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THE EFFECT OF SLOWLY VARYING DYNAMICS ON MILLING STABILITY

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Abstract

In order to predict the quality and the stability properties of milling processes, the relevant dynamics reduced to the cutting edges needs to be known. However, this dynamics varies through the workspace along the tool path during a given machining operation. This is the case for large heavy duty milling operations, where the main source of the relevant dynamics is related to the otherwise slowly varying machine structure rather than to the fairly steady milling tool dynamics. The effect of slowly varying dynamic parameters is presented on milling stability when the cutting process takes place in a region of the working space where the steady-state cutting would change from stable to unstable. After the separation of the slow and fast time scales, the governing non-autonomous delay differential equation is frozen in slow-time in order to determine the time-periodic stationary cutting solution of the milling operation for different ram extensions.

Keywords: milling; regeneration; chatter; varying dynamics

1 Introduction

The aim of this work is to point out the effect of slowly changing parameters on milling dynamics. It is well known that machining processes like drilling, turning or milling are subjected to regenerative effect when the past relative motion of the workpiece-tool system influences the present behavior of the operation. By modelling the geometric arrangement of the cutting edges, the corresponding regenerative delays can be identified [1]. Combining with an empirical cutting force charasteristics and with the dynamic model of the machine tool structure, the governing equation can be derived [2, 3]. This results in delay differential equation (DDE) [4] of autonomous (time-independent) or non-autonomous (time-dependent) kinds for different machining operations. In case of milling, the governing equations are time-periodic due to the non-regular cutter-workpiece-engagement (CWE) and the rotation of the tool [5]. In this time-periodic case, the instability of corresponding time-periodic stationary solution refers to unstable milling operation that leads to chatter [6]. By using the Floquet theory on the linearized variational system [7], stability charts can be constructed usually in the parameter space of spindle speed n and depth of cut a. Between the stable and unstable domains, the stability boundaries correspond to either (secondary) Hopf or period doubling losses of stability.

The above mentioned methodology is capable to predict chatter for constant parameters; however, in reality, one or more parameters may be slowly varying during the machining operations. For example, large machines are well known to have varying dynamic behavior, thus, slowly moving cutter through their

workspace is subjected to slowly varying dynamic properties. In five axis milling, even rigid compact machines operate in slowly changing environment during complex 3D tool motions, while the varying geometry along the tool path also affects the CWE in time.

In mathematical terms, the slowly changing variable introduces a permanent non-cyclic time dependency in the originally time-periodic milling model. This means that the DDE cannot be handled using Floquet theory, or at least, not in a straightforward manner. The slowly changing ordinary differential equation (ODE) models have already revealed the effect of slowly changing parameters on the corresponding stability loss and bifurcation [8, 9]. In case of Hopf bifurcations, by using slow time scale, a shift of the stability boundary can be identified by considering the accumulative effect of the variational dynamics around the slowly changing stationary solution [10].

This work intends to apply the slow time scale methodology in the time-periodic DDE model of milling operations to show how those slowly varying parameters affect the classical chatter predictions. The paper considers a simple 1Dof model of the milling operation as a demonstrative example. After having a mathematical assumption for the slow-time deviation from the frozen-parameter case, a first order delay differential equation is derived for the amplitude and phase deviations. Stability of the slowly changing dynamics is determined by considering the exponential growth of the amplitude deviation from the frozen-time solutions. The single degree of freedom model describes 'naively' the cantilever-like structural arrangement of a heavy-duty milling machine. The model represents the slowly changing dynamics during milling processes performed in ram-axis-direction. It is shown that ram-directional motion can shift stability boundaries only because of the varying dynamics.

