Recover Dynamic Utility from Observable Process: Application to the economic equilibrium

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Abstract

Decision making under uncertainty is often viewed as an optimization problem under choice criterium, but its calibration raises the "inverse" problem to recover the criterium from the data. An example is the theory of "revealed preference" by Samuelson in the 40s, [Sam38].

The observable is a so-called increasing characteristic process $\mathfrak{X} = (\mathcal{X}_t(x))$ and the objective is to recover a dynamic stochastic utility $\mathbf{U}$ "revealed" in the sense where "$U(t, \mathcal{X}_t(x))$ is a martingale". A linearized version is provided by the first order conditions $U_x(t, \mathcal{X}_t(x)) = Y_t(u_z(x)$, and the additional martingale conditions of the processes $\mathcal{X}_x(t,x)Y_t(u_z(x))$ and $\mathcal{X}_t(x)Y_t(u_z(x))$. When $\mathfrak{X}$ and $\mathbf{Y}$ are regular solutions of two SDEs with random coefficients, under strongly martingale condition, any revealed utility is solution of a non linear SPDE, and is the stochastic value function of some optimization problem. More interesting is the dynamic equilibrium problem as in He and Leland [HL93], where Y is coupled with $\mathcal{X}$ so that the monotonicity of $Y_t(z, u_z(z))$ is lost. Nevertheless, we solve the He & Leland problem (in random environment), by characterizing all the equilibria: the adjoint process still linear in y (GBM in Markovian case) and the conjugate utilities are a deterministic mixture of stochastic dual power utilities. Besides, the primal utility is the value function of an optimal wealth allocation in the Pareto problem.

1 Introduction

Decision making under uncertainty is often viewed as an optimization problem under choice criterium, and the available observed data as the result of the decision process and its evolution over the time. Most theories focus on the derivation of the "optimal decision" and its out-comes, but poor information is available on the preference criterium generating (yielding to) these observed data. The problem to recover the criterium from the data may be viewed as an inverse problem. Similar question was addressed by the economist Samuelson in the 40s, [Sam38, Sam48], with the theory of "revealed preference". Since then, the theory has been growing in interest and the economic reality created new incentives for different approaches, see Chambers & Echenique [CE16]. A renewed interest is provided by the evolutionary economics by W.B.Arthur [Art99], studying economies as complex evolutionary systems, where the agents try to predict the out-comes of their actions, and how the market would be modified by their decisions.

In this forward-looking viewpoint, the agents also need to adjust their (random) preferences over time, following an "inverse thinking" approach as suggested by J.Gomez-Ramirez [GR13]. The *forward modeling*

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allows anticipations on the future values of observables, and the inverse problem uses those "predictions" to infer the values of the parameters that characterize the system. Thus, the robustness of the method is obtained from a family of forward model solutions, consistent with the data rather that one prediction.
Recently, similar ideas have been also developed in *Preference learning* which is a subfield in machine learning in which the goal is to learn a predictive preference model from observed preference informations, see [FH11, FSS06]. Depending on the approaches used to modeling preferences, this give rise to two kinds of learning problems: learning utility functions and learning preference relations. ([Sta04, CKO01, Hib12])).
As well, reasoning with preferences has been recognized as a particularly promising research direction for artificial intelligence see [NJ04, QXL14]. For example, computerized methods for discovering the preferences of individuals are useful in e-commerce and various other fields, see [WTKD04], Ortuzar & Willumsen (1994), Nechyba & Strauss (1998). Other learning problems can also be studied from the viewpoint of an expected utility maximizing as learning probabilistic models, see [FS16, FS03].
Our approach is therefore a learning approach, based on the observation of the behavior of a "player" from several initial conditions, at many dates in the future, and the question is what can be deduced about his utility at any time? The answer to this question suggests to work in a general probabilistic framework which includes discrete frame, or semimartingale's frame (with jumps). Thus our results could apply to the "Utility learning theory, Preference learning theory, Algorithmic Decision theory" (much considered nowadays), see [CKO01, WTKD04, QXL14, NJ04] and many others.

Let us introduce, more in details, the reference framework. By definition, in the revealed dynamic utility problem, the observable is a so-called dynamic characteristic process $\mathfrak{X} = (\mathcal{X}_t(x))$ considered for different values of its initial condition $\mathcal{X}_0(x) = x$, and assumed to be increasing in x, (to be coherent with the expected utility criterium).
The goal is to recover, from a given initial utility function u, a stochastic dynamic utility $\mathbf{U} = \{U(t,z), z > 0\}$, "revealed optimally" in the sense that at any (stopping) time, the preference for the observable process is at least equal to its value at time 0, $\mathbb{E}(U(t,\mathcal{X}_t(x))) = u(x)$; from the probabilistic point of view, the performance process "$U(t,\mathcal{X}_t(x))$ is a martingale".
Focusing on the concavity of the criterium $\mathbf{U}$, tools of convex analysis play a key role, in particular the convex Fenchel-Legendre transform $\widetilde{U}(t,y)$ of $U(t,x)$. These stochastic processes are linked by the "Master equation" based on the marginal utility $U_z(t,z)$, i.e. $U(t,z) - z\,U_z(t,z) = \widetilde{U}(t,U_z(t,z))$. By the first order condition $U_x(t,\mathcal{X}_t(x)) = Y_t(u_z(x))$, we study a "linearized" problem with the help of the processes $(Y, \mathcal{X}, \mathcal{X}_x, u_z)$, defined by the following martingale properties, "$\mathcal{X}_x(t,x)Y_t(u_z(x))$ and $\mathcal{X}_t(x))Y_t(u_z(x))$ are two martingales"; this last condition guarantees the martingale property to the Fenchel conjugate $\widetilde{U}(t,y)$ along the process Y.
Similar dynamic utility was first considered by Zariphopoulou & Musiela [MZ03, MZ10a, MZ10b] (under the name of forward utility) for the study of optimization problem in financial market, see also Henderson & Hobson [HH07]. But here, there is no financial market and no optimization problem a priori.

In Section 2, we recall elementary results on deterministic and dynamic utility process and their dual. Then, we introduce the martingale property, illustrated by simple but interesting examples. In the second part of Section 2, we focus on the first order condition $U_x(t,\mathcal{X}_t(x)) = Y_t(u_z(x))$ based on the so-called *adjoint process* $\mathbf{Y} = (\mathbf{Y_t(y)})$, in order to reduce the problem of the computation of the marginal utility to the use of a change of variable formula. Since the process $x \to \mathcal{X}_t(x)$ is increasing, with values in $[0,\infty)$, we introduce its inverse process $\xi(t,z)$ and deduce a pathwise representation of the marginal utility $U_z(t,z) = Y_t(u_z(\xi(t,z)))$. The first order condition $U_z(t,\mathcal{X}_t(x)) = Y_t(u_z(x))$ associated with the monotonicity assumption induces a one to one correspondance between the compatible marginal utility $U_z(t,z)$ and the adjoint process $Y_t(u_z(z))$. In Section 3, a change of variable gives a way, via a Stieltjes integral, to study the martingale properties of $U(t, \mathcal{X}_t(x)) = \int_0^x Y_t(u_z(z)) d_z \mathcal{X}_t(z)$ and then we establish one of the main results of this paper consisting on a necessary and sufficient orthogonality condition for the solvability of the recovery problem. Along Sections 2 and 3, no regularity in time on the characteristic process or dynamic utility or adjoint processes is assumed. In return, monotonicity for the functions $(\mathcal{X}_t(x)), (-U_z(t,z)), (Y_t(y))$ with respect to their initial condition is fundamental.

Section 4 and 5 illustrate these results in the Itô framework where the dynamics of the revealed utility is established under different orthogonality conditions between the characteristic and the adjoint processes. In Section 4, the two SDEs are decoupled, and the framework is similar to the one developed by the authors in [EKM13]. The monotonicity of the solutions is provided by the regularity of the random coefficients. The coordinates are assumed to be strongly orthogonal, yielding to a framework similar to financial market. Then, any revealed utility is solution of a non linear SPDE, and is the value function of two different optimisation problem, the first one is relatif to the family of SDE's solutions strongly orthogonal to Y, and the second one to the family of linear (in x) processes indexed by their volatility (the portfolios in finance), and orthogonal to Y. In Section 5, we solve the problem of the dynamic equilibrium characterization considered by He and Leland in [HL93], where Y is coupled with X so that the monotonicity of $Y_t(z, u_z(z))$ is lost. Nevertheless, we prove that even if the environment is random, the dual problem at equilibrium is very easy to characterize: the adjoint process is linear in y (geometric Brownian motion (GBM) in Markovian case) and the conjugate utilities are deterministic mixtures of stochastic dual power utilities; the primal utility is the value function of an optimal wealth allocation in sup-convolution Pareto problem. Then, we discuss the consequences of this result.

2 The forward recovery problem

2.1 Dynamic utility

We start by reminding some definitions and properties of static or dynamic utility criterion.

Deterministic utility function In economics and finance, the standard notion of utility function, used as performance measure, refers to a concave function u on $\mathbb{R}^+$, positive, increasing, normalized by $u(0) = 0$, whose range is $\mathbb{R}^+$ ($u(+\infty) = \infty$). An important role is played by its derivative u_z, also called *marginal utility*, assumed to be continuous, positive and decreasing on $]0, +\infty[$, with range $]0, +\infty[$, and satisfying the Inada's conditions $u_z(+\infty) = 0$ and $u_z(0) = +\infty$. whose As usual, the *Fenchel-Legendre convex conjugate function* $\tilde{u}(y)$ highlights some other aspects of the performance measure. The pair $(u, \tilde{u})$ satisfies the following system:

$$\text{(Main equation)} \begin{cases} \tilde{u}(y) = \sup_{z>0} \big(u(z) - yz\big), \quad u(z) = \inf_{y>0} \big(\tilde{u}(y) + yz\big). & (2.1) \\ u(z) - z\, u_z(z) = \tilde{u}(u_z(z)), \;\; u(-\tilde{u}_y(y)) + y\, \tilde{u}_y(y) = \tilde{u}(y)\,. & (2.2) \end{cases}$$

In particular, since $\tilde{u}(y) > 0$, we have $u(z) > z\, u_z(z) > 0$ and $z\, u_z(z) \to u(0) = 0$. The range of the decreasing function $\tilde{u}(y)$ is $]0, +\infty[$ since $\tilde{u}(y) \to \infty$ when $y \to 0$ (since $\sup_{z>0} u(z) = +\infty$) and $\tilde{u}(y) \to 0$ when $y \to +\infty$ ($u(0) = 0$ and $z\, u_z(z) > 0$). In Economics or Finance, utility functions are often assumed to be of class $\mathcal{C}^2(]0, \infty[)$. Some complementary indicators are in common use: the *relative risk aversion* function defined by the ratio $\gamma(z) = -z u_{zz}(z)/u_z(z)$, and its conjugate, the *relative risk tolerance function* $\tilde{\gamma}(y) = -y \tilde{u}_{yy}(y)/\tilde{u}_y(y) = 1/\gamma(-\tilde{u}_y(y))$.

Power utility: A typical example, intensively used by the economists, is the family of *power utility*

$u^{(\alpha)}(z) = \frac{z^{1-\alpha}}{1-\alpha}$ for $\alpha \in [0, 1[$ and $u(z) = (\ln z)^+$ for $\alpha = 1$, whose relative risk aversion is the constant α. Its Fenchel conjugate is the power function $\tilde{u}^{(\beta)}(y) = y^{(1-\beta)}/(\beta - 1), \beta = 1/\alpha > 1$, whose risk tolerance is the constant β.

Dynamic utility and its Fenchel conjugate A dynamic utility should represent, possibly changing over time, individual preferences of an agent starting with a today's specification of his utility, $U(0, z) = u(z)$. The preferences are affected over time by the available information represented by the filtration $(\mathcal{F}_t)_{t\geq 0}$ defined on the probability space $(\Omega, \mathbb{P}, \mathcal{F})$. The filtered probability space $(\Omega, \mathbb{P}, (\mathcal{F}_t))$ is assumed to satisfy usual conditions of right continuity and completeness. The filtration $\mathcal{F}_0$ is not necessarily assumed to be trivial. On the space $(\Omega \times \mathbb{R}^+)$, the σ-fields $\mathcal{O}$ of optional processes or $\mathcal{P}$ of predictable processes are generated by the families of adapted, respectively right-continuous or left-continuous processes.

A dynamic utility $\mathbf{U}$ is a collection of random utility functions $\{U(t, \omega, z)\}$ whose the temporal evolution is "updated" over the time in accordance with the new information $(\mathcal{F}_t)$ from an initial utility value $u(z) = U(0, z)$, eventually random if $\mathcal{F}_0$ is not trivial.

Definition 2.1. *A dynamic utility* $\mathbf{U}$ *is a family of optional processes* $\{U(t, z), z \in \mathbb{R}^+\}$ *(also called optional random field) such that* $\mathbb{P}$ *.a.s., for every* $t \geq 0$*, the function* $(z \to U(t, z))$ *is a standard utility function with* $U(t, 0) = 0$.

- Its marginal utility $\mathbf{U}_z$ *is the decreasing optional random field* $\{U_z(t, z)\}$ *with range* $[\infty, 0]$.
- Its conjugate utility $\widetilde{\mathbf{U}}$ *is the convex optional random field,* $\widetilde{U}(t, y) = \sup_{z>0} \big(U(t, z) - yz\big)$,

$$\text{(Master Equation)} \quad U(t, z) - zU_z(t, z) = \widetilde{U}(t, U_z(t, z)) \text{ and } \widetilde{U}_y(t, U_z(t, z)) = -z. \ \mathbb{P}.a.s., \tag{2.3}$$

2.2 Martingale properties for the forward recovery problem

In this paper, we address the question of the recoverability of a dynamic utility from an observed process; the flexibility introduced by the forward point of view and the dynamic character of the utility allows us to introduce a natural martingale constraint between the observed process and the measure of its performance to be recovered. Let us make two remarks at this stage:

− to recover a family of "utility functions" depending of a "wealth parameter" z, at any time in the future, it is necessary for the family of "observable" processes to have various initial conditions x. Such family will be called *characteristic* and denoted by the calligraphic symbols $\mathfrak{X} = (\mathcal{X}_t(x), (t, x) \in \mathbb{R}^+ \times \mathbb{R}^+)$.

