An Illustrated Guide of the Modern Approches of Hamilton-Jacobi Equations and Control Problems with Discontinuities

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Information: At the end of the book, in addition to the usual index, two appendices gather the main notations and assumptions which are used throughout this book. Also included is a list of the different notions of solutions as a quick guide.

Foreword (I)

This manuscript is a preliminary version of (hopefully) a future book whose aim is not only to describe the state-of-the-art for Hamilton-Jacobi Equations with discontinuities, but also to provide new results and applications, in particular for problems with boundary conditions.

We have decided to put this version online to have some reactions on this probably imperfect first attempt: we are conscious that some points in the presentation can be improved; some ideas can be, at the same time, generalized and simplified; some results may also be added (actually some parts are still missing) and of course, referencing can be improved. Moreover, some sections are just announced and not drafted yet. For all these reasons, please, do not hesitate to send us all the remarks you may have on what is written down or on what we should write; we will try to give credits to all valuable contributions in a suitable way.

What is the aim of this book and how is it written?

Our first aim was to revisit the recent progress made in the study of Hamilton-Jacobi Equations with discontinuities and related topics which had some influence on it, like problems set on networks (a subject which we had to consider a little bit even if we did not want to enter too deeply into it). Revisiting means that we are not merely copy-pasting with few modifications the existing articles. Instead we have tried to emphasize the main common ideas, either technical or more fundamental ones. This is why, while thinking about all the common points in several works, we have decided to dedicate an entire part to the "basic results", which are common bricks, used very often under perhaps slightly different forms, to prove the main results. This has the advantage to lighten the presentation of the main results and their proofs, but that creates a rather technical – and perhaps difficult to read– part, although it is not uninteresting to see some classical ideas revisited in (sometimes) an unusual way.

A second part consists in describing and comparing different notions of solutions for codimension-1 type discontinuities: we begin with the classical Ishii's notion of viscosity solutions but we consider also different approaches used for networks. We have tried to analyze all these different approaches in full detail, trying to give to the reader the most precise comparison of their advantages and disadvantages in terms of the generality of assumptions and results. Even if they are different, they share a lot of common points which partly justifies our first part on common tools. A very intriguing question is the convergence of the vanishing viscosity approximation, for which one has a complete answer in this codimension-1 framework but which remains open in more general situations, like on chessboard-type configuration for example.

The largest part of this book is dedicated to stratified problems where we can have discontinuities of any co-dimensions: this opens a very large range of applications, new ones being for problems with boundary conditions (a part which is not completely drafted here). Some a priori very singular problems can be addressed and even treated, the most fascinating ones being in the boundary conditions case. Here, to our point of view, the main message is the identification of what we believe as being the "right framework" for studying discontinuities in Hamilton-Jacobi Equations, namely the assumptions of "tangential continuity", "normal controllability" and the right notion of solution. The reader who is familiar with either exit time, state-constraint control problems or boundary conditions for Hamilton-Jacobi Equations will recognize some common features. With these assumptions, it is surprising to see how some applications can be treated without major additional difficulty compared to the continuous case.

We hope that the reader of this manuscript will enjoy reading it. Again, please, feel free to react in any possible way on this version: we do not promise to take into account all reactions (except the references, of course) but we promise to study all of them very carefully.

We thank you in advance!

G. Barles E. Chasseigne

Foreword (II)

In this second version, besides of improving few points in the first one, we have added applications to KPP (Kolmogorov-Petrovsky-Piskunov) type problems and started to develop the use of stratified solutions to treat problems with boundary conditions (Dirichlet, Neumann and mixed boundary conditions), where both the boundary may be non-smooth and the data may present discontinuities. Of course, for all these questions, we only provide examples of what can be done since developing a whole theory would be too long, out of the scope of this book and probably a little bit beyond what we are able to do up to now.

In these two directions, we address, in particular, a new question which is interesting when considering applications: under which conditions can one prove that Ishii's viscosity solutions are stratified solutions? The motivation is clear: classical viscosity solutions have nice stability properties and passages to the limit, even in rather complicated situations, are rather simple. On the other hand, we have a general comparison result for stratified solutions. Hence in the cases where these two concepts of solutions coincide, we benefit of all the advantages. Of course, as it is clear from the study of codimension 1 discontinuities where we have a very precise and complete picture, this strategy does not lead to the most general results. But still, it provides interesting results in a rather cheap way and in cases where our understanding of discontinuities is not so satisfactory (any discontinuity of codimension bigger than 1, in fact...).

In the case of problems with boundary conditions, we mainly examine the cases when either the discontinuities appear in the boundary conditions and/or the boundary is non-smooth (but respect the stratification conditions). On the contrary the equation inside the domain is assumed to be continuous. Of course, theoretically, there is no problem to combine discontinuities in the equation and in the boundary condition but this book is already very long...

A priori the third version (next release planned in june 2020) will probably be the last one: besides of improving the second version, we plan to address Large Deviations and homogenization problems. This will be the occasion to test the results we have presented so far and to improve them.

Again, please, do not hesitate to send us all the remarks you may have on what is written down or on what we should write; we will try to give credits to all valuable contributions in a suitable way.

G. Barles E. Chasseigne

Contents

Introduction

Chapter 1

General introduction

In 1983, the introduction of the notion of viscosity solutions by Crandall and Lions [47] solved the main questions concerning first-order Hamilton-Jacobi Equations (HJE in short), at least those set in the whole space $\mathbb{R}^N$, for both stationary and evolution equations: this framework provided the right notion of solutions for which uniqueness and stability hold, allowing to prove (for example) the convergence of the vanishing viscosity method. In this founding article the definition was very inspired by the works of Kružkov [94, 95, 97, 96] and, in fact, viscosity solutions appeared as the L^∞-analogue of the L^1-entropy solutions for scalar conservation laws. This initial, rather complicated Kružkov-type definition, was quickly replaced by the present definition, given in the article of Crandall, Evans and Lions [45], emphasizing the key role of the Maximum Principle and of the degenerate ellipticity, thus preparing the future extension to second-order equations.

The immediate success of the notion of viscosity solutions came from both its simplicity but also universality: only one definition for all equations, no matter whether the Hamiltonian was convex or not. A single theory was providing a very good framework to treat all the difficulties connected to the well-posedness (existence, uniqueness, stability...etc.) but it was also fitting perfectly with the applications to deterministic control problems, differential games, front propagations, image analysis etc.

Of course, a second key breakthrough was made with the first proofs of comparison results for second-order elliptic and parabolic, possibly degenerate, fully nonlinear partial differential equations (pde in short) by Jensen [91] and Ishii [89]. They allow the extension of the notion of viscosity solutions to its natural framework and open the way to more applications. The article of Ishii and Lions [87] was the first one in which the comparison result for second-order equations was presented in the definitive form;

we recommend this article which contains a lot of results and ideas, in particular for using the ellipticity in order to obtain more general comparison results or Lipschitz regularity of solutions.

We refer to the User's guide of Crandall, Ishii and Lions [46] for a rather complete introduction of the theory (See also Bardi and Capuzzo-Dolcetta[9] and Barles [23] for first-order equations, Fleming and Soner [60] for second-order equations together with applications to deterministic and stochastic control, Bardi, Crandall, Evans, Soner and Souganidis [8] ot the CIME course [1] for a more modern presentation of the theory with new applications). This extension definitively clarifies the connections between viscosity solutions and the Maximum Principle since, for second-order equations, the Maximum Principle is a standard tool and viscosity solutions (for degenerate equations) are those for which the Maximum Principle holds when testing with smooth test-functions.

Despite all these positive points, the notion of viscosity solutions had a little weakness: it only applies with the maximal efficiency when solutions are continuous and, this is even more important, when the Hamiltonians in the equations are continuous. This fact is a consequence of the keystone of the theory, namely the comparison result, which is mainly proved by the "doubling of variables" technic, relying more or less on continuity both of the solutions and the Hamiltonians.

Yet, a definition of discontinuous solutions has appeared very early (in 1985) in Ishii [88] and a first attempt to use it in applications to control problems was proposed in Barles and Perthame [16]. The main contribution of [16] is the "half-relaxed limit method", a stability result for which only a L^∞-bound on the solutions is needed. But this method, based on the Ishii's notion of discontinuous viscosity solutions for discontinuous Hamiltonians, uses discontinuous solutions more as an intermediate tool than as an interesting object by itself.

However, in the late 80's, two other types of works considered discontinuous solutions and Hamiltonians, breaking the universality feature of viscosity solutions. The first one was the study of measurable dependence in time in time-dependent equation (cf. Barron and Jensen [30], Lions and Perthame [99], see also the case of second-order equations in Nunziante [104, 105], Bourgoing [34, 35] with Neumann boundary conditions, and Camilli and Siconolfi [40]): in these works, the pointwise definition of viscosity solutions has to be modified to take into account the measurable dependence in time. It is worth pointing out that there was still no difference between convex and non- convex Hamiltonians.

On the contrary, Barron and Jensen [29] in 1990 considered semi-continuous solutions of control problems (See also [22] for a slightly simpler presentation of the ideas of [29] and Frankowska [64], Frankowska and Plaskacz [66], Frankowska and Mazzola

[65] for different approaches): they introduced a particular notion of viscosity solution which differs according to whether the control problem consists in minimizing some cost or maximizing some profit; thus treating differently convex and concave Hamiltonians. This new definition had the important advantage to provide a uniqueness result for lower semi- continuous solutions in the case of convex Hamiltonians, a very natural result when thinking in terms of optimal control.

In the period 1990-2010, several attempts were made to go further in the understanding of Hamilton-Jacobi Equations with discontinuities. A pioneering work is the one of Dupuis [50] whose aim was to construct and study a numerical method for a calculus of variation problem with discontinuous integrand, motivated by a Large Deviations problem. Then, problems with a discontinuous running cost were addressed by Garavello and Soravia [69, 68] and Soravia [117] who highlight some non-uniqueness feature for the Bellman Equations in optimal control, but identify the maximal and minimal solutions. To the best of our knowledge, all the uniqueness results use either a special structure of the discontinuities or different notions solutions, which are introduced to try to tackle the main difficulties as in [48, 49, 71, 72, 77] or an hyperbolic approach as in [5, 44]. For the boundary conditions, Blanc [31, 32] extended the [16] and [29] approaches to treat problems with discontinuities in the boundary data for Dirichlet problems. Finally, even the case of measurability in the state variable was considered for Eikonal type equations by Camilli and Siconolfi [39].

Before going further, we point out that we do not mention here the L^p-viscosity solutions nor viscosity solutions for stochastic pdes, two very interesting subjects but too far from the scope of this book.

In this period, the most general contribution for first-order Hamilton-Jacobi-Bellman Equations was the work of Bressan and Hong [36] who considered the case of control problems in *stratified domains*. In their framework, the Hamiltonians can have discontinuities on submanifolds of $\mathbb{R}^N$ of any codimensions and the viscosity solutions inequalities are disymmetric between sub and supersolutions (we come back on this important point later on). In this rather general setting, they are able to provide comparison results by combining pde and control methods. Of course, we are very far from the context of an universal definition but it seems difficult to have more general discontinuities. Before going further, we refer the reader to Whitney [122, 121] for the notion of *Whitney stratified space*.

In the years 2010's, a lot of efforts have been spent to understand Hamilton-Jacobi Equations on networks and, maybe surprisingly, this had a key impact on the study of discontinuities in these equations. An easy way to understand why is to look at an HJ-equation set on the real line $\mathbb{R}$, with only one discontinuity at $x = 0$. Following this introduction, it seems natural to jump on to Ishii's definition and to address the problem as an equation set on $\mathbb{R}$. But another point of view consists in seeing $\mathbb{R}$ as a network with two branches $\mathbb{R}^-$ and $\mathbb{R}^+$. This way, $x = 0$ becomes the intersection of the two branches and it is conceivable that the test-functions could be quite different in each branch, leading to a different notion of solution. Moreover, a "junction condition" is needed at 0 which might come from the two Hamiltonians involved (one for each branch) but also a specific inequality at 0 coming from the model and the transmission condition we have in mind. Therefore, at first glance, these "classical approach" and "network approach" seem rather different.

Surprisingly (with today's point of view), these two approaches were investigated by different people and (almost) completely independently until Briani, Imbert and the authors of this book made the simple remark which is described in the last above paragraph. But, in some sense, this "mutual ignorance" was a good point since different complementary questions were investigated and we are going to described these questions now.

For the "classical approach", in the case of the simplest codimension 1 discontinuity in $\mathbb{R}$ or $\mathbb{R}^N$ and for deterministic control problems, *i.e.* with convex Hamiltonians, these questions were

(*i*) Is Ishii's definition of viscosity solutions providing a unique solution which is the value-function of an associated control problem?

(*ii*) If not, can we identify the minimal and maximal solutions in terms of value functions of *ad hoc* control problems?

(*iii*) In non-uniqueness cases, is it possible to recover uniqueness by imposing some additional condition on the discontinuity?

(*iv*) Can the limit of the vanishing viscosity method be identified? Is it the maximal or minimal solution? Or can it change depending on the problem?

These questions were investigated by Rao [108, 109], Rao and Zidani [110], Rao, Siconolfi and Zidani [107] by optimal control method, and Barles, Briani and Chasseigne [10, 11] by more pde methods. In [10, 11], there are some complete answers to questions (*i*) and (*ii*), almost complete for (*iii*) and really incomplete for (*iv*).

For the "network approach", in the case of two (or several) $1-$dimensional (or multi-dimensional) branches, the questions were different and the convexity of the Hamiltonians appears as being less crucial

(*v*) What is the correct definition of solution at the junction? What are the different possible junction conditions and their meanings in the applications?

(*vi*) Does a comparison result for such network problems hold?

(*vii*) Does the Kirchhoff condition (involving derivatives of the solution in all branches) differ from tangential conditions (which just involve tangential derivatives)?

(*viii*) What are the suitable assumptions on the Hamiltonians to get comparison?

(*ix*) Can we identify the limit of the vanishing viscosity method?

Questions (*v*)-(*vi*) were investigated under different assumptions in Schieborn [112], Camilli and Marchi [37], Achdou, Camilli, Cutrì and Tchou [2], Schieborn and Camilli [113], Imbert, Monneau and Zidani [85], Imbert and Monneau [83] for 1-dimensional branches and Achdou, Oudet and Tchou [3, 4], Imbert and Monneau [84] for all dimensions; while Graber, Hermosilla and Zidani [74] consider the case of discontinuous solutions. The most general comparison result (with some restrictions anyway) is the one of Lions and Souganidis [102, 103] which is valid with very few, natural assumptions on the Hamiltonians, and not only in the case of Kirchhoff conditions but also for general junction conditions. It allows to answer in full generality to question (*ix*) which is also investigated in Camilli, Marchi and Schieborn [38].

In fact, taking into account the very general ideas of the comparison result of Lions and Souganidis, Question (*viii*) seems to disappear but Question (*vii*) becomes crucial since the junction condition plays a key role in the uniqueness of the associated viscosity solution. Unfortunately, the universality of the Lions-Souganidis proof is in some sense lost here since the junction condition and its form will depend on the convexity or concavity of the Hamiltonians. Imbert and Monneau [83, 84] have studied completely the junction conditions (through the "flux limiter" approach) and proved the connection between general Kirchhoff conditions and flux-limiters, allowing the identification of the vanishing viscosity limit in the quasi-convex case.

In this book, our aim is to consider various problems with different type of discontinuities and to describe the different approaches to treat them. Thinking about all the common points that can be found in the works mentioned above, we have decided to dedicate an entire part to the "basic results", which are common bricks, used very often under perhaps slightly different forms. This has the advantage to lighten the presentation of the main results and their proofs, but that creates a rather technical – and perhaps difficult to read– part. But we also think it is interesting to see some classical ideas revisited in sometimes unusual ways.

Then, the first problems we address are "simple" co-dimension 1 discontinuities (a discontinuity along an hyperplane or an hypersurface, for example) in the whole space $\mathbb{R}^N$. For these problems, we provide in Part II a full description of the "classical approach" and the connections with the "network approach" with different comparison proofs (the Lions-Souganidis one and the Barles, Briani, Chasseigne and Imbert one). We also analyze their advantages and disadvantages.

In this second part, we make a point to emphasize the following important issues which will play a key role in all the other parts and seem to be the key assumptions to be used in problems with discontinuities in order to have a continuous solution and a comparison result between sub and supersolutions

(NC) **Normal controlability (or coercivity):** for control problems, this property means that one should be able to reach the interface (here the codimension 1 manifold where we have the discontinuity) because a more favorable situation (in terms of cost) may exist there. Such assumption ensures that this potentially favorable situation is "seen". This is translated into a coercivity-type assumptions in the normal coordinates on the associated Hamiltonian.

(TC) **Tangential continuity :** with respect to the coordinates of the interface, the Hamiltonians have to satisfy standard comparison $\mathbb{R}^N$-type assumptions.

We insist on the fact that these assumptions will be used for ANY type of results: comparison but also stability and connections with control. These are really key assumptions and we will find them everywhere throughout all the books, expressed in different ways.

The third part is devoted to the case of "stratified problems" in the whole space $\mathbb{R}^N$, *i.e.* to the case where discontinuities of any codimension can appear. In Part III, we describe the extension of Bressan and Hong [36] obtained in [25] with some extensions and applications: we present the main ideas, using in key way (NC)-(TC), and these ideas are also used in Part IV where we consider the "stratified problems" set in a domain with state-constraint boundary conditions. It is worth pointing out that this stratified formulation allows to treat various boundary conditions (Dirichlet, Neumann, sliding boundary conditions,...) in the same framework, without assuming the boundary of the domain to be smooth, and taking also into account some unbounded control features. For this reason, we think that if the formulation may seem a little bit weird or difficult, the range of applications it allows to treat fully justifies its introduction.

Different approaches for control problems in stratified frameworks, more in the spirit of Bressan & Hong have been developed by Hermosilla, Wolenski and Zidani [81] for Mayer and Minimum Time problems, Hermosilla and Zidani [82] for classical state-constraint problems, Hermosilla, Vinter and Zidani [80] for (very general) state-constraint problems (including a network part).

We conclude this introduction by a remark on "how to read this book?" vs "how not to read this book?".

As we already mentioned it above, we have decided to start by an entire part (Part I) gathering basics results which are identified as the key bricks appearing in any type of problems involving Hamilton-Jacobi Bellman Equations and deterministic control problems. This part is unavoidably a "little bit technical" and admittedly hard to read without a serious motivation... Which we hope can be found in the next parts!

We have tried to draft all the proofs by emphasizing the role of the related key bricks but in order to be readable without knowing the details of these bricks: in that way, one can avoid reading the different independent sections of Part I before being completely convinced that it is necessary.

Part II is certainly the most unavoidable one since it describes all the challenges and potential solutions at hand in a rather simple context of a co-dimension 1 discontinuity. Yet the difficulty of this part is to extract a clear global vision and we try to provide our point of view in Section 11.3.

Stratified problems require a non-neglectable investment but we have tried to point out the main ideas to keep in mind and to start from the easiest case and then go to the most sophisticated ones. We hope that the general treatment of singular boundary conditions in non-smooth domains will be a sufficient motivation for suffering all the difficulties! But also the applications of Chapter 16.

Chapter 2

The basic (continuous) framework and the classical assumptions revisited

In order to go further in the presentation of both the results contained in this book and the assumptions we use, let us describe first the most classical continuous framework. Then, we make comments on the general approach we introduce afterwards. As we will only sketch the approach and results in this chapter since they are classical, we refer the reader to well-known references on this subject for more details: Lions [100], Bardi and Capuzzo-Dolcetta [9], Fleming and Soner [60], the CIME courses [8, 1] and Barles [23].

We consider a finite horizon control problem in $\mathbb{R}^N$ on the time interval $[0,T]$ for some $T>0$, where, for $x\in\mathbb{R}^N$ and $t\in[0,T]$, the dynamic is given by

$$\dot{X}(s)=b(X(s),t-s,\alpha(s))\ ,\ X(0)=x\in\mathbb{R}^N\ .$$

Here, $\alpha(\cdot)\in\mathcal{A}:=L^\infty(0,T;A)$ is the control which takes values in the compact metric space A and b is a continuous function of all its variables. More precise assumptions are introduced later on.

For a finite horizon problem, the value function is classically defined by

$$U(x,t)=\inf_{\alpha(\cdot)\in\mathcal{A}}\left\{\int_0^t l(X(s),t-s,\alpha(s))\exp\left(\int_0^s c(X(\tau),t-\tau,\alpha(\tau))d\tau\right)\mathrm{d}s\right.$$
$$\left.+\,u_0(X(t))\exp\left(\int_0^t c(X(\tau),t-\tau,\alpha(\tau))d\tau\right)\right\}\ ,$$

where l is the running cost, c the discount factor and u_0 is the final cost. All these functions are assumed to be continuous on $\mathbb{R}^N \times [0,T] \times A$ (for l and c) and on $\mathbb{R}^N$ (for u_0) respectively.

The most classical framework use the following assumptions which will be refered below as $(\mathbf{H}^{\text{class.}}_{\mathbf{BA-CP}})$ for *Basic Assumptions on the Control Problem – Classical case*:

(i) The function $u_0 : \mathbb{R}^N \to \mathbb{R}$ is a bounded, uniformly continuous function.

(ii) The functions b, c, l are bounded, uniformly continuous on $\mathbb{R}^N \times [0,T] \times A$.

(iii) There exists a constant $C_1 > 0$ such that, for any $x, y \in \mathbb{R}^N$, $t \in [0,T]$, $\alpha \in A$, we have
$$|b(x,t,\alpha) - b(y,t,\alpha)| \leq C_1|x-y| \; .$$

One of the most classical results connecting the value function with the associated Hamilton-Jacobi-Bellman Equation is the

Theorem 2.0.1 *If Assumption* $(\mathbf{H}^{\text{class.}}_{\mathbf{BA-CP}})$ *holds, the value function* U *is continuous on* $\mathbb{R}^N \times [0,T]$ *and is the unique viscosity solution of*

$$u_t + H(x,t,u,D_x u) = 0 \quad \text{in } \mathbb{R}^N \times (0,T) \; , \tag{2.1}$$

$$u(x,0) = u_0(x) \quad \text{in } \mathbb{R}^N \; . \tag{2.2}$$

where
$$H(x,t,r,p) := \sup_{\alpha \in A} \left\{ -b(x,t,\alpha)\cdot p + c(x,t,\alpha)r - l(x,t,\alpha) \right\} \; .$$

In Theorem 2.0.1, we have used the notation u_t for the time derivative of the function $(x,t) \mapsto u(x,t)$ and $D_x u$ for its derivatives with respect to the space variable x. These notations will be used throughtout this book.

Sketch of Proof — Of course, there exists a lot of variants of this result with different assumptions on b, c, l and u_0 but, with technical variants, the proofs use mainly the same steps:

(a) The first one consists in proving that U satisfies a Dynamic Programming Principle (DPP in short), *i.e.* that it satisfies for $0 < h < t$,

$$U(x,t) = \inf_{\alpha(\cdot)\in\mathcal{A}} \Bigg\{ \int_0^h l(X(s), t-s, \alpha(s)) \exp\left(\int_0^s c(X(\tau), t-\tau, \alpha(\tau)) d\tau \right) \mathrm{d}s \\ + U(X(h), t-h) \exp\left(\int_0^h c(X(\tau), t-\tau, \alpha(\tau)) d\tau \right) \Bigg\} \; .$$

This is done by using the very definition of U and taking suitable controls.

(b) If U is smooth, using the DPP on $[0, h]$, after dividing by h and sending $h \to 0$ we deduce that U is a classical solution of (2.1)-(2.2). If U is not smooth, this has to be done with test-functions and we obtain that U is a viscosity solution of the problem.

(c) Finally one proves a comparison result for (2.1)-(2.2), which shows that U is the unique viscosity solution of (2.1)-(2.2).

Q.E.D.

We point out that, in this sketch of proof, the continuity (or uniform continuity) of U can be either obtained directly, by working on the definition of U and maybe using the DPP, or as a consequence of the comparison result. We insist on the fact that in this classical framework, we are mainly interested in cases where U is continuous and therefore in assumptions ensuring this continuity.

Concerning Assumption $(\mathbf{H}^{\text{class.}}_{\mathbf{BA-CP}})$, it is clear that (iii) together with (ii) ensure that we have a well-defined trajectory, for any control $\alpha(\cdot)$, by the Cauchy-Lipschitz Theorem. Moreover, this trajectory $X(\cdot)$ exists for all time by the boundedness of b. On the other hand, the boundedness of l, c allows to show that $U(x,t)$ is well-defined, bounded in $\mathbb{R}^N \times [0,T]$ and even here uniformly continuous. Therefore it gives all the necessary information at the control level.

But Assumption $(\mathbf{H}^{\text{class.}}_{\mathbf{BA-CP}})$ plays also a key role at the pde level, in view of the comparison result: indeed, it implies that the Hamiltonian H satisfies the following property: for any $R \geq 1$

There exists $M > 0$, C_1 and a modulus of continuity $m : [0,+\infty) \to [0,+\infty)$ such that, for any $x, y \in \mathbb{R}^N$, $t, s \in [0,T]$, $-R \leq r_1 \leq r_2 \leq R \in \mathbb{R}$ and $p, q \in \mathbb{R}^N$

$$|H(x,t,r_1,p) - H(y,s,r_1,p)| \leq (C_1|x-y| + m(|t-s|))\,|p| + m\left((|x-y| + |t-s|)R\right),$$

$$H(x,t,r_2,p) - H(x,t,r_1,p) \geq -M(r_2 - r_1)\ ,$$

$$|H(x,t,r_1,p) - H(x,t,r_1,q)| \leq M|p-q|\ .$$

Of course, these properties are satisfied with $M = \max(||b||_\infty, ||c||_\infty, ||l||_\infty)$ and m is the modulus of uniform continuity of b, c, l.

Remarks on the comparison proof – we want to insist on two points here, that are important throughout this book. First point: if one wants to compare a subsolution u and a supersolution v (See Section 3.1), the initial step is to reduce to the case when $r \mapsto H(x,t,r,p)$ is increasing (or even non-decreasing) for any x, t, p. This can be done through the classical change of unknown function

$u(x,t) \to \tilde{u}(x,t) = u(x,t)\exp(-Kt)$, $v(x,t) \to \tilde{v}(x,t) = v(x,t)\exp(-Kt)$ for some $K \geq M$; the Hamiltonian H is changed in

$$\tilde{H}(x,t,r,p) := \sup_{\alpha \in A} \{-b(x,t,\alpha)\exp(-Kt)\cdot p + [c(x,t,\alpha)+K]r - l(x,t,\alpha)\exp(-Kt)\} \ ,$$

thus allowing to assume that we can reduce to the case when either $c(x,t,\alpha) \geq 0$ for any x,t,α or even ≥ 1. *We will always assume in this book that, one way or the other, we can reduce to the case when* $c \geq 0$.

The second point we want to emphasize is the t-dependence of b. It is well-know that, in the comparison proof, the term "$(C_1|x-y| + m(|t-s|))\,|p|$" is playing a key role and to handle the difference in the behavior of b in x and t, one has to perform a proof with a "doubling of variable" technique which is different in x and t, namely to consider the function

$$(x,t,y,s) \mapsto \tilde{u}(x,t) - \tilde{v}(y,s) - \frac{|x-y|^2}{\varepsilon^2} - \frac{|t-s|^2}{\beta^2} - \eta(|x|^2+|y|^2) \ ,$$

where $0 < \beta \ll \varepsilon \ll 1$ and $0 < \eta \ll 1$. We recall that the η-term ensures that this function achieves its maximum and the ε, β-terms ensure (x,t) is close to (y,s) and therefore the maximum of this function looks like $\sup_{\mathbb{R}^N}(\tilde{u}-\tilde{v})$.

The idea of this different doubling in x and t is that we need a term like

$$(C_1|x-y| + m(|t-s|))\,|p|$$

to be small. Since $|p|$ behaves like $o(1)\varepsilon^{-1}$, $|x-y|$ like $o(1)\varepsilon$ and $|t-s|$ like $o(1)\beta$, the product $C_1|x-y||p|$ is indeed small, but in order to ensure that the product $m(|t-s|)|p|$ is also small, we need to choose β small enough compared to ε.

In the following, since we want to handle cases when b,c,l can be discontinuous on submanifolds in $\mathbb{R}^N \times [0,T]$ which may depend on time, there will be no reason to have different assumptions in x and t. On the other hand, we will use in a more central way the Lipschitz continuity of H in p to have a more local comparison proof.

Basic Assumptions – The previous remarks lead us to replace $(\mathbf{H}^{\text{class.}}_{\mathbf{BA-CP}})$ by the following

$(\mathbf{H}_{\mathbf{BA-CP}})$ *Basic Assumptions on the Control Problem*:

(*i*) The function $u_0 : \mathbb{R}^N \to \mathbb{R}$ is a bounded, continuous function.

(ii) The functions b, c, l are bounded, continuous functions on $\mathbb{R}^N \times [0,T] \times A$ and the sets $(b,c,l)(x,t,A)$ are convex compact subsets of $\mathbb{R}^{N+2}$ for any $x \in \mathbb{R}^N$, $t \in [0,T]$ [(1)].

(iii) For any ball $B \subset \mathbb{R}^N$, there exists a constant $C_1(B) > 0$ such that, for any $x, y \in \mathbb{R}^N$, $t \in [0,T]$, $\alpha \in A$, we have

$$|b(x,t,\alpha) - b(y,s,\alpha)| \leq C_1(B)\left(|x-y| + |t-s|\right) .$$

We will explain in Section 17.1 how to handle a more general dependence in time when the framework allows it. In terms of equation, and although the following assumption is not completely equivalent to $(\mathbf{H_{BA-CP}})$, we will use the

$(\mathbf{H_{BA-HJ}})$ *Basic Assumptions on the Hamilton-Jacobi equation*:

There exists a constant $C_2 > 0$ and, for any ball $B \subset \mathbb{R}^N \times [0,T]$, for any $R > 0$, there exists constants $C_1(B,R) > 0, \gamma(R) \in \mathbb{R}$ and a modulus of continuity $m(B,R) : [0,+\infty) \to [0,+\infty)$ such that, for any $x, y \in B$, $t, s \in [0,T]$, $-R \leq r_1 \leq r_2 \leq R$ and $p, q \in \mathbb{R}^N$

$$|H(x,t,r_1,p) - H(y,s,r_1,p)| \leq C_1(B,R)[|x-y|+|t-s|]|p| + m(B,R)(|x-y|+|t-s|) ,$$

$$|H(x,t,r_1,p) - H(x,t,r_1,q)| \leq C_2|p-q| ,$$

$$H(x,t,r_2,p) - H(x,t,r_1,p) \geq \gamma(R)(r_2 - r_1) .$$

In the next part "Tools", we introduce the key ingredients which allow to pass from the above standard framework to the discontinuous one; they are concerned with

a. **Hamilton-Jacobi Equations:** we recall the notion of viscosity solutions and we revisit the comparison proof in order to have an easier generalization to the discontinuous case. We immediately point out that the regularization of sub and supersolutions by sup or inf-convolutions will play a more important role in the discontinuous setting than in the continuous one.

b. **Control problems:** the discontinuous framework leads to introduce **Differential inclusions** in order to define properly the dynamic, discount and cost when b, c, l are discontinuous. We provide classical and less classical results on the DPP in this setting.

[(1)] The last part of this assumption which is not a loss of generality will be used for the connections with the approach by differential inclusions.

c. **Stratifications:** we describe the notion of Whitney's stratification which is the notion used in Bressan and Hong [36] for the structure of the discontinuities of H or the (b, c, l).

Using these tools requires to make some basic assumptions for each of them, which are introduced progressively in this next part. Apart from $(\mathbf{H_{BA-HJ}})$ and $(\mathbf{H_{BA-CP}})$ that we introduced above, we will use $(\mathbf{H_{BCL}})$ and $(\mathbf{H_{ST}})$ respectively for the Differential Inclusion and the Stratification. We have also compiled the various assumptions in this book in an appendix for the reader's convenience.

Part I

Preliminaries: A Toolbox for Discontinuous Hamilton-Jacobi Equations and Control Problems

Chapter 3

PDE tools

3.1 Discontinuous Viscosity Solutions for Equations with Discontinuities, “Half-Relaxed Limits” Method

In this section, we recall the classical definition of discontinuous viscosity solutions introduced by Ishii[88] for equations which present discontinuities. We have chosen to present it in the first-order framework since, in this book, we are mainly interested in Hamilton-Jacobi Equations but it extends without major changes to the case of fully nonlinear elliptic and parabolic pdes. We refer to the Users’ guide of Crandall, Ishii and Lions [46], the books of Bardi and Capuzzo-Dolcetta [9] and Fleming and Soner [60] and the CIME courses [8, 1] for more detailed presentations of the notion of viscosity solutions in this more general setting.

We (unavoidably) complement this definition by the description of the discontinuous stability result (often called “Half-Relaxed Limits Method”) which is certainly its main justification and is clearly needed when dealing with discontinuities. We recall that the “Half-Relaxed Limits Method” allows passage to the limit in fully nonlinear elliptic and parabolic pdes with just an L^∞–bound on the solutions. The “Half-Relaxed Limits Method” was introduced by Perthame and the first author in [16] and developed in a series of works [17, 18]. One of its first striking consequence was the “Perron’s method” of Ishii [116] for proving the existence of viscosity solutions for a very large class of first- and second-order equations (see also the above references for a complete presentation).

The definition of viscosity solutions uses the upper semicontinuous (u.s.c.) envelope and lower semicontinuous (l.s.c.) envelope of both the (sub and super) solutions and of the Hamiltonians and we introduce the following notations: if z is a locally bounded function (possibly discontinuous), we denote by z^* its u.s.c. envelope

$$z^*(y) = \limsup_{\tilde{y} \to y} \; z(\tilde{y}) \, ,$$

and by z_* its l.s.c. envelope

$$z_*(y) = \liminf_{\tilde{y} \to y} \; z(\tilde{y}) \, .$$

Throughout this section, we use $y \in \mathbb{R}^N$ as the generic variable to cover both the stationary and evolution cases where respectively, $y = x \in \mathbb{R}^n$ or $y = (x,t) \in \mathbb{R}^n \times \mathbb{R}$.

3.1.1 Discontinuous Viscosity Solutions

We consider a generic Hamiltonian $G : \overline{\mathcal{O}} \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}$ where $\mathcal{O}$ is an open subset of $\mathbb{R}^N$ and $\overline{\mathcal{O}}$ denotes its closure. We just assume that G is a locally bounded function which is defined pointwise.

The definition of viscosity sub and supersolution is the following

Definition 3.1.1 (Discontinuous Viscosity Solutions) *A locally bounded function u is a viscosity subsolution of the equation*

$$G(y, u, Du) = 0 \qquad on \ \overline{\mathcal{O}} \tag{3.1}$$

if, for any $\varphi \in C^1(\overline{\mathcal{O}})$, at a maximum point $y_0 \in \overline{\mathcal{O}}$ of $u^* - \varphi$, one has

$$G_*(y_0, u^*(y_0), D\varphi(y_0)) \leq 0 \, .$$

A locally bounded function v is a viscosity supersolution of the Equation (3.1) if, for any $\varphi \in C^1(\overline{\mathcal{O}})$, at a minimum point $y_0 \in \overline{\mathcal{O}}$ of $v_* - \varphi$, one has

$$G^*(y_0, v_*(y_0), D\varphi(y_0)) \geq 0 \, .$$

A (discontinuous) solution is a function whose u.s.c. and l.s.c. envelopes are respectively viscosity sub and supersolution of the equation.

Several classical remarks on this definition: first, if the space of "test-functions" φ which is here $C^1(\overline{\mathcal{O}})$ is changed into $C^2(\overline{\mathcal{O}})$, $C^k(\overline{\mathcal{O}})$ for any $k > 1$ or $C^\infty(\overline{\mathcal{O}})$, we obtain an equivalent definition. Then, for a classical stationary equation (say in $\mathbb{R}^n$) like

$$H(x, u, Du) = 0 \quad \text{in } \mathbb{R}^n \, ,$$

the variable y is just x, $N = n$ and Du stand for the usual gradient of u in $\mathbb{R}^n$. But this framework also contains the case of evolution equations

$$u_t + H(x, u, D_x u) = 0 \quad \text{in } \mathbb{R}^n \times (0, T)\,,$$

where $y = (x, t) \in \mathbb{R}^n \times (0, T)$, $N = n + 1$ and $Du = (u_t, D_x u)$ where u_t denotes the time-derivative of u and $D_x u$ is the derivative with respect to the space variables x, and the Hamiltonian reads

$$G(y, u, P) = p_t + H(x, u, p_x)\ ,$$

where $P = (p_x, p_t)$.

In general, the notion of subsolution is given for u.s.c. functions while the notion of super-solution is given for l.s.c. functions: this may appear natural when looking at the above definition where just u^* and v_* play a role. But, for example in control problems, we face functions which are a priori neither u.s.c. nor l.s.c. and still we wish to prove that they are sub and supersolution of some equations. Therefore such a formulation is needed.

Last but not least, this definition is a little bit strange since the equation is set on a closed subset, a very unusual situation. There are two reasons for introducing it this way: the first one is to unify equation and boundary condition in the same formulation as we will see below. With such a general formulation, we avoid to have a different results for each type of boundary conditions. The second one, which provides also a justification of the "boundary conditions in the viscosity sense" is the convergence result we present in the next section.

To be more specific, let us consider the problem

$$\begin{cases} F(y, u, Du) = 0 & \text{in } \mathcal{O}, \\ L(y, u, Du) = 0 & \text{on } \partial\mathcal{O}, \end{cases}$$

where F, L are given continuous functions. If we introduce the function G defined by

$$G(y, u, p) = \begin{cases} F(y, u, p) & \text{if } y \in \mathcal{O}, \\ L(y, u, p) & \text{if } y \in \partial\mathcal{O}. \end{cases}$$

we can just rewrite the above problem as

$$G(y, u, Du) = 0 \quad \text{on } \overline{\mathcal{O}}\ ,$$

where the first important remark is that G is a priori a discontinuous Hamiltonian. Hence, even if we assume F and L to be continuous, we face a typical example which we want to treat in this book!

The interpretation of this new problem can be done by setting the equation in $\overline{\mathcal{O}}$ instead of $\mathcal{O}$. Applying blindly the definition, we see that u is a subsolution if

$$G_*(y,u^*,Du^*) \leq 0 \quad \text{on } \overline{\mathcal{O}}\,,$$

i.e. if

$$\begin{cases} F(y,u^*,Du^*) \leq 0 & \text{in } \mathcal{O}, \\ \min(F(y,^*,Du),L(y,u^*,Du^*)) \leq 0 & \text{on } \partial\mathcal{O}, \end{cases}$$

while v is a supersolution if

$$G^*(y,v_*,Dv_*) \geq 0 \quad \text{on } \overline{\mathcal{O}}\,,$$

i.e. if

$$\begin{cases} F(y,v_*,Dv_*) \geq 0 & \text{in } \mathcal{O}, \\ \max(F(y,v_*,Dv_*),L(y,v_*,Dv_*)) \geq 0 & \text{on } \partial\mathcal{O}, \end{cases} .$$

Indeed, we have just to compute G_* and G^* on $\overline{\mathcal{O}}$ and this is where the "min" and the "max" come from on $\partial\mathcal{O}$.

Of course, these properties have to be justified and this can be done by the discontinuous stability result of the next section which can be applied (for example) to the most classical way to solve the above problem, namely the vanishing viscosity method

$$\begin{cases} -\varepsilon\Delta u_\varepsilon + F(y,u_\varepsilon,Du_\varepsilon) & = 0 \text{ in } \mathcal{O}\,, \\ L(y,u_\varepsilon,Du_\varepsilon) & = 0 \text{ on } \partial\mathcal{O}\,. \end{cases}$$

Indeed, by adding a $-\varepsilon\Delta$ term, we regularize the equation in the sense that one can expect to have more regular solutions for this approximate problem – typically in $C^2(\mathcal{O}) \cap C^1(\overline{\mathcal{O}})$.