2 1DoF milling model

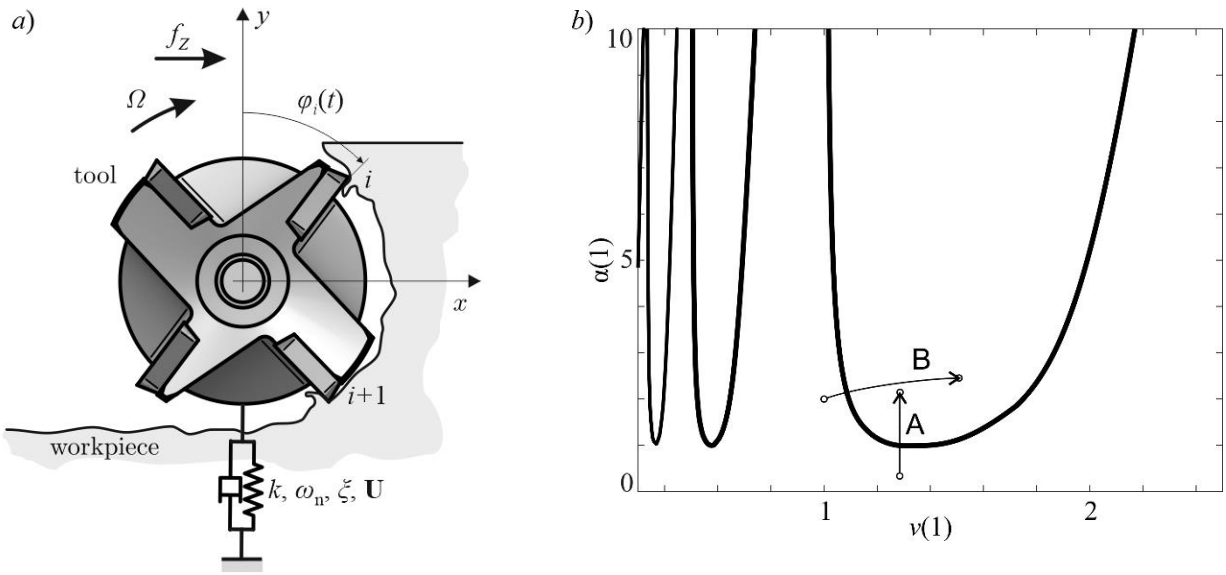

Figure 1: *a*) sketch of the actual mechanical model of milling; *b*) dimensionless stability lobe diagram with dimensionless spindle speed υ and depth of cut α [11]. The paths A and B represent examples related to slowly increasing depth of cut *a* (see subsection 4.1) and ram overhang *L* (see subsection 4.2), respectively.

Simple one DoF model of the regenerative milling process is considered here with the following second order time periodic DDE

$$\ddot{q}(t) + 2\,\xi\,\omega_{\mathrm{n}}\,\dot{q}(t) + \omega_{\mathrm{n}}^2\,q(t) = a\,K_{\mathrm{c},t}\,B(t)\big(q(t-\tau) - q(t)\big) + G(t), \tag{1}$$

where ω_{n}, ξ, τ and $K_{\mathrm{c},t}$ stand for the natural angular frequency, damping ratio, regenerative delay and tangential cutting coefficient, respectively. The regenerative delay is originated in the tooth passing frequency $\Omega_Z = Z\Omega$ as $\tau = 2\pi/\Omega_Z$, where Ω (rad/s) = $2\pi n$ rpm / (60 s/min) is the angular velocity of the tool. The milling cutter is assumed to be equally spaced with cutting edges, that is, the principle period equals with the regenerative delay as $T_{\mathrm{p}} = \tau$ (see Figure 1*a*). The stability diagram is usually depicted in the parameter space spanned by the spindle speed n and axial depth of cut a as demonstrated in Figure 1*b*. As shown in [11] and [12], the time periodicity of (1) appears in

$$B(t) \equiv B\big(t + T_{\mathrm{p}}\big) = \mathbf{U}^{\mathsf{T}} \sum_{i=1}^{Z} \frac{g\big(\varphi_i(t)\big)}{\sin\kappa}\,\mathbf{T}\big(\varphi_i(t)\big)\boldsymbol{\kappa}_{\mathrm{c}} \otimes \mathbf{n}\big(\varphi_i(t)\big)\,\mathbf{U}, \tag{2}$$