− an equivalent problem could have been to recover the Fenchel conjugate dynamic utility from an "adjoint process" $\mathbf{Y} = (Y_t(y), (t, y) \in \mathbb{R}^+ \times \mathbb{R}^+)$. Thanks to the Master equation (2.3), this process must be connected with the characteristic process $\mathfrak{X}$ by the relation $Y_t(u_z(x)) = U_z(t, \mathcal{X}_t(x))$, which can be interpreted as a *first-order condition*. Then, constraints to recover $\mathbf{U}$ become equivalent to conjugate contraints to recover $\widetilde{\mathbf{U}}$.

2.2.1 Martingale properties

The first step is to define the "natural" relationship between the characteristic process and the dynamic utility, $\{U(t, z), (t, z) \in \mathbb{R}^+ \times \mathbb{R}^+\}$ to be recovered.

It is natural to assume the time-stability on the "expected utility" (performance) of the characteristic process in the futur.

− Mathematically, this means that for any bounded *stopping time* τ (such that the positive random variable $U(\tau, \mathcal{X}_\tau(x))$ is integrable), $\mathbb{E}[U(\tau, \mathcal{X}_\tau(x))] = u(x)$. The pathwise point of view, which is more

efficient, is that the process $\{U(t, \mathcal{X}_t)\}$ is assumed to be a *(strong) martingale*. By means of a change of variable $x \to \mathcal{X}_t(x)$, we have made the "curve" $\{U(t, \mathcal{X}_t(x))\}$ constant in expectation, hence the name of characteristic process for $\mathfrak{X}$.
– An other question concerns the time-stability in expectation of the conjugate dynamic utility with respect to the adjoint process $Y_t(u_z(x)) = U_z(t, \mathcal{X}_t(x))$. Once again, thanks to the Master equation (2.3), this property holds true if and only if in addition of the previous martingale property, $\mathbb{E}[U_z(\tau, \mathcal{X}_\tau(x))\mathcal{X}_\tau(x)] = x$, said differently, if and only if the product $Y_t(u_z(x))\mathcal{X}_t(x)$ is a martingale.
Remark: Time-stability is not a new principle, since it is already found in Bachelier in 1900 [**?**], about the mechanism of prices formation, and of course in mathematical finance in relation with the no-arbitrage assumption. Time stability and the use of concave criterium is also the cornerstone of the stochastic control, without any reference to economy or finance as well as to the use of concave criterium in which the martingale property holds true for the *"backward"* value functional of optimization problem, read along the optimal process, ([EKPY12] Saint Flour). This principle is known as the *dynamic programming principle*, or Bellman principle. Observe that the martingale property has no links with the concavity of the criterium.
Martingale properties are taken as central in the definition of revealed dynamic utility from an initial utility function and a characteristic process. Note that, the existence (and of course the uniqueness) of such utilities is not a priori guaranteed.

Definition 2.2 (Revealed dynamic utility). **(i)** *Let $\mathfrak{X}$ be a characteristic process and u a initial utility. A dynamic utility* $\mathbf{U}$ *is a* $(\mathfrak{X}, u)$-revealed dynamic utility *if and only if:*

$$\forall x \in (0, \infty), \quad U(t, \mathcal{X}_t(x)) \text{ is a positive martingale.} \tag{2.4}$$

(ii) *Let us defined the so-called* adjoint process $Y_t(y) = U_z(t, \mathcal{X}_t(-\tilde{u}_y(y)))$. *The pair* $(\mathbf{U}, \widetilde{\mathbf{U}})$ *is said to be* recoverable *from* $(\mathfrak{X}, u)$ *if in addition* $\widetilde{\mathbf{U}}$ *is revealed by* $(\mathbf{Y}, \tilde{u})$.
A necessary and sufficient condition is that $\mathbf{U}$ *is revealed by* $(\mathfrak{X}, u)$, *and* $\{\mathcal{X}_t(x)U_z(t, \mathcal{X}_t(x)) = \mathcal{X}_t(x)Y_t(u_z(x))\}$ *is a family of positive martingales.*

The equivalence in the second property **(ii)** is a simple consequence of the Master equation, and of the definition of the adjoint process.

2.2.2 Examples of Recovery Problem

The following simple examples of characteristic (observed) processes are used to give a first insight of the problem of revealed utility. The case of constant process, corresponding to "to do nothing in finance", is very illustrative. The case of linear characteristic process is the most frequently used in economics; it can be easily extended to convex increasing process.

Proposition 2.1. **(i)** *Assume the characteristic process to be constant in time* $\mathcal{X}_t(z) = z, \forall t$.
A dynamic utility $\{U(t, z)\}$ *is a* revealed utility *if and only if its marginal utility* $\{U_z(t, z), z > 0\}$ *is a martingale. Then, the pair* $(\mathbf{U}, \widetilde{\mathbf{U}})$ *is recoverable from* (z, u).
(i) *Assume the characteristic process to be linear in* x, $\{\mathcal{X}_t(x) = xX_t\}$. *The recovery problem has a solution if and only if there exists an adjoint process* $\mathbf{Y}$ *such that for any* y, $\{X_t\, Y_t(y)\}$ *is a martingale. Then,* $\{Y_t(y)\}$ *is a characteristic process for the conjugate utility process.*
(ii) *Assume the characteristic process* $\{\mathcal{X}_t(x)\}$ *to be* **concave**, *increasing and differentiable in* x *with differential* $\{\mathcal{X}_x(t, x)\}$. *The revealed problem has a solution if and only if there exists a monotonic process* $\mathbf{Y}$ *such that for any* x, $\{\mathcal{X}_x(t, x)\, Y_t(u_x(x))\}$ *is a martingale. If in addition,* $\{\mathcal{X}(t, x)\, Y_t(u_x(x))\}$ *is a martingale,* $(\mathbf{U}, \widetilde{\mathbf{U}})$ *is recoverable from* $(\mathcal{X}, u)$, *and* $\{Y_t(y)\}$ *is a characteristic process for the conjugate dynamic utility* $\widetilde{\mathbf{U}}$.

Proof. **(i)** Assume the utility process $\{U(t,x)\}$ to be a martingale. By the Master equation (2.3), $zU_z(t,z) \leq U(t,z) \leq U(t,z_{\max})$. By Lebesgue's derivative Theorem, the martingale property can be extended to the derivative random field $\{U_z(t,z)\}$. Conversely, if the x-decreasing process $\{U_z(t,x)\}$ is a martingale, by Fubini's Theorem $\{U(t,x)-U(t,x_0)\}$ is also a martingale, with expectation $u(x)-u(x_0)$. Thanks to the monotonicity of utility functions, $U(t,x_0)$ decreases to 0 when x_0 goes to 0, and the martingale property remains valid at the limit for $\{U(t,x)\}$. By the Master Equation (2.3), $(\mathbf{U},\widetilde{\mathbf{U}})$ is recoverable from (z,u).
(ii) If $\mathcal{X}(t,x)=xX_t$, the change of variable (also called change of numéraire in finance) $x\to x/X_t$ yields to a new dynamic utility, $U^X(t,z)=U(t,zX_t)$ which is a martingale if U is revealed, with characteristic process x. By **(i)** $U^X(t,z)$ is a martingale if and only if $U_z^X(t,x)=X_t\,U_z(t,xX_t)=X_t\,Y_t(u_z(x))$ is a martingale. This condition is then sufficient to show that once again, $(\mathbf{U},\widetilde{\mathbf{U}})$ is recoverable from (xX_t,u).
(iii) When the characteristic process is concave, differentiable and increasing in x, ($x\to\mathcal{X}_x(t,x)$ decreasing in x), for any dynamic utility the random field $\{U(t,\mathcal{X}_t(x))\}$ is x-concave. Then, $\{U^X(t,z)=U(t,\mathcal{X}_t(z))\}$ is a dynamic utility martingale or equivalently its derivative is martingale, that is equivalent to ($\{U(t,\mathcal{X}_t(x))\mathcal{X}_x(t,z)=Y_t(u_z(z))\mathcal{X}_x(t,z)\}$ is a martingale). To prove that $\widetilde{\mathbf{U}}$ is revealed by $\{Y_t(y)\}$, the process $\{Y_t(u_z(z))\mathcal{X}_t(z)\}$ must be also martingale. □

Remark . *As the power utilities, a time separable random field $U(t,z)=L_tV(z)$ is a martingale utility iff V(z) is a standard utility and L_t is a positive martingale.*

2.3 Monotonicity and compatibility of adjoint process

The previous examples have in common the monotonicity of the characteristic process in x. The role of this property is not surprising, since monotonicity and concavity of $z\to U(t,z)$ cannot be obtained only from the martingale property of $U(t,\mathcal{X}_t(x))$, even when the initial condition u is concave and increasing. This difficulty comes from the forward point of view for which comparison results are difficult to obtain in general.

2.3.1 Monotonicity and Algebraic Compatibility

The characteristic process $\mathfrak{X}=\{\mathcal{X}_t(x)\}$ is assumed to be an optional random field, increasing in x with range $[0,\infty)$. Then $x\to\mathcal{X}_t(x)$ is continuous in x, and has an optional increasing *inverse flow* $\xi(t,z)$ such that $\mathcal{X}_t(\xi(t,z))=z$.
The same increase property is then satisfied by the so-called *adjoint process* $\mathbf{Y}=\{Y_t(y)\}$, defined by $Y_t(u_z(x))=U_z(t,\mathcal{X}_t(x))$. Observe that $Y_t(u_z(\xi(t,z))(=U_z(t,z))$ is integrable in a neighborhood of 0 as any marginal utility function. The triplet $(u,\mathcal{X}_t(x),Y_t(y))$ is said to be admissible in the class $\mathfrak{C}$.

Definition 2.3 (Algebraic Compatibility)**.** *An admissible increasing triplet $(u,\mathfrak{X},\mathbf{Y})\in\mathfrak{C}$ defines a* unique dynamic utility $\{U(t,z)\}$*, through the* "first order condition"*:*

$$U_z(t,z)=Y_t(u_z(\xi_t(z))\ \textit{so that}\ U(t,\mathcal{X}_t(x))=\int_0^x Y_t(u_z(z))d_z\mathcal{X}_t(z). \tag{2.5}$$

Such dynamic utility is said to be $(u,\mathfrak{X},\mathbf{Y})$-compatible.

Surprisingly, the monotonicity property is rarely highlighted in economics and finance although it is satisfied by all known solutions.

2.3.2 Forward starting dates and time consistency

As usual in dynamic viewpoint, it is appropriate to consider that we can start at another date s in the future. The time consistency is a general principle that we find in many situation, in the theory of stochastic differential equation with respect of their initial condition, in the study of Markov processes, or in optimization problems where it is known as the dynamic programming principle. These forward version requires to define a family of processes, starting from x at any time s, together with a dynamic time-consistency constraint. At the end, the problem is to generalize the stochastic representation of the deterministic function $u_z(x) = \widetilde{\xi}(t, U_z(t, \mathcal{X}_t(x)))$ where $\widetilde{\xi}(t,z)$ is the inverse flow of $Y_t(y)$.
The data at time 0 is a triplet $(u, \mathfrak{X}, \mathbf{Y}) \in \mathfrak{C}$ and its compatible dynamic utility $\mathbf{U}$; the problem is to deduce a new triplet starting with initial condition (s, z), coherent with this data, in particular with the same associated dynamic utility $U(t, z)$. The monotonicity assumption plays a key role, thanks in particular to the inverse processes ξ and $\widetilde{\xi}$ of $\mathfrak{X}$ and $\mathbf{Y}$ respectively.

Proposition 2.2 (Forward and Backward Formulation). **(i)** *The new forward characteristic processes starting from the date s are defined by for any $t \geq s$ by*

$$\mathcal{X}_t(s, x) = \mathcal{X}_t(\xi(s, x)), \quad Y_t(s, y) := Y_t\big(\widetilde{\xi}(s, y)\big), \tag{2.6}$$

They verify the semi-group property, $\mathcal{X}_t(x) := \mathcal{X}_t(s, \mathcal{X}_s(x)), \quad Y_t(y) := Y_t(s, Y_s(y))$.
(ii) *The dynamic utility $\mathbf{U} = \{U(t, z)\}$ which is compatible with $(u, \mathfrak{X}, \mathbf{Y})$ is for $t \geq s$, the dynamic utility compatible with $(U(s, z), \mathfrak{X}(s, .), \mathbf{Y}(s, .))$.*

Proof. The verification of the compatibility of $U(t, z)$ with the new processes relies on the verification of the algebraic identity $U_z(t, \mathcal{X}_t(s, x)) = Y_t(s, U_z(s, x))$; by the semi-group property, this equality becomes $U_z(t, \mathcal{X}_t(x)) = Y_t(s, U_z(s, \mathcal{X}_s(x))) = Y_t(s, Y_s(u_z(z))) = Y_t(u_z(z))$, which still holds thanks to the compatibility condition. □

Remark: Note that this treatment of the time consistency is unusual, since there is no reference to any notion of conditional expectation as in a Markovian problem.

