mathcal{E}(\overline{\mathbb{Q}}^*, \bar{\mathbb{P}}) < \infty$  and then study an exponential utility maximization problem under  $\overline{\mathbb{Q}}^*$ :

$$\inf_{\overline{H} \in \overline{\mathcal{H}}} \log \mathbb{E}^{\overline{\mathbb{Q}}^*} [\exp (\bar{g} + (\overline{H} \circ X)_T)],$$

to establish a duality result. In this case, one has  $\overline{\mathbb{Q}}^* \ll \bar{\mathbb{P}}$  and the classical no-arbitrage condition  $\text{NA}(\{\overline{\mathbb{Q}}^*\})$  holds, i.e. for all  $\overline{H} \in \overline{\mathcal{H}}$ ,

$$(\overline{H} \circ X)_T \geq 0, \overline{\mathbb{Q}}^* \text{-a.s.} \implies (\overline{H} \circ X)_T = 0, \overline{\mathbb{Q}}^* \text{-a.s.}$$



Let us denote

$$\overline{\mathcal{Q}}_\varphi^{**} := \{\overline{\mathcal{Q}} \in \overline{\mathcal{Q}}_\varphi^* : \mathcal{E}(\overline{\mathcal{Q}}, \overline{\mathcal{Q}}^*) < +\infty\}, \quad (\text{A.41})$$

then similar to Lemma 4.4, the weak duality holds

$$\inf_{\overline{H} \in \overline{\mathcal{H}}} \log \mathbb{E}^{\overline{\mathcal{Q}}^*} [\exp(\bar{g} + (\overline{H} \circ X)_T)] \geq \sup_{\overline{\mathcal{Q}} \in \overline{\mathcal{Q}}_0^{**}} \{\mathbb{E}^{\overline{\mathcal{Q}}}[\bar{g}] - \mathcal{E}(\overline{\mathcal{Q}}, \overline{\mathcal{Q}}^*)\}, \quad (\text{A.42})$$

where  $\overline{\mathcal{Q}}_0^{**}$  is defined in (A.41) with  $\varphi \equiv 0$ . In the following, we will prove the reverse inequality (in order to obtain a duality result). The arguments are very similar to that of Proposition 4.2, except that we are in a dominated setting. For completeness, we will rather provide a sketch of the proof.

Let  $\overline{\mathcal{Q}}_t^* : \overline{\Omega}_t \rightarrow \mathfrak{B}(\overline{\Omega}_{t+1})$  be a r.c.p.d. of  $\overline{\mathcal{Q}}^*$  knowing  $\overline{\mathcal{F}}_t^0$ . For any random variable  $\bar{Y}_{t+1} : \overline{\Omega}_{t+1} \rightarrow \mathbb{R}_+$ , we define  $\overline{\mathcal{Q}}_{Y_{t+1}}^{**}(t, \bar{\omega}^t) := \{\overline{\mathcal{Q}} \in \overline{\mathcal{Q}}_{Y_{t+1}}^*(t, \bar{\omega}^t) : \mathcal{E}(\overline{\mathcal{Q}}, \overline{\mathcal{Q}}_t^*(\bar{\omega}^t)) < +\infty\}$ . Given an upper semi-analytic functional  $\bar{g}_{t+1} : \overline{\Omega}_{t+1} \rightarrow \mathbb{R}$ , we define  $\bar{g}_t$  by

$$\bar{g}_t(\bar{\omega}^t) := \sup_{\overline{\mathcal{Q}} \in \overline{\mathcal{Q}}_0^{**}(t, \bar{\omega}^t)} \left\{ \mathbb{E}^{\overline{\mathcal{Q}}}[\bar{g}_{t+1}] - \mathcal{E}(\overline{\mathcal{Q}}, \overline{\mathcal{Q}}_t^*(\bar{\omega}^t)) \right\}, \quad \text{for all } \bar{\omega}^t \in \overline{\Omega}_t.$$

Similar to Remark 4.8, for any random variable  $\bar{Y}_{t+1} : \overline{\Omega}_{t+1} \rightarrow \mathbb{R}_+$ , one has

$$\bar{g}_t(\bar{\omega}^t) = \sup_{\overline{\mathcal{Q}} \in \overline{\mathcal{Q}}_{Y_{t+1}}^{**}(t, \bar{\omega}^t)} \left\{ \mathbb{E}^{\overline{\mathcal{Q}}}[\bar{g}_{t+1}] - \mathcal{E}(\overline{\mathcal{Q}}, \overline{\mathcal{Q}}_t^*(\bar{\omega}^t)) \right\}, \quad \overline{\mathcal{Q}}^* \text{-a.s.}$$

Then similar to Lemma 4.9 and 4.10, we have the following results.