and in the stationary excitation

$$G(t) \equiv G\big(t + T_{\mathrm{p}}\big) = -a\,\mathbf{U}^{\mathsf{T}} \sum_{i=1}^{Z} \frac{g\big(\varphi_i(t)\big)}{\sin\kappa}\,\mathbf{T}\big(\varphi_i(t)\big)\left(\mathbf{K}_{\mathrm{e}} + \mathbf{K}_{\mathrm{c}} \otimes \mathbf{n}\big(\varphi_i(t)\big)\begin{bmatrix} f_Z \\ 0 \\ 0 \end{bmatrix}\right). \tag{3}$$

The mass normalized mode shape vector, lead angle and the feed per tooth are denoted by $\mathbf{U}$, κ and f_Z. The edge normal, the edge coefficients and the cutting coefficients are stored in $\mathbf{n}$, $\mathbf{K}_{\mathrm{e}}$ and $\mathbf{K}_{\mathrm{c}} = K_{\mathrm{c},t}\boldsymbol{\kappa}_{\mathrm{c}}$ vectors defined in local edge (t, r, a) system, while the transformation between (t, r, a) and (x, y, z) is realized by $\mathbf{T}(\varphi)$ [13]. The edge angular position and the effect of CWE are traced by $\varphi_i(t)$ and $g(\varphi)$ [13].

Linear stability of the time periodic stationary solution $Q(t) = Q\big(t + T_{\mathrm{p}}\big)$ of (1) can be easily determined by various methods, like multi-frequency solution [14, 15, 16] and semi-discretization [17]. The presented example with 90 deg lead angle κ and zero helix angle η (see Table 1) was calculated by first order semi-discretization and using triangularization algorithm.

Table 1: Parameters of full immersion milling process used in this paper [19].

Z	κ (deg)	η (deg)	feed direction		mode direction
4	90	0	[1 0 0]		[0 1 0]
$K_{\mathrm{c},t}$ (MPa)	$K_{\mathrm{c},r}$ (MPa)	$K_{\mathrm{c},a}$ (MPa)	ω_{n} (Hz)	ξ (%)	k (N/μm)
1459	259	0	94	0.66	58.38

3 Slowly changing milling model

A real milling process is usually subjected to time-dependent slowly changing parameters. In this manner, we can modify the DDE (1) by considering some slowly changing parameters with respect to the so-called "slow time" $s \coloneqq \varepsilon t$, where the general rate of change is denoted by ε. Thus,

$$\ddot{y}(t) + 2\,\xi(s)\omega_{\mathrm{n}}(s)\dot{y}(t) + \omega_{\mathrm{n}}^2(s)y(t) = a(s)K_{\mathrm{c},t}\,B(t,s)\Big(y\big(t-\tau(s)\big) - y(t)\Big) + G(t,s). \tag{4}$$

Surely, not all parameters are changing at the same rate during a given process. For example, in the general form of (4) one can model a cutting process defined for a ramp-like workpiece [18] with slowly changing $a \coloneqq a(s)$ and $G(t) \coloneqq G(t,s)$ (see path A in Figure 1*b* and description in Subsection 4.1). In the meantime, the changing ram extension can be followed by slowly changing $\omega_{\mathrm{n}} \coloneqq \omega_{\mathrm{n}}(s)$, $\xi \coloneqq \xi(s)$, $B(t) \coloneqq B(t,s)$

and $G(t) \coloneqq G(t,s)$ due to $\mathbf{U} \coloneqq \mathbf{U}(s)$ (see path B in Figure 1b and description in Section 4.2). Note that $y \equiv q$; it has been introduced only to distinguish the solution of (1) and that of (4).

In any case, the governing DDE (4) form of the slowly changing milling operation is no longer exactly time-periodic. It behaves as a general non-autonomous (time-dependent) system with two different time scales described by the (real) 'fast time' t and the (introduced) 'slow time' s.