3 Solvability of the recovery problem

Let $(u, \mathfrak{X}, \mathbf{Y}) \in \mathfrak{C}$ be an admissible triplet, and $\mathbf{U}$ the associated algebraic dynamic utility. The main question arising is to find conditions on $(u, \mathfrak{X}, \mathbf{Y})$ under which $\mathbf{U}$ is a revealed utility, that is $\{U(t, \mathcal{X}_t(x)\}$ is a martingale.

3.1 Conditions for the solvability of the recovery problem

We have seen that for a convex differentiable characteristic process $\mathfrak{X}$ with derivative $\mathfrak{X}_x$ ($\mathcal{X}_x(t, 0) = 1$), the martingale property of $\{\mathcal{X}_x(t, x)Y_t(u_z(x))\}$ is a necessary and sufficient property for the recovery problem. In fact the sufficient condition does not use the convexity of $\mathfrak{X}$, and still holds also for only differentiable characteristic process. When the characteristic process is not differentiable, we have to rewrite the "first order condition" in a more general formulation.

3.1.1 Solvability conditions for recovery problem

In the general case, we use the rate of variation in place of the derivative, and the theorem of intermediate values. For a $(u, \mathfrak{X}, \mathbf{Y})$-compatible dynamic utility $\mathbf{U}$, it is easy to control the x-variation of $U(t, \mathcal{X}_t(x))$ with the help of the Y process.

Lemma 3.1. *There exists an optional process $\{\psi_t(x,x')\}$ such that for any $(x<x')$*

$$U(t,\mathfrak{X}_t(x'))-U(t,\mathfrak{X}_t(x))=\big(\mathfrak{X}_t(x')-\mathfrak{X}_t(x)\big)Y_t\big(u_z(\psi_t(x,x'))\big) \tag{3.1}$$

Proof. The proof is a simple consequence of the theorem of the intermediate values,

$$\begin{cases} U(t,z')-U(t,z)=(z'-z)U_z\big(t,\psi_t(z,z')\big), \quad \psi_t(z,z')\in(z,z'),\ z'>z>0\\ \psi_t(z,z')=(U_z)^{-1}\big(t,\Delta U(t,(z,z'))\big), \quad \Delta U(t,(z,z'))=\dfrac{U(t,z')-U(t,z)}{z'-z}. \end{cases} \tag{3.2}$$

Then, with $x<x'$, and $(z=\mathfrak{X}_t(x), z'=\mathfrak{X}_t(x'))$, there exists an optional process $\xi_t(z,z')\in[\mathfrak{X}_t(x),\mathfrak{X}_t(x')]$ such that, $U(t,\mathfrak{X}_t(x'))-U(t,\mathfrak{X}_t(x))=\big(\mathfrak{X}_t(x')-\mathfrak{X}_t(x)\big)U_z(t,\xi_t(z,z'))$
By a change of variable, $\xi_t(z,z')$ can be sent into $\psi_t(x,x')$ in the interval (x,x') by the formula $\xi_t(z,z')=\mathfrak{X}_t(\psi_t(x,x'))$. So, $U_z\big(t,\xi_t(z,z')\big)=U_z\big(t,\mathfrak{X}_t(\psi_t(x,x'))\big)=Y_t\big(u_z(\psi_t(x,x'))\big)$. □

Then, for any revealed utility, the left side is a martingale and so, by equation (3.1) for $x'>x$, the right side " $\big(\mathfrak{X}_t(x')-\mathfrak{X}_t(x)\big)Y_t\big(u_z(\psi_t(x,x'))\big)$" is a martingale. Our main result is that this condition is also sufficient.

Abstract result: The argument uses the approximation of the Stieltjes integral $\int_{x_0}^x Y_t(u_z(z))d_z\mathfrak{X}_t(z)$ defined on a compact interval $[x_0,x]$ with the help of Darboux sums obtained as follows: we start with a partition of the interval $[x_0,x]$ into N subintervals $]z_n,z_{n+1}]$ where the mesh approaches zero, and we consider the following sequences

$$S_t^N(x_0,x)=\sum_{n=0}^{N-1}Y_t(u_z(\bar z_t^n))\big(\mathfrak{X}_t(z_{n+1})-\mathfrak{X}_t(z_n)\big), \tag{3.3}$$

where $\bar z_t^n$ is a random variable in the interval $[z_n,z_{n+1}]$. Given the continuity in z of $z\to Y_t(u_z(z))$ and $z\to\mathfrak{X}_t(z)$, Darboux's theorem states that all the Darboux sums converge to the Stieltjes integral when the mesh goes to 0.

Theorem 3.2. *Let $(u,\mathfrak{X},\mathbf{Y})\in\mathfrak{C}$ be an admissible triplet generating the dynamic utility* $\mathbf{U}$.
(i) $\mathbf{U}$ *is a* revealed utility *if and only if for any (x,x'), $x<x'$, there exists an optional process $\{\psi_t(x,x')\}$ with $x\le\psi_t(x,x')\le x'$, such that:*

$$\text{the process } \big(\mathfrak{X}_t(x')-\mathfrak{X}_t(x)\big)Y_t\big(u_z(\psi_t(x,x'))\big) \text{ is a martingale.}$$

(ii) *This property is satisfied as soon as for any (x,y), $(\mathfrak{X}_t(x)Y_t(y))$ is a martingale*

Proof. We use the approximations based on Darboux sums centered around the processes $\bar z^n(t)=\psi_t(z_n,z_{n+1})$. By assumption, these Darboux approximations $S_t^N(x_0,x)$ are finite sum of positive martingales, and then also positive martingales. By the positive Fubini's Theorem, we can interchange limit and expectation so that the martingale property is preserved and $\int_{x_0}^x Y_t(u_z(z))d_z\mathfrak{X}_t(x)$ is a martingale, with expectation $\int_{x_0}^x u_z(z)dz=u(x)-u(x_0)$.
Once again, by monotonicity, the random variables $\int_{x_0}^x Y_t(u_z(z))d_z\mathfrak{X}_t(x)$ go to a limit with finite expectation. So, the Stieltjes integral is well-defined up to 0 and $\Psi_0^X(t,x)=\int_0^x Y_t(u_z(z))d_z\mathfrak{X}_t(x)$ is a martingale. The revealed utility process is then given by $U(t,z)=\int_0^z Y_t\big(u_z(\xi(t,x)\big)\,dx$..
Similar arguments can be developed for the conjugate utility function. □

Efficient sufficient conditions In the main result, the existence of a process $\psi_t(z,z')$ with the martingale property can be difficult to establish. The proof based on Darboux sums suggests to relax the martingale assumptions in the *only sufficient condition*: it is sufficient to assume that the common Darboux sums, $S_t^{N,up}(x_0,x) = \sum_{n=0}^{N-1} Y_t(u_z(z_n))\big(\mathfrak{X}_t(z_{n+1}) - \mathfrak{X}_t(z_n)\big)$ and $S_t^{N,down}(x_0,x) = \sum_{n=0}^{N-1} Y_t(u_z(z_{n+1}))\big(\mathfrak{X}_t(z_{n+1}) - \mathfrak{X}_t(z_n)\big)$ are respectively supermartingales and submartingales. Since the both sequences have the same limit, this limit is expected to be a martingale.

Theorem 3.3. *Let $(u,\mathfrak{X},\mathbf{Y}) \in \mathfrak{C}$ be an admissible triplet generating the dynamic utility $\mathbf{U}$. Assume that $x' > x$ so that the following processes are positive.*
(i) *If the process $\{Y_t(u_z(x))\big(\mathfrak{X}_t(x') - \mathfrak{X}_t(x)\big)\}$ is a supermartingale and the process $\{Y_t(u_z(x'))(\mathfrak{X}_t(x') - \mathfrak{X}_t(x))\}$ is a submartingale, then the dynamic utility $\mathbf{U}$ is a revealed utility.*
(ii) *Assume in addition that $\mathfrak{X}_t(x)Y_t(u_z(x))$ is a martingale, so that the dual utility $\widetilde{U}$ is also revealed. Then, the supermartingale condition becomes for $x' > x$, $\mathfrak{X}_t(x')Y_t(u_z(x))$ and $Y_t(u_z(x'))\mathfrak{X}_t(x)$ are supermartingale processes.*

Proof. Let $0 < x_0 < x$ and consider a partition of the interval $[x_0,x]$ into N subintervals $]z_n, z_{n+1}]$ where the mesh approaches zero. We approach the integral $\int_{x_0}^x Y_t(u_z(z))d_z\mathfrak{X}_t(z)$ by above, respectively by below, by the Darboux sums $S_t^{N,up}(x_0,x)$, respectively by $S_t^{N,down}(x_0,x)$. Thanks to the monotonicity of the processes $\mathbf{Y}$ and $\mathfrak{X}$, the Darboux sums $S_t^{N,down}(x_0,x)$ and $S_t^{N,up}(x_0,x)$ are bounded above, and converge a.s. to the Stieltjes integral $\int_{x_0}^x Y_t(u_z(z))d_z\mathfrak{X}_t(z)$.

Furthermore, by assumption, the sum $S_t^{N,up}(x_0,x)$ is a positive supermartingale, while the sum $S_t^{N,down}(x_0,x)$ is a positive submartingale. By the positive Fubini's theorem, for fixed $x_0 > 0$, one can interchange the $\lim_{N\to\infty}$ and the expectation to justify that the sub- and super- martingale properties are preserved at the limit. So, $\int_{x_0}^x Y_t(u_z(z))d_z\mathfrak{X}_t(x)$ is a martingale, with expectation $\int_{x_0}^x u_z(z))dz = u(x) - u(x_0)$.
Once again, by monotonicity, the integrals $\int_{x_0}^x Y_t(u_z(z))d_z\mathfrak{X}_t(x)$ go to a limit with finite expectation. So, the Stieltjes integral is well-defined up to 0 and $\{\Psi^{\mathfrak{X}}(t,x) = \int_0^x Y_t(u_z(z))d_z\mathfrak{X}_t(z)\}$ is a martingale; then the process $\{U(t,x) = \Psi^{\mathfrak{X}}(t,\xi(t,x))\}$ is a revealed dynamic utility. □

3.1.2 Orthogonality conditions: definitions and remarks

In the characterization of revealed utility, we will often encountered properties as "the product of two processes is a martingale or a supermartingale". This kind of assumption is frequently used for martingale processes and associated with a notion of orthogonality, due to the cancellation of their quadratic variation.
The role of initial conditions When the notion of orthogonality concerns the product of two martingales $\{M_t, N_t\}$, it is not important to precise their initial conditions, because $(M_t.N_t)$ is a martingale, if and only if $((M_t - M_0).(N_t - N_0))$ is a martingale. When the orthogonality condition is applied to general optional processes, called $\{\Phi_t(x), \Psi_t(y)\}$ in all generality, the difference is important because the martingale property of $(\Phi(x)\Psi_t(y))$ is no more equivalent to the martingale property of $(\Phi_t(x) - x)(\Psi_t(y) - y))$. In Theorem 3.2, we have seen the role of the assumption "the processes $\Phi_t(x) = \mathfrak{X}_t(x)$ and $\Psi_t(y) = Y_t(y)$ are orthogonal for any $x > 0$ and $y > 0$", which is also true in the linear case as in Section 1; but in general the assumption concerns only processes where y is a monotonic function of x, or in a linked interval to x.

Definition 3.1 (Different notions of orthogonality)**.** *Let $(\mathbf{\Phi},\mathbf{\Psi})$ be two optional positive monotonic random fields, and $v(z)$ be a decreasing function with range $[\infty,0]$. Then, we are concerned by the following notions,*

$$\begin{cases} \text{strong orthogonality} : \{\Phi(t,z)\Psi(t,y)\} \textit{ is a martingale } \forall(z,y) > 0. \\ \text{v-orthogonality} : \{\Phi(t,x)\Psi(t,v(x))\} \textit{ is a martingale for any } x > 0. \end{cases} \tag{3.4}$$

REMARK: The v-suborthogonality properties also make sense: $\forall\, x' > x$,

$$\begin{cases} \{\Psi(t,v(x))\big(\Phi(t,x')-\Phi(t,x)\big)\} \text{ is a supermartingale} \quad \text{and} \\ \{\Psi(t,v(x'))\big(\Phi(t,x')-\Phi(t,x)\big)\} \text{ is a positive submartingale.} \end{cases} \tag{3.5}$$

Obviously the strong orthogonality implies all the other forms of orthogonality in particular that of Theorems 3.2 and 3.3.