**Lemma A.2.** (i) *Given a Borel random variable  $Y_{t+1} : \overline{\Omega}_{t+1} \rightarrow \mathbb{R}_+$ , one has that*

$$\left[ \left[ \overline{\mathcal{Q}}_{Y_{t+1}}^{**}(t) \right] \right] := \left\{ (\bar{\omega}, \overline{\mathcal{Q}}) : \bar{\omega} \in \overline{\Omega}_t, \overline{\mathcal{Q}} \in \overline{\mathcal{Q}}_{Y_{t+1}}^*(t, \bar{\omega}) \text{ and } \mathcal{E}(\overline{\mathcal{Q}}, \overline{\mathcal{Q}}_t^*(\bar{\omega}^t)) < +\infty \right\} \text{ is analytic.}$$

(ii) *Assume that  $\text{NA}(\{\overline{\mathcal{Q}}^*\})$  holds true. Then  $\bar{g}_t$  is upper semi-analytic, and there exists a universally measurable map  $\bar{h}_{t+1} : \overline{\Omega}_t \rightarrow \mathbb{R}^d$ , together with a  $\overline{\mathcal{Q}}^*$ -null set  $\overline{N} \subset \overline{\Omega}_t$  such that, for all  $\bar{\omega}^t \in \overline{N}^c$ , one has*

$$g_t(\bar{\omega}^t) = \log \mathbb{E}^{\overline{\mathcal{Q}}_t^*(\bar{\omega}^t)} \left[ \exp(\bar{g}_{t+1} + \bar{h}_{t+1}(\bar{\omega}^t)(X_{t+1} - X_t)) \right] > -\infty.$$

*Proof.* (i) It follows from the definition of r.c.p.d. that  $\overline{\mathcal{Q}}_t^*(\cdot)$  is a Borel kernel, thus  $(\bar{\omega}^t, \overline{\mathcal{Q}}) \in \overline{\Omega}_t \times \mathfrak{B}(\overline{\Omega}_{t+1}) \mapsto \mathcal{E}(\overline{\mathcal{Q}}, \overline{\mathcal{Q}}_t^*(\bar{\omega}^t))$  is Borel by Lemma 4.2 of [3]. Let us define  $A := \{(\bar{\omega}, \overline{\mathcal{Q}}) \in \overline{\Omega}_t \times \mathfrak{B}(\overline{\Omega}_{t+1}) : \mathcal{E}(\overline{\mathcal{Q}}, \overline{\mathcal{Q}}_t^*(\bar{\omega}^t)) < +\infty\}$ . As in Lemma 4.9, it is easy to see that  $\left[ \left[ \overline{\mathcal{Q}}_{Y_{t+1}}^*(t) \right] \right]$  is analytic for any Borel  $Y_{t+1} : \overline{\Omega}_{t+1} \rightarrow \mathbb{R}_+$ . It follows that  $\left[ \left[ \overline{\mathcal{Q}}_{Y_{t+1}}^{**}(t) \right] \right] = \left[ \left[ \overline{\mathcal{Q}}_{Y_{t+1}}^*(t) \right] \right] \cap A$  is analytic.

(ii) As in (i), we have that  $(\bar{\omega}^t, \overline{\mathcal{Q}}) \in \overline{\Omega}_t \times \mathfrak{B}(\overline{\Omega}_{t+1}) \mapsto -\mathcal{E}(\overline{\mathcal{Q}}, \overline{\mathcal{Q}}_t^*(\bar{\omega}^t))$  is Borel and thus upper semi-analytic. As  $\bar{g}_{t+1}$  is upper semi-analytic, it follows from (i) and a measurable selection argument (see e.g. [9, Proposition 7.26, 7.47, 7.48]) that  $\bar{\omega}^t \mapsto g_t$  is upper semi-analytic. By defining

$$V_t^*(\bar{\omega}^t) := \inf_{\bar{h}_{t+1} \in \mathbb{R}^d} \log \mathbb{E}^{\overline{\mathcal{Q}}_t^*(\bar{\omega}^t)} \left[ \exp(\bar{g}_{t+1} + \bar{h}_{t+1}(X_{t+1} - X_t)) \right],$$

and applying Theorem 3.1 of [3], we obtain

$$g_t(\bar{\omega}^t) = V_t^*(\bar{\omega}^t) > -\infty, \quad \text{if } \text{NA}(\{\overline{\mathcal{Q}}_t^*(\bar{\omega}^t)\}) \text{ holds true.}$$

As  $\text{NA}(\{\overline{\mathbb{Q}}^*\})$  holds, this is valid outside a  $\overline{\mathbb{Q}}^*$ -null set.