3.1 Stationary Solution

It is straightforward to assume that (4) must have a slowly changing but in this case not time-periodic stationary solution $Y(t) \neq Y(t+T_\mathrm{p})$. Perturbation x is introduced in the solution as

$$y(t) := Y(t,s) + x(t). \tag{5}$$

Abusing the notation, the real time and slow time dependencies are dropped for a while. Substituting the assumption (5) into (4) one can get the following form

$$\ddot{Y} + \ddot{x} + 2\xi\omega_\mathrm{n}\left(\dot{Y} + \dot{x}\right) + \omega_\mathrm{n}^2\,(Y + x) = \; a\,K_{\mathrm{c},t}\,B(Y^{\tau,\varepsilon\tau} + x^{\tau,\varepsilon\tau} - Y - x) + G, \tag{6}$$

where

$$Y^{\tau,\varepsilon\tau} := Y(t-\tau, s-\varepsilon\tau) \approx Y(t-\tau,s) - \varepsilon\tau\frac{\partial}{\partial s}Y(t-\tau,s) =: Y^{\tau,0} - \varepsilon\tau\,Y_s^{\tau,0}.$$

This results in

$$\ddot{Y} + \ddot{x} + 2\xi\omega_\mathrm{n}\left(\dot{Y} + \dot{x}\right) + \omega_\mathrm{n}^2\,(Y + x) = \; a\,K_{\mathrm{c},t}\,B\left(Y^{\tau,0} - \varepsilon\tau\,Y_s^{\tau,0} + x^{\tau,\varepsilon\tau} - Y - x\right) + G. \tag{7}$$

Approximating the stationary solution $Y(t,s)$ of (4) with the stationary solution of (1) for fixed (frozen) slow time s, that is, by $Y(t,s) \approx Q(t;s) = Q(t+T_\mathrm{p};s)$, (7) is simplified to

$$\ddot{x} + 2\xi\omega_\mathrm{n}\dot{x} + \omega_\mathrm{n}^2\,x = \; a\,K_{\mathrm{c},t}\,B(x^{\tau,\varepsilon\tau} - x) - a\,K_{\mathrm{c},t}\,\varepsilon\tau\,Q_s^{\tau,0}. \tag{8}$$

Considering the time periodicity in $Q_s^{\tau,0}$, (8) has the same form as (4). By following the same procedure, a frozen-time stationary solution can be determined as $X(t;s)$ herewith introduced by

$$x(t) := \varepsilon\,X(t;s) + u(t). \tag{9}$$

In this successive substitution, actually, an expansion w.r.t. ε of the stationary solution of (4) can be derived, resulting in the following form

$$\ddot{u} + 2\xi\omega_\mathrm{n}\dot{u} + \omega_\mathrm{n}^2\,u = \; a\,K_{\mathrm{c},t}\,B(u^{\tau,\varepsilon\tau} - u) - a\,K_{\mathrm{c},t}\,\varepsilon^2\tau\,X_s^{\tau,0}, \tag{10}$$

with

$$Y(t,s) \approx Q(t;s) + \varepsilon\,X(t;s). \tag{11}$$

Considering ε to be sufficiently small, the last term in (10) can be dropped and the variational equation described by the perturbation u around the stationary solution Y has the form

$$\ddot{u} + 2\xi\omega_\mathrm{n}\dot{u} + \omega_\mathrm{n}^2\,u = \; a\,K_{\mathrm{c},t}\,B(u^{\tau,\varepsilon\tau} - u). \tag{12}$$

3.2 Asymptotic behavior of the stationary solution

The asymptotic behavior of this non-autonomous slow/fast system can be derived by the WKB method [10] considering the original time periodicity slightly depending on the slow time s. According to Floquet theory [7], the general solution of a linear time periodic system is given by an exponential and a time-periodic term