3.2 Application to aggregation problem

The problem of the aggregation of the preferences is a standard problem in convex analysis, especially in economy, where the problem is very intuitive and used as a reference framework.
Consider a group of agents classified into classes (indexed by θ) having the same preferences u^θ for a wealth amount x^θ. The weight of the different classes is quantified by a finite positive measure $\mu(d\theta)$ on a metric space Θ. The global wealth z of the economy is $\int x^\theta\, \mu(d\theta) = z$. So, z is shared between the different agents through the family of functions $z \to x^\theta(z)$. To guarantee the monotonicity of the various aggregated quantities, the functions $z \to x^\theta(z)$ are assumed to be increasing and differentiable, with range $(0,\infty)$. The simplest example is the linear case, $x^\theta(z) = \alpha^\theta z$ with $\int \alpha^\theta \mu(d\theta) = 1$.(as in [EKHM18b]).
The next step is to define the utility of a "representative agent "of the aggregated economy. As usual in convex analysis, the aggregation concerns the marginal utilities and not the utilities themselves. So, under the assumption that $u^\theta_z(x^\theta(z))$ is z-integrable in $z=0$, the marginal utility (and then the utility) of the aggregated problem is defined by:

$$u^{(\mu)}_z(z) = \int_\Theta u^\theta_z(x^\theta(z))\mu(d\theta), \quad z = \int_\Theta x^\theta(z)\mu(d\theta) \tag{3.6}$$

By analogy, put $\bar{y}^\theta(u^{(\mu)}_z(z)) = u^\theta_z(x^\theta(z))$, then

$$\tilde{u}^{(\mu)}_y(y) = \int_\Theta \tilde{u}^\theta_y(\bar{y}^\theta(y))\mu(d\theta), \quad y = \int_\Theta \bar{y}^\theta(y)\mu(d\theta) \tag{3.7}$$

Our aim is to characterize a representative agent of the aggregated population and his representative preference. In the dynamic framework, the agents are classified by their forward dynamic utility $\mathbf{U}^\theta$ and their "characteristic" wealth (for simplicity c is dropped out) $(\mathfrak{X}^\theta, x)$, their initial utility u^θ as well as by their adjoint process $Y^\theta_t(u^\theta_z(x)) = U^\theta_z(t, \mathcal{X}^\theta_t(x))$. The initial properties on wealth sharing, aggregation and other can be generalized without modification to the dynamic framework, into

$$\begin{cases} \mathcal{X}^{(\mu)}_t(z) = \int \mathcal{X}^\theta_t(x^\theta(z))\, \mu(d\theta) & (3.8) \\ U^{(\mu)}_z(t, \mathcal{X}^{(\mu)}_t(z)) = \int_\Theta U^\theta_z(t, \mathcal{X}^\theta_t(x^\theta(z)))\mu(d\theta) & (3.9) \\ Y^{(\mu)}_t(u^{(\mu)}_z(x)) := \int_\Theta Y^\theta_t(u^\theta_z(x^\theta(z)))\mu(d\theta) = \int Y^\theta_t(\bar{y}^\theta(u^{(\mu)}_z(z)))\mu(d\theta) & (3.10) \end{cases}$$

The last question is to specify under which conditions the aggregation of revealed utilities is still a revealed utility. The dynamic utility $\mathbf{U}^{(\mu)}$, which is compatible with $(\mathfrak{X}^{(\mu)}, \mathbf{Y}^{(\mu)}, u^{(\mu)})$, is expected to satisfy that $\{U(t, \mathcal{X}_t(x))\}$ is a martingale. We have seen that a sufficient condition is the strong orthogonality of the processes $(\mathfrak{X}^{(\mu)}, \mathbf{Y}^{(\mu)})$ which will be obtained for instance from the strong orthogonality of the family $(\mathfrak{X}^\theta, \mathbf{Y}^{\theta'})$.

Proposition 3.4. *Consider the characteristic processes of the aggregated economy $(\mathfrak{X}^{(\mu)}, \mathbf{Y}^{(\mu)}, u)$ defined by equations (3.9),(3.6),(3.10). Assume that :*

for any (θ, θ', x, y), $\mathcal{X}_t^\theta(x) Y_t^{\theta'}(y)$ *is a martingale.*

Then, the processes $\mathfrak{X}^{(\mu)}$ *and* $\mathbf{Y}^{(\mu)}$ *are strongly orthogonal and the utility* $\mathbf{U}^{(\mu)}$ *defined in* (3.6) *is a revealed utility with characteristic process* $\mathfrak{X}^{(\mu)}$*.*

Proof. From previous results, it is sufficient to show that $(\mathfrak{X}^{(\mu)}, \mathbf{Y}^{(\mu)}c)$ are strongly orthogonal, or equivalently that $\mathcal{X}_t^{(\mu)}(x)\, Y_t^{(\mu)}(y)$ is a martingale. By (positive) Fubini's Theorem, and the definition of $\mathfrak{X}^{(\mu)}$, and $\mathbf{Y}^{(\mu)}$,

$$\mathcal{X}_t^{(\mu)}(x)\, Y_t^{(\mu)}(y) = \iint \mu(d\theta)\mu(d\theta')\mathcal{X}_t^\theta(x^\theta(x))\, Y_t^{\theta'}(y^{\theta'}(y)).$$

The martingale property of $(\mathcal{X}_t^\theta(z)\, Y_t^{\theta'}(y))$ is extended to the product $\mathcal{X}_t^{(\mu)}(x)\, Y_t^{(\mu)}(y)$, once again thanks to positive Fubini's theorem. □

Links with the Pareto wealth allocation We come back to the definition of the initial marginal utility from the wealth sharing functions $\{x^\theta(x)\}$. The question of the "optimal" choice of these functions is associated with the following optimization problem, known as the Pareto allocation problem: "Find the best allocation $\{x^\theta(x)\}$ such that $\int_\Theta x^\theta(x)\mu(d\theta) = x$, maximizing the sup-convolution criterium $\sup \int_\Theta u^\theta(x^\theta(x))\mu(d\theta)$" see Proposition 5.5 . An optimal solution $\{x^{\theta,*}(x)\}$, (if there exists), must satisfy the first order condition on the marginal rate of substitution, $(\forall x, \forall\theta,\ u_z^\theta(x^{\theta,*}(x)) = u_z(x))$. This optimal choice is Pareto optimal in the sense that the wealth is allocated in the most efficient manner, but this choice does not imply equality or fairness. The extension to dynamic processes and forward utility is discussed in Section 5.2 of [EKM13]

4 Revealed utilities with Itô's characteristics

The purpose of this section is to illustrate the previous results in the framework of Itô's semimartingales. Similar problems has been considered in [EKM13], with a rigorous treatment of the regularity of the solutions of stochastic differential equations (SDEs) with random coefficients; all references are relative to this paper.

We continue to make the distinction between the algebraic point of view of the $(u, \mathfrak{X}, \mathbf{Y})$ compatible dynamic utility and their martingale properties. The algebraic point of view is based on the monotonicity of SDE's solutions, whose proof is obtained by differentiation. Theorem 2.2 and Proposition 2.3 in [EKM13] give sufficient regularity conditions on the coefficients random fields to justify such property. Then, the marginal *compatible* dynamic utility, generated by the regular solutions of a decoupled SDEs system is solution of a SPDE depending on the drift and the volatility parameters of two characteristics SDEs.

The martingale point of view is based on the strong assumption that the characteristic process and the associated adjoint process are strongly orthogonal. We have seen that this assumption implies that the dynamic utility applied to the characteristic process is a martingale. Given these additional specifications on the coefficients, the compatible utility SPDE is simplified.The strong orthogonality of the processes $(\mathfrak{X}, \mathbf{Y})$ is very restrictive, yielding to a framework very closed to the one of a financial market.

4.1 SDEs and Compatible utility SPDE

We start by recalling general regularity results on one dimensional SDEs solutions. All the results are presented and justified in [EKM13]. We use the generic notation X and Y for the SDEs solutions, and continue to keep the notation $\mathcal{X}$ to a characteristic process.

4.1.1 Regularity of SDEs and Itô's-Ventzel formula.

The probability space $(\Omega, (\mathcal{F}_t), \mathbb{P})$ is equipped with a d-dimensional Brownian motion (W_t) driving the different SDEs with random coefficients of type $(\mu_t(z), \sigma_t(z))$,

$$\begin{cases} \text{SDE form:} \quad dX_t(x) = \mu_t(X_t(x))dt + \sigma_t(X_t(x)).dW_t, \quad X_0 = x & (4.1) \\ \text{Random field} \quad dX_t(x) = \beta(t,x)dt + \gamma(t,x).dW_t, \quad X_0 = x & (4.2) \\ \text{Differential parameters} \quad \beta(t,x) = \mu_t(X_t(x)), \quad \gamma(t,x) = \sigma_t(X_t(x)) & (4.3) \end{cases}$$

The second line is an interpretation of the SDE's solution as a random field (RF), with differential parameters $(\beta(t,x), \gamma(t,x))$. In the sequel, the interplay between these two formulations facilitates analysis and interpretations. The SDE point of view is better suited to questions relative to the order structure as positivity, monotonicity, concavity, when the RF point of view is better suited to differentiability issues. In [EKM13], several technical questions were raised and solved in order to carry out the calculations and deduce the dynamics of the compatible utility U and its dual $\widetilde{U}$. The classes $\mathcal{K}^{m,\delta}$ $(m \in \mathbb{N}^*, \delta \in (0,1])$ of random fields played a major role.

Definition 4.1. *An optional random field $(\phi \in \mathbb{R}^d)$ $(d \geq 1)$ is said to be in the class $\mathcal{K}^{m,\delta}_{loc}$ $(m \in \mathbb{N}^*, \delta \in (0,1])$ if ϕ and is of class $\mathcal{C}^m$ in x with "locally" bounded derivatives such that $\partial^m \phi$ is δ-Hölder.*

The following statements are achieved in Theorems 2.2, 2.5 and 2.6 and Proposition 2.3 in [EKM13]. We also recall the main tool of the differential calculus for random fields, the Itô-Ventzel's formula based on an extension of Itô's formula. We refer to Ventzel [Ven65] and Kunita [Kun97] (Theorem 3.3.1) for different variants of this formula and their proofs.
Observe that generally, random fields are designated as $F(t,z)$, when SDE's solutions are designed as $X_t(x)$.

Proposition 4.1. *Let us consider an Itô random field $\{F(t,z)\}$ with differential parameters $\beta^F(t,z)$ and $\gamma^F(t,z)$, that is $dF(t,z) = \beta^F(t,z)dt + \gamma^F(t,z).dW_t$.*

- *if β^F and γ^F are of class $\mathcal{K}^{m,\delta}_{loc}$, then F is of class $\mathcal{K}^{m,\varepsilon}_{loc}$, $\forall \varepsilon < \delta$ and conversely if $F \in \mathcal{K}^{m,\delta}_{loc}$, then β^F and γ^F are in $\mathcal{K}^{m,\varepsilon}_{loc}$, $\forall \varepsilon < \delta$.*
- *If $m \geq 1$, $dF_z(t,z) = \beta^F_z(t,z)dt + \gamma^F_z(t,z).dW_t$*

SDE PROPERTIES*: If $(X_t(x))$ is a solution starting from $X_0 = x$, of the SDE,*

$$dX_t(x) = \mu^X_t(X_t(x))dt + \sigma^X_t(X_t(x)).dW_t,$$

- *When μ^X and σ^X are in the class $\mathcal{K}^{1,\delta}_{loc}$, $\delta > 0$, a strong solution $X_t(x)$ exists, is unique, positive, monotonous and regularly differentiable in x, with differential*

$$dX_x(t,x) = X_x(t,x)\big[\mu^X_x(t, X_t(x)))dt + \sigma^X_x(t, X_t(x)).dW_t\big], \quad X_x(0,x) = 1 \tag{4.4}$$

  In particular, $x \to X_t(x)$ is increasing. Moreover, if μ^X and σ^X are $\mathcal{K}^{m,\delta}_{loc}$, then X is in the class $\mathcal{K}^{m,\varepsilon}_{loc}, \delta < \epsilon$.
- *Moreover, if $m \geq 3$ then the inverse flow $\{\xi(t,z)\}$ of $(X_t(x))$ is a semimartingale random field in the class $\mathcal{K}^{m-1,\varepsilon}_{loc}$, solution of the SPDE*

$$d\xi(t,z) = -\xi_z(t,z)\big[\mu^X_t(z)dt + \sigma^X_t(z).dW_t\big] + \frac{1}{2}\partial_z\big(\|\sigma^X(t,z)\|^2\xi_z(t,z)\big) \tag{4.5}$$

ITÔ-VENTZEL'S FORMULA: *Consider a $\mathcal{K}^2_{loc}$-Itô semimartingale $\mathbf{F}$ with differential parameters (β^F, γ^F). For any continuous Itô semimartingale Z,*

$$\begin{aligned} dF(t,Z_t) &= \beta^F(t,Z_t)dt + \gamma^F(t,Z_t).dW_t \\ &+ F_x(t,Z_t)dZ_t + \frac{1}{2}F_{xx}(t,Z_t)\langle dZ_t\rangle + \langle\gamma^F_z(t,Z_t).dW_t, dZ_t\rangle \end{aligned} \tag{4.6}$$

4.1.2 Decoupled differential system, and compatible marginal utility SPDE

We continue to adopt the notation X rather than $\mathcal{X}$, since the assumptions imply that the SDE's solutions are still increasing. Let us assume the pair $(\mathbf{X},\mathbf{Y})$ to be the *positive* regular solutions of the differential decoupled system,

$$Syst(I)\begin{cases} dX_t(x)=\mu^X_t(X_t(x))dt+\sigma^X_t(X_t(x)).dW_t,\ X_0(x)=x, & (4.7)\\ dY_t(y)=\mu^Y_t(Y_t(y))dt+\sigma^Y_t(Y_t(y)).dW_t,\ Y_0(y)=y. & (4.8)\end{cases}$$

Obviously, the coefficients are assumed nul at $x=0$, and frequently the multiplicative notation $\mu(t,x)=x\rho(t,x),\sigma(t,x)=x\kappa(t,x)$ is used.

By the regularity of the SDEs, the increasing processes $\mathbf{X}$ and $\mathbf{Y}$ may be considered as the stochastic components of an *admissible* triplet associated with a $\mathcal{C}^2$ regular function u_z. So, they define a unique compatible marginal dynamic utility as in Definition 2.3 by the formula $U_z(t,z)=Y_t(u_z(\xi(t,z)))$ where as usual $\{\xi(t,z)\}$ is the inverse process of X.

This point of view has been developed in Theorem 2.7 in [EKM13], where using Itô-Ventzel's formula, under the right assumptions, the SPDE driving the dynamics of $U_z(t,z)$ is established.