Define  $\phi_t(\bar{\omega}^t, \bar{h}_{t+1}) := \log \mathbb{E}^{\overline{\mathbb{Q}}^*}(\bar{\omega}^t) \left[ \exp(\bar{g}_{t+1} + \bar{h}_{t+1}(X_{t+1} - X_t)) \right]$ , we can argue similarly as Lemma 4.10 to see that  $(\bar{\omega}^t, \bar{h}_{t+1}) \mapsto \phi_t$  is in  $\overline{\mathcal{F}}_t \otimes \mathcal{B}(\mathbb{R}^d)$ . Let us now consider the random set

$$\Phi(\bar{\omega}^t) := \{h \in \mathbb{R}^d : \phi(\bar{\omega}^t, h) = g_t(\bar{\omega}^t)\}.$$

The previous arguments yield that  $[[\Phi]]$  is in  $\overline{\mathcal{F}}_t \otimes \mathcal{B}(\mathbb{R}^d)$ . Thus by Lemma 4.11 of [13],  $\Phi$  admits an  $\overline{\mathcal{F}}_t$ -measurable selector  $\bar{h}_{t+1}$  on the universally measurable set  $\Phi(\bar{\omega}^t) \neq \emptyset$ . Moreover, Theorem 3.1 of [3] implies that  $\Phi(\bar{\omega}^t) \neq \emptyset$  holds true outside a  $\overline{\mathbb{Q}}^*$ -null set  $\bar{N}$ , thus  $\bar{h}_{t+1}$  solves the infimum  $\overline{\mathbb{Q}}^*$ -a.s.  $\square$

For any random variable  $\bar{Y}_t : \bar{\Omega}_t \rightarrow \mathbb{R}_+$ , we define  $\overline{\mathcal{Q}}_{\bar{Y}_t, t}^{**} := \{\overline{\mathbb{Q}} \in \overline{\mathcal{Q}}_{\bar{Y}_t, t}^* : \mathcal{E}(\overline{\mathbb{Q}}, \overline{\mathbb{Q}}^* |_{\bar{\Omega}_t}) < +\infty\}$ .

**Lemma A.3.** (i) *Let  $t+1 \leq T$ , then for any universally measurable random variable  $\bar{Y}_{t+1} : \bar{\Omega}_{t+1} \rightarrow \mathbb{R}_+$ , there is a universally measurable random variable  $\bar{Y}_t : \bar{\Omega}_t \rightarrow \mathbb{R}_+$  such that*

$$\sup_{\overline{\mathbb{Q}} \in \overline{\mathcal{Q}}_{\bar{Y}_t, t}^{**}} \left\{ \mathbb{E}^{\overline{\mathbb{Q}}}[\bar{g}_t] - \mathcal{E}(\overline{\mathbb{Q}}, \overline{\mathbb{Q}}^* |_{\bar{\Omega}_t}) \right\} \leq \sup_{\overline{\mathbb{Q}} \in \overline{\mathcal{Q}}_{\bar{Y}_{t+1}, t+1}^{**}} \left\{ \mathbb{E}^{\overline{\mathbb{Q}}}[\bar{g}_{t+1}] - \mathcal{E}(\overline{\mathbb{Q}}, \overline{\mathbb{Q}}^* |_{\bar{\Omega}_{t+1}}) \right\}. \quad (\text{A.43})$$

(ii) *For any random variable  $\varphi : \bar{\Omega} \rightarrow \mathbb{R}_+$ , one has*

$$\inf_{\bar{H} \in \bar{\mathcal{H}}} \log \mathbb{E}^{\overline{\mathbb{Q}}^*} \left[ \exp(\bar{g} + (\bar{H} \circ X)_T) \right] = \sup_{\overline{\mathbb{Q}} \in \overline{\mathcal{Q}}_{\varphi}^{**}} \left\{ \mathbb{E}^{\overline{\mathbb{Q}}}[\bar{g}] - \mathcal{E}(\overline{\mathbb{Q}}, \overline{\mathbb{Q}}^*) \right\}.$$

Moreover, there exists an optimal trading strategy  $\bar{H}^*$ .