[19]. Similarly, using the WKB method, the following general solution can be assumed for the slowly changing equation (12):

$$u(t) := u(t,s) = \mathrm{e}^{\frac{\sigma(s)}{\varepsilon}} A(t,s), \text{ where } A(t,s) = A(t+T_{\mathrm{p}}, s). \tag{13}$$

In (13), the slow time s is also carried by the asymptotic behavior referring to the stability (rapid exponential growth $\sigma(s)$) of the slowly varying solution u on the fast time scale t. Substitution of (13) into (12) leads to a partial differential equation as

$$\varepsilon\, \sigma_{ss} A + \sigma_s^2 A + 2\sigma_s \dot{A} + \ddot{A} + 2\xi\omega_{\mathrm{n}}\left(\sigma A + \dot{A}\right) + \omega_{\mathrm{n}}^2 A = a\, K_{\mathrm{c},t}\, B\, (\mathrm{e}^{-\tau\, \sigma_s} - 1)\, A \tag{14}$$

by using the assumption $\mathrm{e}^{\frac{\sigma(s-\varepsilon\,\tau)}{\varepsilon}} \approx \mathrm{e}^{\frac{\sigma(s)}{\varepsilon}} \mathrm{e}^{-\tau\, \sigma_s(s)}$. In accordance with (13), one can obtain

$$\dot{A} = A_t + \varepsilon\, A_s \quad \text{and} \quad \ddot{A} = A_{tt} + 2\varepsilon\, A_{ts} + \varepsilon^2 A_{ss}. \tag{15}$$

However, keeping ε sufficiently small and consequently having small change on A w.r.t. slow time s, one can assume $A_s = A_{st} = A_{ss} = 0$. Thus, Fourier expansion is applied on the now exactly time-periodic A and B as

$$A(t) := \sum_{l=-\infty}^{\infty} A_l\ \mathrm{e}^{\mathrm{i}\, l\, \Omega_Z t} \quad \text{and} \quad B(t) := \sum_{l=-\infty}^{\infty} B_l\ \mathrm{e}^{\mathrm{i}\, l\, \Omega_Z t}. \tag{16}$$

The multi-frequency approach [15] or Hill type of infinite expansion of the slowly changing milling dynamics can be given after substituting (16) into (14) as

$$\left((\varepsilon\, \sigma_{ss} + \sigma_s^2 + 2\xi\omega_{\mathrm{n}}\sigma_s + \omega_{\mathrm{n}}^2)\mathbf{I} + [2\,\mathrm{i}\, l\, \Omega_Z \sigma_s - l^2\Omega_Z^2 + 2\,\xi\,\mathrm{i}\, l\, \Omega_Z]_{l=-\infty}^{\infty}\right)\begin{bmatrix} \vdots \\ A_{-1} \\ A_0 \\ A_1 \\ \vdots \end{bmatrix} = a\, K_{\mathrm{c},t}\, (\mathrm{e}^{-\tau\, \sigma_s} - 1)\begin{bmatrix} \ddots & \vdots & \vdots & \vdots & \iddots \\ \cdots & B_0 & B_{-1} & B_{-2} & \cdots \\ \cdots & B_1 & B_0 & B_{-1} & \cdots \\ \cdots & B_2 & B_1 & B_0 & \cdots \\ \iddots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}\begin{bmatrix} \vdots \\ A_{-1} \\ A_0 \\ A_1 \\ \vdots \end{bmatrix}. \tag{17}$$

For the sake of simplicity, zeroth order (average) consideration can be derived by picking only the averages of A and B as in [20], resulting in

$$\left(\varepsilon\, \sigma_{ss} + \sigma_s^2 + 2\xi\omega_{\mathrm{n}}\sigma_s + \omega_{\mathrm{n}}^2 - \ a\, K_{\mathrm{c},t}\, (\mathrm{e}^{-\tau\, \sigma_s} - 1) B_0\right) A_0 = 0. \tag{18}$$