Theorem 4.2. *Under the previous regularity assumptions, let $\{X_t(x)\}$ and $\{Y_t(y)\}$ be the monotonic regular solutions of the Syst(I), and $\{\xi(t,z)\}$ the regular inverse of $\{X_t(x)\}$. Let $\widehat{Q}^X_{t,z}=\frac{1}{2}\partial_z(\|\sigma^X_t(z)\|^2\partial_z)$ be the conjugate second order operator of X which appears in the SPDE of $\{\xi(t,z)\}$, equation* (4.5). *The system is completed by a $\mathcal{C}^2$-marginal utility function u_z.*

The random field $Y_t(u_z(\xi(t,z)))=U_z(t,z)$ is a compatible marginal utility *evolving as,*

$$\begin{aligned} dU_z(t,z) &= \sigma^Y_t(U_z(t,z)).dW_t+\mu^Y_t(U_z(t,z))dt+\widehat{Q}^X_{t,z}(U_z)(t,z)dt\\ &- U_{zz}(t,z)[\mu^X_t(z)dt+\sigma^X_t(z).(dW_t+\sigma^Y_y(t,U_z(t,z))dt] \end{aligned}\tag{4.9}$$

The differential parameters of U_z, (β_z,γ_z) are given by:

$$\begin{cases} \gamma_z(t,z)=\sigma^Y_t(U_z(t,z))-U_{zz}(t,z)\sigma^X_t(z) & (4.10)\\ \beta_z(t,z)=\mu^Y_t(U_z(t,z))-U_{zz}(t,z)[\mu^X_t(z)+\langle\sigma^X_t(z),\sigma^Y_y(t,U_z(t,z))\rangle]+\widehat{Q}^X_{t,z}(U_z)(t,z) & (4.11)\end{cases}$$

Comment: The condition on $\gamma_z(t,z)$ is standard, directly derived from the fact that $Y_t(u_z)$ is a monotonic random field of $X_t(x)$, and not too sensitive to the different assumptions.

To recover U from the previous result, formally it is sufficient to take the primitive of the different terms, but the interpretation of the result is not easy. In particular, it is difficult to read on the direct form, under which conditions the compatible utility is a revealed utility. Fortunately, the results of Section 3 suggest different frameworks where the martingale property is satisfied.

4.2 Revealed utility SPDE under the strong orthogonality assumption

We consider now the martingale point of view. In addition to Theorem 4.2 assumptions, we assume herein the strong orthogonality between the solutions $\mathbf{X}$ and $\mathbf{Y}$ of Syst(I) and its consequence on the coefficients

Assumption 4.1 (Strong orthogonality). *The product $\{X_t(x)\,Y_t(y)\}$ is a martingale for any (x,y). This property is equivalent to*

$$x\,\mu^Y_t(y)+y\,\mu^X_t(x)+<\sigma^X_t(x),\sigma^Y_t(y)>=0,\ \forall x,y.\quad dt\otimes\mathbb{P},a.s.,\tag{4.12}$$

and implies by differentiation w.r.t. y that: $\quad\mu^X_t(z)+<\sigma^X_t(z),\sigma^Y_y(t,y)>=-z\,\mu^Y_y(t,y)$

As explained in the previous sections, this condition implies the revealed nature of a compatible utility. But it allows us very simply to deduce the SPDE of the utility U from the SPDE of its derivative U_z given in Theorem 4.2.

Theorem 4.3. *In addition to Assumptions of Theorem 4.2, suppose $(\mathbf{X}, \mathbf{Y})$ strongly orthogonal. Then, the compatible utility, satisfying the first order condition $U_z(t, X_t(x)) = Y_t(u_z(x))$, is a revealed utility whose dynamics is solution of the SPDE,*

$$\begin{cases} d_tU(t,z) = \big(z\mu^Y_t(U_z(t,z)) + \frac{1}{2}U_{zz}(t,z)\|\sigma^X_t(z)\|^2\big)dt + \gamma(t,z)dW_t \\ \gamma_z(t,z) := \sigma^Y_t(U_z(t,z)) - U_{zz}(t,z)\sigma^X_t(z). \end{cases} \tag{4.13}$$

Proof. The proof relies on the strong orthogonality assumption applied to the drift parameter (4.11) of the marginal SPDE, Equation (4.12) implies that

$$\beta_z(t,z) = \mu^Y_t(U_z(t,z)) - U_{zz}(t,z)(\mu^X_t(z) + <\sigma^X_t(z), \sigma^Y_y(t, U_z(t,z))>) + \partial_z \frac{1}{2}\|\sigma^X(t,z)\|^2 U_{zz}(t,z).$$

$$\text{since} \quad \mu^X_t(z) + <\sigma^X_t(z), \sigma^Y_y(t,y)> = -z\,\mu^Y_y(t,y)$$

$$\text{then} \quad \mu^Y_t(U_z(t,z)) - U_{zz}(t,z)(\mu^X_t(z) + <\sigma^X_t(z), \sigma^Y_y(t, U_z(t,z))> = \partial_z(z\mu^Y_t(U_z(t,z))),$$

$$\text{and} \quad \beta_z(t,z) = \partial_z\big[z\mu^Y_t(U_z(t,z)) + \frac{1}{2}\|\sigma^X(t,z)\|^2 U_{zz}(t,z)\big]$$

By integrating with respect to z, the U-SPDE (4.13) follows by positivity assumption $\sigma^X(t,0) \equiv 0$, and by regularity $z\mu^Y_t(U_z(t,z)) \to 0$ if $z \to 0$. □

Since X and Y play similar roles and since the strong orthogonality condition is symmetrical, one can by exchanging the symbols X and Y, establish the dynamics of the dual utility using the first order condition $\widetilde{U}_y(t, Y_t(y)) = -X_t(-\tilde{u}(y))$.

Corollary 4.4. *Under Assumptions of Theorem 4.3, any revealed dynamic conjugate utility is solution of the SPDE,*

$$\begin{cases} d_t\widetilde{U}(t,y) = \big(-y\mu^X_t(-\widetilde{U}_y(t,y)) + \frac{1}{2}\widetilde{U}_{yy}(t,y)\|\sigma^Y_t(y)\|^2\big)dt + \tilde{\gamma}(t,y)dW_t & (4.14) \\ \tilde{\gamma}_y(t,y) := -\sigma^X_t(-\widetilde{U}_y(t,y)) - \widetilde{U}_{yy}(t,y)\sigma^Y_t(y). & (4.15) \end{cases}$$

4.2.1 Optimality in the learning problem

In this paragraph, we reintroduce the distinction between characteristic process $\mathcal{X}$ and the general solution X of SDE, to show using calculations the martingale property of $U(t, \mathcal{X}_t(x))$ under strong orthogonality assumption, proved in Section 3 by a different way. What is new is that, for any other SDE solution $(X_t(x))$ strongly orthogonal to Y, the dynamic utility $U(t, X_t(x))$ is a supermartingale, whose expectation is still less than $u(x)$. Then, $\mathcal{X}$ is the best choice for this expected utility criterium.
We start with a general formula for the dynamics of $U(t, Z_t)$ for an Itô semimartingale Z.

Proposition 4.5. *Under Assumptions of Theorem 4.3 and the strong orthogonality assumption, let U be the $(\mathcal{X}_t(x), Y_t(y))$ compatible utility.*
(i) *For any semimartingale Z with differential parameters (ϕ^Z_t, ψ^Z_t)*

$$\begin{aligned} dU(t,Z_t) = \frac{1}{2}U_{zz}(t,Z_t)||\sigma^{\mathcal{X}}_t(Z_t) - \psi^Z_t||^2 dt + \langle \gamma^U(t,Z_t) + U_z(t,Z_t)\psi^Z_t, dW_t\rangle \\ +[Z_t\,\mu^Y_t(U_z(t,Z_t)) + U_z(t,Z_t)\phi^Z_t + <\sigma^Y_t(U_z(t,Z_t)), \psi^Z_t>]dt \end{aligned} \tag{4.16}$$

(ii) Learning problem *Let $\mathbf{X}$ be the regular solution of SDE(μ^X, σ^X), assumed to be strongly orthogonal to Y, $(x\,\mu^Y_t(y) + y\,\mu^X_t(x) + <\sigma^X_t(x), \sigma^Y_t(y)> = 0)$. By the concavity of $U(t,z)$,*

– *$U(t, X_t(x))$ is a supermartingale, with negative drift* $\frac{1}{2}U_{zz}(t, X_t(x))||\sigma^{\mathcal{X}}_t(X_t(x)) - \sigma^X_t(X_t(x))||^2$,

– *$U(t, X_t(x))$ is a martingale if and only if* $\sigma^{\mathcal{X}}_t(x) = \sigma^X_t(x)$*, and then* $X_t(x) = \mathcal{X}_t(x)$

$(\mathcal{X}_t(x))$ is U-optimal in the class of the regular SDE solutions, strongly orthogonal to Y.

Proof. **(i)** Given the dynamics of U, Equation (4.13), the Itô-Ventzel's formula applied to $U(t, Z_t)$ where the differential parameters of Z are (ϕ^Z_t, ψ^Z_t) gives

$$\begin{cases} dU(t,Z_t) = \big[Z_t\, \mu^Y_t(U_z(t,Z_t)) + \dfrac{1}{2}U_{zz}(t,Z_t)||\sigma^X_t(Z_t)||^2 + \dfrac{1}{2}U_{zz}(t,Z_t)||\psi^Z_t||^2\big]dt \\ +[U_z(t,Z_t)\,\phi^Z_t + < \gamma^U_z(t,Z_t), \psi^Z_t >]dt + \big(\gamma^U(t,Z_t) + K_t\psi^Z_t\big)dW_t \\ \gamma^U_z(t,Z_t) := \sigma^Y_t(U_z(t,Z_t)) - U_{zz}(t,Z_t)\sigma^X_t(Z_t) \end{cases} \tag{4.17}$$

Then, the drift term of $U(t, Z_t)$ is decomposed as the sum of two terms, the first one implies the second derivative of U, as $A^{\phi,\psi}_t = \frac{1}{2}U_{zz}(t, Z_t)||\sigma^{\mathcal{X}}_t(Z_t) - \psi^Z_t||^2$ which is negative since U is concave. The second one is given by, $B^{\phi,\psi}_t = Z_t\, \mu^Y_t(U_z(t, Z_t)) + U_z(t, Z_t)\phi^Z_t + < \sigma^Y_t(U_z(t, Z_t)), \psi^Z_t >$.

(ii) Assume that $(Z_t = X_t(x), \phi^Z_t = \mu^X(t, X_t(x)), \psi^Z_t = \sigma^X(t, X_t(x)))$, where (μ^X, σ^X) satisfy the orthogonality condition, $x\, \mu^Y_t(y) + y\, \mu^X_t(x) + < \sigma^X_t(x), \sigma^Y_t(y) >= 0$. Then, $B^{\phi,\psi}_t(x) = 0, d\mathbb{P} \otimes dt$ a.s., and by the concavity of U, $U(t, X_t(x)$ is a supermartingale, whose drift coefficient $\frac{1}{2}U_{zz}(t, X_t(x))||\sigma^{\mathcal{X}}_t(X_t(x)) - \sigma^X_t(X_t(x))||^2$ is equal to 0 if and only if $\sigma^X = \sigma^{\mathcal{X}}$. In this case, the orthogonality conditions for $\mathcal{X}$ and X imply also that $\mu^X = \mu^{\mathcal{X}}$. Then, by uniqueness of the SDE(μ^X, σ^X), $X_t(x) = \mathcal{X}_t(x)$. □

4.2.2 Strongly orthogonal SDEs and optimization in financial market

In order to obtain the dynamics of the revealed utility in Theorem 4.3, we have introduced the strong orthogonality assumption without exploiting all the consequences of this condition on the SDE's coefficients. In this paragraph, we are going into the details and we show the links on the coefficients of these two equations equivalent to this condition.
For simplicity, we use the multiplicative representation $\mu_t(z) = z\rho_t(z)$, $\sigma_t(z) = z\kappa_t(z)$ (by positivity of the solutions). Thus, the strong orthogonality of $(\mathbf{X}, \mathbf{Y})$ becomes

$$d\mathbb{P} \otimes dt\; a.s., \quad \rho^X_t(x) + \rho^Y_t(y) + < \kappa^X_t(x), \kappa^Y_t(y) >= 0, \ \forall x, y. \tag{4.18}$$

The fact that this identity is true for all x and y has many consequences, on the volatilities of the two processes, and then of course on their drifts.

Lemma 4.6. *Put* $\kappa^X_t(0) = \eta^X_t, \kappa^Y_t(0) = \eta^Y_t$*, and* $\rho^X_t(0) = r_t - < \eta^X_t, \eta^Y_t >$.
The condition "$\rho^X_t(x) + \rho^Y_t(y) + < \kappa^X_t(x), \kappa^Y_t(y) >= 0, \forall x, y$*" is equivalent to,*

$$\begin{cases} \forall x, x', y, y', < \kappa^X(t,x) - \kappa^X(t,x'), \kappa^Y(t,y) - \kappa^Y(t,y') >= 0 & (4.19) \\ \rho^X_t(x) = r_t - < \kappa^X_t(x), \eta^Y_t >, , & (4.20) \\ \rho^Y_t(y) = -r_t - < \kappa^Y_t(y) - \eta^Y_t, \eta^X_t > = -\rho^X_t(0) - < \kappa^Y_t(y), \eta^X_t > & (4.21) \end{cases}$$

Proof. The inner product $< \kappa^X_t(x), \kappa^Y_t(y) >$ is separable in the parameters x and y. After a double differentiation with respect to x and y of Equation (4.18), we obtain that $< \kappa^X_x(t, x), \kappa^Y_y(t, y) >= 0$; this identity, which is stable by linear combinations, is equivalent after integration to the orthogonality of the increments $< \kappa^X(t, x) - \kappa^X(t, x'), \kappa^Y(t, y) - \kappa^Y(t, y') >= 0$.
Making respectively $y = 0$ and $x = 0$ in (4.18), we obtain the value of the relative drift as an affine functional of the volatility. In finance, the volatility vector $\bar{\kappa}^Y_t(y)$ is assumed to be orthogonal to the vector $\kappa^X_t(x)$, or only to the vector $\eta^X_t = \kappa^X_t(0)$, from the strong orthogonality condition. □

Equations (4.20), (4.21) remind us a standard framework of financial market, where the underlying asset, here the characteristic process $(\mathfrak{X}_t(x))$ is strongly orthogonal to an adjoint processes $\{Y_t(y)\}$, often called "pricing kernel", because of the orthogonality relation $\mathbb{E}(\mathfrak{X}_t(x)Y_t(y)/y) = x$. By analogy with the portfolios, we consider the family of semimartingales (Z_t^π) controlled by their volatility π_t and orthogonal to the random field $\{Y_t(y)\}$. The parameter π_t can be viewed as an "open-loop control", when a volatility coefficient $\kappa_t(z)$ can be interpreted as a "feedback control".