*Proof.* (i) Under the reference probability  $\overline{\mathbb{Q}}^*$ , for any universally measurable random variable  $\bar{Y}_{t+1}$ , there exists a Borel measurable random variable  $\bar{Y}_{t+1}^*$ , such that  $\bar{Y}_{t+1} = \bar{Y}_{t+1}^*$ ,  $\overline{\mathbb{Q}}^*$ -a.s. and thus  $\mathbb{E}^{\overline{\mathbb{Q}}}[\bar{Y}_{t+1}] = \mathbb{E}^{\overline{\mathbb{Q}}}[\bar{Y}_{t+1}^*]$ , for all  $\overline{\mathbb{Q}} \in \overline{\mathcal{Q}}_{\bar{Y}_{t+1}, t+1}^{**}$ . So we can assume w.l.o.g. that  $\bar{Y}_{t+1}$  is Borel measurable. By Lemma A.2 (i) together with a measurable selection argument (see e.g. Proposition 7.50 of [9]), for any  $\varepsilon > 0$ , there exists a universally measurable kernel  $\overline{\mathbb{Q}}_t^\varepsilon(\cdot) : \bar{\Omega}_t \rightarrow \mathfrak{B}(\bar{\Omega}_1)$  such that  $\delta_{\bar{\omega}} \otimes \overline{\mathbb{Q}}_t^\varepsilon(\bar{\omega}) \in \overline{\mathcal{Q}}_{\bar{Y}_{t+1}}^{**}(t, \bar{\omega})$  for all  $\bar{\omega} \in \bar{\Omega}_t$ , and

$$\bar{g}_t(\bar{\omega}) \leq \mathbb{E}^{\delta_{\bar{\omega}} \otimes \overline{\mathbb{Q}}_t^\varepsilon(\bar{\omega})}[\bar{g}_{t+1}] - \mathcal{E}(\delta_{\bar{\omega}} \otimes \overline{\mathbb{Q}}_t^\varepsilon(\bar{\omega}), \overline{\mathbb{Q}}^*(\bar{\omega})) + \varepsilon.$$

The rest arguments are exactly the same as in Lemma 4.11 and we shall omit the details.

(ii) We can argue by induction as in the proof of Proposition 4.2. Noticing  $\text{NA}(\{\overline{\mathbb{Q}}^*\})$  holds, the case  $T = 1$  is proved by Theorem 3.1 of [3]. Suppose the case  $T = t$  is verified with optimal strategy  $\hat{H} := (\hat{H}_1, \dots, \hat{H}_t)$ :

$$\log \mathbb{E}^{\overline{\mathbb{Q}}^*} \left[ \exp(\bar{g}_t + (\hat{H} \circ X)_t) \right] = \sup_{\overline{\mathbb{Q}} \in \overline{\mathcal{Q}}_{\bar{Y}_t, t}^{**}} \left\{ \mathbb{E}^{\overline{\mathbb{Q}}}[\bar{g}_t] - \mathcal{E}(\overline{\mathbb{Q}}, \overline{\mathbb{Q}}^* |_{\bar{\Omega}_t}) \right\}, \quad (\text{A.44})$$

and we shall pass to the  $T = t+1$  case. Defining  $\hat{H}_{t+1}(\bar{\omega}^t) := \bar{h}_{t+1}$  as in Lemma A.2 (ii), and setting  $Y_{t+1} := \varphi$  for any universally measurable random variable  $\varphi : \bar{\Omega} \rightarrow \mathbb{R}_+$ , we have