In (18), A_0 has non-trivial solution if the following is satisfied

$$\varepsilon\, \sigma_{ss} + \sigma_s^2 + 2\xi\omega_{\mathrm{n}}\sigma_s + \omega_{\mathrm{n}}^2 - \ a\, K_{\mathrm{c},t}\, (\mathrm{e}^{-\tau\, \sigma_s} - 1) B_0 = 0. \tag{19}$$

Note that, in (19), at least one parameter is slow-time dependent. Also, (19) resembles to the characteristic equation of constant parameter case (1). Moreover, in point of σ_s, (19) has the form of a singular perturbed algebraic equation $\Gamma(\sigma_s)$, which is released by dynamic term σ_{ss} with ε, that is

$$\varepsilon\, \sigma_{ss} + \ \Gamma(\sigma_s) = 0. \tag{20}$$

Obviously, if the rate of parameter change approaches zero, (19) is restricted to the constant parameter characteristic equation for σ_s originated from (13).

3.3 Stability criteria

The general solution for the slow-time system in (13) suggests that, in case of negative real part σ, the perturbation introduced in (9) dies out, while positive real part σ induces rapid explosion w.r.t. s. That means, the first order nonlinear ODE representation of the characteristic equation in (19) has to be integrated by using a proper initial condition when $s = 0$. This can be done by using numerical algorithms like Runge-Kutta method. For the sake of initial condition, we assume at $s = 0$ the system is frozen for a while in slow time s, then

$$\sigma(s) \approx \lambda s \quad \Longrightarrow \quad \sigma_s(s) \approx \lambda \quad \Longrightarrow \quad \sigma_s(0) \approx \lambda, \tag{21}$$

where λ is the frozen time characteristic exponent. The real part of the cumulated value of σ_s over slow time has the form

$$\mathrm{re}\left(\int_0^s \sigma_s(\zeta)\mathrm{d}\zeta\right) \tag{22}$$

that refers to the stability properties of (17) as shown in [10]. If the function (22) crosses zero, the stability property of the slowly changing stationary solution $Y(t,s)$ (11) will flip.

Keeping in mind the 'singular perturbation'-like structure of (20), formula (22) has interesting theoretical consequences. On the one hand, sufficiently small ε induces shifting of the onset of unstable motion independently of ε, as shown by only integrating the solution of $\Gamma(\sigma_s) = 0$ from (20). On the other hand, (22) suggests that the accumulated stability is overtaken by the accumulated instability, which results in the shift on the onset of the unstable motion, which very much depends on the initial values of the slowly changing variables [10].

4 Case Studies

In this section we provide two distinct examples, in which the behavior of the dynamic bifurcation analysis might have a relevance. It is important to emphasize that these are completely artificial examples. Real case study might result in different significance of the explained effect.