Theorem 4.7. *Let $\{(\mathfrak{X}_t(x), Y_t(y)\}$ be a pair of strongly orthogonal SDE, with relative coefficients as in Lemma 4.6, where the optional process (r_t) is interpreted as the short rate and $-\eta_t^Y$ as the risk premium of the market.*
(i) *A (Z_t^π) positive semimartingale controlled by its volatility π_t is orthogonal to the random field $\{Y_t(y)\}$, if and only if*

$$dZ_t^\pi = r_t Z_t^\pi dt + Z_t^\pi < \pi_t, dW_t - \eta_t^Y dt >, \quad \text{and} < \pi_t - \eta_t^X, \kappa_t^Y(y) - \eta_t^Y >= 0 \quad \forall y \tag{4.22}$$

In particular, for a fixed initial wealth x_0, the process $(\mathfrak{X}_t(x_0))$ is controlled by its feedback volatility $\kappa_t^X(X_t(x_0))$ (see (4.19)).
(ii) *The process $U(t, Z_t^\pi)$ is a supermartingale, and a martingale if and only if $\pi_t = \kappa^{\mathfrak{X}}(t, Z_t^\pi)$, that is if Z_t^π is solution of the SDE$(x\rho^{\mathfrak{X}}, x\kappa^{\mathfrak{X}})$.*
(iii) *The revealed utility $U(0, x)$ is the value function of a standard portfolio optimization problem in incomplete market, with stochastic reward at time T, $U(T, Z_T^\pi(x))$, in the sense where $U(0, x) = \sup_\pi\{U(T, Z_T^\pi(x))| < \pi_t - \eta_t^X, \kappa_t^Y(y) - \eta_t^Y >= 0, \forall y\}$. The optimal portfolio is $\mathfrak{X}$ and the optimal pricing kernel is $\mathbf{Y}$.*

Proof. **(i)** The process Z_t^π is strongly orthogonal to $Y_t(y)$, since $\rho_t^\pi = r_t - < \pi_t, \eta_t^Y >, \kappa_t^\pi = \pi_t$, and $\rho_t^\pi + \rho_t^Y(t, y) + < \pi_t, \kappa_t^Y(y) >=< \pi_t - \eta_t^X, \kappa_t^Y(y) - \eta_t^Y >= 0$.
For a given x_0, the characteristic process $\mathfrak{X}_t(x_0)$ is associated with the linear portfolio Z_t^κ whose the strategy is $\pi_t = \kappa^{\mathfrak{X}}(t, x_0)$.
(ii) Then, by Proposition 4.5, $U(t, Z_t^\pi)$ is a supermartingale, and a martingale if and only if $\pi_t = \kappa^{\mathfrak{X}}(t, Z_t^\pi)$. But as before, if this condition holds, then Z_t^π is solution of the same SDE as $\mathfrak{X}$. We have proved that the revealed dynamic utility is the value function of the standard portfolio optimization problem in incomplete market, with optimal solution $\mathfrak{X}$. □

Remark . This presentation of the portfolio optimization problem does not make any reference to the notion of arbitrage and self-financing, but only to the orthogonality with an "optimal" state pricing process.

Nevertheless, in standard financial markets, not only interest rates but also market risk premium η_t^Y are assumed to be exogenous, and equal to the Y- volatility $\kappa_t^Y(y) = \eta_t^Y$.

Corollary 4.8. *Assume X and Y to be strongly orthogonal with given interest rate and risk premium equal to the Y-volatility.*
(i) *Then, $Y_t(y) = y\bar{Y}_t$ is a linear process, known in finance as pricing kernel or stochastic discount factor, and $\bar{Y}_t = \exp(\int_0^t r_s ds + \int_0^t \eta_s^Y.dW_s - \frac{1}{2}\|\eta_s^Y\|^2 ds)$.*
(ii) *The Y-strongly orthogonal processes (Z_t^π) are self-financing portfolios, $(dZ_t^\pi = r_t Z_t^\pi dt + Z_t^\pi < \pi_t, dW_t - \eta_t^Y dt >)$, and for any initial utility, the revealed dynamic utility is the value function of a standard portfolio optimization problem, with linear pricing kernel $y\bar{Y}$.*
(iii) *$y\bar{Y}$ is optimal for the conjugate $\widetilde{U}(t, y)$, solution of*

$$\begin{cases} d_t\widetilde{U}(t, y) = \big(- \widetilde{U}_y(t, y) r_t + \dfrac{y^2}{2}\widetilde{U}_{yy}(t, y)\|\eta_t^Y\|^2\big)dt + \tilde{\gamma}(t, y)dW_t & (4.23) \\ \tilde{\gamma}_y(t, y) := -\widetilde{U}_y(t, y)\kappa_t^X(-\widetilde{U}_y(t, y)) - y\widetilde{U}_{yy}(t, y)\eta_t^Y. & (4.24) \end{cases}$$

5 Revealed non-concave preferences and economic equilibrium

The first motivation of this section is to extend the previous construction to a regular SDE's system with coupled coefficients, in order to solve equilibrium problems. The paper by He and Leland [HL93] on the necessary conditions for equilibrium has been a rich source of inspiration, and we have adapted their ideas in our context.

5.1 Partially coupled SDEs system

We adopt a similar presentation than in Section 4, but we assume now that the pair $(\mathbf{X}, \mathbf{Y})$ is a regular solution of the partially coupled differential system, whose first component $\mathbf{X}$ is assumed to be autonomous.

$$Syst(II)\begin{cases} dX_t(x) = \mu_t^X(X_t(x))dt + \sigma_t^X(X_t(x))dW_t, & X_0(x) = x, \quad (5.1)\\ dY_t(x,y) = \mu_t^Y(X_t(x), Y_t(x,y))dt + \sigma_t^Y(X_t(x), Y_t(x,y))dW_t, & Y_0(x,y) = y. \quad (5.2)\end{cases}$$

Regularity assumptions are made on the coefficients of the system to guarantee existence, uniqueness, regularity and positivity of the solution $(X_t(x), Y_t(x,y))$ of the SDE's system. The multiplicative form of the coefficients is for X, $(\mu_t^X(x) = x\rho_t^X(x), \sigma_t^X(x) = x\kappa_t^X(x))$ and for Y $(\mu_t^Y(x,y) = y\rho_t^Y(x,y), \sigma_t^Y(x,y) = y\kappa_t^Y(x,y))$.

Under the regularity assumptions, $x \to X_t(x)$ is increasing, and the process $y \to Y_t(x,y)$ is increasing for a given x. But the function $x \to Y_t(x,y)$ is more complex to analyze, and is not increasing in general. In [**?**], we have studied the limit behavior of $X_t(x)$; $X_t(x) \to 0$ when $x \to 0$ and to $+\infty$ when $x \to +\infty$. Moreover, the uniform Lipschitz assumption on μ^Y implies that if $(y \to \infty, x \to 0)$ then $\lim \rho_t^Y(x,y) < +\infty$. We also need the more precise result that $xY_t(x,y) \to 0$ if $x \to 0, y \to \infty$, and $x\, y \to 0$. Such result can be in general deduced from the regularity of the coefficients (or taken as an assumption).

5.1.1 SPDE of compound stochastic flows and non concave preferences

Let us consider a deterministic (decreasing) function v, of class $\mathcal{C}^2$ on $(0,\infty)$, with limit ∞ at 0, and 0 at ∞. We are concerned with the random field $\widehat{Y}(t,x) = Y_t(x, v(x)), x > 0$, whose differential characteristics are $\widehat{\beta}^Y(t,x) = \mu_t^Y(X_t(x), \widehat{Y}(t,x))$, and $\widehat{\gamma}^Y(t,x) = \sigma_t^Y(X_t(x), \widehat{Y}(t,x))$. This random field is no longer decreasing in x, but sufficiently regular to apply Itô-Ventzel's formula, and its limit when $x \to 0$ is well-understood.

General SPDE: As previously, we denote by $\{\xi(t,z)\}$ the regular inverse of the monotonic process $\{X_t(x)\}$. For any $\mathcal{C}^2$-function v, we introduce the compound random field $G(t,z) = Y_t(\xi(t,z), v(\xi(t,z)))$, also denoted $\widehat{Y}(t, \xi(t,z))$. The same calculation as in the previous section can be carried out for the study of G. The main difference is that G still a positive regular random field on $(0,\infty)$, but in general not decreasing (not a marginal utility). Indeed, by Itô-Ventzel's formula, the dynamics of the random field $G(t,z)$ is the SPDE, with the second order operator $\widehat{Q}_{t,z}^X = \frac{1}{2}\partial_z(\|\sigma_t^X(z)\|^2\partial_z)$,

$$\begin{aligned} dG(t,z) &= \sigma_t^Y(z,G).dW_t + \mu_t^Y(z,G)dt + \widehat{Q}_{t,z}^X(G)\,dt \\ &-G_z(t,z)[\mu_t^X(z)dt + \sigma_t^X(z).(dW_t + \sigma_y^Y(t,z,G)dt)] - \sigma_t^X(z).\sigma_x^Y(t,z,G)dt \end{aligned} \quad (5.3)$$

whose differential characteristics are $(\beta^G(t,z), \gamma^G(t,z))$.

Compared to the dynamics (4.9), the new G-SPDE (5.3) differs essentially by the **extra term**, coming from the derivative of $\sigma_t^Y(x,y)$ with respect to x, $\sigma_t^X(z).\sigma_x^Y(t,z,G(t,z))$. If $\widehat{Y}(t,x)$ is monotonous with respect to x, then $G(t,z)$ is the derivative $U_z(t,z)$ of a compatible utility $U(t,z)$. It remains to us to

introduce the right orthogonality conditions in view of the martingale properties.
ORTHOGONALITY CONDITIONS: These conditions are of two types, the strong orthogonality of the coordinates and the v-orthogonality of $\widehat{Y}(t,x) = Y_t(x, v(x))$ and $X_x(t,x)$; equivalently, the processes $\{X_t(x)Y_t(x,y),\ x,y > 0\}$ and $\{Y_t(x,v(x))X_x(t,x),\ x > 0\}$ are assumed to be martingales. The martingale condition of the product of the coordinates (going to 0 if $(y \to \infty, x \to 0, xy \to 0)$ is equivalent to the following constraints on the coefficients

$$(\mathrm{I}) \begin{cases} x\mu_t^Y(x,y) + y\mu_t^X(x) + <\sigma_t^X(x), \sigma_t^Y(x,y)> = 0, & (5.4) \\ \text{or } \rho_t^Y(x,y) + \rho_t^X(x) + <\kappa_t^X(z), \kappa_t^Y(x,y)> = 0. & (5.5) \end{cases}$$

Then, the dynamics of $Y_t(x,y)$ take the following simple form where Y_t appears only in the volatility κ^Y,

$dY_t/Y_t = -\rho_t^X(X_t)dt + <\kappa_t^Y(X_t, Y_t), dW_t - \kappa_t^X(X_t)dt>$.

The second assumption concerns the v-orthogonality of $\{Y_t(x,v(x))\}$ and the derivative $\{X_x(t,x)\}$ for any $x > 0$. An equivalent formulation, where y is replaced by $G(t,z)$, is

$$(\mathrm{vII}) \begin{cases} \mu_t^Y(z, G(t,z)) + G(t,z)\mu_z^X(t,z) + <\sigma_z^X(t,z)\sigma_t^Y(z, G(t,z)) = 0, & (5.6) \\ \text{or } \rho_z^X(t,z) + <\kappa_z^X(t,z), \kappa_t^Y(z, G(t,z))> = 0 & (5.7) \end{cases}$$

By taking these two relations in consideration in the G-SPDE, we obtain the same SPDE than in the decoupled case for the primitive J of G.