$$\begin{aligned} & \sup_{\overline{\mathbb{Q}} \in \overline{\mathcal{Q}}_{\varphi}^{**}} \left\{ \mathbb{E}^{\overline{\mathbb{Q}}}[\bar{g}_{t+1}] - \mathcal{E}(\overline{\mathbb{Q}}, \overline{\mathbb{Q}}^* |_{\bar{\Omega}_{t+1}}) \right\} \geq \sup_{\overline{\mathbb{Q}} \in \overline{\mathcal{Q}}_{\bar{Y}_t, t}^{**}} \left\{ \mathbb{E}^{\overline{\mathbb{Q}}}[\bar{g}_t] - \mathcal{E}(\overline{\mathbb{Q}}, \overline{\mathbb{Q}}^* |_{\bar{\Omega}_t}) \right\} \\ &= \log \mathbb{E}^{\overline{\mathbb{Q}}^*} \left[ \exp(\bar{g}_t + (\hat{H} \circ X)_t) \right] \\ &= \log \mathbb{E}^{\overline{\mathbb{Q}}_0^* \otimes \dots \otimes \overline{\mathbb{Q}}_{t-1}^*} \left[ \exp \left( \log \mathbb{E}^{\overline{\mathbb{Q}}^*} \left[ \exp(\bar{g}_{t+1} + (\hat{H} \circ X)_{t+1}) \right] \right) \right] \\ &= \log \mathbb{E}^{\overline{\mathbb{Q}}^*} \left[ \exp(\bar{g}_{t+1} + (\hat{H} \circ X)_{t+1}) \right] \\ &\geq \inf_{H \in \bar{\mathcal{H}}} \log \mathbb{E}^{\overline{\mathbb{Q}}^*} \left[ \exp(\bar{g}_{t+1} + (H \circ X)_{t+1}) \right], \end{aligned}$$

where the first inequality follows by (i) and the third line follows by Lemma A.2 (ii). As the reverse inequality is the weak duality by (A.42), we have proved the case  $T = t+1$ . In particular,  $(\hat{H}_1, \dots, \hat{H}_t, \hat{H}_{t+1})$  is the optimal trading strategy for the case  $T = t+1$ .  $\square$

**Proof of Proposition A.1.** We only need to consider the case where  $\mathcal{E}(\overline{\mathbb{Q}}^*, \overline{\mathbb{P}}) < \infty$ , where in particular, one has  $\overline{\mathbb{Q}}^* \ll \overline{\mathbb{P}}$ . Recall that  $\overline{\mathbb{Q}}_\varphi^{**}$  is defined in (A.41). Then using the weak duality result in (A.42) and then the duality result in Lemma A.3 (ii), it follows that

$$\begin{aligned}
\mathbb{E}^{\overline{\mathbb{Q}}^*}[\bar{g}] - \mathcal{E}(\overline{\mathbb{Q}}^*, \overline{\mathbb{P}}) &\leq \mathbb{E}^{\overline{\mathbb{Q}}^*} \left[ \bar{g} - \log \frac{d\overline{\mathbb{Q}}^*}{d\overline{\mathbb{P}}} \right] - \mathcal{E}(\overline{\mathbb{Q}}^*, \overline{\mathbb{Q}}^*) \\
&\leq \inf_{\overline{H} \in \overline{\mathcal{H}}} \log \mathbb{E}^{\overline{\mathbb{Q}}^*} \left[ \exp \left( \bar{g} - \log \frac{d\overline{\mathbb{Q}}^*}{d\overline{\mathbb{P}}} + (\overline{H} \circ X)_T \right) \right] \\
&= \sup_{\overline{\mathbb{Q}} \in \overline{\mathcal{Q}}_\varphi^{**}} \left( \mathbb{E}^{\overline{\mathbb{Q}}} \left[ \bar{g} - \log \frac{d\overline{\mathbb{Q}}^*}{d\overline{\mathbb{P}}} \right] - \mathcal{E}(\overline{\mathbb{Q}}, \overline{\mathbb{Q}}^*) \right) \\
&\leq \sup_{\overline{\mathbb{Q}} \in \overline{\mathcal{Q}}_\varphi^*} \left( \mathbb{E}^{\overline{\mathbb{Q}}} \left[ \bar{g} - \log \frac{d\overline{\mathbb{Q}}^*}{d\overline{\mathbb{P}}} \right] - \mathcal{E}(\overline{\mathbb{Q}}, \overline{\mathbb{Q}}^*) \right) = \sup_{\overline{\mathbb{Q}} \in \overline{\mathcal{Q}}_\varphi^*} \left( \mathbb{E}^{\overline{\mathbb{Q}}}[\bar{g}] - \mathcal{E}(\overline{\mathbb{Q}}, \overline{\mathbb{P}}) \right).
\end{aligned}$$

$\square$

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