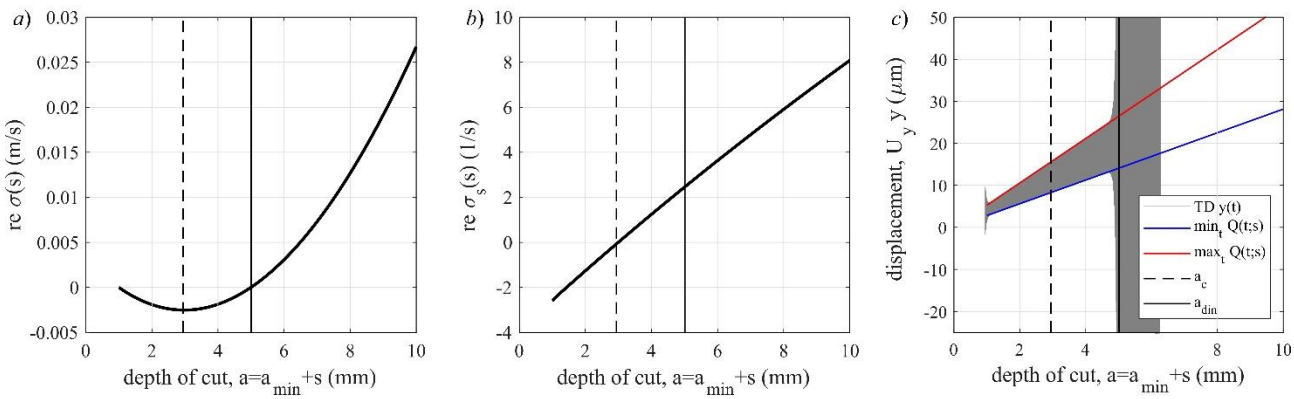

Figure 2. The real part of the critical eigenvalue $\sigma(s)$ *a*) and its derivative $\sigma_s(s)$ *b*) solving (17) w.r.t. s under slowly changing axial depth of cut $a(s)$. In panel *c*) time domain simulation $y(t)$ and the frozen time stationary solution $Q(t;s)$ are shown. Here $\varepsilon = 1 \times 10^{-4}$ (m/s).

4.1 Slowly varying depth of cut

In the literature there are various measurement examples, when the test required pre-manufactured workpiece with a gentle slope [18, 19, 21]. These measurements typically are aimed to present stability limits or so-called nonlinear hysteresis phenomenon, the direct consequence of subcritical Hopf bifurcation of the stationary milling process. The slowly changing parameter in (4) the axial depth of cut is defined as

$$a(s) = a_{\min} + \varepsilon\, t, \tag{23}$$

with $a_{\min} = 1$ mm and $a_{\max} = 10$ mm. In Figure 2*ab*) one can follow the accumulation of the critical eigenvalue that shows according to the criterion (22). The system loses its stability after the constant parameter limit a_c at the dynamic one a_{din}. The loss of stability can be realized as the time domain solution in Figure 2*c*) "escapes" the stationary solution calculated by using simply the frozen time stationary solution $Q(t; s)$. One can realize the linear dependency of the axial depth of cut a on the stationary solution in Figure 2*c*). It can be also realized in Figure 2*c*) the actual onset point where the solution escapes from the stationary solution is a bit ahead of the predicted position. This suggests deeper dependency of the rate of change ε on the dynamics which needs further, more detailed study of the problem.

4.2 Slowly varying ram overhang

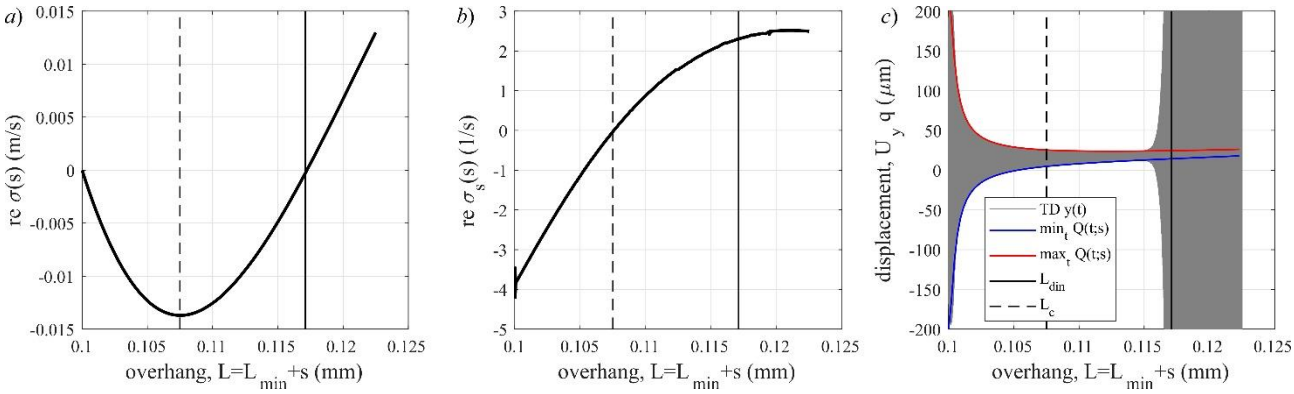

Figure 3. The real part of the critical eigenvalue $\sigma(s)$ *a*) and its derivative $\sigma_s(s)$ *b*) solving (17) w.r.t. *s* under slowly changing ram extension (cantilever overhang) $L(s)$. In panel *c*) time domain simulation $y(t)$ and the frozen time stationary solution $Q(t; s)$ is shown. $\varepsilon = 1 \times 10^{-4}$ (m/s).