To complete this section, we turn to a key example: the case of a two half-spaces problem, which prensents a discontinuity along an hyperplane. We use the following framework: in $\mathbb{R}^N$, we set $\Omega_1 = \{x_N > 0\}$, $\Omega_2 = \{x_N < 0\}$ and $\mathcal{H} = \{x_N = 0\}$. We assume that we are given three continuous Hamiltonians, H_1 on $\overline{\Omega}_1$, H_2 on $\overline{\Omega}_2$ and H_0 on $\mathcal{H}$. Let us introduce

$$G(y,u,p) := \begin{cases} p_t + H_1(x,u,p_x) & \text{if } x \in \Omega_1\,, \\ p_t + H_2(x,u,p_x) & \text{if } x \in \Omega_2\,, \\ p_t + H_0(x,u,p_x) & \text{if } x \in \mathcal{H}\,. \end{cases}$$

Then solving $G(y,u,Du) = 0$ for $y = (x,t) \in \mathbb{R}^{N+1}$ means to solve the equations $u_t + H_i(x,u,Du) = 0$ in each Ω_i $(i = 1,2)$ with the "natural" conditions on $\mathcal{H}$ given by the Ishii's conditions for the sub and super-solutions, namely

$$\begin{cases} \min(u_t + H_1(x,u^*,Du^*), u_t + H_2(x,u^*,Du^*), u_t + H_0(x,u^*,Du^*)) & \leq 0 \quad \text{on } \mathcal{H}\,, \\ \max(u_t + H_1(x,v_*,Dv_*), u_t + H_2(x,v_*,Dv_*), u_t + H_0(x,v_*,Dv_*)) & \geq 0 \quad \text{on } \mathcal{H}\,. \end{cases}$$

Remark 3.1.2 *We have decided to present the definition of viscosity solution on a closed space $\overline{\mathcal{O}}$ for the reasons we explained above. But we can define as well equations set in open subset of $\mathbb{R}^N$ (typically $\mathcal{O}$) or open subsets of $\overline{\mathcal{O}}$ (typically $\overline{\mathcal{O}} \cap B(y,r)$ for some $y \in \overline{\mathcal{O}}$ and $r > 0$). The definition is readily the same, considering local maximum points of $u^* - \varphi$ or minimum points of $v_* - \varphi$ which are in $\mathcal{O}$ or $\overline{\mathcal{O}} \cap B(y,r)$.*

3.1.2 The Half-Relaxed Limit Method

In order to state it we use the following notations: if $(z_\varepsilon)_\varepsilon$ is a sequence of uniformly locally bounded functions, the half-relaxed limits of $(z_\varepsilon)_\varepsilon$ are defined by

$$\limsup{}^* \, z_\varepsilon(y) \; = \; \limsup_{\substack{\tilde{y} \to y \\ \varepsilon \to 0}} z_\varepsilon(\tilde{y}) \quad \text{and} \quad \liminf{}_* \, z_\varepsilon(y) \; = \; \liminf_{\substack{\tilde{y} \to y \\ \varepsilon \to 0}} z_\varepsilon(\tilde{y}) \; .$$

Theorem 3.1.3 *Assume that, for $\varepsilon > 0$, u_ε is a viscosity subsolution [resp. a supersolution] of the equation*

$$G_\varepsilon(y, u_\varepsilon, Du_\varepsilon) = 0 \quad \textit{on } \overline{\mathcal{O}} \, ,$$

where $(G_\varepsilon)_\varepsilon$ is a sequence of uniformly locally bounded functions in $\overline{\mathcal{O}} \times \mathbb{R} \times \mathbb{R}^N$. If the functions u_ε are uniformly locally bounded on $\overline{\mathcal{O}}$, then $\overline{u} = \limsup^* u_\varepsilon$ [resp. $\underline{u} = \liminf_* u_\varepsilon$] is a subsolution [resp. a supersolution] of the equation

$$\underline{G}(y, u, Du) = 0 \quad \textit{on } \overline{\mathcal{O}} \, ,$$

where $\underline{G} = \liminf_* G_\varepsilon$. [resp. of the equation

$$\overline{G}(y, u, Du) = 0 \quad \textit{on } \overline{\mathcal{O}} \, ,$$

where $\overline{G} = \limsup^* G_\varepsilon$].

Of course, the main interest of this result is to allow the passage to the limit in the notion of sub and supersolutions with very weak assumptions on the solutions but also on the equations: only uniform local L^∞–bounds. In particular, phenomenas like *boundary layers* can be handled with such a result and this is a striking difference with the first stability results for viscosity solutions which were requiring some compactness of the u_ε's in the space of continuous functions (typically some gradient bounds).

The counterpart is that we do not have a limit anymore, but two half-limits $\overline{u}$ and $\underline{u}$ which have to be connected in order to obtain a real convergence result. In fact, the complete **Half-Relaxed Limit Method** is performed as follows

1. Get a locally (or globally) uniform L^∞–bound for the $(u_\varepsilon)_\varepsilon$.
2. Apply the above discontinuous stability result.
3. By definition, there holds $\underline{u} \leq \overline{u}$ on $\overline{\mathcal{O}}$.
4. To obtain the converse inequality, use a **Strong Comparison Result**, **(SCR)** in short, i.e a comparison result which is valid for *discontinuous* sub and supersolutions, which yields
$$\overline{u} \leq \underline{u} \quad \text{in } \ \mathcal{O} \text{ (or on } \ \overline{\mathcal{O}}\,) \,.$$
5. From the **(SCR)**, we deduce that $\overline{u} = \underline{u}$ in $\mathcal{O}$ (or on $\overline{\mathcal{O}}$). Setting $u := \overline{u} = \underline{u}$, it follows that u is continuous (because $\overline{u}$ is u.s.c. and $\underline{u}$ is l.s.c.) and it is easy to show that, u is *the unique solution* of the limit equation, by using again the **(SCR)**. Finally, we also get the convergence of u_ε to u in $C(\mathcal{O})$ (or in $C(\overline{\mathcal{O}})$).

It is clear that, in this method, **(SCR)** play a central role and one of the main challenge in this book is to show how to obtain them in various contexts.

Now we give the **Proof of Theorem 3.1.3**. We do it only for the subsolution case, the supersolution one being analogous.

We first remark that $\limsup^* u_\varepsilon = \limsup^* u_\varepsilon^*$ and therefore changing u_ε in u_ε^*, we can assume without loss of generality that u_ε is u.s.c.. The proof is based on the

Lemma 3.1.4 *Let $(w_\varepsilon)_\varepsilon$ be a sequence of uniformly bounded u.s.c. functions on $\overline{\mathcal{O}}$ and $\overline{w} = \limsup^* w_\varepsilon$. If $y \in \overline{\mathcal{O}}$ is a strict local maximum point of $\overline{w}$ on $\overline{\mathcal{O}}$, there exists a subsequence $(w_{\varepsilon'})_{\varepsilon'}$ of $(w_\varepsilon)_\varepsilon$ and a sequence $(y_{\varepsilon'})_{\varepsilon'}$ of points in $\overline{\mathcal{O}}$ such that, for all ε', $y_{\varepsilon'}$ is a local maximum point of $w_{\varepsilon'}$ in $\overline{\mathcal{O}}$, the sequence $(y_{\varepsilon'})_{\varepsilon'}$ converges to y and $w_{\varepsilon'}(y_{\varepsilon'}) \to \overline{w}(y)$.*

We first prove Theorem 3.1.3 by using the lemma. Let $\varphi \in C^1(\overline{\mathcal{O}})$ and let $y \in \overline{\mathcal{O}}$ be a strict local maximum point de $\overline{u} - \varphi$. We apply Lemma 3.1.4 to $w_\varepsilon = u_\varepsilon - \varphi$ and $\overline{w} = \overline{u} - \varphi = \limsup^* (u_\varepsilon - \varphi)$. There exists a subsequence $(u_{\varepsilon'})_{\varepsilon'}$ and a sequence $(y_{\varepsilon'})_{\varepsilon'}$ such that, for all ε', $y_{\varepsilon'}$ is a local maximum point of $u_{\varepsilon'} - \varphi$ on $\overline{\mathcal{O}}$. But $u_{\varepsilon'}$ is a subsolution of the $G_{\varepsilon'}$-equation, therefore
$$G_{\varepsilon'}(y_{\varepsilon'}, u_{\varepsilon'}(y_{\varepsilon'}), D\varphi(y_{\varepsilon'})) \leq 0 \,.$$
Since $y_{\varepsilon'} \to x$ and since φ is smooth $D\varphi(y_{\varepsilon'}) \to D\varphi(y)$; but we have also $u_{\varepsilon'}(y_{\varepsilon'}) \to \overline{u}(y)$, therefore by definition of $\underline{G}$
$$\underline{G}(x, \overline{u}(y), D\varphi(y)) \leq \liminf\, G_{\varepsilon'}(y_{\varepsilon'}, u_{\varepsilon'}(y_{\varepsilon'}), D\varphi(y_{\varepsilon'})) \,.$$

This immediately yields

$$\underline{G}(x, \overline{u}(y), D\varphi(y)) \leq 0 \ ,$$

and the proof is complete.

Proof of Lemma 3.1.4 — Since y is a strict local maximum point of $\overline{w}$ on $\overline{\mathcal{O}}$, there exists $r > 0$ such that

$$\forall z \in \overline{\mathcal{O}} \cap \overline{B}(y, r) \ , \quad \overline{w}(z) \leq \overline{w}(y) \ ,$$

the inequality being strict for $z \neq y$. But $\overline{\mathcal{O}} \cap \overline{B}(y, r)$ is compact and w_ε is u.s.c., therefore, for all $\varepsilon > 0$, there exists a maximum point y_ε of w_ε on $\overline{\mathcal{O}} \cap \overline{B}(y, r)$. In other words

$$\forall z \in \overline{\mathcal{O}} \cap \overline{B}(y, r) \ , \quad w_\varepsilon(z) \leq w_\varepsilon(y_\varepsilon) \ . \tag{3.2}$$

Now we take the $\limsup$ for $z \to y$ and $\varepsilon \to 0$: by the definition of the $\limsup^*$, we obtain

$$\overline{w}(y) \leq \limsup_\varepsilon \ w_\varepsilon(y_\varepsilon) \ .$$

Next we consider the right-hand side of this inequality: extracting a subsequence denoted by ε', we have $\limsup_\varepsilon w_\varepsilon(y_\varepsilon) = \lim_{\varepsilon'} w_{\varepsilon'}(y_{\varepsilon'})$ and since $\overline{\mathcal{O}} \cap \overline{B}(y, r)$ is compact, we may also assume that $y_{\varepsilon'} \to \bar{y} \in \overline{\mathcal{O}} \cap \overline{B}(y, r)$. But using again the definition of the $\limsup^*$ at $\bar{y}$, we get

$$\overline{w}(y) \leq \limsup_\varepsilon \ w_\varepsilon(y_\varepsilon) = \lim_{\varepsilon'} w_{\varepsilon'}(y_{\varepsilon'}) \leq \overline{w}(\bar{y}) \ .$$

Since y is a strict maximum point of $\overline{w}$ in $\overline{\mathcal{O}} \cap \overline{B}(y, r)$ and that $\bar{y} \in \overline{\mathcal{O}} \cap \overline{B}(y, r)$, this inequality implies that $\bar{y} = y$ and that $w_{\varepsilon'}(y_{\varepsilon'}) \to \overline{w}(y)$ and the proof is complete.

Q.E.D.

We conclude this subsection by the

Lemma 3.1.5 *If $\mathcal{K}$ is a compact subset of $\overline{\mathcal{O}}$ and if $\overline{u} = \underline{u}$ on $\mathcal{K}$ then u_ε converges uniformly to the function $u := \overline{u} = \underline{u}$ on $\mathcal{K}$.*

Proof of Lemma 3.1.5 : Since $\overline{u} = \underline{u}$ on $\mathcal{K}$ and since $\overline{u}$ is u.s.c. and $\underline{u}$ is l.s.c. on $\overline{\mathcal{O}}$, u is continuous on $\mathcal{K}$. We first consider

$$M_\varepsilon = \sup_{\mathcal{K}} \ (u^*_\varepsilon - u) \ .$$

The function u^*_ε being u.s.c. and u being continuous, this supremum is in fact a maximum and is achived at a point y_ε. The sequence $(u_\varepsilon)_\varepsilon$ being locally uniformly bounded, the sequence $(M_\varepsilon)_\varepsilon$ is also bounded and, $\mathcal{K}$ being compact, we can extract subsequences such that $M_{\varepsilon'} \to \limsup_\varepsilon M_\varepsilon$ and $y_{\varepsilon'} \to \bar{y} \in \mathcal{K}$. But by the definition of the $\limsup^*$, $\limsup u^*_{\varepsilon'}(y_{\varepsilon'}) \le \overline{u}(\bar{y})$ while we have also $u(y_{\varepsilon'}) \to u(\bar{y})$ by the continuity of u. We conclude that

$$\limsup_{\varepsilon} M_\varepsilon = \lim_{\varepsilon'} M_{\varepsilon'} = \lim_{\varepsilon'} \left(u^*_{\varepsilon'}(y_{\varepsilon'}) - u(y_{\varepsilon'})\right) \le \overline{u}(\bar{y}) - u(\bar{y}) = 0\ .$$

This part of the proof gives half of the uniform convergence, the other part being obtained analogously by considering $\tilde{M}_\varepsilon = \sup_{\mathcal{K}} \,(u - (u_\varepsilon)_*)$.

3.2 Strong Comparison Results: How to cook them?

In the previous section, we have seen that **(SCR)** are key tools which are needed to use the "Half-Relaxed Limit Method". We have used the terminology "strong" because such comparison results have to hold for discontinuous sub and supersolutions: in general it is easier (from a technical point of view) to compare continuous (or uniformly continuous) sub and supersolutions and even some comparison results may be true in the framework of continuous solutions while they are wrong in the discontinuous ones. But, in this book, we mainly prove **(SCR)** and therefore we will use the expression "comparison result" for **(SCR)**.

Of course, in general, comparison results means a global inequality (i.e. on the whole domain) between sub and supersolutions. But, for Hamilton-Jacobi Equations with discontinuities, it is far easier (if not necessary) to argue locally. This is why we explain, in this section, how to reduce the proof of *global* comparison results to the proof of (a priori easier) *local* comparison results. We do not pretend this section to cover all cases but we tried to make it as general as we could.

3.2.1 Stationary Equations

To do so, we consider a general equation

$$G(x, u, Du) = 0 \quad \text{on } \mathcal{F}\ , \tag{3.3}$$

where $\mathcal{F}$ is a closed subset of $\mathbb{R}^N$ and G is a continuous or discontinuous function on $\mathcal{F} \times \mathbb{R} \times \mathbb{R}^N$.

We introduce the following notations: USCS($\mathcal{F}$) is a subset of u.s.c. subsolutions of (3.3) while LSCS($\mathcal{F}$) is a subset of l.s.c. supersolutions of (3.3). We prefer to remain a little bit vague on these subsets but the reader may have in mind that they are generally defined by some growth conditions at infinity if $\mathcal{F}$ is an unbounded subset of $\mathbb{R}^N$. In these definitions, we may replace below $\mathcal{F}$ by a subset (open or closed) of $\mathcal{F}$ and we use below the following notations

$$\mathcal{F}^{x,r} := B(x,r) \cap \mathcal{F} \quad \text{and} \quad \partial\mathcal{F}^{x,r} := \partial B(x,r) \cap \mathcal{F} \; .$$

Finally we denote by $\mathrm{USCS}(\mathcal{F}^{x,r})$ [resp. $\mathrm{LSCS}(\mathcal{F}^{x,r})$] the set of u.s.c. [resp. l.s.c.] functions on $\overline{\mathcal{F}^{x,r}}$ which are subsolutions [resp. supersolutions] of $G = 0$ in $\mathcal{F}^{x,r}$. Notice that, for these sub and supersolutions, no viscosity inequality is imposed on $\partial B(x,r)$.

By "global" and "local" comparison results we mean the following

$(\mathbf{GCR})^{\mathcal{F}}$ *Global Comparison Result in* $\mathcal{F}$:
For any $u \in \mathrm{USCS}(\mathcal{F})$, *for any* $v \in \mathrm{LSCS}(\mathcal{F})$, *we have* $u \leq v$ *on* $\mathcal{F}$ *.*

$(\mathbf{LCR})^{\mathcal{F}}$ *Local Comparison Result in* $\mathcal{F}$:
For any $x \in \mathcal{F}$, *there exists* $r > 0$ *such that, if* $u \in \mathrm{USCS}(\mathcal{F}^{x,r})$, $v \in \mathrm{LSCS}(\mathcal{F}^{x,r})$ *and* $\max_{\overline{\mathcal{F}^{x,r}}}(u - v) > 0$, *then*

$$\max_{\overline{\mathcal{F}^{x,r}}}(u - v) \leq \max_{\partial\mathcal{F}^{x,r}}(u - v).$$

In the rest of this section, we skip the reference to $\mathcal{F}$ in **(LCR)** and **(GCR)** since there is no ambiguity here. It is clear that a proof of **(LCR)** seems much easier because of the compactness of $\mathcal{F}^{x,r}$ since the behavior at infinity of u and v does not play any role but also because we can use only local properties of G.

Now we formulate two assumptions which allow to reduce **(GCR)** to **(LCR)**.

(**LOC1**): *If* $\mathcal{F}$ *is unbounded, for any* $u \in \mathrm{USCS}(\mathcal{F})$, *for any* $v \in \mathrm{LSCS}(\mathcal{F})$, *there exists a sequence* $(u_\alpha)_{\alpha>0}$ *of u.s.c. subsolutions of* (3.3) *such that* $u_\alpha(x) - v(x) \to -\infty$ *when* $|x| \to +\infty$, $x \in \mathcal{F}$. *Moreover, for any* $x \in \mathcal{F}$, $u_\alpha(x) \to u(x)$ *when* $\alpha \to 0$.

In the above assumption, we do not write that $u_\alpha \in \mathrm{USCS}(\mathcal{F})$ because this is not the case in general: typically, $\mathrm{USCS}(\mathcal{F})$ may be the set of *bounded* subsolutions of (3.3) while u_α is not expected to be bounded.

(**LOC2**): *For any* $x \in \mathcal{F}$, *if* $u \in \mathrm{USCS}(\mathcal{F}^{x,r})$, *there exists a sequence* $(u^\delta)_{\delta>0}$ *of functions in* $\mathrm{USCS}(\mathcal{F}^{x,r})$ *such that* $u^\delta(x) - u(x) \geq u^\delta(y) - u(y) + \eta(\delta)$ *if* $y \in \partial\mathcal{F}^{x,r}$, *where*

$\eta(\delta) > 0$ for all δ. Moreover, for any $y \in \mathcal{F}$, $u^\delta(y) \to u(y)$ when $\delta \to 0$.

The role of (**LOC1**) and (**LOC2**) will be clear in the proof of the property "$(\mathbf{LCR})^{\mathcal{F}}$ implies $(\mathbf{GCR})^{\mathcal{F}}$" below: (**LOC1**) allows to consider maximum points of $u_\alpha - v$ (which was impossible for $u - v$ because $\mathcal{F}$ is not compact a priori and u, v can be unbounded) while (**LOC2**) provides the conclusion.

Proposition 3.2.1 *Under Assumptions* (**LOC1**) *and* (**LOC2**)*, then* $(\mathbf{LCR})^{\mathcal{F}}$ *implies* $(\mathbf{GCR})^{\mathcal{F}}$*.*

Proof — Given $u \in \mathrm{USCS}(\mathcal{F})$ and $v \in \mathrm{LSCS}(\mathcal{F})$, we have to prove that $u \leq v$ on $\mathcal{F}$.

Instead of comparing u and v, we are going to compare u_α and v some u_α given by (**LOC1**) and then to let α tend to 0. Arguing in that way and droping the α for simplifying the notations means that we can assume without loss of generality that $u(x) - v(x) \to -\infty$ when $|x| \to +\infty$, $x \in \mathcal{F}$ and therefore we can consider $M := \max_{\mathcal{F}}(u - v)$.

We argue by contradiction assuming that this maximum is strictly positive, otherwise we have nothing to prove.

Since $\mathcal{F}$ is closed, $u - v$ is u.s.c. and tends to $-\infty$ at infinity, this function achieves its maximum at some point $x \in \mathcal{F}$. We apply (**LOC2**) by introducing the u^δ's. Since $u^\delta \in \mathrm{USCS}(\mathcal{F}^{x,r})$ and since **(LCR)** holds the following alternative holds:
(i) either $u^\delta \leq v$ in $\overline{\mathcal{F}^{x,r}}$, but this cannot be the case for δ small enough since $u^\delta(x) - v(x) \to u(x) - v(x) > 0$;
(ii) or $\max_{\overline{\mathcal{F}^{x,r}}}(u^\delta - v) > 0$ and

$$\max_{\overline{\mathcal{F}^{x,r}}}(u^\delta - v) \leq \max_{\partial \mathcal{F}^{x,r}}(u^\delta - v).$$

But the properties of u^δ given in (**LOC2**) would imply that

$$u^\delta(x) - v(x) \leq \max_{\partial \mathcal{F}^{x,r}}(u^\delta - v)(y) \leq \max_{\partial \mathcal{F}^{x,r}}(u - v)(y) + (u^\delta(x) - u(x)) - \eta(\delta)\ ,$$

i.e. $M \leq M - \eta(\delta)$, a contradiction. Therefore M cannot be strictly positive and the proof is complete.

Q.E.D.

Now a first key question is: how can we check (**LOC1**) and (**LOC2**)? We provide some typical examples.

The Lipschitz case — We assume that there exists a constant $c > 0$ such that the function G satisfies, for all $x \in F$, $z_1 \leq z_2$ and $p, q \in \mathbb{R}^N$

$$G(x, z_1, p) - G(x, z_2, p) \geq c^{-1}(z_1 - z_2) \; ,$$
$$|G(x, z_1, p) - G(x, z_1, q) \leq c|p - q| \; .$$

In the case when USCS, LSCS are sets of bounded sub or supersolutions then (**LOC1**) is satisfied with $u_\alpha(x) = u(x) - \alpha[(|x|^2 + 1)^{1/2} + c^2]$, indeed

$$G(x, u_\alpha(x), Du_\alpha(x)) \leq G(x, u(x), Du(x)) - c^{-1}(\alpha[(|x|^2 + 1)^{1/2} + c^2)) + c\alpha\frac{|x|}{(|x|^2 + 1)^{1/2}} \; ,$$
$$\leq -c^{-1}(\alpha c^2)) + c\alpha = 0.$$

For (**LOC2**), we can choose any r and

$$u^\delta(y) = u(y) - \delta(|y - x|^2 + k) \; .$$

for some well-chosen constant k. Indeed

$$G(y, u^\delta(y), Du^\delta(y)) \leq G(y, u(y), Du(y)) - c^{-1}\delta(|y - x|^2 + k) + 2c\delta|y - x| \; ,$$
$$\leq -\frac{\delta}{c}(|y - x|^2 + k - 2c^2|y - x|),$$

and the choice $k = c^4$ gives the answer.

The convex case — Here the advantage is to avoid the restriction due to the Lipschitz continuity of G in p. We do not propose any explicit building of u_α or u^δ but we build them using the following assumptions:

(Subsol1): For any $u \in \mathrm{USCS}(\mathcal{F})$, $v \in \mathrm{LSCS}(\mathcal{F})$, there exists an u.s.c. subsolution $\psi_1 : F \to \mathbb{R}$ such that for any $0 < \alpha < 1$, $u_\alpha(x) := (1 - \alpha)u(x) + \alpha\psi_1(x)$ satisfies (**LOC1**).

(Subsol2): For any $x \in \mathcal{F}$, there exists $r > 0$ and $\psi_2 \in \mathrm{USCS}(B(x, r) \cap \mathcal{F})$ such that for any $0 < \delta < 1$, $u_\delta(x) = (1 - \delta)u(x) + \delta\psi_2(x)$ satisfies (**LOC2**).

3.2.2 The evolution case

There are some key differences in the evolution case due to the fact that the time-variable is playing a particular role since we are mainly solving a Cauchy problem. To describe them, we first write the equation as

$$G(x, t, u, (D_x u, u_t)) = 0 \quad \text{on } \mathcal{F} \times (0, T] \; , \tag{3.4}$$

where $\mathcal{F}$ is a closed subset of $\mathbb{R}^N$ and G is a continuous or discontinuous function on $\mathcal{F} \times [0,T] \times \mathbb{R} \times \mathbb{R}^{N+1}$.

This equation has to be complemented by an initial data at time $t = 0$ which can be of an usual form, namely

$$u(x,0) = u_0(x) \quad \text{on } \mathcal{F}\ , \tag{3.5}$$

where u_0 is a given function defined on $\mathcal{F}$, or this initial value of u can be obtained by solving an equation, namely

$$G_{init}(x,0,u(x,0),D_x u(x,0)) = 0 \quad \text{on } \mathcal{F}\ , \tag{3.6}$$

where G_{init} is a continuous or discontinuous function on $\mathcal{F} \times [0,T] \times \mathbb{R} \times \mathbb{R}^N$.

A strong comparison result for either (3.4)-(3.5) or (3.4)-(3.6) which is denoted below by (GCR-evol) can be defined in an analogous way as (GCR): subsolutions (in a certain class of functions) are below supersolutions (in the same class of functions), USCS($\mathcal{F}$) and LSCS($\mathcal{F}$) being just replaced by USCS($\mathcal{F}\times[0,T]$) and LSCS($\mathcal{F}\times[0,T]$); we just point out that the initial data is included in the equation in this abstract formulation: for example, a subsolution u satisfies either

$$u(x,0) \leq (u_0)^*(x) \quad \text{on } \mathcal{F}\ ,$$

in the case of (3.5) or the function $x \mapsto u(x,0)$ satisfies

$$G_{init}(x,0,u(x,0),D_x u(x,0)) \leq 0 \quad \text{on } \mathcal{F}\ ,$$

in the viscosity sense, in the case of (3.6).

As it is even more clear in the case of (3.6), a **(SCR)** in the evolution case consists in two steps

(i) first proving that, for any $u \in$ USCS($\mathcal{F} \times [0,T]$) and $v \in$ LSCS($\mathcal{F} \times [0,T]$),

$$u(x,0) \leq v(x,0) \quad \text{on } \mathcal{F}, \tag{3.7}$$

(ii) and then to show that this inequality remains true for $t > 0$, i.e.

$$u(x,t) \leq v(x,t) \quad \text{on } \mathcal{F} \times [0,T]\,.$$

Of course, in the case of (3.5), (3.7) is obvious if u_0 is a continuous function; but, in the case of (3.6), the proof of such inequality is nothing but a (GCR) in the stationary case.

Therefore the main additional difficult consists in showing that Property (ii) holds true and we are going to explain now the analogue of the approach of the previous section *assuming that we have* (3.7).

To redefine **(LCR)**, we have to introduce, for $x \in \mathcal{F}$, $t \in (0,T]$, $r > 0$ and $0 < h < t$, the sets

$$Q^{x,t}_{r,h}[\mathcal{F}] := (B(x,r) \cap \mathcal{F}) \times (t-h, t] \ .$$

As in the stationary case, we introduce the set $\mathrm{USCS}(Q^{x,t}_{r,h}[\mathcal{F}])$, $\mathrm{LSCS}(Q^{x,t}_{r,h}[\mathcal{F}])$ of respectively u.s.c. subsolutions and l.s.c. supersolution of $G(x,t,u,(D_x u, u_t)) = 0$ in $Q^{x,t}_{r,h}[\mathcal{F}]$. This means that the viscosity inequalities holds in $Q^{x,t}_{r,h}[\mathcal{F}]$ and not necessarily on its closure, but these sub and supersolutions are u.s.c. or l.s.c. on $\overline{Q^{x,t}_{r,h}[\mathcal{F}]}$.

On the other hand, including $(B(x,r)\cap\mathcal{F})\times\{t\}$ in the set where the subsolution or supersolution inequalities hold is important in order to have the suitable comparison up to time t and we also refer to Proposition 3.2.4 for the connection between sub and supersolutions in $(B(x,r) \cap \mathcal{F}) \times (t-h,t)$ and on $(B(x,r) \cap \mathcal{F}) \times (t-h,t]$.

With this definition we have

(LCR)-evol: *For any $(x,t) \in \mathcal{F} \times (0,T]$, there exists $r > 0$, $0 < h < t$ such that, if $u \in \mathrm{USCS}(Q^{x,t}_{r,h}[\mathcal{F}])$, $v \in \mathrm{LSCS}(Q^{x,t}_{r,h}[\mathcal{F}])$ and $\max\limits_{\overline{Q^{x,t}_{r,h}[\mathcal{F}]}}(u-v) > 0$, then*

$$\max_{\overline{Q^{x,t}_{r,h}[\mathcal{F}]}}(u-v) \leq \max_{\partial_p Q^{x,t}_{r,h}[\mathcal{F}]}(u-v) \, ,$$

where $\partial_p Q^{x,t}_{r,h}[\mathcal{F}]$ stands for the parabolic boundary of $Q^{x,t}_{r,h}[\mathcal{F}]$, namely

$$\partial_p Q^{x,t}_{r,h}[\mathcal{F}] = \Big\{(\partial B(x,r) \cap \mathcal{F}) \times [t-h,t]\Big\} \bigcup \Big\{(\overline{B(x,r)} \cap \mathcal{F}) \times \{t-h\}\Big\} \, .$$

The corresponding evolution versions of (**LOC1**) and (**LOC2**) are given by:

(**LOC1**)-evol: *If $\mathcal{F}$ is unbounded, for any $u \in \mathrm{USCS}(\mathcal{F} \times [0,T])$, for any $v \in \mathrm{LSCS}(\mathcal{F} \times [0,T])$, there exists a sequence $(u_\alpha)_{\alpha>0}$ of u.s.c. subsolutions of* (3.3) *such that $u_\alpha(x,t) - v(x,t) \to -\infty$ when $|x| \to +\infty$, $x \in \mathcal{F}$. Moreover, for any $x \in \mathcal{F}$, $u_\alpha(x,t) \to u(x,t)$ when $\alpha \to 0$.*

(**LOC2**)-evol: *For any $x \in \mathcal{F}$, if $u \in \mathrm{USCS}(Q^{x,t}_{r,h}[\mathcal{F}])$, there exists a sequence $(u^\delta)_{\delta>0}$ of functions in $\mathrm{USCS}(Q^{x,t}_{r,h}[\mathcal{F}])$ such that if $y \in (\partial \mathcal{F}^{x,r}) \times [t, t-h]$, then $u^\delta(y,t-h) \leq u(y,t-h) + \tilde\eta(\delta)$ where $\tilde\eta(\delta) \to 0$ as $\delta \to 0$. Moreover, for any $y \in \mathcal{F}$, $u^\delta(y) \to u(y)$ when $\delta \to 0$.*

With these assumptions, we have the

Proposition 3.2.2 *Under Assumptions* (**LOC1**)*-evol and* (**LOC2**)*-evol, then* (**LCR**)*-evol implies* (**GCR**)*-evol.*

Proof — There is no main change in the proof except the following point: using (**LOC1**)-evol, we may assume that the maximum of $u - v$ is achieved at some point (x, t). Here we choose t as the minimal time such that we have a maximum of $u - v$. A priori $t > 0$ since we know that $u \leq v$ on $\mathcal{F} \times \{0\}$. Then we apply (**LOC2**)-evol: we know that the maximum of $u - v$ on $(\overline{B(x,r)} \cap \mathcal{F}) \times \{t - h\}$ is strictly less than $\max_{\overline{Q^{x,t}_{r,h}[\mathcal{F}]}} (u - v) = (u - v)(x, t)$ because of the minimality of t and using the property of $\tilde{\eta}(\delta)$ we can choose δ small enough in order to have

$$\max_{(\overline{B(x,r)} \cap \mathcal{F}) \times \{t-h\}} \big(u^\delta(y, s) - v(y, s)\big) < u_\delta(x, t) - v(x, t) \; .$$

The rest of the proof follows the same arguments.

Q.E.D.

In the evolution case, where the equation (or part of the equation) contains some u_t-term, building the u_α and u^δ turns out to be easier. For example

$$u_\alpha(x, t) = u(x, t) - \alpha[(|x|^2 + 1)^{1/2} + Kt] \; ,$$

for $K > 0$ large enough. And for u^δ,

$$u^\delta(y, s) = u(y, s) - \delta[(|y - x|^2 + 1)^{1/2} - 1 + K(s - t)] \; ,$$

where K has to be chosen large enough to have a subsolution and h small enough to have the right property on the parabolic boundary.

Remark 3.2.3 *As the proofs show it (both in the stationary and evolution case), in order to have* (**GCR**)*, we do not need* (**LCR**) *to hold on the whole set $\mathcal{F}$: indeed, if we already know that $u \leq v$ on some subset $\mathcal{A}$ of $\mathcal{F}$, then* (**LCR**) *is required only in $\mathcal{F} \setminus \mathcal{A}$ to have* (**GCR**)*.*

3.2.3 Viscosity inequalities at $t = T$ in the evolution case

We conclude this section by examining the viscosity sub and supersolutions inequalities at $t = T$ and their consequences on the properties of sub and supersolutions. To do so, we have to be a little bit more precise on the assumptions on the function G appearing in (3.4). We introduce the following hypothesis

$(\mathbf{H_{BA-}}p_t)$: *For any* $(x,t,r,p_x,p_t) \in \mathcal{F} \times (0,T] \times \mathbb{R} \times \mathbb{R}^N \times \mathbb{R}$*, the function* $p_t \mapsto G\big(x,t,r,(p_x,p_t)\big)$ *is increasing and* $G\big(x,t,r,(p_x,p_t)\big) \to +\infty$ *as* $p_t \to +\infty$*, uniformly for bounded* x,t,r,p_x*.*

This assumption is obviously satisfied in the standard case, *i.e.* for equations like

$$u_t + H(x,t,u,D_x u) = 0 \quad \text{in } \mathbb{R}^N \times (0,T]\ ,$$

provided H is continuous (or only locally bounded) since in this case $G(x,t,r,(p_x,p_t)) = p_t + H(x,t,r,p_x)$.

Proposition 3.2.4 *Under Assumption* $(\mathbf{H_{BA-}}p_t)$*, we have*

(*i*) *If* $u : \mathcal{F} \times (0,T) \to \mathbb{R}$ [*resp.* $v : \mathcal{F} \times (0,T) \to \mathbb{R}$] *is an usc viscosity subsolution* [*resp. lsc supersolution*] *of*

$$G(x,t,w,(D_x w, w_t)) = 0 \quad \text{on } \mathcal{F} \times (0,T)\ ,$$

then for any $0 < T' < T$*,* u [*resp.* v] *is an usc viscosity subsolution* [*resp. lsc supersolution*] *of*

$$G(x,t,w,(D_x w, w_t)) = 0 \quad \text{on } \mathcal{F} \times (0,T']\ .$$

(*ii*) *Under the same conditions on* u *and* v *and if*

$$u(x,T) = \limsup_{(y,s)\to(x,T),\, s<T} u(y,s) \quad [\text{resp.} \quad v(x,T) = \liminf_{(y,s)\to(x,T),\, s<T} v(y,s)]\ , \tag{3.8}$$

then u *and* v *are respectively sub and supersolution of* (3.4)*.*

(*iii*) *If* $u : \mathcal{F} \times (0,T] \to \mathbb{R}$ *is an usc viscosity subsolution of* (3.4)*, then, for any* $x \in \mathcal{F}$*,* (3.8) *holds for* u*.*

(*iv*) *If* G *satisfies* $G(x,t,r,(p_x,p_t)) \to -\infty$ *as* $p_t \to -\infty$*, uniformly for bounded* x,t,r,p_x *and if* $v : \mathcal{F} \times (0,T] \to \mathbb{R}$ *is a lsc viscosity supersolution of* (3.4)*, then* (3.8) *holds for* v*.*

This result clearly shows the particularities of the viscosity inequalities at the terminal time $t = T$ or $t = T'$: sub and supersolutions in $\mathcal{F} \times (0,T)$ are automatically sub and supersolutions on $\mathcal{F} \times (0,T']$ for any $0 < T' < T$ and even for $T' = T$

provided that they are extended in the right way up to time T, according to (3.8). And conversely sub and supersolutions on $\mathcal{F} \times (0,T]$ satisfy (3.8) provided that G has some suitable properties which clearly hold for the standard H-equation above. Here there is a difference between sub and supersolutions due to the disymmetry of Assumption $(\mathbf{H_{BA}}{-}p_t)$. We will come back later on this point with the control interpretation.

Proof — We only prove the first and second part of the result in the subsolution case, the proof for the supersolution being analogous. Let φ be a smooth function (say, in $\mathcal{F} \times [0,T]$) and let (x,T') be a strict local maximum point of $u-\varphi$ in $\mathcal{F} \times [0,T']$. We introduce the function

$$(y,s) \mapsto u(y,s) - \varphi(y,s) - \frac{[(s-T')^+]^2}{\varepsilon} \; .$$

By classical arguments, this function has a local maximum point at $(x_\varepsilon, t_\varepsilon)$ and we have

$$(x_\varepsilon, t_\varepsilon) \to (x,T') \quad \text{and} \quad u(x_\varepsilon, t_\varepsilon) \to u(x,T') \text{ as } \varepsilon \to 0 \; ,$$

because of both the strict maximum point property and the ε-penalisation. Moreover, for ε small enough, the penalization implies that $t_\varepsilon < T'$.

Since u is a subsolution of the G-equation in $\mathcal{F} \times (0,T)$ and as we noticed, $(x_\varepsilon, t_\varepsilon)$ is a local maximum point in $\mathcal{F} \times (0,T)$, we have

$$G_*\Big(x_\varepsilon, t_\varepsilon, u(x_\varepsilon, t_\varepsilon), (D_x\varphi(x_\varepsilon, t_\varepsilon), \varphi_t(x_\varepsilon, t_\varepsilon) + 2\varepsilon^{-1}(s-T)^+)\Big) \leq 0 \; .$$

But, by $(\mathbf{H_{BA}}{-}p_t)$, $G(y,s,r,(p_x,p_t))$ and therefore $G_*(y,s,r,(p_x,p_t))$ is increasing in the p_t-variable and we have

$$G_*(x_\varepsilon, t_\varepsilon, u(x_\varepsilon, t_\varepsilon), (D_x\varphi(x_\varepsilon, t_\varepsilon), D_t\varphi(x_\varepsilon, t_\varepsilon))) \leq 0 \; .$$

The conclusion follows from the lower semicontinuity of G_* by letting ε tend to 0.

For the proof of (ii), we argue in an analogous way: if (x,T) is a strict local maximum point of $u-\varphi$ in $\mathcal{F} \times [0,T]$, we introduce the function

$$(y,s) \mapsto u(y,s) - \varphi(y,s) - \frac{\varepsilon}{(T-s)} \; .$$

By classical arguments, this function has a local maximum point at $(x_\varepsilon, t_\varepsilon)$ and we have

$$(x_\varepsilon, t_\varepsilon) \to (x,T) \quad \text{and} \quad u(x_\varepsilon, t_\varepsilon) \to u(x,T) \text{ as } \varepsilon \to 0 \; .$$

It is worth pointing out that, in this case, the proof of such properties uses not only the strict maximum point property and the fact that the ε-penalisation is vanishing, but also strongly Property (3.8) for u.

We are led to

$$G_*\Big(x_\varepsilon, t_\varepsilon, u(x_\varepsilon, t_\varepsilon), (D_x\varphi(x_\varepsilon, t_\varepsilon), \varphi_t(x_\varepsilon, t_\varepsilon) + \frac{\varepsilon}{(T-s)^2})\Big) \leq 0 \,,$$

and we conclude by similar arguments as in the proof of (i).

Finally we prove (iii) since the supersolution one, (iv), follows again from similar arguments with the additional assumption on G.

We pick some $(x,T) \in \mathcal{F} \times \{T\}$ and we aim at proving (3.8). We argue by contradiction: if this is not the case then $u(x,T) < \limsup u(y,s)$ as $(y,s) \to (x,T)$, with $s < T$. This implies that for any $\varepsilon > 0$ small enough and any $C > 0$, the function

$$(y,s) \mapsto u(y,s) - \frac{|y-x|^2}{\varepsilon^2} - C(s-T)$$

can only have a maximum point for $s = T$, say at $y = x_\varepsilon$ close to x. The viscosity subsolution inequality reads

$$G_*\Big(x_\varepsilon, T, u(x_\varepsilon, T), (\frac{2(x_\varepsilon - x)}{\varepsilon^2}, C)\Big) \leq 0 \,.$$

But if we fix ε (small enough), all the arguments in G_* remains bouded, except C. So, choosing C large enough, we have a contradiction because of $(\mathbf{H_{BA}}\text{–}p_t)$.

Q.E.D.

Remark 3.2.5 *We point out that, even if Proposition 3.2.4 only provides the result for sub or supersolutions inequalities in sets of the form $\mathcal{F} \times (0,T)$, a similar result can be obtained, under suitable assumptions, for sub and supersolution properties at any point (x,T) of $\mathcal{M}$ where $\mathcal{M}$ is the restriction to $\mathbb{R}^N \times (0,T]$ to a submanifold of $\mathbb{R}^N \times \mathbb{R}$. Indeed, it is clear from the proof that only Assumption $(\mathbf{H_{BA}}\text{–}p_t)$ is really needed to have such properties.*

3.2.4 The simplest examples: continuous Hamilton-Jacobi Equations in the whole space $\mathbb{R}^N$

As a simple example, we consider the standard continuous Hamilton-Jacobi Equation

$$u_t + H(x,t,u,D_xu) = 0 \quad \text{in } \mathbb{R}^N \times (0,T)\,, \tag{3.9}$$

where $H : \mathbb{R}^N \times [0,T] \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}$ is a continuous function, u_t denotes the time-derivative of u and $D_x u$ is the derivative with respect to the space variables x. Of course, this equation has to be complemented by an initial data

$$u(x,0) = u_0(x) \quad \text{in } \mathbb{R}^N \, . \tag{3.10}$$

We provide comparison results in the two cases we already consider above, namely the Lipschitz case and the convex case, the later one allowing more general Hamiltonians coming from unbounded control problems.

Our result is the following

Theorem 3.2.6 (Comparison for the Lipschitz case) *Let $USCS(\mathbb{R}^N \times [0,T])$ be the set of bounded usc subsolution u of (3.9) such that $u(x,0) \leq u_0(x)$ in $\mathbb{R}^N$ and $LSCS(\mathbb{R}^N \times [0,T])$ is the set of bounded lsc supersolutions v of (3.9) such that $u_0(x) \leq v(x,0)$ in $\mathbb{R}^N$. Under Assumption $(\mathbf{H_{BA-HJ}})$, there exists a (GCR-evol) for sub and supersolutions of (3.9)-(3.10) in $USCS(\mathbb{R}^N \times [0,T])$ and $LSCS(\mathbb{R}^N \times [0,T])$ respectively.*

Proof — We just sketch it since it is the standard comparison proof that we recast in a little unusual way.