Theorem 5.1. *Let $J(t,z)$ (with $J(t,0) = 0$) be the primitive preferences $G(t,z) = Y_t(\xi(t,z), v(\xi(t,z)))$.*
(i) *Under the two orthogonality conditions $(I), (vII)$,*

$$\begin{cases} dJ(t,z) = \big[z\mu_t^Y(z, J_z(t,z)) + \frac{1}{2}||\sigma_t^X(z)||^2 J_{zz}(t,z)\big]dt + \gamma^J(t,z)dW_t & (5.8) \\ \gamma_z^J(t,z) = \sigma_t^Y(z, J_z(t,z)) - J_{zz}(t,z)\sigma_t^X(z). & (5.9) \end{cases}$$

Nevertheless, the concavity is not guaranteed in general.
(ii) *As before, the process $J(t, \mathcal{X}_t(x))$ is a martingale, but in absence of concavity J is not an optimal choice.*

Proof. Let $\beta^G(t,z)$ the drift parameter of the random field G and $b^G(t,z) = \beta^G(t,z) - \widehat{Q}_{t,z}^X(G)$.
(i) At first, injecting Equation (5.6) in $b^G(t,z)$ and using the notation G in place of $G(t,z)$ in the functions, we obtain that $b^G(t,z)$ is a derivative function,

$$\begin{aligned} b^G(t,z) &= \mu_t^Y(z,G) - \mu_t^X(z)G_z(t,z) - G_z(t,z) <\sigma_t^X(z), \sigma_y^Y(t,z,G)> + <\sigma_t^X(z), \sigma_z^Y(t,z,G)> \\ &= -G(t,z)\mu_z^X(t,z) - <\sigma_z^X(t,z), \sigma_t^Y(z,G)> - \mu_t^X(z)G_z(t,z) \\ &\quad -G_z(t,z) <\sigma_t^X(z), \sigma_y^Y(t,z,G)> - <\sigma_t^X(z), \sigma_z^Y(t,z,G)> \\ &= -\partial_z[G(t,z)\mu_t^X(z) + <\sigma_t^X(z), \sigma_t^Y(z,G)>] \\ &= \partial_z[z\mu_t^Y(z, G(t,z))] \quad \text{from Condition (I).} \end{aligned} \tag{5.10}$$

Since $\widehat{Q}_{t,z}^X(G)$ is also a derivative, that achieved the proof of **(i)** .
(ii) Since the J-SPDE is similar of the SPDE of the decoupled case, equation (4.16) holds,

$$\begin{aligned} dJ(t,Z_t) = \frac{1}{2}J_{zz}(t,Z_t)||\sigma_t^{\mathcal{X}}(Z_t) - \psi_t^Z||^2 dt + \langle\gamma^J(t,Z_t) + J_z(t,Z_t)\psi_t^Z, dW_t\rangle \\ +[Z_t\,\mu_t^Y(Z_t, J_z(t,Z_t)) + J_z(t,Z_t)\phi_t^Z + <\sigma_t^Y(Z_t, J_z(t,Z_t)), \psi_t^Z>]dt \end{aligned}$$

By the strong orthogonality condition, the martingale property of $\mathcal{X} = X$ holds, but even if Z is "strongly orthogonal", the optimality is lost since the sign of $J_{zz}(t,Z_t)$ is not negative. ☐

5.1.2 Discussion on the v-orthogonality constraint

We can say more on the consequences of the orthogonality conditions ((I),(vII)) given in (5.7), in particular on the v-orthogonality constraint. The main problem is the dependence in v and more generally in G of the constraint, that we recall here

$$\text{Orthogonality}\begin{cases}(I) & \rho_t^Y(z,y)+\rho_t^X(z)+<\kappa_t^X(z),\kappa_t^Y(z,y)>=0.\\ (vII) & \rho_z^X(t,z)+<\kappa_z^X(t,z),\kappa_t^Y(z,G(t,z))>=0\end{cases}$$

We use the asymptotic behavior described at the beginning of the section to define the limit in $(0,+\infty)$ of the Y-multiplicative coefficients when the product $x\,y\to 0$, and denote their limits by $\hat{\rho}_t^Y(0)=\lim_{(0,\infty)}\rho_t^Y(x,y)$ and by $\hat{\eta}_t^Y=\lim_{(0,\infty)}\kappa_t^Y(x,y)$; as previously, $\kappa_t^X(0)=\eta_t^X$. Observe that all this parameters are independent of G and v, and that, as before,

$$\rho_t^X(0)+\hat{\rho}_t^Y(0)=-<\eta_t^X,\hat{\eta}_t^Y>.$$

The structure of the problem differs from that of the decoupled case, since the v-orthogonality condition implies a direct link between the different parameters of the problem to be solved by the presence of the G-function. For instance, this condition implies that, $\rho_t^X(0)=r_t-<\eta_t^X,\hat{\eta}_t^Y>$ if $\kappa_t^X(0)=\eta_t^X,\hat{\kappa}_t^Y(0)=\hat{\eta}_t^Y$ and $\rho_t^X(z)=\rho_t^X(0)-\int_{]0,z]}<\kappa_z^X(t,\alpha),\kappa_t^Y(t,\alpha,G(t,\alpha))>d\alpha$.

Comment: To summarize, in Section 4, we have shown that for any strongly orthogonal decoupled system, and for any concave $\mathcal{C}^3$-function u, the construction by composition of two regular random fields yields to a revealed utility U, solution of the SPDE given in Theorem 4.3, with optimality of the characteristic process $X=\mathfrak{X}$ and its orthogonal process Y. When the strongly orthogonal system is partially coupled and v-orthogonal, the same process yields for J to the same SPDE (5.8), and the martingale property for $J(t,\mathfrak{X}_t(x))$, but many differences appear with the loss of the decreasing property of $x\to Y_t(x,v(x))$, and then of the concavity of J. To find a concave solution, the initial condition $v=u_z$ cannot be any, no more than the random multiplicative coefficients $\kappa_t^X(x)$, $\kappa_t^Y(x,y)$, the other being determined by the orthogonality conditions (up to stochastic initial conditions). The problem is to find motivated conditions yielding to the existence of a triple (X,Y,u) for which a revealed utility can exist.

5.2 Random economic equilibrium and their time-finite variation decreasing preferences

Our aim is to introduce additional assumptions to simplify the problem of existence of a concave solution to the J-SPDE (5.8). The new setting is motivated by the standard Markovian point of view in economy, in particular by the framework of He and Leland [HL93] on the economic equilibrium. The J function is assumed to be concave and deterministic, with finite variation in time, (decreasing for concave function). In random environment (our framework) this notion is extended into preferences with finite variation in time, first studied by Zariphopoulou & alii [MZ10a] and Berrier & alii [BRT09] in a more restrictive setting. Moreover, guided by economic considerations in absence of consumption, the short rate r_t is assumed to be exogenous but stochastic, and equal (by assumption) to the opposite of the relative drift coefficient $\rho_t^Y(x,y)=-r_t$ of the Y-process.

5.2.1 A random economic equilibrium setting

We define a random economic equilibrium setting as an universe generated by a partially coupled strongly orthogonal system $(X_t(x),Y_t^e(x,y))$ (e for equilibrium), as before the relative drift of Y^e, $\rho_t^Y(x,y)$ is equal to $-r_t$. In addition, we only consider dynamic preferences with initial condition u, such that $(X_x(t,x)Y_t(x,u_z(x)))$ are orthogonal, that are of finite variation in time, meaning that the G-diffusion random field $\gamma_z(t,z) = 0$. In the following lemma, we try to explain the role of these two different assumptions.

Proposition 5.2. *Under the two orthogonality constraints,*
(i) *If $\rho^Y_t(x,y) = -r_t$, there exists a process η^e_t such that $\kappa^Y_t(z,G)+\eta^e_t$ is orthogonal to $\kappa^X_t(z)$; if G is of finite variation in time, then $G(t,z)\kappa^Y_t(z,G) = zG_z(t,z)\kappa^X_t(z)$ so that the vectors $\kappa^Y_t(z,G)$ and $\kappa^X_t(z)$ are collinear for any z. If the two properties hold together, only the volatility κ^X of X is free and $\forall z>0$,*

$$\kappa^Y_t(z,G) = -\eta^e_t, \quad \rho^X_t(z) = r_t + <\kappa^X_t(z), \eta^e_t>, \quad \rho^Y_t(z,G) = -r_t \tag{5.11}$$

(ii) *The equilibrium setting is similar to the one of a financial market, described in Corollary 4.8:*
a) *The adjoint process $Y^e = y\bar{Y}^e_t$ is independent of x and linear with respect to y,*

$$Y_t(x,y) = y\exp\big(-\int_0^t (r_s + \frac{1}{2}||\eta^e_s||^2)ds - \int_0^t \eta^e_s.dW_s\big) = y\bar{Y}^e_t. \tag{5.12}$$

b) *$(X_t(x), Y^e_t(y)$ defines a decoupled strongly orthogonal system. Consequently, by Corollary 4.8, for any initial regular utility, the revealed marginal utility U is the value function of some portfolio optimization problem. The linear adjoint process is optimal for the conjugate utility $\widetilde{U}$.*
c) *But these solutions are in general not associated with an economic equilibrium, since in most of the case the dynamic utilities and their conjugate transform are not of finite variation in time.*

Proof. **(i)** Taking the difference between the derivative of (I) (w.r.t. z) and (vII) leads to the identity $\partial_z(\rho^Y_t(z,G(t,z))) + <\partial_z(\kappa^Y_t(z,G(t,z)), \kappa^X_t(z))>= 0$, $\forall z$. If $\rho^Y_t(z,G(t,z))$ is independent of z, then $<\kappa^X_t(z), \partial_z(\kappa^Y_t(z,G(t,z))>= 0$, $\forall z$. That is the vector $\partial_z(\kappa^Y_t(z,G(t,z))$ is orthogonal to $\kappa^X_t(z)$ for every z. So there exists a random field η^e_t such that $<\kappa^Y_t(z,G)+\eta^e_t, \kappa^X_t(z)>= 0$.
If J is with finite variation, then its volatility vector is null almost surely for every z, $\gamma^G(t,z) = \sigma^Y_t(z,G(t,z)) - G_z(t,z)\sigma^X_t(z) = 0$, that is $G(t,z)\kappa^Y_t(z,G(t,z)) = zG_z(t,z)\kappa^X_t(z)$ and the vectors $\kappa^Y_t(z,G)$ and $\kappa^X_t(z)$ are collinear. If the two properties hold together, κ^Y is collinear to κ^X implying that $\kappa^Y_t(z,G(t,z)) = -\eta^e_t$ is also independent of z.
(ii) The representation of $Y^e_t(x,y)$ is then standard, and by construction the processes $X_t(x)$ and $\bar{Y}^e$ are strongly orthogonal. All the other properties are shown in Corollary 4.8. □

Then the problem is now to find a pair $(u_x, X_t(x))$ such that the revealed utility associated with a linear adjoint process is decreasing in time. Clearly, it is very natural to consider first the dual problem whose optimal solution $y\bar{Y}^e_t$ is known to be linear.

5.3 Characterization of the equilibrium preferences

With these different assumptions, at the equilibrium, the market parameters (r_t, η^e_t) are exogenous. Then, the forward SPDE given in Corollary 4.8 becomes,

$$\partial_t\widetilde{J}(t,y) + \frac{1}{2}y^2\|\eta^e_t\|^2\widetilde{J}_{yy}(t,y) - yr_t\widetilde{J}_y(t,y) = 0 \tag{5.13}$$

The question is to characterize all solutions of this forward PDE considered ω by ω.

5.3.1 Characterization of the forward solutions of the duale SPDEs (5.14)

It is easier to find a solution to (5.14) in the family of time-separable conjugate utilities, $\widetilde{J}(t,y) = j(y)H_t$ since dual power utilities are obviously good candidates given by

$$\widetilde{J}^{(\beta)}(t,y) = \tilde{H}^{(\beta)}_t \frac{y^{1-\beta}}{\beta-1}, \quad \tilde{H}^{(\beta)}_t = \exp(-(\beta-1)\int_0^t (r_s + \frac{1}{2}\beta\,\|\eta^e_s\|^2)ds), \quad \beta>1 \tag{5.14}$$

But they are not the only ones. We provide a complete characterization of all positive time-decreasing solutions, with the help of the Widder's theorem characterizing the positive space-time harmonic functions of the Brownian motion. This result was first used by (T. Zariphopoulou & alii.[MZ10a], Berrier & alii.[BRT09]), in the context of dynamic utilities:

Theorem 5.3 (Widder 1963)**.** *A function $\Psi : (0,\infty)\times\mathbb{R}\to\mathbb{R}$ is a positive standard solution to the heat equation, $\Psi_t(t,z)+\frac{1}{2}\Psi_{zz}(t,z)=0$, if and only if it can be represented as*

$$\Psi(t,z)=\int_{\mathbb{R}} e^{\beta z-\frac{1}{2}\beta^2 t}m(d\beta), \tag{5.15}$$

where m is a Borel measure such that the above integral is finite for all $(t,z)\in(0,\infty)\times\mathbb{R}$.

The "revealed conjugate utility" version of this theorem is the following,

Theorem 5.4. *A regular function $\widetilde{J}$ is a revealed conjugate utility if and only if there exists a positive Borel measure $\mu(d\beta)$ on $(1,\infty)$ such that $\tilde{u}(y)=\int_1^\infty \frac{y^{1-\beta}}{\beta-1}\mu(d\beta)<\infty$, and that*

$$\widetilde{J}^{(\mu)}(t,y)=\int_1^\infty \widetilde{J}^{(\beta)}(t,y)\mu(d\beta)=\int_1^\infty \tilde{H}_t^\beta\frac{y^{1-\beta}}{\beta-1}\mu(d\beta), \tag{5.16}$$

where $\tilde{H}_t^\beta=\exp(-(\beta-1)\int_0^t(r_s+\frac{1}{2}\beta\,\|\eta_s^e\|^2)ds)$ is decreasing in time.
$\widetilde{J}^{(\mu)}$ is a solution of the PDE (5.14) whose the initial conjugate utility is necessarily an aggregate conjugate power utility. Any revealed conjugate utility is the aggregated dual utility of a family of heterogenous risk averse agents, with the same linear pricing kernel given by (5.12).