Considering the ram as simply a cantilever beam the following dependencies can be determined w.r.t. the ram overhang L by using simple Euler beam theory [22]

$$\omega_{\mathrm{n}} = A_{\omega_{\mathrm{n}}} \frac{1}{L^2}\,, k = A_k \frac{1}{L}, \mathbf{U} = \mathbf{A_U} \frac{1}{\sqrt{L}}, \text{ and consequently } B_0 \sim \frac{1}{L}. \tag{24}$$

Obviously, the presented relations in (24) are far to be true in a real machine. Instead, the real behavior should be characterized by measuring frequency response functions (FRFs) in many overhang positions. In this manner the real dynamics and even the damping can be determined. This artificial example is calculated with

$$L(s) = L_{\min} + \varepsilon\, t, \tag{25}$$

with $L_{\min} = 100$ mm and $L_{\max} = 122.5$ mm. The parameters introduced in (24) are

$$A_{\omega_\mathrm{n}} = 5.9062\,\frac{\mathrm{rad\ m^2}}{\mathrm{s}}, \qquad A_k = 58.8 \times 10^5\ \mathrm{N}, \qquad \mathbf{A_U} = [0 \quad 0.0244 \quad 0]^\mathrm{T}\sqrt{\frac{\mathrm{m}}{\mathrm{kg}}}. \tag{26}$$

In Figure 3*ab*) the accumulated effect of the slowly changing parameter L is shown on the critical eigenvalue. Similarly to Figure 2*ab*), the dynamic bifurcation limit L_{din} and the constant parameter stability limit L_{c} are not the same. A shift appears confirmed in Figure 3*c*) by time domain simulation. Here, the stationary solution changes drastically (blue and red envelope curves in Figure 3*c*) by varying the ram overhang L. The amplitude of the stationary solution shrinks because by changing the L the system goes away from the stability pockets (see path B in Figure 1*b*), where "resonance" causing high but finite gain on the amplitude [23]. Although the stationary solution shrinks, the system becomes more flexible causing the onset of the unstable motion after the constant parameter limit L_{c} at L_{din} and the amplitude goes to infinite (without modelling the threshold fly-over effect [24]).

5 Conclusion

There are several industrial problems, including ones related to cutting technologies, which may involve dynamic processes on (very) different time scales. In the present work, we study milling operations that clearly have fast time-periodic dynamics. In the meantime, there exists a slow rate of change of some system parameters originated in the slowly varying structural dynamics as the tool moves in the working space of the milling machine.

In this paper, the so-called dynamic bifurcation phenomenon has been introduced for the analysis of milling stability. A new generalized governing equation was derived, with which the stability of the slowly changing milling dynamics can be predicted. Two simplified case studies presented slowly varying behavior. The non-trivial, sometimes counter-intuitive theoretical predictions based on the analysis of the new governing equations were confirmed by time domain simulation results, although some parameter domains still need further and deeper study.

The results may have industrial relevance when the milling cutter moving in the workspace has varying reduced dynamics. The results are somewhat counter intuitive. On the one hand, the accumulated stability, in theory, does not or weakly depends on the rate of change, which suggests that shifting of the stability appears even for extremely small rate of change. On the other hand, the shift (e.g., $a_{\mathrm{din}} - a_{\mathrm{c}}$) carries the initial parameter value, since this has direct effect on the accumulated stability behavior.

It is important to emphasize that the above presented results apply only in cases when the cutting operation starts from stable stationary cutting; at this point, the results do not explain the transition backward from chatter to stable stationary cutting, which requires different modelling techniques that are applicable also for quasi-periodic oscillations.

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