By the argument of the previous section, it suffices to prove (LCR-evol). Therefore, we argue in $\overline{Q^{\bar x,\bar t}_{r,h}}$ for some $\bar x \in \mathbb{R}^N$, $0 < \bar t < T$, $r, h > 0$ and we assume that $\max_{\overline{Q^{\bar x,\bar t}_{r,h}}}(u-v) > 0$ where $u \in USCS(Q^{\bar x,\bar t}_{r,h})$, $v \in LSCS(Q^{\bar x,\bar t}_{r,h})$. It is worth pointing out that, in $\overline{Q^{\bar x,\bar t}_{r,h}}$, taking into account the fact that u and v are bounded, we have fixed constants and modulus in $(\mathbf{H}_{BA-HJ})$ (that we denotes below by C_1, γ and m). Moreover, we can assume w.l.o.g. that $\gamma > 0$ through the classical change $u(x,t) \to \exp(Kt)u(x,t)$, $v(x,t) \to \exp(Kt)v(x,t)$ for some lrge enough constant K.

We argue by contradiction, assuming that

$$\max_{\overline{Q^{\bar x,\bar t}_{r,h}}}(u-v) > \max_{\partial_p Q^{\bar x,\bar t}_{r,h}}(u-v) \, ,$$

and we introduce the classical doubling of variable

$$(x,t,y,s) \mapsto u(x,t) - v(y,s) - \frac{|x-y|^2}{\varepsilon^2} - \frac{|t-s|^2}{\varepsilon^2} \, .$$

By classical arguments, this usc function has a maximum point at $(x_\varepsilon, t_\varepsilon, y_\varepsilon, s_\varepsilon)$ with $(x_\varepsilon, t_\varepsilon), (y_\varepsilon, s_\varepsilon) \in Q^{\bar{x},\bar{t}}_{r,h}$ and

$$u(x_\varepsilon, t_\varepsilon) - v(y_\varepsilon, s_\varepsilon) \to \max_{\overline{Q^{\bar{x},\bar{t}}_{r,h}}}(u - v) \quad \text{and} \quad \frac{|x_\varepsilon - y_\varepsilon|^2}{\varepsilon^2} + \frac{|t_\varepsilon - s_\varepsilon|^2}{\varepsilon^2} \to 0 \; .$$

It remains to write the viscosity inequalities which reads

$$a_\varepsilon + H(x_\varepsilon, t_\varepsilon, u(x_\varepsilon, t_\varepsilon), p_\varepsilon) \leq 0 \quad \text{and} \quad a_\varepsilon + H(y_\varepsilon, s_\varepsilon, v(y_\varepsilon, s_\varepsilon), p_\varepsilon) \geq 0 \; ,$$

with

$$a_\varepsilon = \frac{2(t_\varepsilon - s_\varepsilon)}{\varepsilon^2} \quad \text{and} \quad p_\varepsilon = \frac{2(x_\varepsilon - y_\varepsilon)}{\varepsilon^2} \; .$$

Subtracting the two inequalities, we obtain

$$H(x_\varepsilon, t_\varepsilon, u(x_\varepsilon, t_\varepsilon), p_\varepsilon) - H(y_\varepsilon, s_\varepsilon, v(y_\varepsilon, s_\varepsilon), p_\varepsilon) \leq 0 \; ,$$

that we can write as

$$[H(x_\varepsilon, t_\varepsilon, u(x_\varepsilon, t_\varepsilon), p_\varepsilon) - H(x_\varepsilon, t_\varepsilon, v(x_\varepsilon, t_\varepsilon), p_\varepsilon)] \leq [H(x_\varepsilon, t_\varepsilon, v(x_\varepsilon, t_\varepsilon), p_\varepsilon) - H(y_\varepsilon, s_\varepsilon, v(y_\varepsilon, s_\varepsilon), p_\varepsilon)] \; .$$

It remains to apply $(\mathbf{H}_{BA-HJ})$

$$\gamma(u(x_\varepsilon, t_\varepsilon), p_\varepsilon) - v(x_\varepsilon, t_\varepsilon)) - C_1(|x_\varepsilon - y_\varepsilon| + |t_\varepsilon - s_\varepsilon|)|p_\varepsilon| - m(|x_\varepsilon - y_\varepsilon| + |t_\varepsilon - s_\varepsilon|) \leq 0 \; .$$

But, as $\varepsilon \to 0$, $m(|x_\varepsilon - y_\varepsilon| + |t_\varepsilon - s_\varepsilon|) \to 0$ since $|x_\varepsilon - y_\varepsilon| + |t_\varepsilon - s_\varepsilon| = o(\varepsilon)$ and

$$(|x_\varepsilon - y_\varepsilon| + |t_\varepsilon - s_\varepsilon|)|p_\varepsilon| = \frac{2|x_\varepsilon - y_\varepsilon|^2}{\varepsilon^2} + \frac{2|t_\varepsilon - s_\varepsilon||x_\varepsilon - y_\varepsilon|}{\varepsilon^2} \to 0 \; .$$

Therefore we have a contradiction for ε small enough since

$$\gamma(u(x_\varepsilon, t_\varepsilon), p_\varepsilon) - v(x_\varepsilon, t_\varepsilon)) \to \gamma \max_{\overline{Q^{\bar{x},\bar{t}}_{r,h}}}(u - v) > 0 \; .$$

And the proof is complete.

Q.E.D.

It is worth pointing out the simplifying effect of the localization argument in this proof: the core of the proof becomes far simpler since we do have to handle several penalization terms at the same time (the ones for the doubling of variables and the localisation ones).

We have formulated and proved Theorem 3.2.6 in a classical way and in a way which is consistent with the previous sections but in this Lipschitz framework, we may have the stronger result based on *a finite speed of propagation type phenomena* which we present here since it follows from very similar arguments

Theorem 3.2.7 (Finite speed of propagation) *Let $USCS(\mathbb{R}^N \times [0,T])$ be the set of locally bounded usc subsolution u of (3.9) and $LSCS(\mathbb{R}^N \times [0,T])$ is the set of locally bounded bounded lsc supersolutions v of (3.9). Assume that $(\mathbf{H}_{BA-HJ})$ holds with $\gamma(R)$ independent of R; if $u \in USCS(\mathbb{R}^N \times [0,T])$ and $v \in LSCS(\mathbb{R}^N \times [0,T])$ satisfy $u(x,0) \leq v(x,0)$ for $|x| \leq R$ for some $R > 0$, then*

$$u(x,t) \leq v(x,t) \quad \text{for } |x| \leq R - C_2 t \ ,$$

where C_2 is given by $(\mathbf{H}_{BA-HJ})$.

Proof — Let $\chi : (-\infty, R) \to \mathbb{R}$ be a smooth function such that $\chi(t) \equiv 0$ if $t \leq 0$, χ is increasing on $\mathbb{R}$ and $\chi(t) \to +\infty$ when $t \to R^-$. We set

$$\psi(x,t) := \exp(-|\gamma| t)\chi(|x| + C_2 t) \ .$$

This function is well-defined in $\mathcal{C} := \{(x,t) : \ |x| + C_2 t \leq R\}$.

We claim that, for $0 < \alpha \ll 1$, the function $u_\alpha(x,t) := u(x,t) - \alpha\psi(x,t)$ in a subsolution of (3.9) in $\mathcal{C}$ and satisfies $u_\alpha(x,t) \to -\infty$ if $(x,t) \to \partial\mathcal{C} \cap \{t > 0\}$ and $u_\alpha(x,0) \leq u(x,0)$ for $|x| \leq R$.

The second part of the claim is obvious by the properties of ψ. To prove the first one, we first compute formally

$$(u_\alpha)_t + H(x,t,u_\alpha, D_x u_\alpha) \leq u_t + H(x,t,u,D_x u) - \alpha(\psi_t - |\gamma|\psi - C_2|D_x\psi|) \ .$$

But an easy -again formal- computation shows that $\psi_t - |\gamma|\psi - C_2|D_x\psi| \geq 0$ in $\mathcal{C}$ and since the justification of these formal computations is straightforward, the claim is proved.

The rest of the proof consists in comparing u_α and v in $\mathcal{C}$, which follows from the same arguments as in the proof of Theorem 3.2.6.

Q.E.D.

Now we turn to the *convex case* where we may have some more general behavior for H and in particular no Lipschitz continuity in p. To simplify the exposure, we do not formulate the assumption in full generality but in the most readable way (at least, we hope so!)

$(\mathbf{H_{BA-Conv}})$ $H(x,t,r,p)$ is a locally Lipschitz function which is convex in (r,p). Moreover, for any ball $B \subset \mathbb{R}^N \times [0,T]$, for any $R > 0$, there exists constants $L(B,R), K(B,R) > 0$ and a function $G(B,R) : \mathbb{R}^N \to [1,+\infty[$ such that, for any $x,y \in B$, $t,s \in [0,T]$, $-R \leq u \leq v \leq R$ and $p \in \mathbb{R}^N$

$$D_p H(x,t,r,p) \cdot p - H(x,t,u,p) \geq G(B,R)(p) - L(B,R) \ ,$$

$$|D_x H(x,t,r,p)|, |D_t H(x,t,r,p)| \leq K(B,R)G(B,R)(p)(1+|p|) \;,$$
$$D_r H(x,t,r,p) \geq 0 \;.$$

On the hand, we assume the existence of a subsolution

($\mathbf{H_{Sub-HJ}}$) There exists an C^1-function $\psi : \mathbb{R}^N \times [0,T] \to \mathbb{R}$ which is a subsolution of (3.9) and which satisfies $\psi(x,t) \to -\infty$ as $|x| \to +\infty$, uniformly for $t \in [0,T]$ and $\psi(x,0) \leq u_0(x)$ in $\mathbb{R}^N$.

The result is

Theorem 3.2.8 (Comparison for the Convex case) *Assume* ($\mathbf{H}_{BA-HJ-U}$) *and* ($\mathbf{H_{Sub-HJ}}$). *Let* $USCS(\mathbb{R}^N \times [0,T])$ [*resp.* $LSCS(\mathbb{R}^N \times [0,T])$] *be the set of bounded usc subsolution* u *of* (3.9) [*resp. the set of bounded lsc supersolutions* v *of* (3.9)] *such that*

$$\limsup_{|x|\to+\infty} \frac{u(x,t)}{\psi(x,t)} \geq 0 \quad \left[\textit{resp.}\ \liminf_{|x|\to+\infty} \frac{v(x,t)}{\psi(x,t)} \leq 0\right] \quad \textit{uniformly for } t \in [0,T] \;.$$

Then there exists a (GCR-evol) for sub and supersolutions of (3.9)-(3.10) *in* $USCS(\mathbb{R}^N \times [0,T])$ *and* $LSCS(\mathbb{R}^N \times [0,T])$ *respectively.*

Proof — The first step consists as above in replacing u by $u_\alpha := (1-\alpha)u + \alpha\psi$ for $0 < \alpha \ll 1$. The convexity of $H(x,t,r,p$ in (r,p) implies that u_α is still a subsolution of (3.9) and $u_\alpha(x,0) \leq u_0(x)$ in $\mathbb{R}^N$. Moreover, by the definition of $USCS(\mathbb{R}^N \times [0,T])$ and $LSCS(\mathbb{R}^N \times [0,T])$,

$$\lim(u_\alpha(x,t) - v(x,t)) = -\infty \quad \text{as } |x| \to +\infty, \text{ uniformly for } t \in [0,T].$$

Therefore the subsolution ψ plays its localization role.

For (LCR-evol), we argue exactly in the same way as in the proof of Theorem 3.2.6 in $Q^{\bar{x},\bar{t}}_{r,h}$ (and therefore with fixed contants L, K and a fixed function G) but with the following preliminary reductions: changing u, v in $u(x,t) + Lt$ and $v(x,t) + Lt$, we may assume that $L = 0$. Finally we perform the Kruzkov's change of variable

$$\tilde{u}(x,t) := -\exp(-u(x,t)) \quad , \quad \tilde{v}(x,t) := -\exp(-v(x,t)) \;.$$

The function $\tilde{u}, \tilde{v}$ are respectively sub and supersolution of

$$w_t + \tilde{H}(x,t,w,Dw) = 0 \quad \text{in } Q^{\bar{x},\bar{t}}_{r,h} \;,$$

with $\tilde{H}(x,t,r,p) = -rH(x,t,-\log(-r),-p/r)$.

Computing $D_r\tilde{H}(x,t,r,p)$, we find $(D_pH\cdot p - H)(x,t,-\log(-r),-p/r)) \geq G(-p/r)$, while $D_x\tilde{H}(x,t,r,p)$, $D_t\tilde{H}(x,t,r,p)$ are estimated by $|r|D_xH(x,t,-\log(-r),-p/r)$, $|r|D_xH(x,t,-\log(-r),-p/r)$, i.e. by $|r|KG(-p/r)(1+|p/r|)$.

Following the proof of Theorem 3.2.6, we have to examine an inequality like

$$\tilde{H}(x_\varepsilon,t_\varepsilon,\tilde{u}(x_\varepsilon,t_\varepsilon),p_\varepsilon) - \tilde{H}(y_\varepsilon,s_\varepsilon,\tilde{v}(y_\varepsilon,s_\varepsilon),p_\varepsilon) \leq 0 \, .$$

To do so, we argue as if $\tilde{H}$ was C^1 (the justification is easy by a standard approximation argument) and we introduce the function

$$f(\mu) := \tilde{H}(\mu x_\varepsilon + (1-\mu)y_\varepsilon, \mu t_\varepsilon + (1-\mu)s_\varepsilon, \mu\tilde{u}(x_\varepsilon,t_\varepsilon) + +(1-\mu)\tilde{v}(y_\varepsilon,s_\varepsilon),p_\varepsilon) \, ,$$

which is defined on $[0,1]$. The above inequality reads $f(1)-f(0)\leq 0$ while

$$f'(\mu) = D_x\tilde{H}.(x_\varepsilon - y_\varepsilon) + D_t\tilde{H}.(t_\varepsilon - s_\varepsilon) + D_r\tilde{H}.(\tilde{u}(x_\varepsilon,t_\varepsilon) - \tilde{v}(y_\varepsilon,s_\varepsilon)) \, ,$$

where all the $\tilde{H}$ derivatives are computed at the point

$$(\mu x_\varepsilon + (1-\mu)y_\varepsilon, \mu t_\varepsilon + (1-\mu)s_\varepsilon, \mu\tilde{u}(x_\varepsilon,t_\varepsilon) + (1-\mu)\tilde{v}(y_\varepsilon,s_\varepsilon),p_\varepsilon) \, .$$

If we denote by $r_\varepsilon = \mu\tilde{u}(x_\varepsilon,t_\varepsilon) + (1-\mu)\tilde{v}(y_\varepsilon,s_\varepsilon)$, we have, by the above estimates,

$$\begin{aligned} f'(\mu) \geq & -|r_\varepsilon|KG(-p_\varepsilon/r_\varepsilon)(1+|p_\varepsilon/r_\varepsilon|)(|x_\varepsilon - y_\varepsilon| + |t_\varepsilon - s_\varepsilon|) \\ & + G(-p_\varepsilon/r_\varepsilon).(\tilde{u}(x_\varepsilon,t_\varepsilon) - \tilde{v}(y_\varepsilon,s_\varepsilon)) \\ \geq & G(-p_\varepsilon/r_\varepsilon)\Big[-K(|r_\varepsilon| + |p_\varepsilon)(|x_\varepsilon - y_\varepsilon| + |t_\varepsilon - s_\varepsilon|) + (\tilde{u}(x_\varepsilon,t_\varepsilon) - \tilde{v}(y_\varepsilon,s_\varepsilon))\Big] \, . \end{aligned}$$

But if $M := \max_{\overline{Q^{\bar{x},\bar{t}}_{r,h}}}(\tilde{u}-\tilde{v}) > 0$, the arguments of the proof of Theorem 3.2.6 show that the bracket is larger than $M/2$ if ε is small enough. Therefore $f'(\mu) \geq M/2 > 0$, a contradiction with $f(1)-f(0)\leq 0$.

Q.E.D.

We conclude this part by an application of Theorem 3.2.6 and 3.2.8 to the equation

$$u_t + a(x,t)|D_xu|^q - b(x,t)\cdot D_xu = f(x,t) \quad \text{in } \mathbb{R}^N \times (0,T) \, ,$$

where a,b,f are at least continuous function in $\mathbb{R}^N \times [0,T]$ and $q \geq 1$.

Of course, Theorem 3.2.6 applies if $q=1$ and a,b are locally Lipschitz continuous functions and f is a uniformly continuous function on $\mathbb{R}^N \times [0,T]$.

Theorem 3.2.8 is concerned with the case $q > 1$ and $a(x,t) \geq 0$ in $\mathbb{R}^N \times [0,T]$ in order to have a convex Hamiltonian.

Next the computation gives

$$D_pH(x,t,r,p) \cdot p - H(x,t,u,p) = a(x,t)(q-1)|p|^q - b(x,t) \cdot p + f(x,t) \; .$$

and in order to verify $(\mathbf{H}_{BA-HJ-U})$, we have to reinforce the convexity assumption by assuming $a(x,t) > 0$ in $\mathbb{R}^N \times [0,T]$. If B is a ball in $\mathbb{R}^N \times [0,T]$, we set $m(B) = \min_B a(x,t)$ and we have, using Young's inequality

$$D_pH(x,t,r,p) \cdot p - H(x,t,u,p) = m(B)(q-1)|p|^q + 1 - L(B) \; .$$

Here the "+1" is just a cosmetic term to be able to set $G(B,R)(p) := m(B)(q-1)|p|^q + 1 \geq 1$ and $L(B)$ is a constant depending on the L^∞-norm of b and f on B.

Finally, a, b, f being locally Lipschitz continuous, it is clear enough that the estimates on $|D_xH(x,t,r,p)|, |D_tH(x,t,r,p)|$ hold. It is worth pointing out that the behavior at infinity of a, b, f does not play any role since we have the arguments of the comparison proof are local. But, of course, we do not pretend that this strategy of proof is optimal...

The checking of $(\mathbf{H_{Sub-HJ}})$ is more "example-dependent" and we are not going to try to find "good frameworks". If $b = 0$ and if there exists $\eta >$ such that

$$\eta \leq a(x,t) \leq \eta^{-1} \quad \text{in } \mathbb{R}^N \times (0,T) \, ,$$

the Oleinik-Lax Formula suggests subsolutions of the form

$$\psi(x,t) = -\alpha(t+1)(|x|^{q'} + 1) - \beta \; ,$$

where q' is the conjugate exponent of q, i.e. $\dfrac{1}{q} + \dfrac{1}{q'} = 1$ and α, β are large enough constants. Indeed

$$\psi_t + a(x,t)|D_x\psi|^q - f(x,t) \leq -\alpha(|x|^{q'} + 1) + \eta^{-1}[q'\alpha(t+1)]^q|x|^{q'} - f(x,t) \; .$$

If there exists $c > 0$ such that

$$f(x,t) \geq -c(|x|^{q'} + 1) \quad \text{in } \mathbb{R}^N \times (0,T) \, ,$$

then, for large α, namely $\alpha > \eta^{-1}[q'\alpha]^q + c$, one has a subsolution BUT only on a short time interval $[0,\tau]$. Therefore one has a comparison result if, in addition, the initial data satisfies for some $c' > 0$

$$u_0(x) \geq -c'(|x|^{q'} + 1) \quad \text{in } \mathbb{R}^N \, ,$$

in which case, we should also have $\alpha > c'$.

In good cases, the comparison result on $[0,\tau]$ can be iterated on $[\tau, 2\tau]$, $[2\tau, 3\tau]$,…,etc to get a full result on $[0,T]$.

3.3 Whitney's stratifications: a good framework for Hamilton-Jacobi Equations

In this section, we introduce the notion of Whitney stratification (based on the Whitney conditions found in [122, 121]). This yields a well-adapted structure to deal with the general discontinuities we are considering in this book. We first do it in the case of a flat stratification; the non-flat case is reduced to the flat one by suitable local charts.

3.3.1 Admissible Flat Stratification

We consider here the stratification introduced in Bressan and Hong [36] but in the case when the different embedded submanifolds of $\mathbb{R}^N$ are locally affine subspace of $\mathbb{R}^N$. More precisely

$$\mathbb{R}^N = \mathbf{M}^0 \cup \mathbf{M}^1 \cup \cdots \cup \mathbf{M}^N ,$$

where the $\mathbf{M}^k$ ($k = 0..N$) are disjoint submanifolds of $\mathbb{R}^N$. We say that $\mathbb{M} = (\mathbf{M}^k)_{k=0..N}$ is an *Admissible Flat Stratification* (AFS in short) if the following set of hypotheses $(\mathbf{H_{ST}})_{flat}$ is satisfied

(i) For any $x \in \mathbf{M}^k$, there exists $r > 0$ and V_k a k-dimensional linear subspace of $\mathbb{R}^N$ such that $B(x, r) \cap \mathbf{M}^k = B(x, r) \cap (x + V_k)$. Moreover $B(x, r) \cap \mathbf{M}^l = \emptyset$ if $l < k$.

(ii) If $\mathbf{M}^k \cap \overline{\mathbf{M}^l} \neq \emptyset$ for some $l > k$ then $\mathbf{M}^k \subset \overline{\mathbf{M}^l}$.

(iii) We have $\overline{\mathbf{M}^k} \subset \mathbf{M}^0 \cup \mathbf{M}^1 \cup \cdots \cup \mathbf{M}^k$.

We first notice that Condition $(\mathbf{H_{ST}})_{flat}$-(i) implies that the set $\mathbf{M}^0$, if not void, consists of isolated points. Indeed, in the case $k = 0$, $V_k = \{0\}$.

Before providing comments on the difference between the assumptions $(\mathbf{H_{ST}})_{flat}$ and the ones used in Bressan & Hong [36], we consider the simplest relevant example of a flat stratification.

Example 3.3.1 *We consider in $\mathbb{R}^2$ a chessboard-type configuration, see Figure 3.1. In this case, we have the following decomposition:*

$$\mathbf{M}^0 = \mathbb{Z} \times \mathbb{Z} ,$$

$$\mathbf{M}^1 = [(\mathbb{R} \setminus \mathbb{Z}) \times \mathbb{R}] \cup [\mathbb{R} \times (\mathbb{R} \setminus \mathbb{Z})] ,$$

and $\mathbf{M}^2 = \mathbb{R}^2 \setminus (\mathbf{M}^0 \cup \mathbf{M}^1)$. In this simple case, the checking of the $(\mathbf{H_{ST}})_{flat}$-assumptions is straightforward.

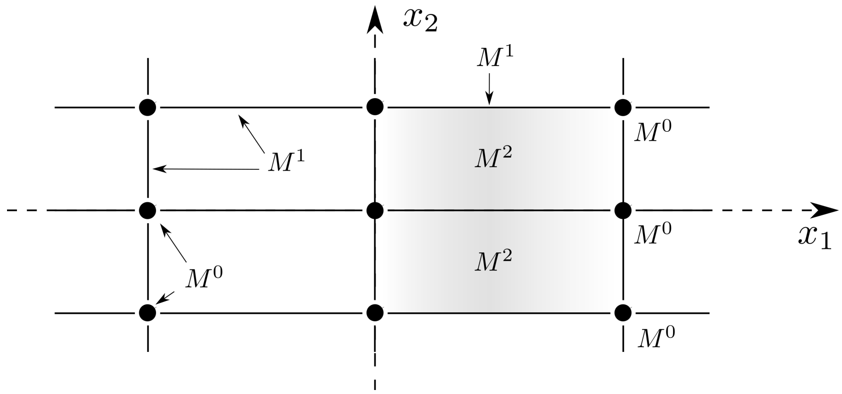


Figure 3.1: The chessboard-type configuration

We point out that, even if the formulation is slightly different, Assumptions $(\mathbf{H_{ST}})_{flat}$ are equivalent (for the flat case) to the assumptions of Bressan & Hong [36]. Indeed, we both assume that we have a partition of $\mathbb{R}^N$ with disjoints submanifolds but we define a different way the submanifolds $\mathbf{M}^k$. The key point is that for us $\mathbf{M}^k$ is here a k-dimensional submanifold while, in [36], the $\mathbf{M}^j$ can be of any dimension. In other words, *our* $\mathbf{M}^k$ is the union of all submanifolds of dimension k in the stratification of Bressan & Hong.

With this in mind it is easier to see that our assumptions $(\mathbf{H_{ST}})_{flat}$-(ii)-(iii) are equivalent to the following assumption of Bressan and Hong: if $\mathbf{M}^k \cap \overline{\mathbf{M}^l} \neq \emptyset$ then $\mathbf{M}^k \subset \overline{\mathbf{M}^l}$ for all indices l, k without asking $l > k$ in our case. But according to the last part of $(\mathbf{H_{ST}})_{flat}$-(i), $\mathbf{M}^k \cap \overline{\mathbf{M}^l} = \emptyset$ if $l < k$: indeed for any $x \in \mathbf{M}^k$, there exists $r > 0$ such that $B(x,r) \cap \mathbf{M}^l = \emptyset$. This property clearly implies $(\mathbf{H_{ST}})_{flat}$-(iii).

In order to be more clear let us consider a stratification in $\mathbb{R}^3$ induced by the upper half-plane $\{x_3 > 0, x_2 = 0\}$ and the x_2-axis (see figure 1. below).

The stratification we use in this case requires first to set $\mathbf{M}^2 = \{x_3 > 0, x_2 = 0\}$. By $(\mathbf{H_{ST}})_{flat}$-(iii), the boundary of $\mathbf{M}^2$ which is the x_1-axis is included in $\mathbf{M}^1 \cup \mathbf{M}^0$ and we also have x_2-axis in the stratification. Hence, $\mathbf{M}^1 \cup \mathbf{M}^0$ contains the cross which is formed by the x_1 and x_2-axis and in order for $\mathbf{M}^1$ to be a manifold, $(0,0,0)$ has to be excluded and we have to set here $\mathbf{M}^0 = \{(0,0,0)\}$. Thus, $\mathbf{M}^1$ consists of four connected components which are induced by the x_1- and x_2-axis (but excluding the origin, which is in $\mathbf{M}^0$). Notice that in this situation, the x_3-axis has no particular status, it is included in $\mathbf{M}^2$.

On the other hand, notice that $(\mathbf{H_{ST}})_{flat}$-(ii) FORBIDS the following decomposi-

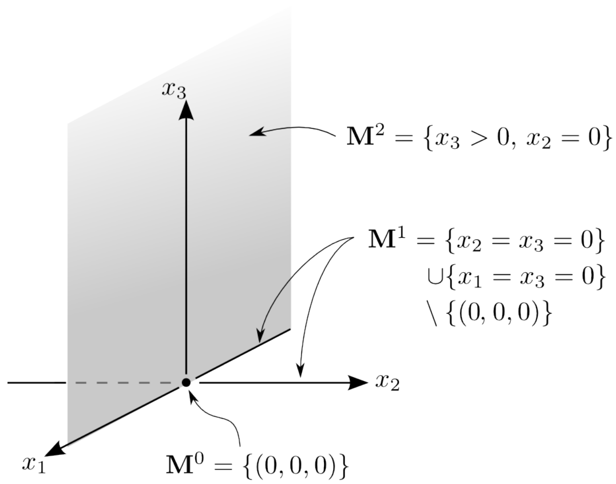


Figure 3.2: Example of a 3-D stratification

tion of $\mathbb{R}^3$

$$\mathbf{M}^2 = \{x_3 > 0, x_2 = 0\},\ \mathbf{M}^1 = \{x_1 = x_3 = 0\} \cup \{x_2 = x_3 = 0\},\ \mathbf{M}^3 = \mathbb{R}^3 - \mathbf{M}^2 - \mathbf{M}^1,$$

because $(0,0,0) \in \mathbf{M}^1 \cap \overline{\mathbf{M}^2}$ but clearly $\mathbf{M}^1$ is not included in $\overline{\mathbf{M}^2}$.

As a consequence of this definition we have following result which will be usefull in a tangential regularization procedure (see Figure 2 below)

Lemma 3.3.2 *Let $\mathbb{M} = (\mathbf{M}^k)_{k=0..N}$ be an $(\mathbf{H_{ST}})_{flat}$ of $\mathbb{R}^N$, let x be in $\mathbf{M}^k$ and r, V_k as in $(\mathbf{H_{ST}})_{flat}$-(i) and $l > k$. Then there exists $r' \leq r$ such that, if $B(x,r') \cap \mathbf{M}^l \neq \emptyset$, then for any $y \in B(x,r') \cap \mathbf{M}^l$, $B(x,r') \cap (y + V_k) \subset B(x,r') \cap \mathbf{M}^l$.*

Proof — We first consider the case when $l = k+1$. We argue by contradiction assuming that there exists $z \in B(x,r') \cap (y+V_k)$, $z \notin \mathbf{M}^{k+1}$. We consider the segment $[y,z] = \{ty + (1-t)z,\ t \in [0,1]\}$. There exists $t_0 \in [0,1]$ such that $x_0 := t_0 y + (1-t_0)z \in \overline{\mathbf{M}^{k+1}} - \mathbf{M}^{k+1}$. But because of the $(\mathbf{H_{ST}})_{flat}$ conditions, $\overline{\mathbf{M}^{k+1}} - \mathbf{M}^{k+1} \subset \mathbf{M}^k$ since no point of $\mathbf{M}^0, \mathbf{M}^1, \cdots \mathbf{M}^{k-1}$ can be in the ball. Therefore x_0 belongs to some $\mathbf{M}^k$, a contradiction since $B(x,r) \cap \mathbf{M}^k = B(x,r) \cap (x+V_k)$ which would imply that $y \in \mathbf{M}^k$.

For $l > k+1$, we argue by induction. If we have the result for l, then we use the same proof as above if $y \in \mathbf{M}^{l+1}$: there exists $z \in B(x,r') \cap (y+V_k)$, $z \notin \mathbf{M}^{l+1}$ and we build in a similar way $x_0 \in \overline{\mathbf{M}^{l+1}} - \mathbf{M}^{l+1} = \mathbf{M}^l$. But this is again a contradiction with the fact that the result holds for l; indeed $x_0 \in \mathbf{M}^l$ and $y \in x_0 + V_k \in \mathbf{M}^{l+1}$.

Q.E.D.

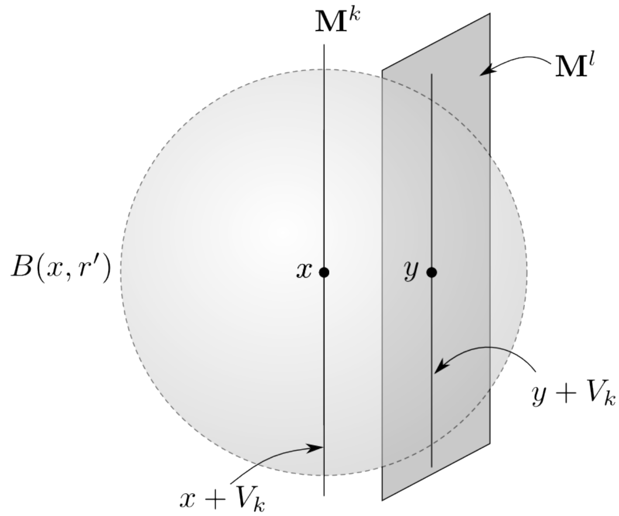


Figure 3.3: local situation

Remark 3.3.3 *In this flat situation, the tangent space at x is $T_x := x+V_k$ while the tangent space at y is $T_y := y+V_l$, where $l > k$. The previous lemma implies that if $(y_n)_n$ is a sequence converging to x, then the limit tangent plane of the T_{y_n} is $x+V_l$ and it contains T_x, which is exactly the Whitney condition —see* [122, 121].

3.3.2 General Regular Stratification

Definition 3.3.4 *We say that $\mathbb{M} = (\mathbf{M}^k)_{k=0..N}$ is a general regular stratification of $\mathbb{R}^N$ (RS in short) if it satisfies the following assumption $(\mathbf{H_{ST}})_{reg}$*

(i) the following decomposition holds: $\mathbb{R}^N = \mathbf{M}^0 \cup \mathbf{M}^1 \cup \cdots \cup \mathbf{M}^N$;

(ii) for any $x \in \mathbb{R}^N$, there exists $r = r(x) > 0$ and a $C^{1,1}$-change of coordinates $\Psi^x : B(x,r) \to \mathbb{R}^N$ such that the $\Psi^x(\mathbf{M}^k \cap B(x,r))$ form an $(\mathbf{H_{ST}})_{flat}$ in $\Psi^x(B(x,r))$.

Remark 3.3.5 *If we need to be more specific, we also say that $(\mathbb{M}, \Psi)$ is a stratification of $\mathbb{R}^N$, keeping the reference Ψ for the collection of changes of variable $(\Psi^x)_x$. This will be usefull in Section 15 when we consider sequences of stratifications.*

The definition of regular stratifications (flat or not) allows to define, for each $x \in \mathbf{M}^k$, the tangent space to $\mathbf{M}^k$ at x, denoted by $T_x\mathbf{M}^k$, which can be identified to $\mathbb{R}^k$. Then, if $x \in \mathbf{M}^k$ and if $r > 0$ and V_k are as in $(\mathbf{H_{ST}})_{flat}$-(i), we can decompose $\mathbb{R}^N = V_k \oplus V_k^\perp$, where $V_k^\perp$ is the orthogonal space to V_k and for any $p \in \mathbb{R}^N$ we have $p = p_\top + p_\perp$ with $p_\top \in V_k$ and $p_\perp \in V_k^\perp$. In the special case $x \in \mathbf{M}^0$, we have $V_0 = \{0\}$, $p = p_\perp$ and $T_x\mathbf{M}^0 = \{0\}$.

The notion of stratification is introduced above as a pure geometrical tool and it remains to connect it with the singularities of Hamilton-Jacobi Equations. In fact, our aim is to define below the “natural framework” to treat Hamilton-Jacobi Equations (or control problems) with discontinuities and this will involve two types of informations: some conditions on the kinds of singularities we can handle and assumptions on the Hamiltonians in a neighborhood of these singularities.

We provide here a first step in this direction by considering the simple example of an equation set in the whole space $\mathbb{R}^N$

$$H(x, u, Du) = 0 \quad \text{in } \mathbb{R}^N ,$$

where the Hamiltonian H has some discontinuities (in the x-variable) located on some set $\Gamma \subset \mathbb{R}^N$. The first question is : what kind of sets Γ can be handled?

The answer is : we always assume below that Γ provides a stratification $\mathbb{M} = (\mathbf{M}^k)_{k=0..N}$ of $\mathbb{R}^N$, which means that $\mathbf{M}^N$ is the open subset of $\mathbb{R}^N$ where H is continuous and $\mathbf{M}^{N-k}$ contains the discontinuities of codimension $k \geq 1$; of course, some of the $\mathbf{M}^{N-k}$ can be empty.

What should be done next is to clarify the structure of the Hamiltonian H in a neighborhood of each point $x \in \mathbf{M}^{N-k}$ and for each $k \geq 1$; this is where the previous analysis on stratifications allows to reduce locally the problem to the following situation: if $x \in \mathbf{M}^{N-k}$, there is a ball $B(x, r)$ for some $r > 0$, and a C^1-diffeomorphism Ψ such that

$$B(x, r) \cap \Psi(\mathbf{M}^{N-k}) = B(x, r) \cap \bigcup_{j=0}^{k} \left(x + V_{N-j}\right).$$

This means that through a suitable C^1 change of coordinates, we are in a *flat* situation where x is only possibly “touched” by $N - j$-dimensional vector spaces for $j \leq k$.

In the next section, we will see how this reduction to flat discontinuities allows us to describe the natural assumptions on H (or more precisely on the Hamiltonian obtained after the Ψ-change) which lead to most of our results.

3.4 Partial Regularity, Partial Regularization

In this section, motivated by Section 3.2 and 3.3, we present some key ingredients in the proof of local comparison results for HJ equations with discontinuities. The assumptions we are going to use are those which are needed everywhere in this book to prove any kind of results and therefore we define at the end of the section a "good local framework for HJ Equations with discontinuities".

Local comparison result leads to consider HJ-Equations in a small ball, namely

$$G(x,u,Du) = 0 \quad \text{in } B_\infty(\bar{x},r) \ , \tag{3.11}$$

where $\bar{x} \in \mathbb{R}^N$ and $r > 0$ are fixed. Because of the previous section, it is natural to assume that the equation has a general *flat stratification-type structure*: the variable $x \in \mathbb{R}^N$ can be decomposed as $(y,z) \in \mathbb{R}^k \times \mathbb{R}^{N-k}$ and G is continuous w.r.t. u, p and y but not with respect to z. In other words, locally around $\bar{x}$ we have in mind that there is a discontinuity for G on $\Gamma = \{(y,z);\ z=0\}$ which, locally, can be identified with $\mathbb{R}^k$.

3.4.1 Regularity and Regularization of Subsolutions

The aim of this section is to study subsolutions of (3.11) and to prove that, under suitable assumptions, they satisfy some "regularity properties" and to construct a suitable approximation by Lipschitz continuous subsolutions which are even C^1 in y in the convex case.

We immediately point out that, for reasons which will clear later on in this book, we are not going to use only subsolutions in the Ishii's sense and therefore, we are not going to use only the lower semi-continuous enveloppe of some Hamiltonian as in the Ishii's definition: to simplify matter, we assume here that the function G contains all the necessary informations for subsolutions. In other words, by subsolution of (3.11), we mean an usc function u which satisfies : *at any maximum point $x \in B_\infty(\bar{x},r)$ of $u-\phi$, where ϕ is a smooth test-function, we have*

$$G(x,u(x),D\phi(x)) \leq 0 \ .$$

In the sequel, we decompose Du as $(D_y u, D_z u)$ (the same convention is used for the test-functions ϕ) and the corresponding variable in G will be $p=(p_y,p_z)$.

For G, we use the following assumptions: for any $R>0$, there exists constants $C_i^R > 0$ for $i=1,...,4$, a modulus of continuity $m^R : [0,+\infty[\to [0,+\infty[$ and either a constant $\lambda^R > 0$ or $\mu^R > 0$ such that

(TC) *Tangential Continuity*: for any $x_1=(y_1,z), x_2=(y_2,z) \in B_\infty(\bar{x},r)$, $|u| \leq R$, $p \in \mathbb{R}^N$, then

$$|G(x_1,u,p) - G(x_2,u,p)| \leq C_1^R|y_1-y_2|.|p| + m^R\big(|y_1-y_2|\big) \ .$$

(NC) *Normal Controllability*: for any $x = (y, z) \in B_\infty(\bar{x}, r)$, $|u| \leq R$, $p = (p_y, p_z) \in \mathbb{R}^N$, then
$$G(x, u, p) \geq C_2^R |p_z| - C_3^R |p_y| - C_4^R\,.$$

Notice that **(NC)** and **(TC)** have counterparts in terms of control elements (dynamic, cost), see **(NC-BCL)**, **(TC-BCL)**, p. 227.

In the last one, if $p_y \in \mathbb{R}^k$, we set $p_y = (p_{y_1}, \cdots, p_{y_k})$

(Mon) *Monotonicity*: for any $R > 0$, there exists $\lambda_R, \mu_R \in \mathbb{R}$, such that we have EITHER $\lambda_R > 0$ and for any $x \in B_\infty(\bar{x}, r)$, $p = (p_y, p_z) \in \mathbb{R}^N$,
$$G(x, u_2, p) - G(x, u_1, p) \geq \lambda^R (u_2 - u_1) \tag{3.12}$$
for any $-R \leq u_1 \leq u_2 \leq R$, OR (3.12) holds with $\lambda_R = 0$, we have $\mu_R > 0$ and
$$G(x, u_1, q) - G(x, u_1, p) \geq \mu^R (q_{y_1} - p_{y_1})\,, \tag{3.13}$$
for any $q = (q_y, p_z)$ with $p_{y_1} \leq q_{y_1}$ and $p_{y_i} = q_{y_i}$ for $i = 2, ..., p$.

We say that (**Mon**-u) is satisfied if (3.12) holds and (**Mon**-p) is satisfied if (3.13) holds.

Before providing results using these assumptions, we give an example showing the type of properties are hidden behind these general assumptions for an equation which is written on the form
$$\mu u_t + H((x_1, x_2), t, u, (D_{x_1} u, D_{x_2} u)) = 0 \quad \text{in } \mathbb{R}^k \times \mathbb{R}^{N-k} \times (0, +\infty)\,,$$
which is seen as an equation in $\mathbb{R}^{N+1}$. Here the constant $\mu \in \mathbb{R}$ satisfies where $0 \leq \mu \leq 1$ and in order to simplify we can assume that H is a continuous function. As above, we can write $x = (t, x_1, x_2) \in (0, +\infty) \times \mathbb{R}^k \times \mathbb{R}^{N-k}$ and we set $y = (t, x_1) \in \mathbb{R}^{k+1}$, $z = x_2 \in R^{N-k}$ and
$$G(x, u, P) = \mu p_t + H((x_1, x_2), t, u, (p_{x_1}, p_{x_2}))\,,$$
where $P = (p_t, (p_{x_1}, p_{x_2}))$.