Proof. The proof is based on the representation of the geometric Brownian motion as a time-dependent function of changed time Brownian motion. In addition of the square integrability of η^e, assume $\|\eta_t^e\|>0$ for every t. After a standard change of time driven by the inverse of the increasing process $A_t=\int_0^t\|\eta_s^e\|\,ds$, we can assume for simplicity that $\|\eta_t^e\|\equiv 1$ for any t, and drop out η_t^e in the PDE (5.14) and replacing r_t by $\tilde{r}_t=r_t/\|\eta_t^e\|^2$. The process $\{\tilde{Y}_t^e(y)\}$ is a time depending function of the Brownian motion $\{\tilde{W}_t=-W_t\}$, $\tilde{Y}_t^e(e^z)=\exp[z+\int_0^t-(\tilde{r}_s+1/2)ds+\tilde{W}_t]$. Then, the martingale $\{\widetilde{J}(t,\tilde{Y}_t^e(e^z))\}$ is a function of the Brownian motion $\{\tilde{W}_t(z)=z+\tilde{W}_t\}$, $\widetilde{J}(t,\tilde{Y}_t^e(e^z))=\Psi(t,\tilde{W}_t(z))$, where the function $\Psi(t,z)=\widetilde{J}(t,e^{-\int_0^t(\tilde{r}_s+1/2)ds}e^z)$. The martingale property of $\{\Psi(t,\tilde{W}_t(z))\}$ implies that $\Psi(t,z)$ is a space-time harmonic solution of the heat equation $\partial_t\Psi(t,z)+\frac{1}{2}\Psi_{zz}(t,z)=0$. By the Widder theorem, there exists a positive measure m such that $\Psi(t,z)=\widetilde{J}(t,e^{-\int_0^t(\tilde{r}_s+1/2)ds}e^z)=\int_{\mathbb{R}}e^{[(1-\beta)z-\frac{1}{2}(1-\beta)^2t]}m(d\beta)$, with $\beta>1$. The proof of Theorem 5.4 when $\tilde{\zeta}=1$ is achieved by taking the inverse change of variable $z=\ln(y)+\int_0^u(\tilde{r}_s+1/2)ds$, and $\mu(d\beta)=(\beta-1)m(d\beta)$; the general case is attained after taking the inverse of the change of time. Monotonicity and convexity implies that $\beta>1$. □

5.3.2 Characterization of the primal SPDE solutions

The primal marginal utility is obtained by inverting the marginal conjugate utility.
Initial power utility For initial conjugate power utility, the solution is simple since the power utility of a conjugate utility $\tilde{H}_t^{(\beta)}\frac{y^{1-\beta}}{\beta-1}$ is a power utility $J^{(\alpha)}(t,x)=H_t^{(\alpha)}\frac{x^{1-\alpha}}{1-\alpha}$, where $\alpha=1/\beta\in(0,1)$ and $H_t^{(\alpha)}=(\tilde{H}_t^{(\beta)})^\alpha$. The characteristic process $\mathcal{X}_t(x)$ is also linear in x and given by $\mathcal{X}_t(x)=x\tilde{H}_t^{(\beta)}(Y_t^e)^{-\beta}$.
In the general aggregate case, the problem is less explicit, but can be formulated as a *pure problem of convex analysis*: *what is the inverse of the mixture of marginal conjugate power functions?* Here, the randomness of the problem does not play any role.
The answer is given via an other optimization problem associated to the sup-convolution of concave functions with initial budget constraint; the main characteristic of the solution is that the derivative of the

conjugate is the sum of the derivative of the conjugate functions.
We denote by $\nu(d\alpha)$ the Borel measure defined, using the notation $\alpha = 1/\beta$, by $\int_0^1 k(\alpha)\nu(d\alpha) = \int_1^\infty k(1/\beta)\mu(d\beta)$. Let us recall the solution of the problem in the deterministic case with easier notations.

Proposition 5.5 (Optimal Pareto allocation problem)**.** *Let us consider a family of conjugate power utilities $(\tilde{j}^{(\beta)})_\beta$ and the conjugate utility $\tilde{j}^{(\mu)}$ which is a mixture of them. Then, the primal utility $j^{(\nu)}(x)$ read along the wealth $x^{(\nu)}(y) = -\tilde{j}^{(\mu)}_y(y)$ is still a mixture, and*

$$\begin{cases} \tilde{j}^{(\mu)}(y) = \int_1^\infty \tilde{j}^{(\beta)}(y)\mu(d\beta), \quad x^{(\nu)}(y) = \int_0^1 x^{(\alpha)}(y)\nu(d\alpha) & (5.17) \\ j^{(\nu)}(x^{(\nu)}(y)) = \int_0^1 j^{(\alpha)}(x^{(\alpha)}(y))\nu(d\alpha) = \int_1^\infty \beta\tilde{j}^{(\beta)}(y)\mu(d\beta) & (5.18) \end{cases}$$

Furthermore, the utility $j^{(\nu)}$ is the sup-convolution of power utilities $j^{(\alpha)}$

$$j^{(\nu)}(x) = \sup\{\int_0^1 j^{(\alpha)}(z^\alpha(x))\nu(d\alpha);\ \int_0^1 z^\alpha(x)\nu(d\alpha) = x\} \tag{5.19}$$

The "Pareto" supremum is achieved at the family $\{z^{(\alpha)}(x) := x^{(\alpha)}(j_z^{(\nu)}(x)),\ \alpha\}$.

This proposition can be applied to $\widetilde{J}^{(\mu)}(t,y)$ at any time t, but the characteristic process $\mathfrak{X}_t^{(\nu)}$ must satisfy some time-coherency. This constraint is easy to verify since the characteristic processes $\mathfrak{X}_t^{(\alpha)}(x)$ associated with power primal utility are also linear. Moreover, since the power utility are time separable, the Pareto-optimal initial allocation is propagated with the same rule than at time 0, using an aggregating measure evolving randomly over the time. Indeed, using the notations of the previous Proposition, the first order condition allows to write the equilibrium risky asset as a mixture of linear process.

$$\text{(Mixing)} \begin{cases} \mathfrak{X}_t^{(\nu)}(x^{(\nu)}(y)) = \int_1^\infty y^{-\beta}\tilde{H}(t,\beta)(Y_t^e)^{-\beta}\mu(d\beta) = \int_1^\infty x^{(\alpha)}(y)\mathfrak{X}_t^{(\alpha)}\nu(d\alpha). \\ J^{(\nu)}(t,\mathfrak{X}_t^{(\nu)}(x^{(\nu)}(y))) = \int_0^1 J^{(\alpha)}(t,x^{(\alpha)}(y)\mathfrak{X}_t^{(\alpha)})\nu(d\alpha), & (5.20) \end{cases}$$

Then we have,

Theorem 5.6. *A economic equilibrium holds if and only if there exists a positive Borel measure ν on $(0,1)$ such that,*
(i) *The utility process $J^{(\nu)}$ is given as the sup-convolution:*

$$J^{(\nu)}(t,x) = \sup\{\int_0^1 J^{(\alpha)}(t,z_\alpha)\nu(d\alpha);\ \int_0^1 z_\alpha\nu(d\alpha) = x\}$$

The supremum is achieved at the family $\{x^\alpha(t,x) := (J_z^{(\alpha)})^{-1}(t,J_z^{(\nu)}(t,x)),\ \alpha\}$ satisfying the condition $\int_0^1 x^\alpha(t,x)\nu(d\alpha) = x$.
(ii) Economic interpretation*: Assume the initial wealth optimally Pareto allocated, then at any time the allocation generated by the individual optimal wealth processes $\mathfrak{X}^{(\alpha)}(t,x^{(\alpha)}(x))$ is Pareto optimal for the aggregated utility $J^{(\nu)}(t,\mathfrak{X}_t^{(\nu)}(x))$ and the optimal wealth at time t.*

The proof of this Theorem is given in the Appendix.

6 General conclusion

6.1 Dynamic Utility in Equilibrium Theory

The last section has been motivated by different works [Sam38, Sam48, Wan93, DR97, Art99, GR13, CE16] on the neo-classical economic equilibrium theory, whose two pillars are the representative agent, and the individual rationality. At the equilibrium (see also Dumas & ali [Dum17]), the best strategy is the technology, whose dynamics is known, and the problem is to find the preferences (utility) of the principal agent, as its utility. But, a little attention is paid to the existence of equilibrium. The He and Leland [HL93] paper considers this question, but only from the viewpoint of the primal problem. They obtain a strongly non linear PDE on the coefficient of X, in the Markovian case, and can only give some examples of solutions. Unfortunately, the solution is quasi-trivial since in the Markovian case, the technology is a monotonic function of a geometric Brownian motion. The result is the same in random environment. The Markovian case gives an easy interpretation of the randomness. Assume the randomness generated by a factor Θ_t, solution of a SDEs system such that (X, Y, Θ) admits a unique solution. The assumption is that Θ can appear in the X coefficient only, but X cannot appear in Θ-coefficients. Then the previous results remain valid provided to look for dynamic utilities which are not functions of the parameter Θ, because the finite variation assumption. In this case, the dynamic utility at time t will depend on the entire past of $\Theta_s; s \leq t$ through the processes X and Y.

6.2 Conclusion

In this work we have provided a necessary and sufficient condition for the existence of a solution to the general problem of revealed utility, using very basic tools of analysis and the theory of integration. We have made almost no assumptions about the regularity of the processes in time t and the constructed utilities are only differentiable in x. To be aware of the efficiency of our method and the extent of our results, it is enough to refer to existing work in the semimartingale framework where calculations are tedious and assumptions are numerous. Requiring to treat the problem in an abstract way has clarified many subtleties including the role of the initial conditions and how to deal with the Stieltjes integral near to zero. The different notions of orthogonality introduced for the first time in this type of problem are the keys of this work. These difficulties are particularly well-illustrated in the Markovian economic equilibrium problem. By approaching the problem from the conjugate point of view to be concentrated on the pricing kernel process only (as in the first part) and exploiting the necessary and sufficient orthogonality condition of our main result yields to the complete resolution of this equilibrium problem, until now without a satisfactory answer.

Also, this condition of orthogonality undoubtedly plays an important role in Markov framework, because they give us the necessary and sufficient conditions in the form of PDEs. Solving them is still an open question that will be studied in a forthcoming paper. Finally, since we have no regularity assumptions with respect to the time, our results can be applied in the discrete frame, and in different settings, as preference learning in which the goal is to learn a predictive preference model from observed preference information, see [FH11, FSS06]. As well, reasoning with preferences has been recognized as a particularly promising research direction for artificial intelligence, see [NJ04, QXL14]. Other learning problems can also be studied from the viewpoint of an expected utility maximizing as learning a probabilistic models, see for example [FS16, FS03] for more details.

7 Appendix

Proof of Theorem 5.6. Using the notation $x^{(\nu)}(y) = -\tilde{j}_y^{(\mu)}(y) = x$ in equation (5.20), follows

$$J^{(\nu)}(t, X_t^{(\nu)}(x)) = \int_0^1 J^{(\alpha)}(t, x^{(\alpha)}(j_z^{(\nu)}(x))\mathfrak{X}_t^{(\alpha)})\nu(d\alpha).$$

As $\mathfrak{X}_t^{(\nu)}(x) = -\widetilde{J}_y^{(\mu)}(t, j_z^{(\nu)}(x)Y_t)$, it is a monotonic function in x with inverse $(j_z^{(\nu)})^{-1}\big(J_z^{(\nu)}(t,x)/Y_t\big)$, we obtain

$$J^{(\nu)}(t, x) = \int_0^1 J^{(\alpha)}\big(t, x^{(\alpha)}\big(J_z^{(\nu)}(t,x)/Y_t\big)\mathfrak{X}_t^{(\alpha)}\big)\nu(d\alpha).$$

Now, as $x^{(\alpha)}(y)\mathfrak{X}_t^{(\alpha)} = -\widetilde{U}_y^{\beta}(t, yY_t)$ $(\beta = 1/\alpha)$, the quantity $x^{(\alpha)}\big(J_z^{(\nu)}(t,x)/Y_t\big)\mathfrak{X}_t^{(\alpha)}$ is equal to $-\widetilde{J}_y^{\beta}(t, J_z^{(\nu)}(t,x))$. In other words,

$$J^{(\nu)}(t, x) = \int_0^1 J^{(\alpha)}\big(t, -\widetilde{J}_y^{\beta}(t, J_z^{(\nu)}(t,x))\big)\nu(d\alpha).$$

Moreover, identical reasoning as in the proof of the previous result, using the inequality

$$J^{(\alpha)}(t, -\widetilde{J}_y^{\beta}(t,y)) \geq J^{(\alpha)}(t, z_\alpha) + (\widetilde{J}_y^{\beta}(t,y) - z_\alpha)y, \ \forall z_\alpha > 0, \tag{7.1}$$

integrating and replacing y by $J_z^{(\nu)}(t,x)$, follows

$$J^{(\nu)}(t,x) \geq \sup\{\int_0^1 J^{(\alpha)}(t, z_\alpha)\nu(d\alpha) + J_z^{(\nu)}(t,x)\int_0^1 (\widetilde{J}_y^{\beta}(t, J_z^{(\nu)}(t,x)) - z_\alpha)\nu(d\alpha)\}, \tag{7.2}$$

with equality iff $\int_0^1 \widetilde{J}_y^{1/\alpha}(t, J_z^{(\nu)}(t,x))\nu(d\alpha) = \int_0^1 z_\alpha \nu(d\alpha)$, in this case the supremum is achieved at $z_\alpha = -\widetilde{J}_y^{\beta}(t, J_z^{(\nu)}(t,x)) = (J_z^{(\alpha)})^{-1}(t, J_z^{(\nu)}(t,x))$. But, using the definition of $\widetilde{J}^{(\mu)}$ as a mixture, one can observes that the integral $\int_0^1 \widetilde{J}_y^{1/\alpha}(t, J_z^{(\nu)}(t,x))\nu(d\alpha)$ is equal to x. This achieves the proof. $\square$

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