The simplest way to have **(TC)**, **(NC)** and **(Mon)** (with an easy way of checking them!) is to assume that $(x_1, t, u) \mapsto H((x_1, x_2), t, u, (p_{x_1}, p_{x_2}))$ is locally Lipschitz continuous for any x_2, p_{x_1}, p_{x_2} and for **(TC)** that one has
$$|D_{x_1} H((x_1, x_2), t, u, (p_{x_1}, p_{x_2}))|, |D_t H((x_1, x_2), t, u, (p_{x_1}, p_{x_2}))| \leq C_1^R(|(p_{x_1}, p_{x_2})| + 1)\,,$$

when $|u| \leq R$; here we are in the simple case when $m^R(\tau) = C_1^R \tau$ for any $\tau \geq 0$. In fact, one easily check that these assumptions implies the right property for G with $y = (t, x_1)$.

Next since $p_{y_1} = p_t$, **(Mon)** reduces to either $\mu > 0$ or $D_u H((x_1, x_2), t, u, (p_{x_1}, p_{x_2})) \geq \lambda_R > 0$ if $|u| \leq R$. Hence we are either in real time evolution context ($\mu > 0$) or $\mu = 0$ and the standard assumption of having H strictly increasing in u should hold.

Finally **(NC)** holds if H satisfies the following coercivity assumption in p_{x_2}

$$H((x_1, x_2), t, u, (p_{x_1}, p_{x_2})) \geq C_2^R |p_{x_2}| - C_3^R |p_{x_1}| - C_4^R ,$$

if $|u| \leq R$. In fact, in order to check **(NC)** for G, the C_3^R may have to be changed in order to incorporate the μp_t-term.

Before stating our main result, we give the following proposition on the "regularity of subsolutions".

Proposition 3.4.1 *Let u be a bounded subsolution of (3.11) and assume that* **(TC)**,**(NC)** *hold. Then, for any $x = (y, z) \in B_\infty(\bar{x}, r)$*

$$u(x) = \limsup\{u(y', z') \; ; (y', z') \to x, \; z' \neq z\} \; . \tag{3.14}$$

Moreover, if $N - k = 1$, we also have

$$u(x) = \limsup\{u(y', z') \; ; (y', z') \to x, \; z' > z\} = \limsup\{u(y', z') \; ; (y', z') \to x, \; z' < z\} \; . \tag{3.15}$$

This proposition means that the subsolutions cannot have "singular values" on affine subspaces $z = \textit{constant}$ where, by singular values, we mean values which are not the limit of values of these subsolutions outside such affine subspace.

Proof — In order to prove (3.14), we argue by contradiction assuming that

$$u(x) > \limsup\{u(y', z') \; ; (y', z') \to x, \; z' \neq z\} \; .$$

Therefore there exists some δ small enough such that $u(y', z') < u(x) - \delta$ if $|(y', z') - x| < \delta$, with $z' \neq z$. Next, for $\varepsilon > 0$, we consider the function

$$y' \mapsto u(y', z) - \frac{|y - y'|^2}{\varepsilon} \; .$$

If ε is small enough, this function has a local maximum point at y_ε which satisfies $|y_\varepsilon - y| < \delta$ and $u(y_\varepsilon, z) > u(x)$. But because of the above property, there exists a neighborhood $\mathcal{V}$ of (y_ε, z) such that, if $(y', z') \in \mathcal{V}$ and $z' \neq z$, $u(y', z') < u(y_\varepsilon, z) - \delta$.

This implies that (y_ε, z) is also a local maximum point of the function

$$(y', z') \mapsto u(y', z') - \frac{|y-y'|^2}{\varepsilon} - Ce\cdot(z'-z)\ .$$

for any positive constant C and for any unit vector e of $\mathbb{R}^{N-k}$. But, by the subsolution property, we have

$$G\left((y_\varepsilon, z), u(y_\varepsilon, z), \left(\frac{2(y_\varepsilon - y)}{\varepsilon}, Ce\right)\right) \leq 0\ ,$$

which using **(NC)** implies for $R = ||u||_\infty$

$$C_2^R.C - C_3^R\frac{2|y_\varepsilon - y|}{\varepsilon} - C_4^R \leq 0\ ,$$

which is a contradiction if we have chosen C large enough, typically $C = \varepsilon^{-1}$ with ε small enough since $\dfrac{2|y_\varepsilon - y|}{\varepsilon} = o(1)/\varepsilon$.

To prove the second part of the proposition, we remark that, if $N-p=1$, we can choose either $e=+1$ and $e=-1$.

If $u(x) > \limsup\{u(y', z')\ ; (y', z') \to x,\ z' > z\}$, we argue as above but looking at a local maximum point of the function

$$(y', z') \mapsto u(y', z') - \frac{|y-y'|^2}{\varepsilon} + \varepsilon^{-1}(z'-z)\ ,$$

therefore with the choice $e=-1$. We first look at a maximum point of this function in compact set of the form

$$\{(y', z'); |y'-y| + |z'-z| \leq \delta,\ z' \leq z\}\ .$$

Notice that, in this set, the term $\varepsilon^{-1}\cdot(z'-z)$ is negative (therefore it has the right sign) and this function has a local maximum point point at $(y_\varepsilon, z_\varepsilon)$ with $u(y_\varepsilon, z_\varepsilon) \geq u(x)$ by the maximum point property and $(y_\varepsilon, z_\varepsilon) \to x$ as $\varepsilon \to 0$.

Using that $u(x) > \limsup\{u(y', z')\ ; (y', z') \to x,\ z' > z\}$, we clearly have the same property at $(y_\varepsilon, z_\varepsilon)$ and $(y_\varepsilon, z_\varepsilon)$ is also a maximum point of the above function for all (y', z') such that $|y'-y| + |z'-z| \leq \delta$ if δ is chosen small enough. And we reach a contradiction as in the first part of the proof using **(NC)**.

Q.E.D.

Now we turn to our second main result which is the

Proposition 3.4.2 *Let u be a bounded subsolution of* (3.11) *and assume that* **(TC)**,**(NC)** *and* **(Mon)** *hold. Then there exists a sequence of Lipschitz continuous functions $(u^\varepsilon)_\varepsilon$ defined in $B_\infty(\bar{x}, r - a(\varepsilon))$ where $a(\varepsilon) \to 0$ as $\varepsilon \to 0$ such that*

(i) the u^ε are subsolutions of (3.11) *in $B_\infty(\bar{x}, r - a(\varepsilon))$,*

(ii) the u^ε are semi-convex in the y-variable

(iii) $\limsup^* u^\varepsilon = u$ *as $\varepsilon \to 0$.*

Remark 3.4.3 *The equations of the form*

$$\max(u_t + G_1(x, D_x u); G_2(x, u, D_x u)) = 0 \ ,$$

do not satisfy **(Mon)** *even if G_2 satisfies* (**Mon**-u) *and the Hamiltonian $p_t + G_1(x, p_x)$ satisfies* (**Mon**-p). *To overcome this difficulty, we have to use a change of variable of the form $v = \exp(Kt)u$ in order that both Hamiltonians satisfy* (**Mon**-u), *which is a natural change (cf. Section 5.5). Of course, suitable assumptions on G_1 and G_2 are needed in order to have* **(TC)**,**(NC)**.

Proof — First we can drop the R in all the constants appearing in the assumptions by remarking that, u being bounded, we can use the constants with $R = ||u||_\infty$.

In the case, when **(Mon)** holds with $\lambda > 0$ we set for $x = (y, z)$

$$u^\varepsilon(x) := \max_{y' \in \mathbb{R}^k} \Big\{ u(y', z) - \frac{(|y - y'|^2 + \varepsilon^4)^{\alpha/2}}{\varepsilon^\alpha} \Big\},$$

for some (small) $\alpha > 0$ to be chosen later on, while, in the other case we set

$$u^\varepsilon(x) := \max_{y' \in \mathbb{R}^k} \Big\{ u(y', z) - \exp(K y_1) \frac{|y - y'|^2}{\varepsilon^2} \Big\},$$

for some constant K to be chosen later on.

In both cases, the maximum is achieved for some y' such that $|y - y'| \leq O(\varepsilon)$ and therefore u^ε is well-defined (and with a point $(y', z) B_\infty(\bar{x}, r)$ in $B_\infty(\bar{x}, r - a(\varepsilon))$ for $a(\varepsilon) > O(\varepsilon)$). On the other hand, it is clear that the u^ε's are continuous in y AND z by applying Proposition 3.4.1.

To prove that u^ε is a subsolution in $B_\infty(\bar{x}, r - a(\varepsilon))$, we consider a smooth test-function ϕ and we assume that $x \in B_\infty(\bar{x}, r - a(\varepsilon))$ is a maximum point of $u^\varepsilon - \phi$. We first consider the "$\lambda > 0$" case : if

$$u^\varepsilon(x) = u(y', z) - \frac{(|y - y'|^2 + \varepsilon^4)^{\alpha/2}}{\varepsilon^\alpha},$$

then (y', z) is a maximum point of $(\tilde{y}, \tilde{z}) \mapsto u(\tilde{y}, \tilde{z}) - \dfrac{(|y-\tilde{y}|^2+\varepsilon^4)^{\alpha/2}}{\varepsilon^\alpha} - \phi(y, \tilde{z})$, and therefore, by the subsolution property for u

$$G((y', z), u(y', z), (p_y, D_z\phi(y, z))) \leq 0 ;$$

where

$$p_y := \alpha(y'-y)\frac{(|y-y'|^2+\varepsilon^4)^{\alpha/2-1}}{\varepsilon^\alpha} .$$

On the other hand the maximum point property in y, implies that $p_y = D_y\phi(y, z)$.

To obtain the right inequality, we have to replace (y', z) by $x = (y, z)$ in this inequality and $u(y', z)$ by $u^\varepsilon(x)$. To do so, we have to use **(TC)**; in order to do it, we need to have a precise estimate on the term $|y-y'||(p_y, D_z\phi(y, z))|$. The explicit form of p_y gives it for $|y-y'||p_y|$ but this is not the case for $|y-y'|.|D_z\phi(y, z)|$ since we have not such a precise information on $D_z\phi(y, z)$. Instead we have to use **(NC)** which implies

$$C_2|D_z\phi(y, z)| - C_3|p_y| - C_4 \leq 0 .$$

(remember that we have dropped the dependence in R for all the constants). On the other hand, we have combining **(TC)** and **(Mon)**

$$G(x, u^\varepsilon(x), (D_y\phi(y, z), D_z\phi(y, z))) \leq G((y', z), u(y', z), (p_y, D_z\phi(y, z)))+ \\ C_1|y-y'||D\phi(x)| + m(|y-y'|) - \lambda\frac{(|y-\tilde{y}|^2+\varepsilon^4)^{\alpha/2}}{\varepsilon^\alpha} .$$

It remains to estimate the right-hand side of this inequality: we have seen above that $|y-y'| = O(\varepsilon)$ and **(NC)** implies that

$$|D\phi(x)| \leq \bar{K}(|p_y|+1) ,$$

for some large constant $\bar{K}$ depending only on C_2, C_3, C_4. Finally

$$|y-y'||p_y| = \alpha|y'-y|^2\frac{(|y-y'|^2+\varepsilon^4)^{\alpha/2-1}}{\varepsilon^\alpha} \leq \alpha\frac{(|y-\tilde{y}|^2+\varepsilon^4)^{\alpha/2}}{\varepsilon^\alpha} .$$

By taking $\alpha < \bar{K}$, we finally conclude that

$$G(x, u^\varepsilon(x), (D_y\phi(y, z), D_z\phi(y, z)) \leq O(\varepsilon) + m(O(\varepsilon)) ,$$

and changing u^ε in $u^\varepsilon - \lambda^{-1}(O(\varepsilon) + m(O(\varepsilon)))$, we have the desired property.

In the μ-case, the equality $p_y = D_y\phi(y, z)$ is replaced by

$$D_y\phi(y, z) = -K\exp(Ky_1)\frac{|y-y'|^2}{\varepsilon^2}e_1 + \exp(Kt)\frac{(y'-y)}{\varepsilon^2} ,$$

where e_1 is the vector $(1, 0, \cdots, 0)$ in $\mathbb{R}^k$. The viscosity subsolution inequality for u at (y', z) reads

$$G((y', z), u(y', z), (\tilde{p}_y, D_z\phi(y, z)) \leq 0 ,$$

where $\tilde{p}_y = \exp(Kt)\dfrac{(y' - y)}{\varepsilon^2}$.

We first use **(NC)**, which implies

$$|D\phi(x)| \leq \bar{K}(|\tilde{p}_y| + 1) = \bar{K}(\exp(Kt)\frac{|y' - y|}{\varepsilon^2} + 1) .$$

Then we combine **(TC)** and **(Mon)** to obtain

$$G(x, u^\varepsilon(x), (D_y\phi(y, z), D_z\phi(y, z)) \leq G((y', z), u(y', z), (\tilde{p}_y, D_z\phi(y, z))+$$
$$C_1|y - y'||D\phi(x)| + m(|y - y'|) - \mu K \exp(Ky_1)\frac{|y - y'|^2}{\varepsilon^2} .$$

We conclude easily as in the first case choosing K such that $\mu K > C_1\bar{K}$.

Properties *(ii)* and *(iii)* are classical properties which are easy to obtain and we drop the proof.

We conclude this proof by sketching the proof of the Lipschitz continuity of u^ε in z. To do so, we write $\bar{x} = (\bar{y}, \bar{z})$ and for any fixed y such that $|y - \bar{y}| < r - a(\varepsilon)$, we consider the function $z \mapsto u^\varepsilon(y, z)$. By using **(NC)** and the Lipschitz continuity of u^ε in the y-variable, it is easy to prove that this function is a subsolution of

$$C_2|D_z w| \leq C_3 K_\varepsilon + C_4 ,$$

where $K_\varepsilon = ||D_y u^\varepsilon||_\infty$ and the estimates of $D_z u^\varepsilon$ follows.

Q.E.D.

Now we turn to the "convex case", where we use the following assumption

$(\mathbf{H_{Conv}})$: For any $x \in B_\infty(\bar{x}, r)$, the function $(u, p) \mapsto G(x, u, p)$ is convex.

Our result is the

Proposition 3.4.4 *Under the assumptions of Proposition 3.4.2 and if* $(\mathbf{H_{Conv}})$ *hold, then the sequence* $(u^\varepsilon)_\varepsilon$ *of Lipschitz continuous subsolutions of* (3.11) *can be built in such a way that they are* C^1 *(and even* C^∞*) in* y.

Proof — By Proposition 3.4.2, we can assume without loss of generality that u is Lipschitz continuous and to obtain further regularity, we are going to use a standard convolution with a sequence of mollifying kernels but only in the y-variable.

To do so, we introduce a sequence $(\rho_\varepsilon)_\varepsilon$ of positive, C^∞-functions on $\mathbb{R}^k$, ρ_ε having a compact support in $B_\infty(0,\varepsilon)$ and with $\int_{\mathbb{R}^k} \rho_\varepsilon(y)dy = 1$. Then we set, for $x = (y,z) \in B_\infty(\bar{x}, r-\varepsilon)$

$$u^\varepsilon(x) := \int_{|e|_\infty < \varepsilon} u(y-e,z)\rho_\varepsilon(e)de \; .$$

By standard arguments, it is clear that u^ε is smooth in y. Moreover, using **(TC)** and $(\mathbf{H_{Conv}})$, it is easy to show that the u^ε are approximate subsolutions of (3.11), i.e.

$$G(x,u^\varepsilon,Du^\varepsilon) \leq \eta(\varepsilon) \quad \text{in } B_\infty(\bar{x}, r-\varepsilon) \; ;$$

indeed, one can use (for example) an approximation argument, approximating the convolution integral by a Riemann's sum.

To drop the $\eta(\varepsilon)$, we can either consider $u^\varepsilon - \lambda^{-1}\eta(\varepsilon)$ if $\lambda > 0$ or $u^\varepsilon - \mu^{-1}\eta(\varepsilon)y_1$ in the other case, and the proof is complete.

Q.E.D.

3.4.2 And what about regularization for supersolutions?

The previous section shows how to regularize subsolutions and we address here the question: is it possible to do it for supersolutions, changing (of course) the sup-convolution into an inf-convolution?

Looking at the proof of Theorem 3.4.2, the answer is not completely obvious: on one hand, the arguments for an inf-convolution may appear as being analogous but, on the other hand, we use in a key way Assumption **(NC)** which allows to control the derivatives in z of the sup-convolution (or the test-function), *an argument which is, of course, valid only for subsolutions.*

Actually, regularizing a supersolution v of (3.11) –a notion which is defined exactly in the same way as for subsolutions– requires additional assumptions on either v or G. For G, we introduce the following stronger version of **(TC)**

for any $R > 0$, there exists a constants $C_1^R > 0$ and a modulus of continuity $m^R : [0,+\infty[\to [0,+\infty[$ such that

(TC-s) *Strong Tangential Continuity*: for any $x_1 = (y_1,z), x_2 = (y_2,z) \in B_\infty(\bar{x}, r)$, $|u| \leq R$, $p = (p_y,p_z) \in \mathbb{R}^N$, then

$$|G(x_1,u,p) - G(x_2,u,p)| \leq C_1^R|y_1 - y_2|.|p_y| + m^R\big(|y_1 - y_2|\big) \; .$$

We point out that, compared to **(TC)**, the "$|p|$" is replaced by "$|p_y|$". This assumption is typically satisfied by equations of the form

$$G(x,u,p) = G_1(x,u,p_y) + G_2(z,u,p) \ ,$$

since, for G_1, **(TC-s)** reduces to **(TC)** and G_2 readily satisfies **(TC-s)**.

An other possibility is to assume that $v(x) = v((y,z))$ is Lipschitz continuous in z in $B_\infty(\bar{x},r)$, uniformly in y, i.e. there exists a constant $K > 0$ such that, for any $x_1 = (y,z_1), x_2 = (y,z_2) \in B_\infty(\bar{x},r)$

$$|v(x_1) - v(x_2)| \leq K|z_1 - z_2| \ . \tag{3.16}$$

The result for the supersolutions is the

Proposition 3.4.5 *Let v be a bounded subsolution of (3.11) and assume that*
(a) either **(TC-s)** *and* **(Mon)** *hold*
(b) or **(TC)***,* **(Mon)** *and (3.16) hold.*
Then there exists a sequence $(v^\varepsilon)_\varepsilon$ defined in $B_\infty(\bar{x},r-a(\varepsilon))$ where $a(\varepsilon) \to 0$ as $\varepsilon \to 0$ such that

(i) the v^ε are supersolutions of (3.11) in $B_\infty(\bar{x},r-a(\varepsilon))$,

(ii) the v^ε are semi-concave in the y-variable

(iii) $\liminf_* v^\varepsilon = v$ as $\varepsilon \to 0$.

Two remarks on this proposition: first, the proof is readily the same as for subsolutions, the only difference is that we do not need to control the z-derivative in the case (a) because of **(TC-s)** and this derivative is clearly bounded in the case (ii) because of (3.16). The second remark is that, a priori, the v^ε are NOT continuous in z in the case (a). Of course, they are Lipschitz continuous in y and z in the case (b).

Chapter 4

Control tools

Of course, the key ingredients used in this chapter are not new, we just try to revisit them in a more modern way: we refer the reader to the founding article of Filippov [59] and to Aubin and Cellina [6], Aubin and Frankowska [7], Clarke [42], Clarke, Ledyaev, Stern and Wolenski [43] for the classical approach of deterministic control problems by non-smooth analysis methods.

4.1 How to define deterministic control problems with discontinuities?

4.1.1 To the most simple problem with a discontinuity: the two half-spaces problem

As in the basic example of a half-space discontinuity that was introduced in Section 3.1, we consider a partition of $\mathbb{R}^N$ into

$$\mathcal{H} = \{x_N = 0\}\,,\ \Omega_1 = \{x_N > 0\}\,,\ \Omega_2 = \{x_N < 0\}\,,$$

and we assume that we have three different control problems in each of these subsets given by $(b_\mathcal{H}, c_\mathcal{H}, l_\mathcal{H}), (b_1, c_1, l_1), (b_2, c_2, l_2)$. For the sake of simplicity, we can assume that they are all defined on $\mathbb{R}^N \times [0,T] \times A_i$ for $i = \mathcal{H}, 1, 2$ and even that they all satisfy $(\mathbf{H_{BA-CP}})$.

For such problems, the first question consists in defining properly the dynamic since, when the trajectory reaches $\mathcal{H}$, we have a discontinuity in b and the controller may have access to dynamics b_1 and b_2, but also to the specific dynamics $b_\mathcal{H}$. But how? And of course, a similar question holds for the cost and discount factor.

The natural tool consists in using the theory of *differential inclusions* that we introduce on the simple example of the previous section. The idea consists in looking at the set valued map

$$\mathbf{BCL}(x,t) := \{(b(x,t,\alpha), c(x,t,\alpha), l(x,t,\alpha)) \ : \ \alpha \in A\} ,$$

and to solve the differential inclusion

$$(\dot{X}(s), \dot{D}(s), \dot{L}(s)) \in \mathbf{BCL}(X(s), t-s) \ , \ (X,D,L)(0) = (x,0,0) \ ,$$

which only required that the set valued map **BCL** is upper-semicontinuous, with values in compact, convex sets (which is almost satisfied here, at least, adding the assumptions that the $\mathbf{BCL}(x,t)$ are convex or solving with their convex hull). Then

$$\tilde{U}(x,t) = \inf_{(X,D,L)} \left(\int_0^t \dot{L}(s) \exp(D(s)) \, \mathrm{d}s + u_0(X(t)) \exp(D(t)) \right) ,$$

The advantage of this approach is to allow to define the dynamic, discount and cost without any regularity in b, c, l and we are going to define **BCL** in the same way for $x \in \Omega_1$ and $x \in \Omega_2$ by just setting

$$\begin{cases} (b(x,t,\alpha), c(x,t,\alpha), l(x,t,\alpha)) = (b_1(x,t,\alpha_1), c_1(x,t,\alpha_1), l_1(x,t,\alpha_1)) & \text{if } x \in \Omega_1 \\ (b(x,t,\alpha), c(x,t,\alpha), l(x,t,\alpha)) = (b_2(x,t,\alpha_2), c_2(x,t,\alpha_2), l_2(x,t,\alpha_2)) & \text{if } x \in \Omega_2 \end{cases}$$

where $\alpha \in A = A_{\mathcal{H}} \times A_1 \times A_2$, the "extended control space".

For $x \in \mathcal{H}$ and $t \in [0,T]$, we just follow the theory of differential inclusions: by the upper semi-continuity of **BCL**, we necessarely have in $\mathbf{BCL}(x,t)$ all the $(b_i(x,t,\alpha_i), c_i(x,t,\alpha_i), l_i(x,t,\alpha_i))$ for $i = \mathcal{H}, 1, 2$ but we have also to take the convex hull of all these elements, namely all the convex combinations of them. In particular, for the dynamic, we have (a priori) all the $b = \mu_1 b_1 + \mu_2 b_2 + \mu_3 b_{\mathcal{H}}$ such that $\mu_1 + \mu_2 + \mu_3 = 1$, $\mu_i \geq 0$ but we will show that such b play a role only if the trajectory stays on $\mathcal{H}$ and therefore if we have $b \cdot e_N = 0$. A more precise statement will be given in Section 6.1.

4.2 Statement of the Deterministic Control Problem and Dynamic Programming Principle

Based on the ideas that we sketched in last section, we consider a general approach of finite horizon control problems with differential inclusions. We use an *extended* *trajectory* (X, T, D, L) in which we also embed the running time variable T, pointing out that, in the basic example we introduced in the previous section, we just have $T(s) = t - s$.

This framework may seem complicated but we made this choice because it allows us to consider all the applications we have in mind: on one hand, time and space will play analogous role when we will have time-dependent discontinuities or for treating some unbounded control type features and, on the other hand, discount factors will be required when dealing with boundary conditions (see Part 4).

In this part, we present general and classical results which do not require any particular assumption concerning the structure of the discontinuities, nor on the control sets.

In the following, we denote by $\mathcal{P}(E)$ the set of all subsets of E.

4.2.1 Dynamics, discount and costs

The first hypothesis we make is

$(\mathbf{H_{BCL}})_{fund}$: We are given a set-valued map $\mathbf{BCL} : \mathbb{R}^N \times [0, T] \to \mathcal{P}(\mathbb{R}^{N+3})$ satisfying

(i) The map $(x, t) \mapsto \mathbf{BCL}(x, t)$ has compact, convex images and is upper semi-continuous;

(ii) There exists $M > 0$, such that for any $x \in \mathbb{R}^N$ and $t > 0$,

$$\mathbf{BCL}(x, t) \subset \left\{ (b, c, l) \in \mathbb{R}^{N+1} \times \mathbb{R} \times \mathbb{R} : |b| \leq M; |c| \leq M; |l| \leq M \right\},$$

where $| \cdot |$ stands for the usual euclidian norm in any euclidean space $\mathbb{R}^p$ (which reduces to the absolute value in $\mathbb{R}$, for the c and l variables). If $(b, c, l) \in \mathbf{BCL}(x, t)$, b corresponds to the dynamic (in space and time), c to the discount factor and l to the running cost, and Assumption $(\mathbf{H_{BCL}})_{fund}$-$(ii)$ means that dynamics, discount factors and running costs are uniformly bounded. In the following, we sometimes have to consider separately dynamics, discount factors and running costs and to do so, we set

$$\mathbf{B}(x, t) = \left\{ b \in \mathbb{R}^{N+1};\ \text{there exists } c, l \in \mathbb{R} \text{ such that } (b, c, l) \in \mathbf{BCL}(x, t) \right\},$$

and analogously for $\mathbf{C}(x, t), \mathbf{L}(x, t) \subset \mathbb{R}$. Finally, we decompose any $b \in \mathbf{B}(x, t)$ in (b^x, b^t), b^x and b^t being respectively the dynamics in space and time

We recall what upper semi-continuity means here: a set-valued map $x \mapsto F(x)$ is upper-semi continuous at x_0 if for any open set $\mathcal{O} \supset F(x_0)$, there exists an open set ω containing x_0 such that $F(\omega) \subset \mathcal{O}$. In other terms, $F(x) \supset \limsup_{y \to x} F(y)$.

4.2.2 The control problem

We look for trajectories $(X, T, D, L)(\cdot)$ of the following inclusion

$$\begin{cases} \dfrac{\mathrm{d}}{\mathrm{d}t}(X, T, D, L)(s) \in \mathbf{BCL}\big(X(s), T(s)\big) & \text{for a.e. } s \in [0, +\infty)\,, \\ (X, T, D, L)(0) = (x, t, 0, 0)\,. \end{cases} \tag{4.1}$$

Then we have the important

Theorem 4.2.1 *Assume that* $(\mathbf{H_{BCL}})_{fund}$ *holds. Then*

(i) for any $(x, t) \in \mathbb{R}^N \times [0, T)$ *there exists a Lipschitz function* $(X, T, D, L) : [0, T] \to \mathbb{R}^N \times \mathbb{R}^3$ *which is a solution of the differential inclusion (4.1).*

(ii) for each solution (X, T, D, L) *of (4.1) there exist measurable functions* $(b, c, l)(\cdot)$ *such that for a.e. any* $s \in (t, T)$,

$$(\dot{X}, \dot{T}, \dot{D}, \dot{L})(s) = (b, c, l)(s) \in \mathbf{BCL}(X(s), T(s))\,.$$

Throughout this chapter we prefer to write this way

$$\begin{aligned} (\dot{X}(s), \dot{T}(s)) &= b\big(X(s), T(s)\big) \\ \dot{D}(s) &= c\big(X(s), T(s)\big) \\ \dot{L}(s) &= l\big(X(s), T(s)\big) \end{aligned}$$

in order to remember that b, c and l correspond to a specific choice in $\mathbf{BCL}(X(s), T(s))$. Later on, we will also introduce a control $\alpha(\cdot)$ to represent the (b, c, l) as $(b, c, l)(X(s), T(s), \alpha(s))$.

In order to simplify the notations, we just use the notation X, T, D, L when there is no ambiguity but we may also use the notations $X^{x,t}, T^{x,t}, D^{x,t}, L^{x,t}$ when the dependence in x, t plays an important role.

Anticipating our definition of the value-function, we point out that the final cost (and exit/reflexion cost in the case of boundary conditions — see Part IV) will just come from a particular choice of the dynamic since at $t = 0$, the only possibility will be to choose $b^t = 0$. Before describing the value function, we are going to make the following structure assumptions on the $\mathbf{BCL}$-set valued map

$(\mathbf{H_{BCL}})_{struct}$: There exists $\underline{c}, K > 0$ such that

(*i*) For all $x \in \mathbb{R}^N$, $t \in [0, T]$ and $b = (b^x, b^t) \in \mathbf{B}(x, t)$, $-1 \le b^t \le 0$. Moreover, there exists $b = (b^x, b^t) \in \mathbf{B}(x, t)$ such that $b^t = -1$.

(ii) For all $x \in \mathbb{R}^N$, $t \in [0,T]$, if $((b^x, b^t), c, l) \in \mathbf{BCL}(x,t)$, then $-Kb^t + c \geq 0$.

(iii) For any $x \in \mathbb{R}^N$, there exists an element in $\mathbf{BCL}(x,0)$ of the form $((0,0),c,l)$ with $c \geq \underline{c}$.

(iv) For all $x \in \mathbb{R}^N$, $t \in [0,T]$, if $(b,c,l) \in \mathbf{BCL}(x,t)$ then $\max(-b^t, c, l) \geq \underline{c}$.

In the present framework, our aim is to gather different type of control problems: in classical finite horizon problems, $b^t = -1$ which indicates a time direction; in this case, $T(s) = t - s$. Here we choose $-1 \leq b^t \leq 0$ to respect this monotone dynamic in time but we also allow $b^t = 0$.

Assumption (iii) and a part of (iv) concern the final cost (u_0 in the example of the previous section) which is (in general) the initial data for the Hamilton-Jacobi-Bellman Equation: as the value function we define below will be associated to a state-constraint in $\mathbb{R}^N \times [0,T]$, it is necessary that we have strategies with $b^t = 0$ for any point $(x,0) \in \mathbb{R}^N \times \{0\}$. Assumption (iii) means that we can stop the trajectory at any point $(x,0)$ as for classical initial data, but we may also have strategies with $b^t = 0$, $b^x \neq 0$ for which (iv) requires to have either a positive discount factor (to ensure that the associated cost is bounded) or a nonnegative cost (to avoid a long use of such strategy). Such situation may also happen for $t > 0$, either to model a possible stopping time (obstacle type problem) or an exit cost (see in Part 4, Dirichlet boundary condition).

On the other hand, since the formulation below will lead to a stationary type equation, it is well-known that the change of unknown function $u \to \exp(-Kt)u$ allows to reduce to the easiest case of a positive discount factor. This is the meaning of (ii): we can reduce to the case of a positive discount factor. Finally, and we will come back on this point later on but the fact that b^t can be 0 or to be close from 0 includes the unbounded control case; in particular if $b^t = 0$, the trajectory can stay at a constant time $\bar{t}$ for, say, $s \in [s_1, s_2]$ which can be seen as a jump from the point $X(s_1)$ to the point $X(s_2)$.

In all the rest of the book, $(\mathbf{H_{BCL}})$ means that both $(\mathbf{H_{BCL}})_{fund}$ and $(\mathbf{H_{BCL}})_{struct}$ are fulfilled.

Before introducing the value-function, we state a result allowing to reduce to the case when $c \geq 0$ for any $(b,c,l) \in \mathbf{BCL}(x,t)$ and for any $(x,t) \in \mathbb{R}^N$.

Lemma 4.2.2 *Assume that* $(\mathbf{H_{BCL}})$ *hold and let* (X,T,D,L) *be a solution of* (4.1) *associated to* $(b,c,l)(\cdot)$ *such that*

$$J(X,T,D,L) = \int_0^{+\infty} l\big(X(s),T(s)\big) \exp(-D(s))\, \mathrm{d}t \ ,$$

exists. Then we have

$$\exp(-Kt)J(X,T,D,L) = J(\tilde{X},\tilde{T},\tilde{D},\tilde{L})\ ,$$

where K is given by $(\mathbf{H_{BCL}})_{struct}$ and $(\tilde{X},\tilde{T},\tilde{D},\tilde{L})$ is the solution of (4.1) associated to $(b, c-Kb_t, l\exp(-KT(s)))(\cdot)$. In particular $\tilde{X}=X$, $\tilde{T}=T$, $\tilde{D}=D+K(T-t)$.

The use of this lemma will be more clear in the next section but it is clear from $(\mathbf{H_{BCL}})_{struct}$-$(ii)$ that the replacement of c by $c-Kb_t$ allows as we wish to reduce to the case when $c\geq 0$.

4.2.3 Value function

Now we introduce the value function which is defined on $\mathbb{R}^N\times[0,T]$ by

$$U(x,t) = \inf_{\mathcal{T}(x,t)}\Big\{\int_0^{+\infty} l\big(X(s),T(s)\big)\exp(-D(s))ds\Big\}\,, \tag{4.2}$$

where $\mathcal{T}(x,t)$ stands for all the Lipschitz trajectories (X,T,D,L) of the differential inclusion which start at $(x,t)\in\mathbb{R}^N\times[0,T]$ and such that $(X(s),T(s))\in\mathbb{R}^N\times[0,T]$ for all $s>0$. We point out that (a priori) $\mathcal{T}(x,t)\neq\emptyset$ for all $(x,t)\in\mathbb{R}^N\times(0,T]$: indeed, there is no problem with the boundary $\{t=0\}$ since Assumption $(\mathbf{H_{BCL}})_{struct}$ implies that we can stop there with some $b=0$ and with either $c\geq\underline{c}$ (which is expected to provide a very small discount term $\exp(-D(s))$) or $l\geq\underline{c}>0$ (which provides a positive cost, certainly non-optimal if the discount term is not small enough) but a rigourous proof of this claim will be given below in the proof of the

Lemma 4.2.3 *Assume that $(\mathbf{H_{BCL}})$ holds. Then*
(i) The value-function U is locally bounded on $\mathbb{R}^N\times[0,T]$.
(ii) For any trajectory (X,T,D,L) of the differential inclusion such that $\int_0^{+\infty} l\big(X(s),T(s)\big)\exp(-D(s))ds$ is bounded, we have $D(s)\to+\infty$ as $s\to+\infty$.

Proof — Of course, we first use Lemma 4.2.2 to do the proof in the case when c is positive.

To prove (i), we first have to show that actually $\mathcal{T}(x,t)\neq\emptyset$: we just sketch the easy proof. We first solve the differential inclusion (4.1) but replacing **BCL** by $\mathbf{BCL}\cap\{(b,c,l)\in\mathbb{R}^{N+3};\ b^t=-1\}$. Since this new set-valued map satisfies all the required assumptions (for any (x,t) we have a non-empty, convex compact subset of $\mathbb{R}^{N+3}$ by Assumption $(\mathbf{H_{BCL}})_{struct}$-$(i)$): we have $T(t)=0$ and for $s\geq t$, we use Assumption $(\mathbf{H_{BCL}})_{struct}$-$(iii)$ at $x = X(t)$ and extend the trajectory by solving $(\dot{X}, \dot{T}, \dot{D}, \dot{L})(s) = ((0,0), c, l)$ where $((0,0), c, l)$ is given by Assumption $(\mathbf{H_{BCL}})_{struct}$-$(iii)$ at $x = X(t)$. This gives a trajectory defined for $s \in [0, +\infty)$ which is in $\mathcal{T}(x,t)$.

Next we have to show that all the quantity $\int_0^{+\infty} l\big(X(s), T(s)\big) \exp(-D(s)) ds$ are bounded from below. To do so, we use $(\mathbf{H_{BCL}})_{struct}$-$(iv)$ and introduce the sets

$$E_1 := \{s : -b^t \geq \underline{c}\} \ , \ E_2 := \{s \notin E_1 : c \geq \underline{c}\} \ , \ E_3 = [0, +\infty) \setminus (E_1 \cup E_2) \ .$$

We can part the integral on $[0, +\infty)$ in three parts: the one on E_1 is bounded since the measure of E_1 is less that $t/\underline{c}$, l is bounded and $0 \leq \exp(-D(s)) \leq 1$. The one on E_3 is positive and we have nothing to do. For the one on E_2, since $\dot{D}(s) = c(s) \geq \underline{c}$, we have

$$\begin{aligned}
\int_{E_2} l\big(X(s), T(s)\big) \exp(-D(s)) ds \geq & - M \int_{E_2} \exp(-D(s)) ds \\
\geq & - M \int_{E_2} \frac{\dot{D}(s)}{\underline{c}} \exp(-D(s)) ds \\
\geq & - M \int_{[0,+\infty)} \frac{\dot{D}(s)}{\underline{c}} \exp(-D(s)) ds \geq -\frac{M}{\underline{c}} \ .
\end{aligned}$$

This completes the proof of (i).

To prove (ii), we examine carefully the three above sets E_1, E_2, E_3. We know that the measure of E_1 ($|E_1| \leq t/\underline{c}$), if the increasing function $s \mapsto D(s)$ does not tend to $+\infty$ when $s \to +\infty$, this means that $|E_2| < +\infty$ and $\exp(D(s)) \geq \gamma > 0$ on $[0, +\infty)$. But on E_3, we have $l(s) \geq \underline{c}$ and

$$\int_{E_3} l\big(X(s), T(s)\big) \exp(-D(s)) ds \geq \int_{E_3} \underline{c}.\gamma ds = \underline{c}.\gamma |E_3| \ ,$$

a contradiction since $|E_3| = +\infty$ while this integral is bounded.

Q.E.D.

4.2.4 DPP and Supersolutions properties

The first result is the

Theorem 4.2.4 (Dynamic Programming Principle)
Under $(\mathbf{H_{BCL}})$, *the value function* U *satisfies*

$$U(x,t) = \inf_{\mathcal{T}(x,t)} \Big\{ \int_0^\theta l\big(X(s), T(s)\big) \exp(-D(s)) ds + U\big(X(\theta), T(\theta)\big) \exp(-D(\theta))\Big\} ,$$

for any $(x,t) \in \mathbb{R}^N \times (0,T]$, $\theta > 0$.

Next we introduce the "usual" Hamiltonian $\mathbb{F}(x,t,r,p)$ for $x \in \mathbb{R}^N$, $t \in [0,T]$, $r \in \mathbb{R}$ and $p = (p_x, p_t) \in \mathbb{R}^N \times \mathbb{R}$ defined as

$$\mathbb{F}(x,t,r,p) = \sup_{(b,c,l) \in \mathbf{BCL}(x,t)} \big\{ -b \cdot p + cr - l \big\} . \tag{4.3}$$

Using $(\mathbf{H_{BCL}})_{fund}$, it is easy to prove that $\mathbb{F}$ is upper semi-continuous (w.r.t. all variables) and is convex and Lipschitz continuous as a function of r, p only.

The second (classical) result is the

Theorem 4.2.5 (Supersolution's Property) *Under Assumptions* $(\mathbf{H_{BCL}})$*, the value function* U *is a viscosity supersolution of*

$$\mathbb{F}(x,t,U,DU) = 0 \quad \text{on } \mathbb{R}^N \times [0,T] \, , \tag{4.4}$$

where we recall that $DU = (D_x U, D_t U)$.

In Theorem 4.2.5, we use the classical definition of viscosity supersolution in the sense of Ishii which unifies different situations in $\mathbb{R}^N \times (0,T]$ and for $\mathbb{R}^N \times \{t\}$ which is slightly different in general. It is worth pointing out that both Theorem 4.2.4 and 4.2.5 hold in a complete general setting, independently of the type of discontinuities we may have in mind.

We continue by a converse result showing that supersolutions always satisfy a super-dynamic programming principle: again we remark that this result is independent of the possible discontinuities for the dynamic, discount factor and cost. But to prove it, we have to add the following ingredient in which we assume that we have already used Lemma 4.2.2 to reduce to the case when $c \geq 0$

Lemma 4.2.6 *Under Assumptions* $(\mathbf{H_{BCL}})_{struct}$*, for* $K > 0$ *large enough, the function* $\chi(t) = -K(t+1)$ *satisfies, for any* $(x,t) \in \mathbb{R}^N \times [0,T]$

$$-b \cdot D\chi(t) + c\chi(t) - l \leq -\underline{c} < 0 \quad \text{for any } (b,c,l) \in \mathbf{BCL}(x,t) \, .$$

Lemma 4.2.6, which is valid both for $t > 0$ and $t = 0$, provides a very classical properties since it means that the underlying HJB equations has a strict subsolution which is a key point in order to have a comparison result. Of course, in this time-dependent case, one could say that such property is obvious but we are not completely in a standard time-dependent case since we can have $b^t = 0$.

Our next result is the

Lemma 4.2.7 *Under Assumptions* $(\mathbf{H_{BCL}})$*, if* v *is a bounded lsc supersolution of* (4.4) *in* $\mathbb{R}^N \times (0,T]$*, then, for any* $(\bar{x},\bar{t}) \in \mathbb{R}^N \times (0,T]$ *and any* $\sigma > 0$*,*

$$v(\bar{x},\bar{t}) \geq \inf_{\mathcal{T}(x,t)} \Big\{ \int_0^\sigma l\big(X(s),T(s)\big)\exp(-D(s))\,\mathrm{d}s + v\big(X(\sigma),T(\sigma)\big)\exp(-D(\sigma))\Big\} \tag{4.5}$$

Proof — Of course, because of Lemma 4.2.2, we can assume that $c \geq 0$ for any $(b,c,l) \in \mathbf{BCL}(x,t)$ and for any (x,t).

We are going to prove Inequality (4.5) for fixed $(\bar{x},\bar{t})$ and σ, and to do so, we are going to argue in the domain $B(\bar{x},M\sigma)\times[0,\bar{t}]$ where M is given by $(\mathbf{H_{BCL}})_{fund}$, thus in a bounded domain.

Next, we consider the sequence of Hamiltonians

$$\mathbb{F}_\delta(x,t,r,p) := \sup_{(b_\delta,c_\delta,l_\delta)\in\mathbf{BCL}_\delta(x,t)} \big\{ -b_\delta\cdot p + c_\delta r - l_\delta \big\}\,,$$

where $\mathbf{BCL}_\delta(x,t)$ is the set of all $(b_\delta,c_\delta,l_\delta) \in \mathbb{R}^{N+1}\times\mathbb{R}\times\mathbb{R}$ where $|b_\delta^x| \leq M$, $-1 \leq b_\delta^t \leq 0$, $0 \leq c_\delta \leq M$ and

$$l_\delta = l + \delta^{-1}\psi\Big(b_\delta, c_\delta, l, x, t\Big)\,,$$

for some $|l| \leq M$ and with

$$\psi(b,c,l,x,t) = \inf_{(y,s)\in\mathbb{R}^N\times[0,T]} \Big(\mathrm{dist}\,\big((b,c,l),\mathbf{BCL}(y,s)\big) + |y-x| + |t-s|\Big)\,,$$

$\mathrm{dist}(\cdot,\mathbf{BCL}(y,s))$ denoting the distance to the set $\mathbf{BCL}(y,s)$. We notice that ψ is Lipschitz continuous and that $\psi(b,c,l,x,t) = 0$ if $(b,c,l) \in \mathbf{BCL}(x,t)$.

We have

(i) For any $\delta > 0$, $\mathbb{F}_\delta \geq \mathbb{F}$ and therefore v is a lsc supersolution of $\mathbb{F}_\delta \geq 0$ on $B(\bar{x},M\sigma)\times(0,t]$
(ii) The Hamiltonians $\mathbb{F}_\delta$ are (globally) Lipschitz continuous w.r.t. all variables.
(iii) $\mathbb{F}_\delta \downarrow \mathbb{F}$ as $\delta \to 0$, all the other variables being fixed.

On the other hand, v being lsc on $\overline{B(\bar{x},M\sigma)}\times[0,\bar{t}]$, there exists a increasing sequence $(v_\delta)_\delta$ of Lipschitz continuous functions such that $v_\delta \leq v$ and $\sup_\delta v_\delta = v$ on $\overline{B(\bar{x},M\sigma)}\times[0,\bar{t}]$.

For $(x,t) \in \overline{B(\bar{x}, M\sigma)} \times [0,\bar{t}]$, we now introduce the function

$$u_\delta(x,t) := \inf \Big\{ \int_0^{\sigma\wedge\theta} l_\delta\big(X_\delta(s), T_\delta(s)\big) \exp(-D_\delta(s))\,\mathrm{d}s \\ + v_\delta\big(X_\delta(\sigma\wedge\theta), T_\delta(\sigma\wedge\theta)\big) \exp(-D_\delta(\sigma\wedge\theta))\Big\} ,$$

where $(X_\delta, T_\delta, D_\delta, L_\delta)$ is a solution of the differential inclusion

$$(\dot{X}_\delta, \dot{T}_\delta, \dot{D}_\delta, \dot{L}_\delta)(s) \in \mathbf{BCL}_\delta(X_\delta(s), T_\delta(s)) \quad , \quad (X_\delta, T_\delta, D_\delta, L_\delta)(0) = (x,t,0,0) .$$

The infimum is taken over all trajectories X_δ which stay in $\overline{B(\bar{x}, M\sigma)}$ till time $\sigma\wedge\theta$ and on any stopping time θ such that either $X_\delta(\theta)$ on $\partial B(\bar{x}, M\sigma)$ or $T_\delta(\theta) = 0$.

By classical arguments, u_δ is continuous (since all the data are continuous), $u_\delta \le v_\delta$ on $(\partial B(\bar{x}, M\sigma) \times [0,\bar{t}]) \cup (B(\bar{x}, M\sigma) \times \{0\}$ (for the same reason) and u_δ satisfies

$$\mathbb{F}_\delta(x,t,u,Du) = 0 \quad \text{in } B(\bar{x}, M\sigma) \times (0,\bar{t}] .$$

Note that this equation, and the one for v_δ, holds up to time $\bar{t}$, a consequence of the fact that $b^t \le 0$ for all $b \in \mathbf{B}(x,t)$ and for all (x,t).

It remains to show that $u_\delta \le v$ in $\overline{B(\bar{x}, M\sigma)} \times [0,\bar{t}]$ and we argue by contradiction assuming that $\max_{\overline{B(\bar{x},M\sigma)}\times[0,\bar{t}]}(u_\delta - v) > 0$.

We consider the function χ given by Lemma 4.2.6: using the definition of l_δ, it is easy to show that

$$\mathbb{F}_\delta(x,t,\chi,D\chi) \le -\underline{c} < 0 \quad \text{in } B(\bar{x}, M\sigma) \times (0,\bar{t}] ,$$

and, by convexity, for any $0<\mu<1$, $u_{\delta,\mu} = \mu u_\delta + (1-\mu)\chi$ is a subsolution of

$$\mathbb{F}_\delta(x,t,u_{\delta,\mu},Du_{\delta,\mu}) \le -(1-\mu)\underline{c} < 0 \quad \text{in } B(\bar{x}, M\sigma) \times (0,\bar{t}] .$$

Moreover, if $\mu<1$ is close enough to 1, we still have $\max_{\overline{B(\bar{x},M\sigma)}\times[0,\bar{t}]}(u_{\delta,\mu} - v) > 0$ and we can choose K large enough in order to have $u_{\delta,\mu} \le v_\delta$ on $(\partial B(\bar{x}, M\sigma) \times [0,\bar{t}]) \cup (B(\bar{x}, M\sigma) \times \{0\}$.

If $(\tilde{x},\tilde{t}) \in \overline{B(\bar{x}, M\sigma)} \times [0,\bar{t}]$ is a maximum point of $u_{\delta,\mu} - v$, we remark that $(\tilde{x},\tilde{t})$ cannot be on $(\partial B(\bar{x}, M\sigma) \times [0,\bar{t}]) \cup (B(\bar{x}, M\sigma) \times \{0\})$ since on these parts of the boundary $u_{\delta,\mu} \le v$.

Now we perform the standard proof using the doubling of variables with the test-function

$$u_{\delta,\mu}(x,t) - v(y,s) - \frac{|x-y|^2}{\epsilon^2} - \frac{|t-s|^2}{\epsilon^2} - (x-\tilde{x})^2 - (t-\tilde{t})^2 .$$

By standard arguments, this function has a maximum point $(x_\varepsilon, t_\varepsilon, y_\varepsilon, s_\varepsilon)$ which converges to $(\tilde{x}, \tilde{t}, \tilde{x}, \tilde{t})$ since $(\tilde{x}, \tilde{t})$ is a strict global maximum point of $(y,s) \mapsto u_{\delta,\mu}(y,s) - v(y,s) - (y-\tilde{x})^2 - (s-\tilde{t})^2$ in $\overline{B(\bar{x}, M\sigma)} \times [0, \bar{t}]$.

We use now the $\mathbb{F}_\delta$-supersolution inequality for v, the strict subsolution inequality for $u_{\delta,\mu}$ and the regularity of $\mathbb{F}_\delta$ together with the fact that $c \geq 0$ for all $(b,c,l) \in \mathbf{BCL}(y,s)$ [or $\mathbf{BCL}_\delta(y,s)$] and any $(y,s) \in \overline{B(\bar{x}, M\sigma)} \times [0,T]$. We are led to the inequality

$$o(1) \leq -(1-\mu)\exp(-K\bar{t})\eta < 0\,,$$

which yields a contradiction. Sending $\mu \to 1$, we get that $u_\delta \leq v$, hence $u_\delta \leq v$ in $\overline{B(x,M)} \times [0,t]$.

To conclude the proof, we use the inequality $u_\delta(\bar{x}, \bar{t}) \leq v(\bar{x}, \bar{t})$ and we first remark that, in the definition of $u_\delta(\bar{x}, \bar{t})$, we necessarely have $\sigma \wedge \theta = \sigma$ since the trajectory X_δ cannot exit $B(\bar{x}, M\sigma)$ before time σ. Then, we have to let δ tend to 0 in this inequality. To do so, we pick an optimal or δ-optimal trajectory $(X_\delta, T_\delta, D_\delta, L_\delta)$.

By the uniform bounds on $\dot{X}_\delta, \dot{T}_\delta, \dot{D}_\delta, \dot{L}_\delta$, Ascoli-Arzela' Theorem implies that up to the extraction of a subsequence, we may assume that $X_\delta, T_\delta, D_\delta, L_\delta$ converges uniformly on $[0,t]$ to (X,T,D,L). And we may also assume that they derivatives converge in L^∞ weak-* topology (in particular $\dot{L}_\delta = l_\delta$).

We use the above property for the δ-optimal trajectory, namely

$$\int_0^\sigma l_\delta\big(X_\delta(s), T_\delta(s)\big)\exp(-D_\delta(s))\,\mathrm{d}s + v_\delta\big(X_\delta(\sigma), T_\delta(\sigma)\big)\exp(-D_\delta(\sigma)) - \delta \leq v(\bar{x}, \bar{t})\,, \tag{4.6}$$

in two ways. First, by multiplying by δ and using that v and v_δ are bounded. With a slight abuse of notations, writing $h(s) = h(X_\delta(s), T_\delta(s))$ for $h = b_\delta, c_\delta, l_\delta$, we obtain

$$\int_0^\sigma \psi\Big(b_\delta(s), c_\delta(s), l_\delta(s), X_\delta(s), T_\delta(s)\Big)\exp(-D_\delta(s))ds = O(\delta)\,.$$

By classical results on weak convergence, since the functions $(b_\delta, c_\delta, l_\delta)$ converge weakly to (b,c,l), there exists $\mu_s \in L^\infty\big(0,t; \mathbb{P}(B(0,M)\times[-M,M]^2)\big)$ where $\mathbb{P}(B(0,M)\times[-M,M]^2)$ is the set of probability measures on $B(0,M)\times[-M,M]^2$ such that, taking into account the uniform convergence of X_δ, T_δ and D_δ, we have

$$\int_0^\sigma \int_{B(0,M)\times[-M,M]^2} \psi\Big(b,c,l,X(s),T(s)\Big)\exp(-D(s))\,\mathrm{d}\mu_s(b,c,l)\,\mathrm{d}s =$$

$$\lim_\delta \int_0^\sigma \psi\Big(b_\delta(s), c_\delta(s), l_\delta(s), X_\delta(s), T_\delta(s)\Big)\exp(-D_\delta(s))ds = 0\,.$$

Finally we remark that $\psi \geq 0$ and $\psi(b,c,l,x,t) = 0$ if and only if $(b,c,l) \in \mathbf{BCL}(x,t)$, therefore (X,T,D,L) is a solution of the **BCL**-differential inclusion.

In order to conclude, we come back to (4.6) and we remark that $l_\delta\big(X_\delta(s),T_\delta(s)\big) \geq l\big(X_\delta(s),T_\delta(s)\big)$ since $\psi \geq 0$. Therefore

$$\int_0^\sigma l\big(X_\delta(s),T_\delta(s)\big)\exp(-D_\delta(s))\,\mathrm{d}s + v_\delta\big(X_\delta(\sigma),T_\delta(\sigma)\big)\exp(-D_\delta(\sigma)) - \delta \leq v(x,t)\ ,$$

and we pass to the limit in this inequality using the lower-semicontinuity of v, together with the uniform convergence of $X_\delta, T_\delta, D_\delta$ and the dominated convergence theorem for the l-term, which provides in particular the property

$$\liminf_\delta [v_\delta\big(X_\delta(\sigma),T_\delta(\sigma)\big)] \geq v\big(X(\sigma),T(\sigma)\big)\ .$$

This yields

$$\int_0^\sigma l(X(s),T(s))\exp(-D(s))\,\mathrm{d}s + v\big(X(\sigma),T(\sigma)\big)\exp(-D(\sigma)) \leq v(\bar{x},\bar{t})\ .$$

Recalling that (X,T,D,L) is a solution of the **BCL**-differential inclusion, taking the infimum in the left-hand side over all solutions of this differential inclusion gives the desired inequality.

Q.E.D.

An easy consequence of Lemma 4.2.7 is the

Corollary 4.2.8 *Under Assumptions* $(\mathbf{H_{BCL}})$*, the value function* U *is the minimal supersolution of* (4.4)*.*

Proof — It suffices to use (4.5) letting σ tend to $+\infty$ and using Lemma 4.2.3.

Q.E.D.

Now it remains to look at the subsolution condition, and in particular for $t = 0$. Indeed the constraint for the trajectories to stay in $\mathbb{R}^N \times [0,T]$ implies that we can use at $t = 0$, the dynamic with $b^t = 0$. This justifies the

Theorem 4.2.9 (Subsolution's Properties) *Under Assumptions* $(\mathbf{H_{BCL}})$*, the value function* U *is a viscosity subsolution of*

$$\mathbb{F}_*(x,t,U,DU) \leq 0 \quad on\ \mathbb{R}^N \times]0,T]\ , \tag{4.7}$$

where we recall that $DU = (D_xU, D_tU)$ *and, for* $t = 0$*, we have*

$$(\mathbb{F}_{init})_*(x,U(x,0),D_xU(x,0)) \leq 0 \quad in\ \mathbb{R}^N, \tag{4.8}$$

where $\mathbb{F}_{init}(x,u,p_x) := \sup_{((b^x,0),c,l)\in\mathbf{BCL}(x,0)} \big\{ -b^x \cdot p_x + cu - l \big\}$.

The proof is standard and therefore we omit it.

As a by-product of this section, taking also into account the ideas of Ishii's notion of viscosity solutions (i.e. the underlying stability properties), we can define viscosity sub and supersolution or the Bellman equation in $\mathbb{R}^N \times [0,T]$.

Definition 4.2.10 *A subsolution of the Bellmann Equation* $\mathbb{F} = 0$ *in* $\mathbb{R}^N \times [0,T]$ *is an usc function* $u : \mathbb{R}^N \times [0,T] \to \mathbb{R}$ *which satisfies*

$$\mathbb{F}_*(x,t,u,Du) \leq 0 \quad \textit{on } \mathbb{R}^N \times]0,T] \ ,$$

and, for $t = 0$

$$\min(\mathbb{F}_*(x,0,u,Du), (\mathbb{F}_{init})_*(x,u(x,0),D_x u(x,0))) \leq 0 \quad \textit{in } \mathbb{R}^N \ .$$

A supersolution of the Bellmann Equation $\mathbb{F} = 0$ *in* $\mathbb{R}^N \times [0,T]$ *is a lsc function* $v : \mathbb{R}^N \times [0,T] \to \mathbb{R}$ *which satisfies*

$$\mathbb{F}(x,t,v,Dv) \geq 0 \quad \textit{on } \mathbb{R}^N \times [0,T] \ .$$

For the supersolution property, we have nothing to do since $\mathbb{F}$ is usc in $\mathbb{R}^N \times [0,T] \times \mathbb{R} \times \mathbb{R}^N$. But for the subsolution definition, we have to take into account inequalities like (4.7) and (4.8) but respecting the fact that we should have a global lsc Hamiltonian on $\mathbb{R}^N \times [0,T]$.

In the next section, we show how to simplify both formulations at $t = 0$ and we also look at the $t = T$ case.

Chapter 5

Mixed tools

5.1 Initial condition for sub and supersolutions of the Bellman Equation

In this section, we consider a little bit more precisely the conditions satisfied by sub and supersolutions of the Belmann Equation at time $t = 0$. In the classical cases, these conditions just reduces to either $u \leq u_0$ in $\mathbb{R}^N$ if u is a subsolution and $v \geq u_0$ in $\mathbb{R}^N$ if v is a supersolution, but here we have a more general setting.

We recall that the definition of sub/supersolutions of $\mathbb{F} = 0$ is not completely symmetric, see Definition 4.2.10 and involves an initial Hamiltonian defined as follows: for any $x \in \mathbb{R}^N$, $u \in \mathbb{R}$ and $p_x \in \mathbb{R}^N$,

$$\mathbb{F}_{init}(x, u, p_x) := \sup_{((b^x,0),c,l)\in \mathbf{BCL}(x,0)} \big\{ - b^x \cdot p_x + cu - l \big\}\,.$$

In the present situation, our result is the

Proposition 5.1.1 *Under Assumptions* $(\mathbf{H_{BCL}})$*, if* $u : \mathbb{R}^N \times [0,T] \to \mathbb{R}$ *is an u.s.c. viscosity subsolution of the Bellman Equation* $\mathbb{F} = 0$*, then* $u(x,0)$ *is a subsolution in* $\mathbb{R}^N$ *of*

$$(\mathbb{F}_{init})_*\big(x, u(x,0), D_x u(x,0)\big) \leq 0 \quad \text{in } \mathbb{R}^N\,.$$

Similarly, if $v : \mathbb{R}^N \times [0,T] \to \mathbb{R}$ *is a l.s.c. supersolution of the Bellman Equation, then* $v(x,0)$ *is a supersolution of* $\mathbb{F}_{init}(x, v(x,0), D_x v(x,0)) \geq 0$ *in* $\mathbb{R}^N$.

Proof — We provide the full proof in the supersolution case and we will add additional comments in the subsolution one. Let $\phi : \mathbb{R}^N \to \mathbb{R}$ be a smooth function and let

x be a local strict minimum point of the function $y \mapsto v(y,0) - \phi(y)$. In order to use the supersolution property of v, we consider for $0 < \varepsilon \ll 1$ the function $(y,t) \mapsto v(y,t) - \phi(y) + \varepsilon^{-1}t$.

By classical arguments, this function has a local minimum point at $(x_\varepsilon, t_\varepsilon)$ and we have at the same time $(x_\varepsilon, t_\varepsilon) \to (x,0)$ and $v(x_\varepsilon, t_\varepsilon) \to v(x,0)$ as $\varepsilon \to 0$. The viscosity supersolution inequality reads

$$\sup_{(b,c,l)\in \mathbf{BCL}(x_\varepsilon,t_\varepsilon)} \left\{\varepsilon^{-1}b^t - b^x \cdot D_x\phi(x_\varepsilon) + cv(x_\varepsilon,t_\varepsilon) - l\right\} \geq 0\ .$$

We denote by $(b_\varepsilon, c_\varepsilon, l_\varepsilon)$ the (b,c,l) for which the supremum is achieved. By Assumptions $(\mathbf{H_{BCL}})$, we may assume that $(b_\varepsilon, c_\varepsilon, l_\varepsilon) \to (\bar b, \bar c, \bar l) \in \mathbf{BCL}(x,0)$; moreover, since $b^t_\varepsilon \leq 0$ and since the other terms are bounded, the above inequality implies that $\varepsilon^{-1}b^t_\varepsilon$ is also bounded independently of ε. In other words, $b^t_\varepsilon = O(\varepsilon)$ and $\bar b = (\bar b^x, 0)$.

Dropping the negative $\varepsilon^{-1}b^t_\varepsilon$-term in the supersolution inequality, we obtain

$$-b^x_\varepsilon \cdot D_x\phi(x_\varepsilon) + c_\varepsilon v(x_\varepsilon,t_\varepsilon) - l_\varepsilon \geq 0\ ,$$

and letting $\varepsilon \to 0$, we end up with $-\bar b^x \cdot D_x\phi(x) + \bar c v(x,0) - \bar l \geq 0$, which implies

$$\sup_{((b^x,0),c,l)\in \mathbf{BCL}(x,0)} \left\{ - b^x \cdot D_x\phi(x) + cv(x,0) - l\right\} \geq 0\,,$$

since $(\bar b, \bar c, \bar l) \in \mathbf{BCL}(x,0)$; that is, $\mathbb{F}_{init}(x,u,p_x) \geq 0$.

In the subsolution case, the proof is analogous but we consider local strict maximum point of the function $y \mapsto u(y,0) - \phi(y)$. Introducing the function $(y,t) \mapsto u(y,t) - \phi(y) - \varepsilon^{-1}t$ for $0 < \varepsilon \ll 1$, we have a sequence of local maximas $(x_\varepsilon, t_\varepsilon)$ such that $(x_\varepsilon, t_\varepsilon) \to (x,0)$ and $u(x_\varepsilon, t_\varepsilon) \to u(x,0)$ as $\varepsilon \to 0$.

If $t_\varepsilon > 0$, the subsolution inequality reads

$$\mathbb{F}_*(x_\varepsilon, t_\varepsilon, u(x_\varepsilon,t_\varepsilon), (D_x\phi(x_\varepsilon), \varepsilon^{-1})) \leq 0\ .$$

This time, we cannot bound $\varepsilon^{-1}b^t$ as we did for the supersolution case, but because of $(\mathbf{H_{BCL}})_{struct}$-$(i)$, in all $\mathbf{BCL}(x,t)$ for $t \geq 0$, there exists an element with $b^t = -1$. This implies that the $\mathbb{F}_*$-term in the above inequality is larger than $\varepsilon^{-1} + O(1)$ and therefore, for ε small enough, the $\mathbb{F}_*$-inequality above cannot hold.

Hence we have necessarily $t_\varepsilon = 0$ and, as a consequence of the strict maximum point property for $u - \phi$, we have also $x_\varepsilon = x$. Applying the same argument to drop the $\mathbb{F}_*$-term in the initial condition of Definition 4.2.10 for ε small enough, we are left with

$$(\mathbb{F}_{init})_*(x, u(x,0), D_x\phi(x)) \leq 0\,,$$

the inequality we wanted to prove.

Q.E.D.

The above result means that, in order to compute the initial data, one has to solve an equation. A fact which is already known in the case of unbounded control. In the case of classical problems, a typical situation is when the elements of $\mathbf{BCL}(x,t)$ for $t>0$ are of the form $((b^x,-1),c,l)$ and on $t=0$ we have to add (thanks to the upper semicontinuity of $\mathbf{BCL}$) elements of the form $((0,0),1,u_0(x))$ where the cost u_0 is lsc in $\mathbb{R}^N$. In that typical situation, we have $\mathbb{F}_{init}(x,u,p_x)=u-u_0(x)$ and $(\mathbb{F}_{init})_*(x,u,p_x)=u-(u_0)^*(x)$. The above result gives back the standard initial data conditions

$$u(x,0)\leq (u_0)^*(x) \quad \text{and} \quad v(x,0)\geq u_0(x) \quad \text{in } \mathbb{R}^N\ .$$

5.2 A second relevant example involving unbounded control

We want to consider here a problem that we first write as

$$\max(u_t+H(x,t,u,D_xu),|D_xu|-1)=0 \quad \text{in } \mathbb{R}^N\times(0,T), \tag{5.1}$$

with an "initial data" g is a bounded, continuous function in $\mathbb{R}^N$ (we are going to make more precise what we mean by initial data). Here the Hamiltonian H is still given by

$$H(x,t,r,p):=\sup_{\alpha\in A}\left\{-b(x,t,\alpha)\cdot p+c(x,t,\alpha)r-l(x,t,\alpha)\right\}\ ,$$

but the functions b,c,l may be discontinuous. Our first aim is to connect this problem with the above framework and deduce the key assumptions which have to be imposed on b,c,l in order to have our assumptions being satisfied.

First we have to give the sets $\mathbf{BCL}$ and to do so, we set, for $x\in\mathbb{R}^N$, $t\in(0,T]$

$$\mathbf{BCL}_1(x,t):=\{((b(x,t,\alpha),-1),c(x,t,\alpha),l(x,t,\alpha))\ :\ \alpha\in A\}\ ,$$

and

$$\mathbf{BCL}_2(x,t):=\{((\beta,0),0,1)\ :\ \beta\in\overline{B(0,1)}\}\ .$$

Then we introduce

$$\mathbf{BCL}(x,t)=\overline{\mathrm{co}}\left(\mathbf{BCL}_1(x,t)\cup\mathbf{BCL}_2(x,t)\right)\ ,$$

where, if $E \subset \mathbb{R}^k$ for some k, $\overline{\mathrm{co}}(E)$ denotes the closed convex of E; computing $\mathbb{F}(x,t,r,p) = \sup_{(b,c,l)\in \mathbf{BCL}(x,t)} \big\{ -b\cdot p + cr - l \big\}$, we actually find that, for any x, t, r, p_x, p_t

$$\mathbb{F}(x,t,r,(p_x,p_t)) = \max(p_t + H(x,t,u,p_x), |p_x| - 1)\ .$$

For $t = 0$, we have to add the following

$$\mathbf{BCL}_0(x,0) := \{((0,0),1,g(x))\}\ .$$

and $\mathbf{BCL}(x,0) = \overline{\mathrm{co}}\,(\mathbf{BCL}_0(x,0) \cup \mathbf{BCL}_1(x,0) \cup \mathbf{BCL}_2(x,0))$.

We first consider Assumption $(\mathbf{H_{BCL}})_{fund}$ which is satisfied if the three functions $b(x,t,\alpha)$, $c(x,t,\alpha)$, $l(x,t,\alpha)$ are bounded on $\mathbb{R}^N \times [0,T] \times A$ and if $\mathbf{BCL}_1(x,t)$ has compact, convex images and is upper semi-continuous. Next we remark that $(\mathbf{H_{BCL}})_{struct}$ obviously holds and we are going to assume in addition that $c(x,t,\alpha) \geq 0$ for all x, t, α (this is not really an additional assumption since we can reduce to this case by the $\exp(-Kt)$- change).

Since all these assumptions hold, this means that all the results of Section 4.2 also holds. Moreover we have for the initial data $\mathbb{F}_{init}(x,u,p_x) := \max\big\{|p_x| - 1, u - g(x)\big\}$ and therefore the computation of the "real" initial data comes from the resolution of the stationary equation

$$\max(|D_x u| - 1, u - g(x)) = 0 \quad \text{in } \mathbb{R}^N. \tag{5.2}$$

Remark 5.2.1 *Of course, this example remains completely standard as long as we are in the continuous case (typically under the assumptions* $(\mathbf{H_{BA-CP}})$*); it will be more interesting when we will treat examples in which we have discontinuities in the dynamics, discount factors and costs or when the term "*$|D_x u| - 1$*" will be replaced by, for instance, "*$|D_x u| - a(x)$*" where* $a(\cdot)$ *is a discontinuous functions satisfying suitable assumptions and in particular* $a(x) \geq \eta > 0$ *in* $\mathbb{R}^N$*.*

5.3 Dynamic Programming Principle for Subsolutions

In this section, we provide a sub-dynamic programming principle for subsolutions of Bellman Equations, but in a more general form than usual, due to the very general framework we use in Section 4.1 allowing dynamics to have some $b^t = 0$. Roughly speaking, we show that if a **(LCR)** holds in a suitable subdomain $\mathcal{O}$ of $\mathbb{R}^N \times [0,T]$ and for a suitable equation, then subsolutions satisfy a sub-dynamic programming principle inside $\mathcal{O}$.

This formulation is needed in order to get sub-dynamic principles away from the various manifolds on which the singularities are located, and to deal with situations where the definition of "subsolution" may be different from the standard one: even if, to simplify matter, we write below the equation in a usual form (cf. (5.3)), the notion of "subsolution" can be either an Ishii's subsolution or a stratified subsolution, depending on the context. These specific sub-dynamic programming principles will play a key role in the proofs of most of our global comparison results.

In order to be more specific, we consider $(x_0, t_0) \in \mathbb{R}^N \times (0, T]$ and the same equation as in the previous section set in $Q^{x_0,t_0}_{r,h}$ for some $r > 0$ and $0 < h < t_0$, namely

$$\mathbb{F}(x, t, u, Du) = 0 \quad \text{on } Q^{x_0,t_0}_{r,h} \, , \tag{5.3}$$

where $\mathbb{F}$ is defined by (4.3), and we recall that $Du = (D_x u, u_t)$. We point out that we assume that **BCL** and $\mathbb{F}$ are defined in the whole domain $\mathbb{R}^N \times [0, T]$.

In the sequel, $\mathcal{M}$ is a closed subset of $\overline{Q^{x_0,t_0}_{r,h}}$ such that $(x_0, t_0) \notin \mathcal{M}$ and $\mathcal{O} = Q^{x_0,t_0}_{r,h} \setminus \mathcal{M} \neq \emptyset$. We denote by $\mathcal{T}^h_{\mathcal{O}}(x_0, t_0)$ the set of trajectories starting from (x_0, t_0), such that $(X(s), T(s)) \in \mathcal{O}$ for all $s \in [0, h]$. For simplicity here, we assume that the size of the cylinder satisfies $Mh < r$. This is not restrictive at all since when we use the following sub-dynamic programming principle, we can always apply it in situations where r is fixed and we can choose a smaller h.

Our result is the

Theorem 5.3.1 *Let $h, r > 0$ be such that $Mh < r$. Let u be a subsolution of (5.3) and let us assume that for any continuous function ψ such that $\psi \geq u$ on $\overline{Q^{x_0,t_0}_{r,h}}$, a* **(LCR)** *holds in $\mathcal{O}$ for*

$$\max(\mathbb{F}(x, t, u, Du), u - \psi) = 0 \quad \text{in } \mathcal{O} \, . \tag{5.4}$$

If $\mathcal{T}^h_{\mathcal{O}}(x_0, t_0) \neq \emptyset$, then for any $\eta \leq h$

$$u(x_0, t_0) \leq \inf_{X \in \mathcal{T}^h_{\mathcal{O}}(x_0,t_0)} \Big\{ \int_0^\eta l\big(X(s), T(s)\big) \exp(-D(s))\, \mathrm{d}s + u\big(X(\eta), T(\eta)\big) \exp(-D(\eta)) \Big\} \, . \tag{5.5}$$

Proof — In order to prove (5.5), the strategy is the following: we build suitable value functions $v^{\varepsilon,\delta}$, depending on two small parameters ε, δ which are supersolutions of some problems of the type $\max(\mathbb{F}(x, t, v, Dv), v - \psi^\delta) \geq 0$, for some function $\psi^\delta \geq u$ on $\overline{Q^{x_0,t_0}_{r,h}}$. Then, comparing the supersolutions $v^{\varepsilon,\delta}$ with the subsolution u and choosing properly the parameters ε, δ we obtain (5.5) after using the dynamic programming principle satisfied by $v^{\varepsilon,\delta}$.

The main difficulty is that we have a comparison result which is not valid up to $\mathcal{M}$, only in $\mathcal{O}$. Therefore we need to make sure that the supersolution enjoys suitable properties not only on $\partial Q^{x_0,t_0}_{r,h}$ but also on $\mathcal{M}$.

To do so, we introduce a control problem in $\mathbb{R}^N \times [t_0 - h, t_0]$ with a large penalization both in a neighborhood of $\partial Q^{x_0,t_0}_{r,h}$ and outside $\overline{Q^{x_0,t_0}_{r,h}}$, but also in a neighborhood of $\mathcal{M}$. Unfortunately, the set valued map $\mathbf{BCL}$ does not necessarily satisfy assumption $(\mathbf{H_{BCL}})_{struct}$-$(iii)$ at time $t = t_0 - h$, which plays the role of the initial time $t = 0$ here. We need also to take care of the possibility that b^t vanishes inside $\overline{Q^{x_0,t_0}_{r,h}}$. For these reasons, we need to enlarge not only the "restriction" of $\mathbf{BCL}$ to $\mathbb{R}^N \times \{t_0 - h\}$ in order to satisfy $(\mathbf{H_{BCL}})_{struct}$, but also on the whole domain $\mathbb{R}^N \times [t_0 - h, t_0]$.

For doing so, since u is u.s.c., it can be approximated a decreasing sequence $(u^\delta)_\delta$ of bounded continuous functions and we enlarge $\mathbf{BCL}(x,t)$ for $t \in [t_0 - h, t_0]$ by adding elements of the form

$$((b^x, b^t), c, l) = ((0,0), 1, u^\delta(x,t) + \delta) \quad \text{for } 0 \leq \delta \ll 1\,.$$

On the other hand, we introduce, for $0 < \varepsilon \ll 1$, the penalization function

$$\psi_\varepsilon(x,t) := \frac{1}{\varepsilon^4}\Big[\big(2\varepsilon - d((x,t),\mathcal{M})\big)^+ + (2\varepsilon - (r - |x - x_0|))^+ + (2\varepsilon - (t - t_0 + h))^+\Big],$$

so that $\psi_\varepsilon(x,t) \geq \varepsilon^{-3}$ if $d((x,t),\mathcal{M}) \leq \varepsilon$, $d(x, \partial B(x_0, r)) \leq \varepsilon$ or $t - (t_0 - h) \leq \varepsilon$.

We use this penalization in order to modify the original elements in $\mathbf{BCL}(x,t)$, where $l(x,t)$ is replaced by $l(x,t) + \psi_\varepsilon(x,t)$. We denote by $\mathbf{BCL}^{\delta,\varepsilon}$ this new set-valued map where, at the same time, $\mathbf{BCL}$ is enlarged and modified; the elements of $\mathbf{BCL}^{\delta,\varepsilon}$ are referenced as $(b^{\delta,\varepsilon}, c^{\delta,\varepsilon}, l^{\delta,\varepsilon})$. We recall that we can assume that for the original $\mathbf{BCL}$, we have $c \geq 0$ and therefore we also have $c^{\delta,\varepsilon} \geq 0$ for all (x,t) and $(b^{\delta,\varepsilon}, c^{\delta,\varepsilon}, l^{\delta,\varepsilon}) \in \mathbf{BCL}^{\delta,\varepsilon}(x,t)$.

In $\mathbb{R}^N \times [t_0 - h, t_0]$, we introduce the value-function $v^{\varepsilon,\delta}$ given by

$$v^{\varepsilon,\delta}(x,t) = \inf_{\mathcal{T}^{\delta,\varepsilon}(x,t)} \left\{ \int_0^{+\infty} l^{\delta,\varepsilon}\big(X^{\delta,\varepsilon}(s), T^{\delta,\varepsilon}(s)\big) \exp(-D^{\delta,\varepsilon}(s))ds \right\},$$

where $(X^{\delta,\varepsilon}, T^{\delta,\varepsilon}, D^{\delta,\varepsilon}, L^{\delta,\varepsilon})$ are solutions of the differential inclusion associated with $\mathbf{BCL}^{\delta,\varepsilon}$, constrained to stay in $\mathbb{R}^N \times [t_0 - h, t_0]$, $\mathcal{T}^{\delta,\varepsilon}(x,t)$ standing for the set of such trajectories.

Borrowing arguments from Section 4.1 and computing carefully the new Hamiltonian, we see that $v^{\varepsilon,\delta}$ is a l.s.c. supersolution of the HJB-equation

$$\max(\mathbb{F}(x,t,w,Dw), w-(u^\delta+\delta)) = 0 \quad \text{in } \mathbb{R}^N \times (t_0-h,t_0]\,,$$

and we notice that u is a subsolution of this equation since $u \leq u^\delta+\delta$ in $\mathbb{R}^N\times(t_0-h,t_0]$. Notice also that, due to the enlargement of **BCL**, $v^{\varepsilon,\delta}(x,t) \leq u^\delta(x,t)+\delta$, which is the value obtained by solving the differential inclusion with $(b,c,l) = ((0,0),1,u^\delta(x,t)+\delta)$. We want to show that $v^{\varepsilon,\delta} \geq u$ in $\overline{\mathcal{O}}$. In order to do so, we have to examine the behavior of $v^{\varepsilon,\delta}$ in a neighborhood of $\partial\mathcal{O}$ first, which is provided by the

Lemma 5.3.2 *For $\varepsilon > 0$ small enough, $v^{\varepsilon,\delta}(x,t) \geq u^\delta(x,t)$ on $\partial\mathcal{O}$.*

We postpone the proof of this result and finish the argument. Since $v^{\varepsilon,\delta} \geq u^\delta \geq u$ on the boundary of $\mathcal{O}$, we have just to look at maximum points of $u - v^{\varepsilon,\delta}$ in $\mathcal{O}$ but, in this set, **(LCR)** holds for (5.4) with $\psi := u^\delta + \delta$. Therefore the comparison is valid and we end up with $v^{\varepsilon,\delta} \geq u$ everywhere in $\overline{\mathcal{O}}$.

Ending the proof and getting the sub-dynamic principle is done in three steps as follows.

Step 1 – at the specific point (x_0,t_0) we have $u(x_0,t_0) \leq v^{\varepsilon,\delta}(x_0,t_0)$, and using the Dynamic programming Principle for $v^{\varepsilon,\delta}$ at (x_0,t_0) gives that for any $\eta > 0$,

$$u(x_0,t_0) \leq \inf_{\mathcal{T}^{\varepsilon,\delta}(x_0,t_0)} \Big\{ \int_0^\eta l^{\delta,\varepsilon}\big(X(s),T(s)\big)\exp(-D(s))ds + v^{\varepsilon,\delta}\big(X(\eta),T(\eta)\big)\exp(-D(\eta))\Big\}\,. \tag{5.6}$$

we want to get the same inequality, but for trajectories in $\mathcal{T}^h_{\mathcal{O}}(x_0,t_0)$. This relies on the following step.

Step 2 – *Claim: if (X,T,D,L) is a given trajectory in $\mathcal{T}^h_{\mathcal{O}}(x_0,t_0)$ and if $\eta < h$, then, for $\varepsilon > 0$ small enough, (X,T,D,L) coincides with a trajectory in $\mathcal{T}^{\varepsilon,\delta}(x_0,t_0)$ on $[0,\eta]$.*

The main argument in order to prove this claim is to notice that for ε small enough, such trajectories satisfy $\psi_\varepsilon(X(s),T(s)) = 0$ on $[0,\eta]$.

Indeed, let us fix $\eta < h$ and take ε small enough such that $t_0 - h + 2\varepsilon < t_0 - \eta$. Then, for any trajectory (X,T,D,L) in $\mathcal{T}^h_{\mathcal{O}}(x_0,t_0)$, $T(s) \in [t_0-\eta,t_0]$ for $s \in [0,\eta]$, so that $T(s) > t_0 - h + 2\varepsilon$. Similarly, since $Mh < r$ and $|b| \leq M$, we get that $d(X(s);\partial B(x_0,r)) > 2\varepsilon$ for $s \in [0,\eta]$. Of course, by definition of $\mathcal{T}^h_{\mathcal{O}}(x_0,t_0)$, the trajectory does not reach $\mathcal{M}$ hence, if ε is small enough, $d((X(s),T(s));\mathcal{M}) > 2\varepsilon$ for any $s \in [0,\eta]$. In other words, for each fixed trajectory in $\mathcal{T}^h_{\mathcal{O}}(x_0,t_0)$, if we take ε small enough (depending on the trajectory) we have $\psi_\varepsilon(X(s),T(s)) = 0$ on $[0,\eta]$.

Therefore, for any trajectory $(X,T,D,L)\in\mathcal{T}^h_{\mathcal{O}}(x_0,t_0)$, $l^{\delta,\varepsilon}(X(s),T(s))=l(X(s),T(s))$ if $\varepsilon>0$ is small enough and $0\leq s\leq\eta<h$. This means that (X,T,D,L) can be seen as a trajectory associated to the extended $\mathbf{BCL}^{\delta,\varepsilon}$, with initial data $(x_0,t_0,0,0)$. Hence it belongs to $\mathcal{T}^{\delta,\varepsilon}(x,t)$, which proves the claim.

Step 3 – *Passing to the limit in ε and δ.*

We take a specific trajectory $(X,T,D,L)\in\mathcal{T}^h_{\mathcal{O}}(x_0,t_0)$ and take ε small enough so that we can use it in (5.6). As we already noticed, $v^{\varepsilon,\delta}\leq(u^\delta+\delta)$ everywhere in $Q^{x_0,t_0}_{r,h}$ due to the enlargement of **BCL**. Passing to the limit as $\varepsilon\to 0$ yields

$$u(x_0,t_0)\leq\int_0^\eta l\big(X(s),T(s)\big)\exp(-D(s))ds+(u^\delta+\delta)\big(X(\eta),T(\eta)\big)\exp(-D(\eta))\Big\}\,.$$

Then, we can let $\delta\to 0$ in this inequality, using that $u=\inf_{\delta\to 0}(u^\delta+\delta)$ and that the trajectory (X,T,D,L) and η are fixed.

Therefore $(u^\delta+\delta)\big(X(\eta),T(\eta)\big)\to u\big(X(\eta),T(\eta)\big)$ and we get

$$u(x_0,t_0)\leq\int_0^\eta l\big(X(s),T(s)\big)\exp(-D(s))ds+u\big(X(\eta),T(\eta)\big)\exp(-D(\eta))\,.$$

Taking the infimum over all trajectories in $\mathcal{T}^h_{\mathcal{O}}(x_0,t_0)$ yields the conclusion when $\eta<h$. The result for $\eta=h$ is obtained by letting η tend to h, arguing once more trajectory by trajectory.

Q.E.D.

Proof of Lemma 5.3.2 — We need to consider three portions of $\partial\mathcal{O}$: $t=t_0-h$, $x\in\partial B(x_0,r)$ and $(x,t)\in\mathcal{M}$. We detail the initial estimate which is technically involved, then the last two parts are done with similar arguments. In the following, we use an optimal trajectory for $v^{\delta,\varepsilon}$, denoted by $(X^{\delta,\varepsilon},T^{\delta,\varepsilon},D^{\delta,\varepsilon},L^{\delta,\varepsilon})$.

Part A. Initial estimates – if $t=t_0-h$, we have to consider

– the running costs $l\big(X^{\delta,\varepsilon}(s),T^{\delta,\varepsilon}(s)\big)+\psi_\varepsilon(X^{\delta,\varepsilon}(s),T^{\delta,\varepsilon}(s))$, with (perhaps) a non-zero dynamic b^x.

– the running costs $u^\delta(X^{\delta,\varepsilon}(s),T^{\delta,\varepsilon}(s))+\delta$ coming from the enlargement with a zero dynamic;

– and the convex combinations of the two above possibilities, obtained by using a weight $\mu^{\delta,\varepsilon}(s)\in[0,1]$.

We first notice that since $t=t_0-h$, we have $T^{\delta,\varepsilon}(s)=t_0-h$ for any $s\geq 0$ since $b^t\leq 0$ and the trajectories have the constraint to stay in $\mathbb{R}^N\times[t_0-h,t_0]$. In the following, we make various estimates (for ε small enough) involving constants

$\kappa_0, \kappa_1, \kappa_2, \kappa_3$ depending on the datas of the problem and $\delta > 0$ but neither on ε nor on $x \in \overline{B(x_0, r)}$.

Next we set

$$E := \left\{ s \in [0, +\infty) : l^{\delta,\varepsilon}\big(X^{\delta,\varepsilon}(s), T^{\delta,\varepsilon}(s)\big) = l^{\delta,\varepsilon}\big(X^{\delta,\varepsilon}(s), t_0 - h\big) \geq \varepsilon^{-3/2} \right\} ,$$

where $l^{\delta,\varepsilon}$ is given by the convex combination

$$\begin{aligned} l^{\delta,\varepsilon}\big(X^{\delta,\varepsilon}(s), t_0 - h\big) = \mu^{\delta,\varepsilon}(s) \Big\{ l\big(X^{\delta,\varepsilon}(s), t_0 - h\big) + \psi_\varepsilon\big(X^{\delta,\varepsilon}(s), t_0 - h\big) \Big\} \\ + \big(1 - \mu^{\delta,\varepsilon}(s)\big) \Big\{ (u^\delta + \delta)\big(X^{\delta,\varepsilon}(s), t_0 - h\big) \Big\} . \end{aligned}$$

By definition of $l^{\delta,\varepsilon}$ and in particular because of the ψ_ε-term, we have, for any $s \geq 0$, if ε is small enough

$$l\big(X^{\delta,\varepsilon}(s), t_0 - h\big) + \psi_\varepsilon\big(X^{\delta,\varepsilon}(s), t_0 - h\big) \geq \kappa_0 \varepsilon^{-3} ,$$

while $(1 - \mu^{\delta,\varepsilon})(u^\delta + \delta)\big(X^{\delta,\varepsilon}(s), t_0 - h\big)$ is bounded uniformly with respect to ε, s and x. Therefore, on E^c, we necessarily have $\mu^{\delta,\varepsilon}(s) \leq \kappa_1 \varepsilon^{3/2}$ for some $\kappa_1 > 0$.

Estimates on E – As we noticed in the proof of Theorem 5.3.1, $v^{\varepsilon,\delta} \leq u^\delta + \delta$. In particular,

$$\begin{aligned} (u^\delta + \delta)(x, 0) &\geq v^{\varepsilon,\delta}(x, 0) \\ &\geq \int_0^{+\infty} l^{\delta,\varepsilon}\big(X^{\delta,\varepsilon}(s), T^{\delta,\varepsilon}(s)\big) \exp(-D^{\delta,\varepsilon}(s)) ds \\ &\geq \int_E l^{\delta,\varepsilon}\big(X^{\delta,\varepsilon}(s), T^{\delta,\varepsilon}(s)\big) \exp(-D^{\delta,\varepsilon}(s)) ds \\ &+ \int_{E^c} l^{\delta,\varepsilon}\big(X^{\delta,\varepsilon}(s), T^{\delta,\varepsilon}(s)\big) \exp(-D^{\delta,\varepsilon}(s)) ds \end{aligned}$$

By definition of E, the first integral is estimated by

$$\int_E l^{\delta,\varepsilon}\big(X^{\delta,\varepsilon}(s), T^{\delta,\varepsilon}(s)\big) \exp(-D^{\delta,\varepsilon}(s)) ds \geq \int_E \varepsilon^{-3/2} \exp(-D^{\delta,\varepsilon}(s)) ds ,$$

while, using the boundedness of l and $(u^\delta + \delta)$ there exists $C > 0$ such that

$$\int_{E^c} l^{\delta,\varepsilon}\big(X^{\delta,\varepsilon}(s), T^{\delta,\varepsilon}(s)\big) \exp(-D^{\delta,\varepsilon}(s)) ds \geq -C \int_{E^c} \exp(-D^{\delta,\varepsilon}(s)) ds .$$

To get an estimate on the Lebesgue measure of E, we need an upper estimate of $\int_{E^c}\exp(-D^{\delta,\varepsilon}(s))ds$. Notice that on E^c, because of the estimate on $\mu^{\delta,\varepsilon}(s)$ we have

$$\dot{D}^{\delta,\varepsilon}(s) = c^{\delta,\varepsilon}\big(X^{\delta,\varepsilon}(s), T^{\delta,\varepsilon}(s)\big) = \mu^{\delta,\varepsilon}(s)c\big(X^{\delta,\varepsilon}(s), T^{\delta,\varepsilon}(s)\big) + \big(1-\mu^{\delta,\varepsilon}(s)\big) = 1 + O(\varepsilon^{3/2}) \;,$$

where the $|O(\varepsilon^{3/2})| \leq M\kappa_1\varepsilon^{3/2}$ is independent of x. Hence, since $\dot{D}^{\delta,\varepsilon}(s) \geq 0$ for any $s \geq 0$,

$$\begin{aligned}\int_{E^c}\exp(-D^{\delta,\varepsilon}(s))ds &= \int_{E^c}\frac{\dot{D}^{\delta,\varepsilon}(s)}{(1+O(\varepsilon^{3/2}))}\exp(-D^{\delta,\varepsilon}(s))ds \\ &\leq (1+O(\varepsilon^{3/2}))^{-1}\int_0^{+\infty}\dot{D}^{\delta,\varepsilon}(s)\exp(-D^{\delta,\varepsilon}(s))ds \\ &\leq (1+O(\varepsilon^{3/2}))^{-1} \;.\end{aligned}$$

Gathering all the above informations, we finally conclude that

$$\int_E \varepsilon^{-3/2}\exp(-D^{\delta,\varepsilon}(s))ds \leq \kappa_2 \;,$$

for some constant κ_2 which is independent of ε and x.

We introduce now a parameter $S > 0$ and denote by $E_S := E \cap [0, S]$. Since $0 \leq \dot{D}^{\delta,\varepsilon}(s) \leq M$ for any $s \geq 0$, we have

$$\exp(-MS)|E_S| \leq \int_{E_S}\exp(-D^{\delta,\varepsilon}(s))ds \leq \int_E \exp(-D^{\delta,\varepsilon}(s))ds \leq \kappa_2\varepsilon^{3/2} \;,$$

where $|E_S|$ denotes the Lebesgue measure of E_S. We choose $S = S_\varepsilon$ such that $\exp(MS_\varepsilon) = \varepsilon^{-1/6}$ which yields

$$|E_{S_\varepsilon}| \leq \kappa_2\varepsilon^{3/2}\exp(MS_\varepsilon) = \kappa_2\varepsilon^{4/3} \;.$$

We remark that S_ε behaves like $\ln(\varepsilon^{-1/6})$, uniformly in x. The reason why we choose S_ε in order to get a power $4/3 > 1$ in $|E_{S_\varepsilon}|$ will become clear in the lateral estimates. For Part A, any power in $(0, 3/2)$ is convenient.

Consequences on $v^{\varepsilon,\delta}$ – We first apply the Dynamic Programming Principle for $v^{\varepsilon,\delta}$ which gives

$$\begin{aligned}v^{\varepsilon,\delta}(x, t_0 - h) = &\int_0^{S_\varepsilon} l^{\delta,\varepsilon}\big(X^{\delta,\varepsilon}(s), t_0 - h\big)\exp(-D^{\delta,\varepsilon}(s))ds \\ &+ v^{\varepsilon,\delta}(X^{\delta,\varepsilon}(S_\varepsilon), t_0 - h)\exp(-D^{\delta,\varepsilon}(S_\varepsilon)) \;.\end{aligned}$$

Now we have to examine each term carefully. We first come back to the equation of $D^{\delta,\varepsilon}$: we have seen above that $|\dot{D}^{\delta,\varepsilon}(s) - 1| \leq M\kappa_1\varepsilon^{3/2}$ on E^c, while $|E_{S_\varepsilon}| \leq \kappa_2\varepsilon^{4/3}$. We deduce that, for $s \in [0, S_\varepsilon]$

$$|D^{\delta,\varepsilon}(s) - s| \leq M(\kappa_1\varepsilon^{3/2}S_\varepsilon + \kappa_2\varepsilon^{4/3}) \leq \kappa_3\varepsilon^{4/3} \tag{5.7}$$

for some $\kappa_3 > 0$. In particular, since $S_\varepsilon \to +\infty$ as $\varepsilon \to 0$, $\exp(-D^{\delta,\varepsilon}(S_\varepsilon)) \to 0$ as $\varepsilon \to 0$ and

$$\liminf_{\varepsilon\to 0} \big(v^{\varepsilon,\delta}(X^{\delta,\varepsilon}(S), t_0 - h)\exp(-D^{\delta,\varepsilon}(S_\varepsilon))\big) \geq 0 \ ,$$

uniformly w.r.t. x since $v^{\varepsilon,\delta}$ is bounded from below.

On an other hand, for the $X^{\delta,\varepsilon}$-equation, we also have, on E^c (in fact only the b^x part is useful here)

$$b^{\delta,\varepsilon}(X^{\delta,\varepsilon}(s), t_0 - h) = \mu^{\delta,\varepsilon}(s)b(X^{\delta,\varepsilon}(s), t_0 - h) + (1 - \mu^{\delta,\varepsilon}(s))(0,0) = O(\varepsilon^{3/2}) \ ,$$

more precisely the bound takes the form $M\kappa_2\varepsilon^{3/2}$. Using the decomposition with E_{S_ε} and its complementary $E^c_{S_\varepsilon} = E^c \cap [0, S_\varepsilon]$ as in (5.7), it follows that

$$\begin{aligned}\int_0^{S_\varepsilon} |b^{\delta,\varepsilon}(\tau)|d\tau &= \int_0^{S_\varepsilon} |b^{\delta,\varepsilon}(\tau)|\mathbb{1}_{\{E_{S_\varepsilon}\}}(s)d\tau + \int_0^{S_\varepsilon} |b^{\delta,\varepsilon}(\tau)|\mathbb{1}_{\{E^c_{S_\varepsilon}\}}(s)\,d\tau \\ &\leq M(\kappa_2\varepsilon^{4/3} + \kappa_1\varepsilon^{3/2}S_\varepsilon) \leq \kappa_3\varepsilon^{4/3}\,.\end{aligned}$$

We deduce that if $s \in [0, S_\varepsilon]$, $X^{\delta,\varepsilon}(s) - x = O(\varepsilon^{4/3})$ and since u^δ is continuous,

$$(u^\delta + \delta)(X^{\delta,\varepsilon}(s), t_0 - h) = (u^\delta + \delta)(x, t_0 - h) + o_\varepsilon(1) \geq (u^\delta + \delta/2)(x, t_0 - h) \ .$$

For a similar reason, on $E^c_{S_\varepsilon}$ we can absorb the $o_\varepsilon(1)$-term by a $\delta/2$ for ε small enough

$$l^{\delta,\varepsilon}\big(X^{\delta,\varepsilon}(s), t_0 - h\big) \geq (u^\delta + \delta/2)\big(x, t_0 - h\big) \ .$$

Gathering all these informations, using (5.7) and that $(l + \psi_\varepsilon) \geq 0$ on E_{S_ε} we get

$$\begin{aligned} I_\varepsilon &:= \int_0^{S_\varepsilon} l^{\delta,\varepsilon}\big(X^{\delta,\varepsilon}(s), t_0 - h\big)\exp(-D^{\delta,\varepsilon}(s))ds \\ &\geq \int_{E^c_{S_\varepsilon}} \Big((u^\delta + \delta/2)\big(x, t_0 - h\big)\Big)\exp\big(-s + O(\varepsilon^{4/3})\big)ds\,. \end{aligned}$$

Then, since S_ε behaves like $\ln(\varepsilon^{-1/6})$ and $|E_{S_\varepsilon}| \leq \kappa_2\varepsilon^{4/3}$, we get

$$\begin{aligned} I_\varepsilon &\geq (u^\delta + \delta/2)\big(x, t_0 - h\big)\int_{E^c_{S_\varepsilon}} \exp(-s)ds + o_\varepsilon(1) \\ &\geq (u^\delta + \delta/2)\big(x, t_0 - h\big) + o_\varepsilon(1) \ . \end{aligned}$$

Hence $v^{\varepsilon,\delta}(x,t_0-h) \geq (u^\delta+\delta/2)\big(x,t_0-h\big)+o_\varepsilon(1)$ where the "$o_\varepsilon(1)$" is independent of x and for ε small enough, we have $v^{\varepsilon,\delta}(x,t_0-h) \geq u^\delta\big(x,t_0-h\big)$ on $\overline{B(x_0,r)}$.

Part B. Lateral estimates – Essentially, the proof is the same as for the initial estimates: the only difference is that the trajectory may exit the region where ψ_ε is large. But, if $d((x,t),\mathcal{M}) \leq \varepsilon$ or if $d(x,\partial B(x_0,r)) \leq \varepsilon$, the running cost satisfies again the estimate $l\big(X^{\delta,\varepsilon}(s),T^{\delta,\varepsilon}(s)\big)+\psi_\varepsilon(X^{\delta,\varepsilon}(s),T^{\delta,\varepsilon}(s)) \geq \kappa_0\varepsilon^{-3} \geq 0$.

Assume that $(x,t)\in\mathcal{M}$, the proof being the same if $(x,t)\in\partial B(x_0,r)$. Since the dynamic b is bounded by M, $d((X(s),T(s)),\mathcal{M}) \leq Ms$ and therefore, a trajectory starting at (x,t) stays in an ε-neighborhood of $\mathcal{M}$ for $s<\varepsilon/M$.

For an optimal trajectory, we repeat the same proof as in Part A, but on $E\cap[0,\tau_\varepsilon\wedge S_\varepsilon]$, where τ_ε is the first time for which $d((X^{\delta,\varepsilon}(s),T^{\delta,\varepsilon}(s)),\mathcal{M})=\varepsilon$ and $a\wedge b=\min(a,b)$.

If we set as above

$$E := \big\{s\in[0;\infty) : l^{\delta,\varepsilon}\big(X^{\delta,\varepsilon}(s),T^{\delta,\varepsilon}(s)\big) \geq \varepsilon^{-3/2}\big\}\,,$$

then the Lebesgue measure of $E\cap[0,\tau_\varepsilon\wedge S_\varepsilon]$ is less than $\kappa_3\varepsilon^{4/3}$ for some $\kappa_3>0$, while on $E^c\cap[0,\tau_\varepsilon\wedge S_\varepsilon]$ we have $\mu^{\delta,\varepsilon}(s)\leq\kappa_4\varepsilon^{3/2}$ for some $\kappa_4>0$. As in Part A, using the decomposition on $E\cap[0,\tau_\varepsilon\wedge S_\varepsilon]$ and its complementary we deduce that

$$\int_0^{\tau_\varepsilon\wedge S_\varepsilon} |b^{\delta,\varepsilon}(s)|\,ds \leq\; M\big\{\kappa_3\varepsilon^{4/3}+\kappa_4\varepsilon^{3/2}(\tau_\varepsilon\wedge S_\varepsilon)\big\}\,,$$

while by definition the distance between (x,t) and $(X^{\delta,\varepsilon}(\tau_\varepsilon),T^{\delta,\varepsilon}(\tau_\varepsilon))$ is ε (if τ_ε is finite, of course).

We claim that for ε small enough, $\tau_\varepsilon\wedge S_\varepsilon=S_\varepsilon$. Indeed, assume on the contrary that for some subsequence $\varepsilon_n\to0$, $\tau_{\varepsilon_n}<S_{\varepsilon_n}$. From the previous estimate it follows that

$$\varepsilon_n \leq M\big\{\kappa_3\varepsilon_n^{4/3}+\kappa_4\varepsilon_n^{3/2}\tau_{\varepsilon_n}\big\}\,.$$

The fact that the power in the first term is greater than 1 implies that τ_{ε_n} goes to infinity, at least like $\varepsilon_n^{-1/2}$. But since by construction S_{ε_n} behaves like $\ln(\varepsilon_n^{-1/6})$, we reach a contradiction.

We deduce that necessarily $\tau_\varepsilon>S_\varepsilon$ as $\varepsilon\to0$, and that on $[0,S_\varepsilon]$, the trajectory remains "trapped" in an ε-neighborhood of $\mathcal{M}$. We end the proof exactly as in Part A, sending $\varepsilon\to0$.

The proof if $x\in\partial B(x_0,r)$ being the same, in conclusion we have shown that $v^{\delta,\varepsilon}\geq u^\delta$ on $\partial\mathcal{O}$ for ε small enough.

Q.E.D.

In the case when b^t is not allowed to vanish, obtaining the sub-dynamic principle is a bit easier since we do not need to consider an obstacle-type problem like (5.4).

Theorem 5.3.3 *Let $h, r > 0$ be such that $Mh < r$ and assume that, for any $(x,t) \in \overline{Q_{r,h}^{x_0,t_0}}$ and any $(b,c,t) \in \mathbf{BCL}(x,t)$, $b^t = -1$. If u is a subsolution of (5.3), if $\mathcal{T}_{\mathcal{O}}^h(x_0,t_0) \neq \emptyset$ and if a* **(LCR)** *holds in $\mathcal{O}$ for the equation $\mathbb{F} = 0$, then for any $\eta \leq h$*

$$u(x_0,t_0) \leq \inf_{X \in \mathcal{T}_{\mathcal{O}}^h(x_0,t_0)} \left\{ \int_0^\eta l\big(X(s),T(s)\big) \exp(-D(s))\,\mathrm{d}s + u\big(X(\eta),T(\eta)\big) \exp(-D(\eta)) \right\}. \tag{5.8}$$

Proof — The difference between the two cases comes from the fact that, under the assumption of Theorem 5.3.3, we could have $T(h) > t_0 - h$ in (5.6) (Step 1) for a trajectory starting from (x_0,t_0) since b^t was allowed to be different from -1: this is why the strategy of the proof of this theorem uses $\eta < h$ and, for handling this situation, we need to have $v^{\varepsilon,\delta}(x,t) \leq u^\delta(x,t) + \delta$ in the whole domain to conclude after using the Dynamic Programming Principle for $v^{\varepsilon,\delta}$ (cf. Step 3).

Here on the contrary we are sure that $T(h) = t_0 - h$ for any such trajectory and we are going the Dynamic Programming Principle for $v^{\varepsilon,\delta}$ up to time $t_0 - h$, i.e. with $s = h$.

For this reason, we are going to prove (5.8) for $\eta = h$, the inequality for $\eta < h$ being obtained by applying the result with h replaced by η.

For all these reasons the proof is similar to that of Theorem 5.3.1 but there are substantial simplifications.

(a) We enlarge **BCL** in the same way BUT ONLY at time $t = t_0 - h$. The consequence is that $v^{\varepsilon,\delta}$ is a supersolution for the HJB-equation $\mathbb{F} = 0$ and not of (5.4), since we have no enlargement for $t \in (t_0 - h, t_0)$. Hence we just have to deal with the comparison results for the $\mathbb{F}$-equation, we do not need to assume some obstacle-type comparison property.

(b) The penalization function we use here does not require a specific penalization for the initial time and we just write it as

$$\psi_\varepsilon(x,t) := \frac{1}{\varepsilon^4}\Big[\big(2\varepsilon - d((x,t),\mathcal{M})\big)^+ + (2\varepsilon - (r - |x - x_0|))^+\Big].$$

The initial inequality $v^{\varepsilon,\delta}(x,t_0-h) \geq (u^\delta + \delta)(x,t_0-h)$ for any $x \in B(x_0,r)$ follows from the following argument: since $b^t = -1$ in **BCL**, the only possibility for a constrained trajectory $(X^{\delta,\varepsilon}, T^{\delta,\varepsilon}, D^{\delta,\varepsilon}, L^{\delta,\varepsilon}) \in \mathcal{T}^{\delta,\varepsilon}(x,t_0-h)$ to remain in $\mathbb{R}^N \times [t_0 -$

$h, t_0]$ is to solve the differential inclusion by using the elements $((0,0), 1, (u^\delta+\delta)(x, t_0-h))$ of $\mathbf{BCL}^{\delta,\varepsilon}$. This implies directly that $v^{\varepsilon,\delta}(x, t_0-h) \geq (u^\delta+\delta)(x, t_0-h)$.

(c) With these simplications, the proof remains the same as in the general case $b^t \in [-1, 0]$: we first get that $v^{\varepsilon,\delta} \geq u$ on $t = t_0 - h$, for $x \in \partial B(x_0, r)$ and for $(x,t) \in \mathcal{M}$. Using that we have a **(LCR)** in $\mathcal{O}$ implies that $v^{\varepsilon,\delta} \geq u$ on $\overline{\mathcal{O}}$. Then we proceed as above using the dynamic programming principle for $v^{\varepsilon,\delta}$. For $\eta \leq h$[(1)], taking $\varepsilon > 0$ small enough allows to restrict this dynamic principle to the trajectories in $\mathcal{T}^h_{\mathcal{O}}(x_0, t_0)$, which avoid $\mathcal{M}$. Sending $\varepsilon \to 0$ and $\delta \to 0$ is done "trajectory by trajectory".

Q.E.D.

5.4 Local comparison for discontinuous HJB equations

The aim of this section is to provide an argument which is a keystone in several comparison results we give for HJB equations with discontinuities, and in particular for stratified problems.

To do so, we consider a C^1-manifold $\mathcal{M} \subset \mathbb{R}^N \times (0,T)$ (which will be in the sequel a set of discontinuity for the HJB equation) and for any $(x,t) \in \mathcal{M}$, we denote by $T_{(x,t)}\mathcal{M}$, the tangent space of $\mathcal{M}$ at (x,t). Then we define the *tangential Hamiltonian* associated with $\mathcal{M}$ by setting

$$\mathbb{F}^{\mathcal{M}}(x,t,u,p) := \sup_{(b,c,l) \in \mathbf{BCL}_T(x,t)} \big\{ -b \cdot p + cu - l \big\}, \tag{5.9}$$

where $\mathbf{BCL}_T(x,t) := \big\{(b,c,l) \in \mathbf{BCL}(x,t) : b \in T_{(x,t)}\mathcal{M}\big\}$. This tangential Hamiltonian is defined for any $(x,t) \in \mathcal{M} \times [0,T]$, $u \in \mathbb{R}$ and $p \in T_{(x,t)}\mathcal{M}$. But by a slight abuse of notation, we also write $\mathbb{F}^{\mathcal{M}}(x,t,u,p)$ when $p \in \mathbb{R}^{N+1}$, meaning that only the projection of p onto $T_{(x,t)}\mathcal{M}$ is used for the computation. We also recall that $Du = (D_x u, u_t)$.

Our main argument comes from the

Lemma 5.4.1 *We assume that* $(\mathbf{H_{BCL}})$ *holds, that* $v : \overline{Q^{x,t}_{r,h}} \to \mathbb{R}$ *is a lsc supersolution of* $\mathbb{F}(x,t,v,Dv) = 0$ *in* $Q^{x,t}_{r,h}$ *where* $(x,t) \in \mathcal{M}$ *and* $0 < t-h < t \leq T$, *and that* $u : \overline{Q^{x,t}_{r,h}} \to \mathbb{R}$ *has the following properties*

(1) Here we do not have to treat separately the cases when $\eta < h$ and $\eta = h$ since we have dropped the penalization term in a neighborhood of $t = t_0 - h$ and we know that $v^{\varepsilon,\delta}(x, t_0-h) \geq (u^\delta + \delta)(x, t_0 - h)$.

(i) $u \in C^0(\overline{Q^{x,t}_{r,h}}) \cap C^1(\mathcal{M})$,

(ii) $\mathbb{F}^{\mathcal{M}}(y,s,u,Du) < 0$ *on* $\mathcal{M}$,

(iii) u *satisfies a "strict" subdynamic principle in* $Q^{x,t}_{r,h}[\mathcal{M}^c] = (B(x,r)\times(t-h,t])\setminus\mathcal{M}$, *i.e. there exists* $\eta > 0$, *such that, for any* $(\bar{x},\bar{t}) \in Q^{x,t}_{r,h}[\mathcal{M}^c]$, *for any solution* (X,T,D,L) *of the differential inclusion such that* $X(0)=\bar{x}$, $T(0)=\bar{t}$ *and* $(X(s),T(s)) \in Q^{x,t}_{r,h}[\mathcal{M}^c]$ *for* $0 < s \leq \bar{\tau}$, *we have, for any* $0 < \tau \leq \bar{\tau}$

$$u(y,s) \leq \int_0^\tau (l(X(s),T(s)) - \eta)\exp(-D(s))\,\mathrm{d}s + u(X(\tau)\exp(-D(\tau))\,, \quad (5.10)$$

Then for any $(y,s) \in \overline{Q^{x,t}_{r,h}} \setminus \partial_p Q^{x,t}_{r,h}$,

$$(u-v)(y,s) < m := \max_{\partial_p Q^{x,t}_{r,h}} (u-v)\,.$$

Proof — Using $(\mathbf{H_{BCL}})_{struct}$, we can assume without loss of generality that $c \geq 0$ for all $(b,c,l) \in \mathbf{BCL}(y,s)$ and for all $(y,s) \in Q^{x,t}_{r,h}$.

We assume by contradiction that $(u-v)$ reaches its maximum on $\overline{Q^{x,t}_{r,h}}$ at a point $(\bar{x},\bar{t}) \in Q^{x,t}_{r,h}$. If $(\bar{x},\bar{t}) \in Q^{x,t}_{r,h} \setminus \mathcal{M}$, we easily reach a contradiction: by Lemma 4.2.7, v satisfies (4.5) and for sufficiently small σ, all the trajectories (X,T,D,L) are such that $(X(s),T(s)) \in Q^{x,t}_{r,h}[\mathcal{M}^c]$. We consider an optimal trajectory for v at $(\bar{x},\bar{t})$, (X,T,D,L) and we gather the information given by (4.5) and (5.10) for some time σ small enough: substracting these inequalities, we get

$$u(\bar{x},\bar{t}) - v(\bar{x},\bar{t}) \leq -\eta\tau + (u(X(\tau)) - v(X(\tau)))\exp(-D(\tau))\,, \quad (5.11)$$

which is a contradiction since $(\bar{x},\bar{t})$ is a maximum point of $u-v$ in $\overline{Q^{x,t}_{r,h}}$ and therefore $u(\bar{x},\bar{t}) - v(\bar{x},\bar{t}) \geq u(X(\tau)) - v(X(\tau))$ while $D(\tau) \geq 0$.

If $(u-v)$ reaches its maximum on $\overline{Q^{x,t}_{r,h}}$ at a point $(\bar{x},\bar{t}) \in Q^{x,t}_{r,h}$ where $(\bar{x},\bar{t}) \in \mathcal{M}$, we face two cases

A. – In (4.5) for $(\bar{x},\bar{t})$, there exists a trajectory (X,T,D,L) and $\tau > 0$ such that $X(0) = \bar{x}$ and

$$v(\bar{x},\bar{t}) \geq \int_0^\tau l\big(X(s),T(s)\big)\exp(-D(s))\,\mathrm{d}s + v\big(X(\tau),T(\tau)\big)\exp(-D(\tau))\,, \quad (5.12)$$

AND $(X(s),T(s)) \in Q^{x,t}_{r,h}\backslash\mathcal{M}$ for $s \in (0,\tau]$. In this case we argue essentially as above: we use as a starting point $(x_\varepsilon,t_\varepsilon) := (X(\varepsilon),T(\varepsilon)) \in Q^{x,t}_{r,h}[\mathcal{M}^c]$ for $0<\varepsilon \ll 1$ and we use (5.10) for the specific trajectory (X,T,D,L) but on the time interval $[\varepsilon,\tau]$

$$u(x_\varepsilon,t_\varepsilon) \leq \int_\varepsilon^\tau (l(X(s),T(s))-\eta)\exp(-D(s))\,\mathrm{d}s + u(X(\tau)\exp(-D(\tau))\,.$$

But in this inequality, we can send ε to 0, using the continuity of u and finally get, combining it with the above inequality for v to obtain (5.11) and a contradiction.

B. – If Case A cannot hold, this means that, for any τ and for any trajectory (X,T,D,L) such that (5.12) holds, then there exists a sequence $t_n \searrow 0$ such that $X(t_n) \in \mathcal{M}$ for any $n \in \mathbb{N}$. We first use the dynamic programming inequality for v between $s=0$ and $s=t_n$, which yields

$$v(\bar{x},\bar{t}) \geq \int_0^{t_n} l(X(s),T(s))\exp(-D(s))\,\mathrm{d}s + v(X(t_n))\exp(-D(t_n))\,.$$

Since $u-v$ reaches a maximum at $(\bar{x},\bar{t})$ we can replace v by u in this inequality which leads to

$$\frac{u(\bar{x},\bar{t})-u(X(t_n))\exp(-D(t_n))}{t_n} \geq \frac{1}{t_n}\int_0^{t_n} l(X(s),T(s))\exp(-D(s))\,\mathrm{d}s\,.$$

Now, since u is C^1-smooth on $\mathcal{M}\times(t-h,t)$, we have (recall that $Du=(D_xu,u_t)$ and that here we use only derivatives which are in the tangent space of $\mathcal{M}$)

$$\begin{aligned} u(X(t_n),T(t_n)) =& u(\bar{x},\bar{t}) + Du(\bar{x},\bar{t})(X(t_n)-\bar{x},T(t_n)-\bar{t}) + o(|X(t_n)-\bar{x}|+|T(t_n)-\bar{t}|)\\ =& u(\bar{x},\bar{t}) + Du(\bar{x},\bar{t})(X(t_n)-\bar{x},T(t_n)-\bar{t}) + o(t_n)\;, \end{aligned}$$

and writing

$$(X(t_n)-\bar{x},T(t_n)-\bar{t}) = \int_0^{t_n} b(s)ds\;,\;\exp(-D(t_n)) = \int_0^{t_n} -c(s)\exp(-D(s))ds$$

we obtain

$$\frac{1}{t_n}\int_0^{t_n}\{-b(s)\cdot Du(\bar{x},\bar{t}) + c(s)u(\bar{x},\bar{t}) - l(X(s),T(s))\}\exp(-D(s))ds \geq 0\;.$$

But **BCL** being usc, we have

$$-b(s)\cdot Du(\bar{x},\bar{t}) + c(s)u(\bar{x},\bar{t}) - l(X(s),T(s)) \leq \mathbb{F}^{\mathcal{M}}(\bar{x},\bar{t},u(\bar{x},\bar{t}),Du(\bar{x},\bar{t})) + o_n(1)\;,$$

for any $s \in [0, t_n]$ and therefore, letting n tends to infinity, we finally obtain

$$\mathbb{F}^{\mathcal{M}}(\bar{x}, \bar{t}, u(\bar{x}, \bar{t}), Du(\bar{x}, \bar{t})) \geq 0 ,$$

a contradiction with the fact that $\mathbb{F}^{\mathcal{M}}(x, t, u, Du) < 0$ on $\mathcal{M} \times (t - h, t]$.

Q.E.D.

Remark 5.4.2 *There are possible variants for this lemma. In particular, in Part II, we use one of them where the sub and supersolution properties for u and v are defined in a slightly different way, namely with taking a more restrictive set of control on $\mathcal{M}$. Of course, in that case, $\mathbb{F}^{\mathcal{M}}$ is replaced by an Hamiltonian which defined in a different way. The proof is still valid if the Dynamic Programming argument of **B.** leads to the right inequality.*

5.5 The "Good Framework for HJ-Equations with Discontinuities"

We conclude this section by defining "good framework for HJ-Equations with discontinuities".

Definition 5.5.1 *We say that we are in the "good framework for HJ-Equations with discontinuities" for the equation*

$$G(y, u, Du) = 0 \quad \textit{in } \mathcal{O} \subset \mathbb{R}^N ,$$

if there exists a set-valued map $\mathbf{BCL} : \mathcal{O} \to \mathbb{R}^N \times \mathbb{R} \times \mathbb{R}$ satisfying $(\mathbf{H_{BCL}})$ such that

$$G(y, r, p) := \sup_{(b,c,l) \in \mathbf{BCL}(y)} \big\{ -b \cdot p + cr - l \big\} ,$$

and a stratification $\mathbb{M} = (\mathbf{M}^k)_{k=0..N}$ of $\mathbb{R}^N$ such that: for any $\bar{y} \in \mathbf{M}^k \cap \mathcal{O}$, there is a ball $B(\bar{y}, r) \subset \mathcal{O}$ for some $r > 0$, and a C^1-diffeomorphism Ψ such that

$$B(\bar{y}, r) \cap \Psi(\mathbf{M}^k) = B(\bar{y}, r) \cap \bigcup_{j=0}^{k} \big(\bar{y} + \mathbb{R}^k \times \{0_{\mathbb{R}^{N-k}}\} \big) ,$$

and if,

$$\tilde{G}(x, v, p) = G(\Psi^{-1}(x), v, [(\Psi^{-1})']^T(x)p) ,$$

where $[(\Psi^{-1})']^T$ denotes the transpose matrix of $(\Psi^{-1})'$ such that **(TC)**, **(NC)** *hold for $\tilde{G}$.*

In this case, we will say that $\mathbb{M}$ is associated to Equation 5.5.1.

We first point out that, as it will be clear in the proof of Theorem 13.2.1, Assumption ($\mathbf{H_{BCL}}$) allows to reduce to the case when (**Mon**-u) holds, bringing the usual monotonicity assumption.

Using this monotonicity together with the tangential continuity and the normal controllability, we are in the framework of Section 3.4 up to some suitable change of variables. This allows us to regularize subsolutions in order to be able to apply Lemma 5.4.1.

In Definition 5.5.1, the diffeomorphism Ψ is assumed to be C^1 but, in non-coercive cases, i.e. when G is not coercive in p, $C^{1,1}$-diffeomorphisms are needed in general to get **(TC)**.

The two extreme cases have also to be commented : if $k = N$, then there is no normal directions, **(TC)** has to be satisfied by all coordinates, G is continuous in a neighborhood of $\bar{y}$, no change Ψ is really needed and, through **(TC)**, we just recover the classical assumption for the uniqueness of viscosity solutions for a standard HJ-Equations without discontinuity. If $k = 0$, $\bar{y}$ is an isolated point, we have no "tangent coordinates" and **(TC)** is void but G is coercive in p in a neighborhood of $\bar{y}$.

Part II

Deterministic Control Problems and Hamilton-Jacobi Equations for Codimension 1 Discontinuities

Chapter 6

Ishii Solutions for the Hyperplane Case

In this part, we consider the simplest possible case of discontinuity for an equation or a control problem, namely the case when this discontinuity is an hyperplane and, to fix ideas, this hyperplane will be $\mathcal{H} = \{x_N = 0\}$. In terms of stratification, as introduced in Section 3.3, this is one of the simplest examples of stratification of $\mathbb{R}^N$ for which $\mathbf{M}^N = \Omega_1 \cup \Omega_2$, $\mathbf{M}^{N-1} = \mathcal{H}$ and $\mathbf{M}^k = \emptyset$ for any $k = 0..(N-2)$, where

$$\Omega_1 = \{x_N > 0\}\,,\ \Omega_2 = \{x_N < 0\}\,.$$

For simplicity of notations, we also write $\Omega_0 = \mathcal{H}$ and we take the convention to denote by $e_N(0,\dots,0,1)$ the unit vector pointing inside Ω_1, so that e_N is also the outward unit normal to Ω_2, see figure 6.1 below.

Two types of questions can be addressed whether we choose the pde or control point of view and, in this part, both will be very connected since we mainly consider Hamilton-Jacobi-Bellman type equations.

From the pde viewpoint, the main question concerns the existence and uniqueness of solutions to the problem

$$\begin{cases} u_t + H_1(x,t,u,Du) = 0 & \text{for } x \in \Omega_1\,, \\ u_t + H_2(x,t,u,Du) = 0 & \text{for } x \in \Omega_2\,, \\ u(x,0) = u_0(x) & \text{for } x \in \mathbb{R}^N\,, \end{cases} \tag{6.1}$$

under some standard assumptions on H_1, H_2 and u_0. It is also very natural to consider a specific control problem or pde on $\mathcal{H}$, which amounts to adding an equation

$$u_t + H_0(x,t,u,D_T u) = 0 \text{ for } x \in \mathcal{H}\,, \tag{6.2}$$

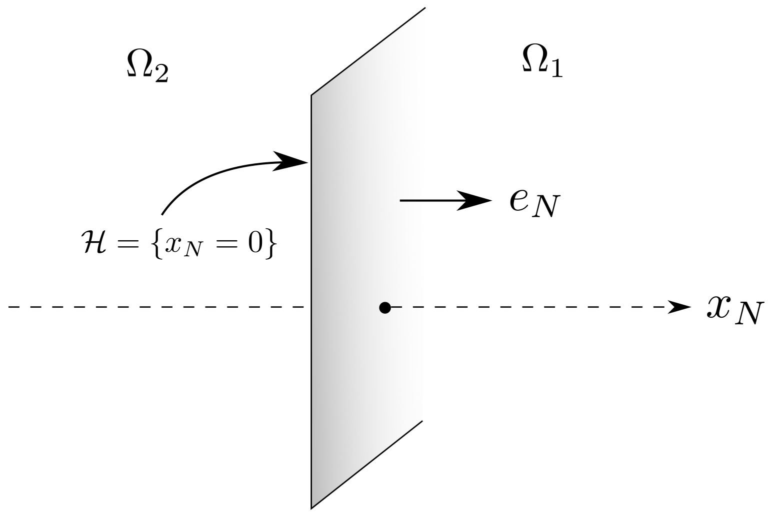


Figure 6.1: Setting of the codimension one case

where $D_T u$ stands for the *tangential derivative* of u, *i.e.* the $(N-1)$ first components of the gradient, leaving out the normal derivative. However, for reasons that will be exposed later in Section 6.4, adding such a condition is not completely tractable in the context of Ishii solutions and is more relevant in the context of flux-limited solutions or junction conditions. So, except for Section 6.4 we restrict ourselves to problem (6.1).

As we explained in Section 3.1, the conditions on $\mathcal{H}$ for those equations have to be understood in the relaxed (Ishii) sense, namely

$$\begin{cases} \max\Big(u_t + H_1(x,t,u,Du), u_t + H_2(x,t,u,Du)\Big) \geq 0\,, \\ \min\Big(u_t + H_1(x,t,u,Du), u_t + H_2(x,t,u,Du)\Big) \leq 0\,, \end{cases} \tag{6.3}$$

meaning that for the supersolution [resp. subsolution] condition, at least one of the inequation has to hold.

From the control viewpoint, we assume to have different dynamics, discount factors and costs on Ω_1, Ω_2 and a double question arises: how to define a global control problem in $\mathbb{R}^N$? and, once this is done, if each Hamiltonian in (6.1) is associated to the control problem in the corresponding domain, is the "usual" value function the unique solution of (6.1)?

In this chapter, we are going to combine the tools we introduced in Part I in order to address these problems. Assuming moreover that each Hamiltonian satisfies

(NC), **(TC)** and **(Mon)**, since the present stratification of $\mathbb{R}^N$ is obviously a typical AFS, we are in the situation of what we called a "good" framework for treating discontinuities in the sense of Definition 5.5.1 (here, no diffeomorphism is needed since the stratification is flat).

But, as we will see, Ishii's notion of solution is not strong enough to ensure comparison (and uniqueness) in general: this is already true for Equation (6.1) but the situation is even worse when adding (6.2) on $\mathcal{H}$. Let us give a brief overview of this story here.

The general formulation of control problems described in Chapter 4 provides a "natural" control solution of (6.1) obtained by minimizing a cost over all the possible trajectories, denoted by $\mathbf{U}^-$. Moreover, by Corollary 4.2.8, $\mathbf{U}^-$ is the minimal supersolution (and solution) of (6.1).

But there exists another value function denoted by $\mathbf{U}^+$ where we minimize over a subset of those trajectories, that are called *regular*. It can be shown that $\mathbf{U}^+$ is also an Ishii solution of (6.1), it is even the maximal Ishii (sub)solution of (6.1). In general $\mathbf{U}^- \neq \mathbf{U}^+$ and we provide an explicit example of such a configuration. Finally both $\mathbf{U}^-$ and $\mathbf{U}^+$ can be characterized by means of an additional "tangential" Hamiltonian on $\mathcal{H}$. Later in this part, we will also see that $\mathbf{U}^+$ is the limit of the vanishing viscosity method.

At this point, the reader may think that there is no difference when adding (6.2) to problem (6.1), after modifying in a suitable way the specific control problem on $\mathcal{H}$. It is, of course, the case for $\mathbf{U}^-$ where again the general results of Chapter 4 apply.

But the determination of the maximal Ishii (sub)solution is more tricky: to understand why, we refer the reader to the Dirichlet/exit time problem for deterministic control problem in a domain; it is shown in [17] that, if the minimal solution of the Dirichlet problem is actually given by an analogue of the value function $\mathbf{U}^-$ for such problems, the maximal one is obtained by considering the "worse stopping time" on the boundary (see also [23]). This *differential game* feature arises here in a more complicated way and we give some elements to understand it in Section 6.4.

In the next three sections, we give a complete study of (6.1): we first introduce the control problem; then we define and characterize $\mathbf{U}^-$ and finally we construct $\mathbf{U}^+$. We discuss the problem of adding (6.2) in the last Section 6.4.

6.1 The Control Problem and the "Natural" Value Function

Assuming that (6.1) is associated to a control problem and recalling that we also denote by $\Omega_0 = \mathcal{H}$, means that there exists some triplets dynamics-discount-cost $(b_i, c_i, l_i) : \bar{\Omega}_i \times [0,T] \times A_i \to \mathbb{R}^{N+3}$ for $i = 1, 2$, such that for any $(x,t,u,p) \in \bar{\Omega}_i \times (0,T] \times \mathbb{R} \times \mathbb{R}^N$,

$$H_i(x,t,u,p) = \sup_{\alpha_i \in A_i} \left\{ -b_i(x,t,\alpha_i) \cdot p + c_i(x,t,\alpha_i)u - l_i(x,t,\alpha_i) \right\}.$$

In the following, we assume that all these (b_i, c_i, l_i) which can be assumed as well to be defined on $\mathbb{R}^N \times [0,T] \times A_i$, satisfy the basic assumptions $(\mathbf{H_{BA-CP}})$ and the normal controllability assumption

$(\mathbf{NC}_\mathcal{H})$ For any $(x,t) \in \mathcal{H} \times [0,T]$, there exists $\delta = \delta(x,t)$ and a neighborhood $\mathcal{V} = \mathcal{V}(x,t)$ such that, for any $(y,s) \in \mathcal{V}$

$$[-\delta, \delta] \subset \{b_1(y,s,\alpha_1) \cdot e_N,\ \alpha_1 \in A_1\} \quad \text{if } (y,s) \in \overline{\Omega_1}\,,$$

$$[-\delta, \delta] \subset \{b_2(y,s,\alpha_2) \cdot e_N,\ \alpha_2 \in A_2\} \quad \text{if } (y,s) \in \overline{\Omega_2}\,,$$

where $e_N = (0, 0 \cdots, 0, 1) \in \mathbb{R}^N$.

It is easy to check that Assumption $(\mathbf{NC}_\mathcal{H})$ implies **(NC)** for H_1 and H_2 and we refer below to assumptions $(\mathbf{H_{BA-CP}})$ for (b_i, c_i, l_i), $i = 1, 2$ and $(\mathbf{NC}_\mathcal{H})$ as the "standard assumptions in the co-dimension-1 case".

6.1.1 Finding Trajectories by Differential Inclusions

In order to introduce the sets-valued map $\mathbf{BCL}$, we notice that all the equations in (6.1) and (6.23) have the form "$u_t + H(x,t,u,p)$" and therefore we are in the case when $b_i^t(x,s,\alpha_i) = -1$ for all $i = 1, 2$, for all $(x,s,\alpha_i) \in \bar{\Omega}_i \times (0,T] \times A_i$. Then we introduce following two set-valued maps $\mathbf{BCL}_i(x,t) := ((b_i, -1), c_i, l_i)(x,t,A_i)$ for $i = 1, 2$ and $t \geq 0$:

$$\mathbf{BCL}(x,t) := \begin{cases} \mathbf{BCL}_1(x,t) & \text{if } x \in \Omega_1\,, \\ \mathbf{BCL}_2(x,t) & \text{if } x \in \Omega_2\,, \\ \overline{\mathrm{co}}(\mathbf{BCL}_1, \mathbf{BCL}_2)(x,t) & \text{if } x \in \mathcal{H}\,, \end{cases}$$

where $\overline{\mathrm{co}}(E_1, E_2)$ denotes the convex hull of the sets E_1, E_2, *i.e.* the union of all possible convex combinations of elements in E_1, E_2.

For $t = 0$ we need to add more information: since we consider a finite horizon problem, we have to be able to stop the trajectory at time $s = 0$, and we want the initial condition $u(0) = u_0$ to be encoded through the Hamiltonian $H_{\mathrm{init}}(x, u, Du) = u - u_0$. So, setting $\mathrm{Init}(x) := \{(0,0), 1, u_0(x)\}$, we are led to define

$$\mathbf{BCL}(x,0) := \begin{cases} \overline{\mathrm{co}}(\mathbf{BCL}_1(x,0), \mathrm{Init}(x)) & \text{if } x \in \Omega_1\,, \\ \overline{\mathrm{co}}(\mathbf{BCL}_2(x,0), \mathrm{Init}(x)) & \text{if } x \in \Omega_2\,, \\ \overline{\mathrm{co}}(\mathbf{BCL}_1(x,0), \mathbf{BCL}_2(x,0), \mathrm{Init}(x)) & \text{if } x \in \mathcal{H}\,. \end{cases} \tag{6.4}$$

We have defined rigourously **BCL** following the general framework described in Part I- Chapter 4 but, since we are mainly in a case where $b^t = -1$, we are going to drop from now on the b^t-part in **BCL** and, in order to simplify the notations, we just write $b = b^x$. In fact, the only place where b^t plays a role is $t = 0$ because **BCL** contain all the time dynamics $b^t \in [-1, 0]$ because of the convex hull. But, in our case, the initial conditions reduce to

$$u(x,0) \leq (u_0)^*(x) \quad \text{and} \quad v(x,0) \geq u_0(x) \quad \text{in } \mathbb{R}^N\,,$$

for a subsolution u and a supersolution v, hence they produce no additional difficulty.

The very first checking in order to solve the control problem is the

Lemma 6.1.1 *The set-valued map* **BCL** *satisfies* $(\mathbf{H_{BCL}})$*.*

Proof — Concerning $(\mathbf{H_{BCL}})_{fund}$, the proof is quite straightforward by construction: first notice that since all the b_i, l_i, c_i are bounded by some constant $M > 0$, then it is the same for all the elements in **BCL**. Then, by construction $\mathbf{BCL}(x,t)$ is closed, hence compact, and it is convex. It remains to see that $(x,t) \mapsto \mathbf{BCL}(x,t)$ is upper semi-continuous which is clear since each $\mathbf{BCL}_i(x,t)$ is upper semi-continuous and we just make a convex hull of them.

We turn now to $(\mathbf{H_{BCL}})_{struct}$, which follows almost immediatly from (6.4): (i) is obviously satisfied by our choice for b^t which always belongs to $[-1, 0]$. Point (ii) clearly holds if $s > 0$. Indeed, if we choose $K = M$ (the constant appearing in $(\mathbf{H_{BCL}})_{fund}$), since $b^t = -1$ for $s > 0$ we get the inequality. Now, if $s = 0$ the inequality comes from the fact that $-Kb^t + c \geq c = 1$. Point (iii) is included in (6.4) and point (iv) follows from the fact that this condition can only happen for $s = 0$ here (otherwise $b^t = -1$), in which case we have $\underline{c} = c = 1 > 0$.

Q.E.D.

Thanks to Theorem 4.2.1 (and recalling that we have dropped the $b^t = -1$ term), we solve the differential inclusion

$$\begin{cases} \dfrac{\mathrm{d}}{ds}(X,D,L)(s) \in \mathbf{BCL}\big(X(s), t-s\big) & \text{for a.e. } s \in [0,+\infty)\,, \\ (X,D,L)(0) = (x,0,0)\,, \end{cases} \tag{6.5}$$

Notice that we have used the fact that $T(s) = t - s$ when the starting point of the (X,T)-trajectory is (x,t).

Now the aim is to give a more precise description of each trajectory. For the sake of clarity, we denote by $(b_\mathcal{H}, c_\mathcal{H}, l_\mathcal{H})$ the (b,c,l) when $X(s) \in \mathcal{H}$. Here there is a slight and notice that of course, on $\mathcal{H}$ we make a convex combination of all the (b_i, c_i, l_i), $i = 1, 2$.

In order to take into account these convex combinations, we introduce the "extended control space", $A := A_1 \times A_2 \times [0,1]^2$ and $\mathcal{A} := L^\infty(0,T;A)$. The extended control takes the form $a = (\alpha_1, \alpha_2, \mu_1, \mu_2)$ and if $x \in \mathcal{H}$,

$$(b_\mathcal{H}, c_\mathcal{H}, l_\mathcal{H}) = \sum_{i=1}^{2} \mu_i (b_i, c_i, l_i)\,,$$

with $\mu_1 + \mu_2 = 1$.

Lemma 6.1.2 *For any trajectory (X,D,L) of (6.5) there exists a control $a(\cdot) = (\alpha_1, \alpha_2, \mu_1, \mu_2)(\cdot) \in \mathcal{A}$ such that*

$$\begin{aligned}(\dot{X}, \dot{D}, \dot{L})(s) = &(b_1, c_1, l_1)(X(s), t-s, \alpha_1(s))\mathbb{1}_{\{X(s)\in\Omega_1\}} \\ &+ (b_2, c_2, l_2)(X(s), t-s, \alpha_2(s))\mathbb{1}_{\{X(s)\in\Omega_2\}} \\ &+ (b_\mathcal{H}, c_\mathcal{H}, l_\mathcal{H})(X(s), t-s, a(s))\mathbb{1}_{\{X(s)\in\mathcal{H}\}}\end{aligned}$$

and $b_\mathcal{H}(X(s), t-s, a(s)) \cdot e_N = 0$ for almost any $s \in (t,T)$ such that $X(s) \in \mathcal{H}$.

Proof — Given a trajectory, we apply Filippov's Lemma (cf. [7, Theorem 8.2.10]). To do so, we define the map $g : \mathbb{R}^+ \times A \to \mathbb{R}^N$ as follows

$$g(s,a) := \begin{cases} b_1\big(X(s), t-s, \alpha_1\big) & \text{if } X(s) > 0 \\ b_2\big(X(s), t-s, \alpha_2\big) & \text{if } X(s) < 0 \\ b_\mathcal{H}\big(X(s), t-s, a\big) & \text{if } X(s) = 0\,, \end{cases}$$

where $a = (\alpha_1, \alpha_2, \mu_1, \mu_2) \in A$.

We claim that g is a Caratheodory map. Indeed, it is first clear that, for fixed s, the function $a \mapsto g(s,a)$ is continuous. Then, in order to check that g is measurable with respect to its first argument we fix $a \in A$, an open set $\mathcal{O} \subset \mathbb{R}^N$ and evaluate

$$g_a^{-1}(\mathcal{O}) = \big\{ s > 0 : g(s,a) \cap \mathcal{O} \neq \emptyset \big\}$$

that we split into three components, the first one being

$$g_a^{-1}(\mathcal{O}) \cap \{ s > 0 : X(s) < 0 \} = \big\{ s > 0 : b_1(X(s), t-s, \alpha_1) \in \mathcal{O} \big\} \cap \{ s > 0 : X(s) < 0 \} .$$

Since the function $s \mapsto b_1(X(s), t-s, \alpha_1)$ is continuous, this set is the intersection of open sets, hence it is open and therefore measurable. The same argument works for the other components, namely $\{s > 0 : X(s) < 0\}$ and $\{s > 0 : X(s) = 0\}$ which finishes the claim.

The function $s \mapsto \dot{X}(s)$ is measurable and, for any s, the differential inclusion implies that

$$\dot{X}(s) \in g(s, A) ,$$

therefore, by Filippov's Lemma, there exists a measurable map $a(\cdot) = (\alpha_1, \alpha_2, \mu_1, \mu_2)(\cdot) \in \mathcal{A}$ such that (6.6) is fulfilled. In particular, by the definition of g, we have for a.e. $s \in [0,T]$

$$\dot{X}(s) = \begin{cases} b_1\big(X(s), t-s, \alpha_1(s)\big) & \text{if } X(s) > 0 \\ b_2\big(X(s), t-s, \alpha_2(s)\big) & \text{if } X(s) < 0 \\ b_{\mathcal{H}}\big(X(s), t-s, a(s)\big) & \text{if } X(s) = 0. \end{cases} \tag{6.6}$$

The last property is a consequence of Stampacchia's theorem (see for instance [73]): setting $y(s) := X_N(s)$, then $\dot{y}(s) = 0$ almost everywhere on the set $\{y(s) = 0\}$. But $\dot{y}(s) = b_{\mathcal{H}}(X(s), t-s, a(s)) \cdot e_N$ on this set, so the conclusion follows.

Q.E.D.

6.1.2 The $\mathbf{U}^-$ Value-Function

Solving (6.5) with **BCL** gives us the set $\mathcal{T}(x,t)$ of all admissible trajectories, without specific condition on $\mathcal{H}$ for (6.1) (see Section 4.2.3). Changing slightly the notations of this section to emphasize the role of the control $a(\cdot)$, we can define the two value functions

$$\mathbf{U}^-(x,t) := \inf_{\mathcal{T}(x,t)} \left\{ \int_0^t l(X(s), t-s, a(s)) \exp(-D(s))\, \mathrm{d}s + u_0(X(t)) \exp(-D(t)) \right\} ,$$

The aim is now to prove that $\mathbf{U}^-$ is a viscosity solution of (6.1).

To do so, we use the control approach described in Section 4.2: we introduce the "global" Hamiltonians[(1)] given, for any $x \in \mathbb{R}^N$, $t \in (0,T)$, $u \in \mathbb{R}$, $p \in \mathbb{R}^N$ by

$$H(x,t,u,p) := \sup_{(b,c,l)\in \mathbf{BCL}(x,t)} \big(- b \cdot p + cu - l \big) \, .$$

Of course $H(x,t,u,p) = H_i(x,t,u,p)$ if $x \in \Omega_i$ for $i = 1,2$ and with the notations of Section 4.2, we have for any $x \in \mathbb{R}^N$, $t \in (0,T)$, $u \in \mathbb{R}$, $p_x \in \mathbb{R}^N$, $p_t \in \mathbb{R}$

$$\mathbb{F}(x,t,u,(p_x,p_t)) := p_t + H(x,t,u,p) \, .$$

By the upper-semicontinuity of $\mathbf{BCL}$, H, and $\mathbb{F}$ are upper-semi-continuous and we have the

Lemma 6.1.3 *If $x \in \mathcal{H}$ then, for all $t \in (0,T)$, $r \in \mathbb{R}$, $p \in \mathbb{R}^N$*

$$H(x,t,r,p) = \max \Big(H_1(x,t,r,p), H_2(x,t,u,p) \Big) \, .$$

Therefore, for any $x \in \mathbb{R}^N$, $t \in (0,T)$, $u \in \mathbb{R}$, $p_x \in \mathbb{R}^N$, $p_t \in \mathbb{R}$

$$\mathbb{F}(x,t,u,(p_x,p_t)) = \max \big(p_t + H_1(x,t,u,p), p_t + H_2(x,t,u,p) \big) \, .$$

Proof — If $(b,c,l) \in \mathbf{BCL}(x,t)$ then (b,c,l) can be written as a convex combination of some $(b_i,c_i,l_i)\mathbf{BCL}_i(x,t)$, $i = 1,2$, and thefore the same is true for $-b \cdot p + cu - l$, namely

$$-b \cdot p + cr - l = \sum_i \mu_i(-b_i \cdot p + c_i r - l_i) \, ,$$

for some $0 \le \mu_i \le 1$ with $\sum_i \mu_i = 1$. This easily gives that $-b \cdot p + cr - l \le \max \Big(H_1(x,t,r,p), H_2(x,t,u,p) \Big)$ since $(-b_i \cdot p + c_i r - l_i) \le H_i(x,t,u,p)$, and therefore

$$H(x,t,r,p) \le \max \Big(H_1(x,t,r,p), H_2(x,t,u,p) \Big) \, .$$

But we also have $H(x,t,r,p) \ge (-b_i \cdot p + c_i r - l_i)$ for any $(b_i,c_i,l_i)\mathbf{BCL}_i(x,t)$ and therefore $H(x,t,r,p) \ge H_i(x,t,r,p)$. And the max property follows. Of course the equality for $\mathbb{F}$ follows immediately.

Q.E.D.

From all these properties on $\mathbb{F}$ we easily deduce

$$(\mathbb{F})_*(x,t,u,(p_x,p_t)) \ge \min \big(p_t + H_1(x,t,u,p), p_t + H_2(x,t,u,p) \big) \, .$$

By using all the results of Section 4.2, we have the

[(1)] where we have dropped the b^t–p_t part since $b^t \equiv -1$.

Proposition 6.1.4 *Assume that the "standard assumptions in the co-dimension-*1 *case" are satisfied. Then the value function* $\mathbf{U}^-$ *is an Ishii viscosity solutions of* (6.1)*. Moreover* $\mathbf{U}^-$ *is the minimal supersolution of* (6.1)*.*

We leave the proof of the reader since it immediately follows from Theorem 4.2.5, Corollary 4.2.8 and Theorem 4.2.9. This result gives almost all the informations we wish to have on $\mathbf{U}^-$ (but not all of them...). To go further, we have to examine more carefully the viscosity inequality on $\mathcal{H}$.

6.1.3 The Complemented Equations

This section is motivated in particular by Lemma 6.1.2 where the term $(b_\mathcal{H}, c_\mathcal{H}, l_\mathcal{H})$ plays a key role as a coupling between the control problems in Ω_1, Ω_2. It seems therefore essential in the study of the discontinuity on $\mathcal{H}$ in both the control problem and the equation(s).

We introduce the tangential elements in **BCL** which maintain the trajectories on $\mathcal{H}$: for any $x \in \mathcal{H}$, $t \in (0,T)$, we set

$$\mathbf{BCL}_T(x,t) := \Big\{(b,c,l) \in \mathbf{BCL}(x,t) : b \cdot e_N = 0\Big\}\,.$$

Similarly we define $\mathbf{B}_T(x,t)$ for the set-valued map of tangential dynamics.

A tangential dynamic $b \in \mathbf{B}_T(x,t)$ can be expressed as a convex combination

$$b = \mu_1 b_1 + \mu_2 b_2 \tag{6.7}$$

for which $\mu_1 + \mu_2 = 1$, $\mu_1, \mu_2 \in [0,1]$ and $(\mu_1 b_1 + \mu_2 b_2) \cdot e_N = 0$

Using these definitions, we introduce tangential Hamiltonian which was already considered in Section 5.4:

$$H_T(x,t,u,p) := \sup_{\mathbf{BCL}_T(x,t)} \Big\{ -b \cdot p + cu - l \Big\}\,. \tag{6.8}$$

The interest of this tangential Hamiltonian is for the

Proposition 6.1.5 *Assume that the "standard assumptions in the co-dimension-*1 *case" are satisfied. Then the value function* $\mathbf{U}^-$ *satisfies on* $\mathcal{H}$ *the inequality*

$$(\mathbf{U}^-)^*_t + H_T(x,t,(\mathbf{U}^-)^*, D_T(\mathbf{U}^-)^*) \le 0 \quad \textit{on } \mathcal{H} \times (0,T)\,.$$

We point out that in Proposition 6.1.5, the viscosity inequalities are $\mathcal{H} \times (0,T)$-viscosity inequalities, which means that we look at maximum points of $(\mathbf{U}^-)^* - \phi$ or $(\mathbf{U}^-)^* - \phi$ on $\mathcal{H} \times (0,T)$ where ϕ is a smooth test-function on $\mathcal{H} \times (0,T)$.

Remark 6.1.6 *In other words,* $\mathbf{U}^-$ *is an Ishii solution satisfying a complemented* H_T*-inequality on* $\mathcal{H}$*, so it is a stratified solution of the problem (actually, we will prove that it is the unique stratified solution).*

Proof — If ϕ is a smooth test-function, we have to prove that, if $(x,t) \in \mathcal{H} \times (0,T)$ is a maximum points on $\mathcal{H} \times (0,T)$ of $(\mathbf{U}^-)^* - \phi$, then (assuming without loss of generality that $(\mathbf{U}^-)^*(x,t) = \phi(x,t)$)

$$\phi_t(x,t) + H_T(x,t,\phi(x,t), D_T\phi(x,t)) \leq 0 \quad \text{on } \mathcal{H} \times (0,T)\,.$$

The first point concerns the computation of $(\mathbf{U}^-)^*(x,t)$ for which we have the

Lemma 6.1.7 *Assume that the "standard assumptions in the co-dimension-*1 *case" are satisfied, then*

$$((\mathbf{U}^-)_{|\mathcal{H}\times(0,T)})^* = (\mathbf{U}^-)^* \quad \text{on } \mathcal{H} \times (0,T)\,,$$

where $(\mathbf{U}^-_{(\mathcal{H})})_{|\mathcal{H}\times(0,T)}$ *denotes the restriction to* $\mathcal{H} \times (0,T)$ *of* $\mathbf{U}^-$*.*

Proof — By definition of $(\mathbf{U}^-)^*$, there exists a sequence $(x_n,t_n) \to (x,t)$ such that $\mathbf{U}^-(x_n,t_n) \to (\mathbf{U}^-)^*(x,t)$. The statement of Lemma 6.1.7 means that we can assume that $x_n \in \mathcal{H}$. Indeed, if $x_n \in \Omega_1$, we use the normal controllability assumption $(\mathbf{NC}_\mathcal{H})$ at (x,t): there exists $\delta > 0$ and a control α_1 such that $b_1(x,t,\alpha_1) \cdot e_N = -\delta < 0$. Considering the trajectory with the constant control α_1

$$\dot{Y}(s) = b_1(Y(s), t_n - s, \alpha_1) \quad , \quad Y(0) = x_n, \tag{6.9}$$

it is easy to show that τ_n^1, the first exit time of the trajectory Y from Ω_1 tends to 0 as $n \to +\infty$. By the Dynamic Programming Principle, denoting $(\tilde{x}_n, \tilde{t}_n) = (X(\tau_n^1), t - \tau_n^1)$, we have

$$\mathbf{U}^-(x_n,t_n) \leq \int_0^{\tau_n^1} l\big(Y(s), t_n - s, \alpha_1\big)\, e^{-D(s)}\, \mathrm{d}s + \mathbf{U}^-(\tilde{x}_n,\tilde{t}_n)\, e^{-D(\tau_n^1)} = \mathbf{U}^-(\tilde{x}_n,\tilde{t}_n) + o_n(1),$$

where $o_n(1) \to 0$. Therefore $(\tilde{x}_n,\tilde{t}_n) \to (x,t)$, $\mathbf{U}^-(\tilde{x}_n,\tilde{t}_n) \to (\mathbf{U}^-)^*(x,t)$ and $\tilde{x}_n \in \mathcal{H}$, which is exactly what we wanted to prove. The same results holds if $x_n \in \Omega_2$ using a control such that $b_2(x,t,\alpha_2) \cdot e_N = \delta > 0$.

Q.E.D.

Therefore, we can pick a sequence $(x_n, t_n) \to (x, t)$ such that $\mathbf{U}^-(x_n, t_n) \to (\mathbf{U}^-)^*(x, t)$ with $x_n \in \mathcal{H}$. Using the maximum point property we can insert the test-function ϕ in the dynamic programming principle and get

$$\phi(x_n, t_n) + o_n(1) \leq \inf_{\mathcal{T}(x_n,t_n)} \Big(\int_0^\tau \ell\big(X_n(s), t_n - s, a(s)\big)\, e^{-D_n(s)}\, \mathrm{d}s + \phi(X_n(\tau), t_n - \tau)\, e^{-D_n(\tau)} \Big) . \tag{6.10}$$

Our aim is to show that this inequality implies

$$\phi_t(x,t) - b \cdot D\phi(x,t) + c\phi(x,t) - l \leq 0 \ ,$$

for any $(b, c, l) \in \mathbf{BCL}_T(x, t)$.

By definition of $\mathbf{BCL}_T(x, t)$, (b, c, l) can be expressed as a convex combination of the (b_i, c_i, l_i) for $i = 1, 2$, namely

$$(b, c, l) = \mu_1(b_1, c_1, l_1) + \mu_2(b_2, c_2, l_2)$$

with $\mu_1 + \mu_2 = 1$, $\mu_1, \mu_2 \in [0, 1]$ and $(\mu_1 b_1 + \mu_2 b_2) \cdot e_N = 0$. We denote by α_i the control which is associated to (b_i, c_i, l_i)

Slightly modifying b_1 and b_2 by using the normal controllability on $\mathcal{H}$, we may assume without loss of generality that $b_1 \cdot e_N \neq 0$ and $b_2 \cdot e_N \neq 0$. But $(\mu_1 b_1 + \mu_2 b_2) \cdot e_N = 0$ and therefore we have two cases either $b_1 \cdot e_N < 0 < b_2 \cdot e_N$ or $b_1 \cdot e_N > 0 > b_2 \cdot e_N$.

We have the

Lemma 6.1.8 *For any $(x, t) \in \mathcal{H} \times (0, T)$ and for any $(b, c, l) \in \mathbf{BCL}_T(x, t)$ defined as above, if $(b_1(x, t, \alpha_1) \cdot e_N).(b_2(x, t, \alpha_1) \cdot e_N) < 0$, there exists a neighborhood $\mathcal{V}$ of (x, t) in $\mathcal{H} \times (0, T)$ and a Lipschitz continuous map $\psi : \mathcal{V} \to \mathbb{R}^N \times \mathbb{R} \times \mathbb{R}$, such that $\psi(x, t) = (b, c, l)$ and $\psi(y, s) = (\tilde{b}(y, s), \tilde{c}(y, s), \tilde{l}(y, s)) \in \mathbf{BCL}_T(y, s)$ for any $(y, s) \in \mathcal{V}$.*

Proof — Our assumption means that

$$(\mu_1 b_1(x, t, \alpha_1) + \mu_2 b_2(x, t, \alpha_2)) \cdot e_N = 0 \ .$$

If (y, s) is close enough to (x, t), we set

$$\mu_1^\sharp(y, s) := \frac{b_2(y, s, \alpha_2) \cdot e_N}{(b_2(y, s, \alpha_1) - b_1(y, s, \alpha_1)) \cdot e_N} \ , \quad \mu_2^\sharp := 1 - \mu_1^\sharp \ .$$

By this choice we have $0 \leq \mu_1^\sharp, \mu_2^\sharp \leq 1$ and $\Big(\mu_1^\sharp(y, s) b_1(y, s, \alpha_1) + \mu_2^\sharp(y, s) b_2(y, s, \alpha_2)\Big) \cdot e_N = 0$, hence we have a tangential dynamic which is well-defined as long as $(b_2(y, s, \alpha_1) - b_1(y, s, \alpha_1)) \cdot e_N \neq 0$, and in particular in a neighborhood of (x, t).

The function ψ given

$$\psi(y,s) := \mu_1^\sharp(y,s)(b_1,c_1,l_1) + \mu_2^\sharp(y,s)(b_2,c_2,l_2)\ ,$$

satisfies all the desired properties; it is Lipschitz continuous since b_1, b_2 are Lipschitz continuous in x and t and $\mu_1^\sharp(x,t) = \mu_1$, $\mu_2^\sharp(x,t) = \mu_2$.

Q.E.D.

In order to conclude, we first solve the ode

$$(\dot{X}_n(s), \dot{D}_n(s), \dot{L}_n(s)) = \psi(X_n(s), t_n - s)\ ,$$

with $(X_n(0), D_n(0), L_n(0)) = (x_n, 0, 0)$, where ψ is given by Lemma 6.1.8. Because of the properties of ψ, Cauchy-Lipschitz Theorem implies that there exists a unique solution which, for (x_n, t_n) close enough to (x,t), is defined on a small but fixed interval of time and $(X_n, D_n, L_n) \in \mathcal{T}(x_n, t_n)$ for any n. Therefore, (6.10) implies

$$\phi(x_n,t_n) + o_n(1) \le \int_0^\tau \ell\big(X_n(s), t_n - s, a(s)\big)\, e^{-D_n(s)}\, \mathrm{d}s + \phi(X_n(\tau), t_n - \tau)\, e^{-D_n(\tau)}\,.$$

In this inequality, we can let n tend to infinity, using the continuity of the trajectory with respect to (x_n, t_n)

$$\phi(x,t) \le \int_0^\tau \ell\big(X(s), t - s, a(s)\big)\, e^{-D(s)}\, \mathrm{d}s + \phi(X(\tau), t - \tau)\, e^{-D(\tau)}\,,$$

and the conclusion follows from the fact that $\psi(x,t) = (b,c,l)$.

Q.E.D.

6.1.4 A Characterization of $\mathbf{U}^-$

In the previous section, we have seen that $\mathbf{U}^-$ satisfies an additional subsolution inequality on $\mathcal{H} \times (0,T)$. The aim of this section is to show that this additional inequality is enough to characterize it.

The precise result is the

Theorem 6.1.9 *Assume that the "standard assumptions in the co-dimension-1 case" are satisfied. Then $\mathbf{U}^-$ is the unique Ishii solution of (6.1) such that*

$$u_t + H_T(x,t,u,D_T u) \le 0 \quad on \quad \mathcal{H} \times (0,T)\,. \tag{6.11}$$

Proof — The proof is obtained by a combination of arguments which will also be used in the next part for the stratified problems.

We recall that we already know (cf. Proposition 6.1.4) that $\mathbf{U}^-$ is the minimal Ishii supersolution of (6.1). Therefore we only need to compare $\mathbf{U}^-$ with subsolutions u such that $u_t + H_T(x,t,u,D_T u) \leq 0$ on $\mathcal{H} \times (0,T)$. Our aim is therefore to prove that $\mathbf{U}^- \geq u$ in $\mathbb{R}^N \times [0,T]$ and this is obtained through the conjonction of several arguments exposed in Part I.

Step 1: *Reduction to a local comparison result* **(LCR)** – As already noticed in Part I (see Remarks on page 21), setting $\tilde{u}(x,t) := \exp(Kt)u(x,t)$ for $K > 0$ large enough allows to reduce the proof to the case where $c_i \geq 0$ for any $(b_i, c_i, l_i) \in \mathbf{BCL}_i(x,t)$, $i = 1,2$. As a consequence, we can assume that the H_i $(i = 1,2)$ are nondecreasing in the u-variable, and that H_T enjoys the same property.

Then, rewriting here some arguments already given in Section 3.2 and using that the c_i's are positive, we notice that, for $\delta > 0$ small enough, $\psi(x,t) = -\delta(1+|x|^2)^{1/2} - \delta^{-1}(1+t)$ is a $\delta/2$-strict subsolution (6.1) but also for the H_T-equation on $\mathcal{H} \times (0,T)$ and we can also assume that $\psi \leq u$ in $\mathbb{R}^N \times [0,T]$. Then we set, for $\mu \in (0,1)$

$$u_\mu(x,t) := \mu u(x,t) + (1-\mu)\psi(x,t)$$

yields a η-strict subsolution u_μ for some $\eta(\mu,\delta) > 0$. By this, we means that each inequality in (6.1) is η-strict for u_μ but also that $(u_\mu)_t + H_T(x,t,u_\mu,Du_\mu) \leq \eta < 0$ on $\mathcal{H} \times (0,T)$. This claim is obvious for the initial data, let us prove it for instance for H_1.

Using the convexity property of H_1 in r,p, we get successively

$$\begin{aligned}
&(u_\mu)_t + H_1(x,t,u_\mu,Du_\mu)\\
&= \mu u_t + (1-\mu)\psi_t + H_1(x,t,\mu u + (1-\mu)\psi, \mu Du + (1-\mu)D\psi)\\
&\leq \mu u_t + (1-\mu)\psi_t + \mu H_1(x,t,u,Du) + (1-\mu)H_1(x,t,\psi,D\psi)\\
&\leq \mu\big\{u_t + H_1(x,t,u,Du)\big\} + (1-\mu)\big\{\psi_t + H_1(x,t,\psi,D\psi)\big\}\\
&\leq \mu\big\{u_t + H_1(x,t,u,Du)\big\} - (1-\mu)(\delta/2) \leq -(1-\mu)(\delta/2) < 0\,.
\end{aligned}$$

The same is valid for H_2 and H_T for similar reasons. Moreover, by construction $u_\mu - \mathbf{U}^- \to -\infty$ as $|x| \to +\infty$ since $\psi(x,t) \to -\infty$ as $|x| \to +\infty$, so that (**LOC1**) is satisfied for any of those Hamiltonians.

Checking (**LOC2**) is easier: if we are looking for a comparison result around the point (x_0,t_0), it is enough to use

$$u_{\delta'}(x,t) := u(x,t) - \delta'(|x-x_0|^2 + |t-t_0|^2)$$

for $\delta' > 0$ small enough. Thus we are in the situation where a **(LCR)** is enough to ensure a **(GCR)**.

So let us introduce $Q^{x,t}_{r,h}$, a (small) cylinder around (x, t) where we want to perform the **(LCR)**. Notice that of course, if $x \in \Omega_1$ or Ω_2, then taking r small enough reduces the proof to the standard comparison result since in this case, $Q^{x,t}_{r,h}$ does not intersect with $\mathcal{H}$. Thus, we assume in the following that $x \in \mathcal{H}$. Our aim is to use Lemma 5.4.1 with $\mathcal{M} := (\mathcal{H} \times [0, T]) \cap \overline{Q^{x,t}_{r,h}}$ and $\mathbb{F}^{\mathcal{M}}(x, t, r, (p_x, p_t)) := p_t + H_T(x, t, r, p_x)$.

Step 2: *Approximation of the subsolution* – We wish to use an approximation by convolutions (inf-convolution and usual convolution with a smoothing kernel) for the subsolution as in Proposition 3.4.4; to do so, we introduce a slightly larger cylinder $Q^{x,t}_{r',h'}$ where $r' > r$ and $h' > h$ are fixed in order to have some "room" for those convolutions. From Step 1, we know that u_μ is an η-strict subsolution of (6.1) in $Q^{x,t}_{r',h'}$ for some $\eta = \eta(\mu, \delta)$.

Since $(\mathbf{H_{Conv}})$, **(NC)**, **(TC)** and (**Mon**-u) are satisfied for all the Hamiltonians, we deduce from Proposition 3.4.4 that there exists a sequence $(u_{\mu,\varepsilon})_\varepsilon$ of $C^0(\overline{Q^{x,t}_{r,h}}) \cap C^1(\mathcal{M})$ functions which are all $(\eta/2)$-strict subsolutions of (6.1) in some smaller cylinder $Q(\varepsilon) \subset Q^{x,t}_{r',h'}$, and $Q(\varepsilon) \to Q^{x,t}_{r',h'}$ as $\varepsilon \to 0$ in the sense of the euclidian distance in $\mathbb{R}^{N+1}$. So, for ε small enough, we can assume with no restriction that $Q^{x,t}_{r,h} \subset Q(\varepsilon) \subset Q^{x,t}_{r',h'}$ so that $u_{\mu,\varepsilon}$ is an $(\eta/2)$-strict subsolution in $Q^{x,t}_{r,h}$. This has two consequences

- (a) for any $\varepsilon > 0$ small enough, $(u_{\mu,\varepsilon})_t + H_T(x, t, u_{\mu,\varepsilon}, D_T u_{\mu,\varepsilon}) \leq -\eta/2 < 0$ in $\mathcal{M}$ and in a classical sense since $u_{\mu,\varepsilon}$ is C^1 on $\mathcal{M}$;
- (b) since $u_{\mu,\varepsilon}$ is an $(\eta/2)$-strict subsolution in $\mathcal{O} := Q^{x,t}_{r,h} \setminus \mathcal{M}$ (for the Hamiltonians H_1, H_2) and a **(LCR)** holds there, we use the subdynamic programming principle for subsolutions (cf. Theorem 5.3.3) which implies that each $u_{\mu,\varepsilon}$ satisfies an $(\eta/2)$-strict dynamic programming principle in $Q^{x,t}_{r,h}[\mathcal{M}^c]$.

These two properties allow us to make a **(LCR)** in $Q^{x,t}_{r,h}$ in the final step.

Step 3: *Performing the local comparison* – From the previous step we know that for each $\varepsilon > 0$, $u = u_{\mu,\varepsilon}$ satisfies the hypotheses of Lemma 5.4.1. Using $v := \mathbf{U}^-$ as supersolution in this lemma, we deduce that

$$\forall (y, s) \in \overline{Q^{x,t}_{r,h}} \setminus \partial_P Q^{x,t}_{r,h}\,, \quad (u_{\mu,\varepsilon} - \mathbf{U}^-)(y, s) < \max_{\overline{Q^{x,t}_{r,h}}} (u_{\mu,\varepsilon} - \mathbf{U}^-)\,.$$

Using that $u_\mu = \limsup^* u_{\mu,\varepsilon}$, this yields a local comparison result (with inequality in the large sense) between u_μ and $\mathbf{U}^-$ as $\varepsilon \to 0$. By step 1, we deduce that the **(GCR)** holds: $u_\mu \leq \mathbf{U}^-$ in $\mathbb{R}^N \times [0, T]$, and sending finally $\mu \to 1$ gives that $u \leq \mathbf{U}^-$.

The conclusion is that if u is an Ishii solution such that $u_t + H_T(x,t,u,D_T u) \leq 0$ on $\mathcal{H}$, necessarily $u \equiv \mathbf{U}^-$, which ends the proof.

Q.E.D.

6.2 A Less Natural Value-Function: Regular and Singular Dynamics

In the study of $\mathbf{U}^-$, we have introduced the set $\mathbf{BCL}_T$ which are the subset of $\mathbf{BCL}$ where the dynamics are tangent to $\mathcal{H}$ and therefore we were examining the trajectories X which remains on $\mathcal{H}$. The new point in this section is to remark that there are two different kinds of dynamics that allow to stay on $\mathcal{H}$ as the following definition shows it.

Definition 6.2.1 *We say that $b \in \mathbf{B}_T(x,t)$ is regular if $b = \mu_1 b_1 + \mu_2 b_2$ with $b_1 \cdot e_N \leq 0 \leq b_2 \cdot e_N$. We denote by*

$$\mathbf{BCL}_T^{\mathrm{reg}}(x,t) := \Big\{(b,c,l) \in \mathbf{BCL}_T(x,t) : b \textit{ is regular } \Big\}$$

the set containing the regular tangential dynamics, and $\mathcal{T}^{\mathrm{reg}}(x,t)$ the set of controlled trajectories with regular dynamics on $\mathcal{H}$, i.e.

$$\mathcal{T}^{\mathrm{reg}}(x,t) := \Big\{(X,T,D,L) \textit{ solution of } (6.5) \textit{ such that} \\ \dot{X}(s) \in \mathbf{B}_T^{\mathrm{reg}}(X(s), t-s) \textit{ when } X(s) \in \mathcal{H}\Big\}.$$

In other terms, a regular dynamic corresponds to a "push-push" strategy: the trajectory is maintained on $\mathcal{H}$ because it is pushed on $\mathcal{H}$ from both sides, using only dynamics coming from Ω_1 and Ω_2 (not $\mathcal{H}$). On the contrary, the dynamic is said *singular* if $b_1 \cdot e_N > 0$ and $b_2 \cdot e_N < 0$, which is a "pull-pull" strategy, a quite instable situation where the trajectory remains on $\mathcal{H}$ because each side pulls in the opposite direction.

We remark that, by $(\mathbf{NC}_\mathcal{H})$, the sets $\mathbf{BCL}_T(x,t)$ and $\mathbf{BCL}_T^{\mathrm{reg}}(x,t)$ are non-empty for any $(x,t) \in \mathcal{H}$ (see Lemma 6.1.8) and, for $(x,t) \in \mathcal{H} \times (0,T)$, $r \in \mathbb{R}$ and $p \in \mathbb{R}^N$, we can defined the Hamiltonian

$$H_T^{\mathrm{reg}}(x,t,r,p) := \sup_{\mathbf{BCL}_T^{\mathrm{reg}}(x,t)} \Big\{ -b \cdot p + cu - l \Big\}. \tag{6.12}$$

We can now define another value-function obtained by minimizing only on regular trajectories

$$\mathbf{U}^+(x,t) := \inf_{\mathcal{T}^{\rm reg}(x,t)} \left\{ \int_0^t l(X(s),t-s,a(s))\exp(-D(s))\,\mathrm{d}s + u_0(X(t))\exp(-D(t)) \right\} .$$

Of course it is clear that $\mathbf{U}^- \leq \mathbf{U}^+$ in $\mathbb{R}^N \times [0,T]$ and we are going to study $\mathbf{U}^+$.

Our first result is the

Proposition 6.2.2 *Assume that the "standard assumptions in the co-dimension-1 case" are satisfied. Then the value function $\mathbf{U}^+$ is an Ishii solution of (6.1). Moreover $\mathbf{U}^+$ satisfies on $\mathcal{H} \times (0,T)$ the inequality*

$$(\mathbf{U}^+)^*_t + H^{\rm reg}_T(x,t,(\mathbf{U}^+)^*, D_T(\mathbf{U}^+)^*) \leq 0 \quad \text{on } \mathcal{H} \times (0,T)\,.$$

Before prooving this result, we want to make the following remark: most of the results we provide above for $\mathbf{U}^-$ were more or less direct consequences of results given in Chapter 4, in particular all the supersolution inequalities using Lemma 6.1.3. But this is not the case for $\mathbf{U}^+$.

Proof — Of course, the only difficulties comes from the discontinuity on $\mathcal{H} \times (0,T)$, therefore we concentrate on this case.

A priori $\mathbf{U}^+$ is not continuous, we have to use semi-continuous envelopes. In order to prove that $(\mathbf{U}^+)_*$ is a supersolution we consider a point $(x,t) \in \mathcal{H} \times (0,T)$ a strict local minimum point of $(\mathbf{U}^+)_* - \phi$, ϕ being a smooth test-function, and we can assume with no restriction that $(\mathbf{U}^+)_*(x,t) = \phi(x,t)$.

By definition of $(\mathbf{U}^+)_*$, there exists a sequence (x_n,t_n) which converges to (x,t) such that $\mathbf{U}^+(x_n,t_n) \to (\mathbf{U}^+)_*(x,t)$ and by the dynamic programming principle[(2)],

$$\mathbf{U}^+(x_n,t_n) = \inf_{\mathcal{T}(x_n,t_n)} \left\{ \int_0^\tau l\big(X_n(s),t_n - s,a(s)\big)\, e^{-D(s)}\,\mathrm{d}s + \mathbf{U}^+\big(X_n(\tau),t_n - \tau\big)\, e^{-D(\tau)} \right\},$$

where $\tau \ll 1$ and the notation X_n is to recall that this trajectory is associated with $X_n(0) = x_n$. Using that (i) $\mathbf{U}^+(x_n,t_n) = (\mathbf{U}^+)_*(x,t) + o_n(1)$ where $o_n(1) \to 0$, (ii) $\mathbf{U}^+\big(X_n(\tau),t_n-\tau\big) \geq \mathbf{U}^+_*\big(X_n(\tau),t_n-\tau\big)$ and the maximum point property, we obtain

$$\phi(x_n,t_n) + o_n(1) \geq \inf_{\mathcal{T}(x_n,t_n)} \left\{ \int_0^\tau l\big(X_n(s),t_n - s,a(s)\big)\, e^{-D(s)}\,\mathrm{d}s + \phi\big(X_n(\tau),t_n - \tau\big)\, e^{-D(\tau)} \right\}.$$

(2) There is no difficulty for showing that actually $\mathbf{U}^+$ satisfies the dynamic programming principle since all the arguments works with only regular trajectories.

Now we use the expansion of ϕ along the trajectory of the differential inclusion

$$\begin{aligned}\phi(X_n(\tau), t_n - \tau)\, e^{-D(\tau)} = \phi(x_n, t_n) + \int_0^\tau \Big(&- \partial_t \phi(X_n(s), t_n - s) \\ &+ b(X(s), t_n - s, a(s)) \cdot D\phi(X_n(s), t_n - s) \\ &+ c(X(s), t_n - s, a(s))\phi(X_n(s), t_n - s)\Big)\, \mathrm{d}s\end{aligned} \tag{6.13}$$

Pluging this expansion in the dynamic programming principle and using that the global Hamiltonian H is the sup over all the (b, c, l), we are led to

$$o_n(1) \leq \int_0^\tau \Big(\partial_t \phi(X_n(s), t_n - s) + H(X_n(s), t_n - s, \phi(X_n(s), t_n - s), D\phi(X_n(s), t_n - s))\Big)\, \mathrm{d}s\,.$$

Using the smoothness of ϕ and the upper semicontinuity of H together with the facts that $|X_n(s) - x|, |(t_n - s) - t| = o_n(1) + O(s)$, we can replace $X_n(s)$ by x and $t_n - s$ by t in the integral; hence we have, for τ small enough

$$o_n(1) \leq \tau\Big(\partial_t \phi(x, t) + H(x, t, \phi(x, t), D\phi(x, t))\Big)\,.$$

It remains to let first $n \to \infty$ and then, we divide by $\tau > 0$ and send $\tau \to 0$, which yields that $\partial_t \phi(x, t) + H(x, t, \phi, D\phi) \geq 0$. Using Lemma 6.1.3, we deduce that $\mathbf{U}^+$ satisfies the Ishii supersolution condition on $\mathcal{H} \times (0, T)$ and it is clear that the same proof works for $\mathbf{U}^+$.

For the subsolution condition, we have to consider $(x, t) \in \mathcal{H} \times (0, T)$ a local maximum points of $(\mathbf{U}^+)^* - \phi$, ϕ being a smooth function and we assume again that $(\mathbf{U}^+)^*(x, t) = \phi(x, t)$.

Ny definition of the upper semicontinuous envelope, there exists a sequence $(x_n, t_n) \to (x, t)$ such that $\mathbf{U}^+(x_n, t_n) \to (\mathbf{U}^+)^*(x, t)$ and we first claim that we can assume $x_n \in \mathcal{H}$. To prove it, we use exactly the same argument as in the proof of Lemma 6.1.7 for $\mathbf{U}^-$ since it realy only on the normal controllability assumption $(\mathbf{NC}_\mathcal{H})$ at (x, t).

Therefore, assuming that $x_n \in \mathcal{H}$, using the maximum point property we insert the test-function ϕ in the dynamic programming principle and get

$$\phi(x_n, t_n) + o_n(1) \leq \inf_{\mathcal{T}(x_n, t_n)} \Big(\int_0^\tau \ell\big(X_n(s), t_n - s, a(s)\big)\, e^{-D_n(s)}\, \mathrm{d}s + \phi(X_n(\tau), t_n - \tau)\, e^{-D_n(\tau)}\Big)\,. \tag{6.14}$$

We argue by contradiction: if

$$\min\big\{\phi_t(x, t) + H_1\big(x, t, \phi(x, t), D\phi(x, t)\big), \phi_t(x, t) + H_2\big(x, t, \phi(x, t), D\phi(x, t)\big)\big\} > 0\,,$$

then there exists some $(\alpha_1, \alpha_2) \in A_1 \times A_2$, such that, for all $i = 1, 2$

$$\phi_t(x,t) - b_i(x,t,\alpha_i) \cdot D\phi(x,t) + c_i(x,t,\alpha_i)\phi(x,t) - l_i(x,t,\alpha_i) > 0\,, \tag{6.15}$$

and the same is true, for n large enough, if we replace (x,t) by (x_n, t_n). Now we separate the proof in three cases according to the different configurations, using those constant controls in (6.14). Notice, as we already remarked above, that in what follows we derive the subsolution condition in the Ishii sense without using α_0 at all. For the sake of simplicity of notations, we just note below by b_i the quantity $b_i(x,t,\alpha_i)$.

Case 1 – assume that $b_1 \cdot e_N > 0$ or $b_2 \cdot e_N < 0$. In the first case, we use the trajectory (X_n, D_n, L_n) defined by with the constant control α_1. In particular

$$\dot{X}_n(s) = b_1(X_n(s), t_n - s, \alpha_1) \quad , \quad X_n(0) = x_n. \tag{6.16}$$

Then there exists a time $\tau > 0$ such that $X_n(s) \in \Omega_1$ for $s \in (0, \tau]$. Choosing such constant control α_1 in (6.14) and arguing as above, we are led to

$$\phi_t(x,t) - b_1(x,t,\alpha_1) \cdot D\phi(x,t) + c_1(x,t,\alpha_1)\phi(x,t) - l_1(x,t,\alpha_1) \leq 0\,,$$

which yields a contradiction with (6.15). And the proof is the same in the second case, considering the trajectory associated with the constant control α_2 in b_2.

We point out that this case could have been covered by arguments of Proposition 7.4.3, later in this book.

Case 2 – if $b_1 \cdot e_N < 0 < b_2 \cdot e_N$, then borrowing arguments of the proof of Lemma 6.1.8, for (y,s) close enough to (x,t), we can set

$$\mu_1^\sharp(y,s) := \frac{b_2(y,s,\alpha_2) \cdot e_N}{(b_2(y,s,\alpha_2) - b_1(y,s,\alpha_1)) \cdot e_N}\,, \quad \mu_2^\sharp := 1 - \mu_1^\sharp\,.$$

By this choice we have $0 \leq \mu_1^\sharp, \mu_2^\sharp \leq 1$ and $\Big(\mu_1^\sharp(y,s) b_1(y,s,\alpha_1) + \mu_2^\sharp(y,s) b_2(y,s,\alpha_2)\Big) \cdot e_N = 0$, hence we have a regular dynamic.

Then we consider the ode

$$\dot{X}^\sharp(s) = \mu_1^\sharp(X^\sharp(s), t_n - s) b_1(X^\sharp(s), t_n - s, \alpha_1) + \mu_2^\sharp(X^\sharp(s), t_n - s) b_2(X^\sharp(s), t_n - s, \alpha_2)\,.$$

By our hypotheses on b_1 and b_2, the right-hand side is Lipschitz continuous so that the Cauchy-Lipschitz theorem applies and gives a solution $X^\sharp(\cdot)$ which remains on $\mathcal{H}$, at least until some time $\tau > 0$.

Using $X^\sharp(\cdot)$ in (6.14) together with the associated discount and cost and arguing as above, we are led to

$$\mu_1^\sharp\Big(\phi_t(x,t) - b_1(x,t,\alpha_1)\cdot D\phi(x,t) + c_1(x,t,\alpha_1)\phi(x,t) - l_2(x,t,\alpha_1)\Big)$$
$$+\mu_2^\sharp\Big(\phi_t(x,t) - b_2(x,t,\alpha_2)\cdot D\phi(x,t) + c_2(x,t,\alpha_2)\phi(x,t) - l_2(x,t,\alpha_2)\Big) \leq 0\ ,$$

a contradiction. We point out that, since we have used a regular strategy the proof of this case is valid as well for $\mathbf{U}^+$.

Case 3 – The last case is when we have either $b_1\cdot e_N = 0 < b_2\cdot e_N$ or $b_1\cdot e_N < 0 = b_2\cdot e_N$. But using ($\mathbf{NC}_\mathcal{H}$), we can slightly modify b_1 or b_2 by a suitable convex combination in order to be in the framework of Case 1 or Case 2.

Finally the H_T^{reg}-inequality can be obtained as the H_T-inequality for $\mathbf{U}^-$ using in a key way Lemma 6.1.8 (we already did it above). This concludes the proof.

Q.E.D.

6.3 A Detailed Study of $\mathbf{U}^+$

The aim of this Section is to prove that $\mathbf{U}^+$ is the maximal Ishii solution. The strategy is similar to that of $\mathbf{U}^-$ but we need first to derive an equivalent result to Lemma 5.4.1.

6.3.1 More on Regular Trajectories

Our first result shows that regular trajectories satisfy stability properties which allow to extend usual arguments.

Lemma 6.3.1 *Assume that all the (b_i, c_i, l_i) satisfy ($\mathbf{H_{BA-CP}}$). For any $\varepsilon > 0$, let $(X,T,D,L)^\varepsilon \in \mathcal{T}^{\mathrm{reg}}(x,t)$ be a sequence of regular trajectories converging uniformly to (X,T,D,L) on $[0,t]$. Then $(X,T,D,L) \in \mathcal{T}^{\mathrm{reg}}(x,t)$.*

Though it may seem quite natural, this result is not so easy to obtain. It is a direct corollary of Proposition 6.3.7 (with constant **BCL** and initial data) which we prove in Subsection 6.3.5 below. Let us focus now on the immediate consequences.

Corollary 6.3.2 *Assume that all the (b_i, c_i, l_i) satisfy $(\mathbf{H_{BA-CP}})$. Then for any $(x,t) \in \mathbb{R}^N \times (0,T)$ there exists a regular trajectory $(X,T,D,L) \in \mathcal{T}^{\mathrm{reg}}(x,t)$ such that*

$$\mathbf{U}^+(x,t) = \int_0^t l\big(X(s), t-s, a(s)\big)e^{-D(s)}\,\mathrm{d}s + u_0(X(\tau), t-\tau)e^{-D(\tau)} \ ,$$

therefore there is an optimal trajectory. Moreover, the value-function $\mathbf{U}^+$ satisfies the sub-optimality principle, i.e., for any $(x,t) \in \mathbb{R}^N \times [0,T]$ and $0 < \tau < t$, we have

$$(\mathbf{U}^+)^*(x,t) \leq \inf_{\mathcal{T}^{\mathrm{reg}}(x,t)} \left\{ \int_0^\tau l\big(X(s), t-s, a(s)\big)e^{-D(s)}\,\mathrm{d}s + (\mathbf{U}^+)^*(X(\tau), t-\tau)e^{-D(\tau)} \right\} \ ,$$

and the super-optimality principle, i.e.

$$(\mathbf{U}^+)_*(x,t) \geq \inf_{\mathcal{T}^{\mathrm{reg}}(x,t)} \left\{ \int_0^\tau l\big(X(s), t-s, a(s)\big)e^{-D(s)}\,\mathrm{d}s + (\mathbf{U}^+)_*(X(\tau), t-\tau)e^{-D(\tau)} \right\} \ .$$

Corollary 6.3.2 provides slightly different (and maybe more direct) arguments to prove that $\mathbf{U}^+$ is an Ishii solution of (6.1) but it relies on the extraction of regular trajectories, is a rather delicate result to prove.

Proof — We just sketch it since it is a straightforward application of Lemma 6.3.1. For the existence of an optimal trajectory, we consider ε-optimal trajectories $(X,T,D,L)^\varepsilon$ associated with controls $a^\varepsilon \in \mathcal{A}$, *i.e.* trajectories and controls which satisfy

$$\mathbf{U}^+(x,t) \leq \int_0^t l\big(X^\varepsilon(s), t-s, a^\varepsilon(s)\big)e^{-D^\varepsilon(s)}\,\mathrm{d}s + u_0(X^\varepsilon(\tau), t-\tau)e^{-D^\varepsilon(\tau)} + \varepsilon \ .$$

By applying Ascoli's Theorem on the differential inclusion, we can assume without loss of generality that $(X,T,D,L)^\varepsilon \to (X,T,D,L)$ in $C([0,t])$ and $\int_0^t l\big(X^\varepsilon(s), t-s, a^\varepsilon(s)e^{-D^\varepsilon(s)}\big)\,\mathrm{d}s \to \int_0^t l\big(X(s), t-s, a(s)\big)e^{-D^\varepsilon(s)}\,\mathrm{d}s$ for some $a \in \mathcal{A}$. Applying Lemma 6.3.1 shows that (X,T,D,L) is actually a regular trajectory.

The proof of the sub and super-optimality principle follows from similar arguments considering, for example, a sequence $(x_k, t_k) \to (x,t)$ such that $\mathbf{U}^+(x_k,t_k) \to (\mathbf{U}^+)_*(x,t)$ and passing to the limit in an analogous way.

Q.E.D.

6.3.2 A Magical Lemma for $\mathbf{U}^+$

Now we turn a key result in the proof that $\mathbf{U}^+$ is the maximal Ishii solution of (6.1).

Theorem 6.3.3 *Assume that the "standard assumptions in the co-dimension-1 case" are satisfied. Let $\phi \in C^1\big(\mathcal{H}\times[0,T]\big)$ and suppose that (x,t) is a local minimum point of $(z,s)\mapsto (\mathbf{U}^+)_*(z,s)-\phi(z,s)$ in $\mathcal{H}\times[0,T]$. Then we have either*

A) *There exist $\eta>0$, $i\in\{1,2\}$ and a control $\alpha_i(\cdot)$ such that the associated trajectory (X,T,D,L) satisfies $X(s)\in\bar\Omega_i$ with $\dot X(s)=b_i(X(s),t-s,\alpha_i(s))$ for all $s\in]0,\eta]$ and*

$$(\mathbf{U}^+)_*(x,t) \geq \int_0^\eta l_i(X(s),t-s,\alpha_i(s))e^{-D(s)}\,\mathrm{d}s + (\mathbf{U}^+)_*(X(\eta),t-\eta)e^{-D(\eta)}\,, \quad (6.17)$$

or

B) *it holds*

$$\partial_t\phi(x,t) + H_T^{\mathrm{reg}}\big(x,t,(\mathbf{U}^+)_*(x,t),D_{\mathcal{H}}\phi(x,t)\big) \geq 0. \quad (6.18)$$

Proof — Using the result and the proof of Corollary 6.3.2, for any $0<\eta<t$, there exists a regular trajectory X and a control a such that

$$(\mathbf{U}^+)_*(x,t) \geq \int_0^\eta l\big(X(s),t-s,a(s)\big)e^{-D(s)}\,\mathrm{d}s + (\mathbf{U}^+)_*(X(\eta),t-\eta)e^{-D(\eta)}\,.$$

Indeed, for any η the infimum in the sub-optimality principle is achieved.

Then there are two cases: either there exists $\eta>0$ and $i\in\{1,2\}$ such that $X(s)\in\bar\Omega_i$ with $\dot X(s)=b_i(X(s),t-s,\alpha_i(s))$ for all $s\in]0,\eta]$, and then **A)** follows.

Or this is not the case, which means that there exists a sequence $(\eta_k)_k$ converging to 0 such that $\eta_k>0$ and $X(\eta_k)\in\mathcal{H}$. We then have

$$(\mathbf{U}^+)_*(x,t) \geq \int_0^{\eta_k} l\big(X(s),t-s,a(s)\big)e^{-D(s)}\,\mathrm{d}s + (\mathbf{U}^+)_*(X(\eta_k),t-\eta_k)e^{-D(\eta_k)}\,,$$

and the minimum point property on $\mathcal{H}$ yields

$$\phi(x,t) \geq \int_0^{\eta_k} l\big(X(s),t-s,a(s)\big)e^{-D(s)}\,\mathrm{d}s + \phi(X(\eta_k),t-\eta_k)e^{-D(\eta_k)}\,.$$

We rewrite this inequality as

$$\int_0^{\eta_k} A[\phi](s)\,\mathrm{d}s \geq 0$$

where

$$\begin{aligned}A[\phi](s) :=\Big\{&\phi_t(X(s),t-s) - \dot X(s)\cdot D_x\phi(X(s),t-s)\\ &+ c\big(X(s),t-s,a(s)\big)\phi(X(s),t-s) - l\big(X(s),t-s,a(s)\big)\Big\}e^{-D(s)}\,.\end{aligned}$$

In order to prove **B)**, we argue by contradiction, assuming that

$$\partial_t\phi(x,t) + H_T^{\mathrm{reg}}\big(x,t,(\mathbf{U}^+)_*(x,t), D_{\mathcal{H}}\phi(x,t)\big) < 0\,, \tag{6.19}$$

and to get a contradiction we have to examine the sets $\mathcal{E}_i := \{s \in (0,\eta_k) : X(s) \in \Omega_i\}$ and $\mathcal{E}_{\mathcal{H}} := \{s \in (0,\eta_k) : X(s) \in \mathcal{H}\}$.

Reaching a contradicton $\mathcal{E}_{\mathcal{H}}$ is easy: since $\dot X(s) = b_{\mathcal{H}}(X(s), t-s, a(s))$ if $X(s) \in \mathcal{H}$, by definition of H_T^{reg} as the supremum we get directly

$$\int_0^{\eta_k} A[\phi](s)\,\mathrm{d}s \le \int_0^{\eta_k} \Big\{\partial_t\phi(X(s),t-s) \\ + H_T^{\mathrm{reg}}\big(X(s),t-s,(\mathbf{U}^+)_*(X(s),t-s), D_{\mathcal{H}}\phi(X(s),t-s)\big)\Big\}\,\mathrm{d}s\,,$$

which is stricly negative provided η_k is small enough, thanks to (6.19) and the continuity of H_T^{reg}.

On the other hand, the sets $\mathcal{E}_i$ are open and therefore $\mathcal{E}_i = \cup_k(a_{i,k}, b_{i,k})$ with $a_{i,k}, b_{i,k} \in \mathcal{H}$. On each interval $(a_{i,k}, b_{i,k})$, $\dot X(s) = b_i\big((X(s), t-s, a(s)\big)$ and if $d(x) = x_N$, we have

$$0 = d(X(b_{i,k})) - d(X(a_{i,k})) = \int_{a_{i,k}}^{b_{i,k}} e_N \cdot b_i\big((X(s),t-s,a(s)\big)\,\mathrm{d}s\,. \tag{6.20}$$

By the regularity of (b_i, c_i, l_i) with respect to $X(s)$ we have

$$\int_{a_{i,k}}^{b_{i,k}} (b_i,c_i,l_i)\big((X(s),t-s,a(s)\big)\,\mathrm{d}s = \int_{a_{i,k}}^{b_{i,k}} (b_i,c_i,l_i)\big(x,t,a(s)\big)\,\mathrm{d}s + O(\eta_k)\,.$$

Then, using the convexity of the images of $\mathbf{BCL}_i$, there exists a control $a^\flat_{i,k}$ such that

$$\int_{a_{i,k}}^{b_{i,k}} (b_i,c_i,l_i)\big((X(s),t-s,a(s)\big)\,\mathrm{d}s = (b_{i,k}-a_{i,k})\,(b_i,c_i,l_i)\big(x,t,a^\flat_{i,k}\big)\,\mathrm{d}s + O(\eta_k)\,,$$

and (6.20) implies that $b_i\big(x,t,a^\flat_{i,k}\big)\cdot e_N = O(\eta_k)$. In terms of **BCL**, we have a $(b_i,c_i,l_i)^\flat \in \mathbf{BCL}_i(x,t)$ such that $b_i^\flat \cdot e_N = O(\eta_k)$. Using the normal controllabilty and regularity properties of $\mathbf{BCL}_i$, for η_k small enough, there exists a $(b_i,c_i,l_i)^\sharp \in \mathbf{BCL}_i(x,t)$ close to $(b_i,c_i,l_i)^\flat$ such that $b_i^\sharp \cdot e_N = 0$. This means that there exists a control $\alpha^\sharp_{i,k} \in A_i$ such that still

$$\int_{a_{i,k}}^{b_{i,k}} (b_i,c_i,l_i)\big((X(s),t-s,a(s)\big)\,\mathrm{d}s = (b_{i,k}-a_{i,k})\,(b_i,c_i,l_i)\big(x,t,a^\sharp_{i,k}\big)\,\mathrm{d}s + O(\eta_k)\,,$$

holds, and $b_i\big(x,t,a^\flat_{i,k}\big)\cdot e_N=0$. In other words, we have a regular dynamic for this specific control.

Hence, using the regularity of ϕ, since $a^\sharp_{i,k}$ is regular we get

$$
\begin{aligned}
&\int_{a_{i,k}}^{b_{i,k}} A[\phi](s)\,\mathrm{d}s\\
&=(b_{i,k}-a_{i,k})(\phi_t(x,t)-b_i(x,t,a^\sharp_{i,k})\cdot D_x\phi(x,t)+c(x,t,a^\sharp_{i,k})\phi(x,t)-l_i(x,t,a^\flat_{i,k})+O(\eta_k))\ ,\\
&\le (b_{i,k}-a_{i,k})(\partial_t\phi(x,t)+H_T^{\mathrm{reg}}\big(x,t,(\mathbf{U}^+)_*(x,t),D_{\mathcal{H}}\phi(x,t)\big)+O(\eta_k)<0\ .
\end{aligned}
$$

Therefore, for η_k small enough, on each part $\mathcal{E}_1$, $\mathcal{E}_2$ and $\mathcal{E}_{\mathcal{H}}$ of $(0,\eta_k)$, the integral is strictly negative and we have the desired contradiction.

Q.E.D.

Remark 6.3.4 *Notice that the alternative above with H_T^{reg} only holds for $\mathbf{U}^+$, and not for any arbitrary supersolution (see Theorem 6.1.9 where H_T is used and not H_T^{reg}).*

6.3.3 Maximality of $\mathbf{U}^+$

In order to prove that $\mathbf{U}^+$ is the maximal subsolution, we need the following result on subsolutions

Lemma 6.3.5 *Assume that the "standard assumptions in the co-dimension-1 case" are satisfied. If $u:\mathbb{R}^N\times(0,T)\to\mathbb{R}$ is an usc subsolution of (6.1), then it satisfies*

$$
u_t+H_T^{\mathrm{reg}}(x,t,u,D_Tu)\le 0\quad \text{on } \mathcal{H}\times(0,T)\,. \tag{6.21}
$$

Proof of Lemma 6.3.5 — Let ϕ be a C^1 test-function on $\mathcal{H}\times(0,T)$ – therefore we can assume that ϕ is just a function of x' and t – and $(\bar x,\bar t)\in\mathcal{H}\times(0,T)$ be a strict local maximum point of $u(x,t)-\phi(x',t)$ on $\mathcal{H}\times(0,T)$. We have to show that

$$
\phi_t(\bar x',\bar t)+H_T^{\mathrm{reg}}(\bar x,\bar t,u(\bar x,\bar t),D_T\phi(\bar x,\bar t))\le 0\ ,
$$

where $D_T\phi(\bar x,\bar t)$ is nothing but $D_{x'}\phi(\bar x',\bar t)$ and we also identify it below with $(D_{x'}\phi(\bar x',\bar t),0)$.

To do so, setting $a=\phi_t(\bar x',\bar t)$ and $p_T=D_T\phi(\bar x,\bar t)$, we have to prove that

$$
\mathcal{I}:=a-b\cdot p_T+cu(\bar x,\bar t)-l\le 0\ ,
$$

for any $(b,c,l) \in \mathbf{BCL}_T^{\mathrm{reg}}(\bar{x},\bar{t})$.

By definition of $\mathbf{BCL}_T^{\mathrm{reg}}(\bar{x},\bar{t})$, we can write

$$(b,c,l) = \mu_1(b_1,c_1,l_1) + \mu_2(b_2,c_2,l_2)\ ,$$

with $b_1 \cdot e_N \leq 0 \leq b_2 \cdot e_N$. Using the normal controllability and an easy approximation argument, we can assume without loss of generality that $b_1 \cdot e_N < 0 < b_2 \cdot e_N$. Of course, even if we do not write it to have simpler notations, (b_1,c_1,l_1) is associated to a control α_1 and (b_2,c_2,l_2) to a control α_2.

For $i=1,2$, we consider the affine functions

$$\psi_i(\delta) := a - b_i \cdot (p_T + \delta e_N) + c_i u(\bar{x},\bar{t}) - l_i\ .$$

By the above properties we have (i) ψ_1 is strictly increasing, (ii) ψ_2 is strictly decreasing and (iii) $\mu_1\psi_1(\delta) + \mu_2\psi_2(\delta) = \mathcal{I}$ which is independent of δ.

We argue by contradiction, assuming that $\mathcal{I} > 0$ and we choose $\bar{\delta}$ such that $\psi_1(\bar{\delta}) = \psi_2(\bar{\delta})$, which is possible since by the strict monotonicity properties and the fact that $\psi_1(\mathbb{R}) = \psi_2(\mathbb{R}) = \mathbb{R}$. We have therefore $\psi_1(\bar{\delta}) = \psi_2(\bar{\delta}) = \mathcal{I} > 0$.

Next, for $0 < \varepsilon \ll 1$, we consider the function

$$(x,t) \mapsto u(x,t) - \phi(x',t) - \bar{\delta} x_N - \frac{x_N^2}{\varepsilon^2}\ .$$

Since $(\bar{x},\bar{t})$ is a strict local maximum point of $u-\phi$ on $\mathcal{H} \times (0,T)$, there exists a sequence $(x_\varepsilon,t_\varepsilon)$ of local maximum point of this function which converges to $(\bar{x},\bar{t})$, with $u(x_\varepsilon,t_\varepsilon)$ converging to $u(\bar{x},\bar{t})$.

Since u is a viscosity subsolution we have either a H_1 or H_2 inequality for u and the aim is to show that none of these inequality can hold. Indeed, if the H_1-inequality holds, this means that $(x_\varepsilon)_N \geq 0$ and this implies, using the regularity of ϕ

$$a - b_1(x_\varepsilon,t_\varepsilon,\alpha_1) \cdot (p_T + \bar{\delta} e_N + \frac{2(x_\varepsilon)_N}{\varepsilon^2} e_N) + c_1(x_\varepsilon,t_\varepsilon,\alpha_1) u(\bar{x},\bar{t}) - l_1(x_\varepsilon,t_\varepsilon,\alpha_1) \leq o_\varepsilon(1)\ .$$

But since $(x_\varepsilon,t_\varepsilon) \to (\bar{x},\bar{t})$, $b_1(x_\varepsilon,t_\varepsilon,\alpha_1) \to b_1(\bar{x},\bar{t},\alpha_1)$ and therefore $b_1(x_\varepsilon,t_\varepsilon,\alpha_1) \cdot e_N < 0$ for ε small enough. Therefore, using that $(x_\varepsilon)_N \geq 0$, this inequality implies

$$a - b_1(x_\varepsilon,t_\varepsilon,\alpha_1) \cdot (p_T + \bar{\delta} e_N) + c_1(x_\varepsilon,t_\varepsilon,\alpha_1) u(\bar{x},\bar{t}) - l_1(x_\varepsilon,t_\varepsilon,\alpha_1) \leq o_\varepsilon(1)\ .$$

By the definition and properties of $\bar{\delta}$ and the fact that $\mathcal{I} > 0$, this inequality cannot hold for ε small enough, showing that the H_1 inequality cannot hold neither. A similar argument being valid for the H_2 inequality, we have a contradiction and therefore $\mathcal{I} \leq 0$.

Q.E.D.

Theorem 6.3.6 *Assume that the "standard assumptions in the co-dimension-*1 *case" are satisfied. Then* $\mathbf{U}^+$ *is continuous and it is the maximal Ishii solution of* (6.1).

Proof — Let u be any subsolution of (6.1), we want to show that $u \leq (\mathbf{U}^+)_*$ in $\mathbb{R}^N \times [0, T)$.

To do so, we first notice that, as we did in the proof of the characterization of $\mathbf{U}^-$ (Theorem 6.1.9), we can reduce the proof to a local comparison argument since (**LOC1**) and (**LOC2**) are satisfied. Hence, let $Q^{x,t}_{r,h}$ be a cylinder in which we want to perform the **(LCR)** between u and $(\mathbf{U}^+)_*$.

Using again the arguments of the proof of Theorem 6.1.9, we may assume without loss of generality that u is a strict subsolution of (6.1) and in particular a strict subsolution of (6.11). Finally we can regularize u in order that it is C^1 on $\mathcal{H} \times (0, T)$.

Using Theorem 5.3.3 to show that u satisfies a sub-dynamic programming principle with trajectories in $\mathcal{T}(x, t)$, we see that we are (almost) in the framework of Lemma 5.4.1, the usual $\mathbb{F}^{\mathcal{M}}$-inequality for u being replaced by (6.11).

Using in an essential way Theorem 6.3.3[(3)], it is easy to see that the result of Lemma 5.4.1 still holds in this slightly different framework and yields

$$\max_{\overline{Q^{x,t}_{r,h}}} (u - (\mathbf{U}^+)_*) \leq \max_{\partial Q^{x,t}_{r,h}} (u - (\mathbf{U}^+)_*) \,,$$

and the **(GCR)** follows: $u \leq (\mathbf{U}^+)_*$ in $\mathbb{R}^N \times [0, T]$.

Taking $u = (\mathbf{U}^+)^*$ which is a subsolution, we deduce that in the end, $\mathbf{U}^+ = (\mathbf{U}^+)_* = (\mathbf{U}^+)^*$ so that $\mathbf{U}^+$ is continuous and is maximal among Ishii subsolutions.

Q.E.D.

6.3.4 The One Dimensional Case, a Typical Example where $\mathbf{U}^+ \not\equiv \mathbf{U}^-$

We consider a one-dimensional finite horizon problem where

$$\Omega_1 = \{x > 0\},\ \Omega_2 = \{x < 0\},\ \mathcal{H} = \{x = 0\}$$

without any specific control problem on $\mathcal{H}$, so that the (b_0, c_0, l_0) and H_0 are not considered here. The reader will find in [10] a detailed study of this situation for infinite horizon control problems with a general description of the structure of solutions

(3) which replaces the arguments for the supersolution v in the proof of Lemma 5.4.1 (cf. Remark 5.4.2).

and the link between the minimal and maximal Ishii solutions with state-constraint solutions as well as several explicit examples. Here we restrict ourselves to exposing an explicit example of non-uniqueness for illustration purposes.

We consider the dynamics

$$\dot{X}(t) = \alpha_1(t) \text{ in } \Omega_1 , \quad \dot{X}(t) = \alpha_2(t) \text{ in } \Omega_2 ,$$

where $\alpha_1(\cdot), \alpha_2(\cdot) \in L^\infty\big(0, +\infty; [-1, 1]\big)$ are the controls. As for the costs, we choose

$$l_1(x, \alpha_1) = 1 - \alpha_1 + \min(|x|, 1) \text{ in } \Omega_1 , \quad l_2(x, \alpha_2) = 1 + \alpha_2 + \min(|x|, 1) \text{ in } \Omega_2 ,$$

where $\alpha_1(\cdot), \alpha_2(\cdot) \in L^\infty\big(0, +\infty; [-1, 1]\big)$. Finally, we set $c_1(x, \alpha_1) = c_2(x, \alpha_2) = 1$ for the discount factor and also $g = \min(|x|, 1)$ for the final cost. Therefore,

$$\mathbf{U}^-(x,t) = \inf_{\mathcal{T}(x)} \left\{ \int_0^t l(X(s), a(s)) e^{-s} \, \mathrm{d}s + g(X(t)) e^{-t} \right\} ,$$

where l is either l_1, l_2 or a convex combination of both for $x = 0$, and $a(\cdot) = (\mu, \alpha_1, \alpha_2)$ is the extended control. The definition for $\mathbf{U}^+$ is similar, the infimum being taken over $\mathcal{T}^{\text{reg}}(x)$.

It is clear that $l_1(x, \alpha_1), l_2(x, \alpha_2) \geq 0$ and these running costs are even strictly positive for $x \neq 0$. Therefore, we have $\mathbf{U}^-(x,t) \geq 0$ for any $x \in \mathbb{R}$ and $t \geq 0$. On the other hand, for $x = 0$, we have a 0-cost by choosing the singular "pull-pull" strategy $a = (\mu, \alpha_1, \alpha_2) = (1/2, 1, -1)$ which gives

$$b(0, a) = \mu\alpha_1 + (1 - \mu)\alpha_2 = 0 ,$$

$$l(0, a) = \mu(1 - \alpha_1) + (1 - \mu)(1 + \alpha_2) = 0 .$$

As a consequence, it is clear that this is the best strategy for $x = 0$ and $\mathbf{U}^-(0,t) = 0$ for any $t \geq 0$.

For $\mathbf{U}^+(0,t)$, we just want to show that $\mathbf{U}^+(0,t) > 0$. For simplicity, we are going to do it only for $t \leq 1$ since, in that way, any trajectory satisfies $|X(s)| \leq 1$ for any $0 \leq s \leq t$ and $\min(|X(s)|, 1)$ can be replaced by $|X(s)|$ everywhere (in the running cost and terminal cost).

Let X be any trajectory starting from $X(0) = 0$ and associated to a regular control. If $X(s) > 0$, then

$$\begin{aligned} l(X(s), a(s)) e^{-s} =& (1 - \dot{X}(s) + X(s)) e^{-s} \\ =& e^{-s} - \operatorname{sign}(X(s)) [X e^{-s}]'(s) \end{aligned}$$

In the same way, we also have the same formula if $X(s) < 0$. Therefore in both cases, $l(X(s), a(s))e^{-s} = e^{-s} - [|X|e^{-s}]'(s)$.

It remains to examine the case when $X(t) = 0$ and it is easy to see that, if $b(0, a) = 0$ is a regular dynamic, then $l(0, a) \geq 1$. Therefore, for $X(s) = 0$, the above formula is changed into $l(X(s), a(s))e^{-s} \geq e^{-s} - [|X|e^{-s}]'(s)$ (the equality being obtained for the above mentioned choice of a). Therefore

$$\begin{aligned}\int_0^t l(X(s), a(s))e^{-s}\,\mathrm{d}s + g(X(t))e^{-t} \geq& \int_0^t \left(e^{-s} - [|X|e^{-s}]'(s)\right)\,\mathrm{d}s + g(X(t))e^{-t}\\ \geq& 1 - e^{-t} > 0\,,\end{aligned}$$

proving that $\mathbf{U}^+(0, t) \geq 1 - e^{-t} > 0$ (and in fact the equality holds by the right choice of a).

6.3.5 Extraction of Trajectories

We end this section with proving in particular the convergence property of regular trajectories, Lemma 6.3.1. Actually we prove a more general result here:

Proposition 6.3.7 *Let $t > 0$ be fixed and for each $\varepsilon > 0$ let $\mathbf{BCL}^\varepsilon$ be a set-valued map satisfying $(\mathbf{H_{BCL}})_{fund}$. and let $(X, T, D, L)^\varepsilon$ be solution of the differential inclusion*

$$\forall s \in (0, t)\,, \quad (X, T, D, L)^\varepsilon(s) \in \mathbf{BCL}^\varepsilon(X^\varepsilon(s), t - s)\,.$$

(*i*) *If $\mathbf{BCL}^\varepsilon$ converges to $\mathbf{BCL}$ locally unformly in $\mathbb{R}^N \times (0, t)$ (for the Haussdorf distance on sets) and $(X, T, D, L)^\varepsilon(0) \to (x, t, d, l)$, then up to extraction, $(X, T, D, L)^\varepsilon$ converges to some trajectory (X, T, D, L) which satisfies*

$$\forall s \in (0, t)\,, \quad (X, T, D, L)(s) \in \mathbf{BCL}(X(s), t - s)$$

with inital value $(X, T, D, L)(0) = (x, t, d, l)$.

(*ii*) *If moreover each trajectory X^ε is regular, then the limit trajectory X is also regular.*

This result is obtained through several lemmas. The first one proves part (*i*) of the proposition, which is not very difficult.

Lemma 6.3.8 *If $\mathbf{BCL}^\varepsilon$ converges to $\mathbf{BCL}$ locally unformly in $\mathbb{R}^N \times (0, t)$ (for the Haussdorf distance on sets) and $(X, T, D, L)^\varepsilon(0) \to (x, t, d, l)$, then up to extraction, $(X, T, D, L)^\varepsilon$ converges to some trajectory (X, T, D, L) which is a solution of the differential inclusion associated with $\mathbf{BCL}$ with the corresponding initialization.*

Proof — Notice first that since the $\mathbf{BCL}^\varepsilon$ all satisfy $(\mathbf{H_{BCL}})_{fund}$ with constants independent of ε, and the initial value converges, the trajectories $(X,T,D,L)^\varepsilon$ are equi-Lipschitz and equi-bounded on $[0,t]$. Hence we can extract a subsequence $(X,T,D,L)^{\varepsilon_n} \to (X,T,D,L)$ uniformly on $[0,t]$. Moreover, for any $\kappa > 0$ small enough, if n is big enough we have

$$\forall s \in (0,t)\,, \quad \mathbf{BCL}^\varepsilon(X^{\varepsilon_n}, t-s) \subset \mathbf{BCL}(X, t-s) + \kappa B_{N+3}$$

where B_{N+3} is the unit ball of $\mathbb{R}^{N+3}$. Passing to the limit as $\varepsilon_n \to 0$, we deduce that (X,T,D,L) satisfies the differential inclusion associated with $\mathbf{BCL}$, and of course its initial data is $(X,T,D,L)(0) = (x,t,d,l)$.

Q.E.D.

Now we need several results in order to prove part (ii) which is much more involved. Before proceeding, let us comment a little bit: using the control representation of the differential inclusion (Lemma 6.1.2), there exist some controls $\alpha_i^\varepsilon, a^\varepsilon$ such that

$$\dot{X}^\varepsilon(s) = \sum_{i=1,2} b_i^\varepsilon\big(X^\varepsilon(s), t-s, \alpha_i^\varepsilon(s)\big)\mathbb{1}_{\{X^\varepsilon\in\Omega_i\}}(s) + b_{\mathcal{H}}^\varepsilon\big(X^\varepsilon(s), t-s, a^\varepsilon(s)\big)\mathbb{1}_{\{X^\varepsilon\in\mathcal{H}\}}(s)\,.$$

Recall that the control a^ε is actually complex since it involves $\alpha_1^\varepsilon, \alpha_2^\varepsilon$ but also α_0^ε. In other words, $b_{\mathcal{H}}$ is a mix of b_0, b_1, b_2 with weights $\mu_0^\varepsilon, \mu_1^\varepsilon, \mu_2^\varepsilon$. However, notice that focusing on regular dynamics, the b_0-term is not a problem since it is already tangential (hence, regular).

In order to send $\varepsilon \to 0$ we face two difficulties: the first one is that we have to deal with weak convergences in the $b_i^\varepsilon, b_{\mathcal{H}}^\varepsilon$-terms. But the problem is increased by the fact that some pieces of the limit trajectory $X(\cdot)$ on $\mathcal{H}$ can be obtained as limits of trajectories $X^\varepsilon(\cdot)$ which lie either on $\mathcal{H}$, Ω_1 or Ω_2. In other words, the indicator functions $\mathbb{1}_{\{X^\varepsilon\in\mathcal{H}\}}(\cdot)$ do not necessarily converge to $\mathbb{1}_{\{X\in\mathcal{H}\}}(\cdot)$, and similarly the $\mathbb{1}_{\{X^\varepsilon\in\Omega_i\}}(\cdot)$ do not converge to $\mathbb{1}_{\{X\in\Omega_i\}}(\cdot)$.

From Lemma 6.3.8 we already know that $\dot{X}^\varepsilon$ converges weakly on $(0,t)$ to some $\dot{X}$ which can be represented as for X^ε above, by means of some controls (α_1, α_2, a). The question is to prove that this control a yields regular dynamics on $\mathcal{H}$. In order to to do, we introduce several tools. The first one is a representation of X by means of some regular controls $(\alpha_1^\sharp, \alpha_2^\sharp, a^\sharp)$. Those controls may differ from (α_1, α_2, a), but they are an intermediate step which will help us to prove the final result.

Lemma 6.3.9 *For any $s \in (0,t)$ there exists three measures $\nu_1(s,\cdot), \nu_2(s,\cdot), \nu_{\mathcal{H}}(s,\cdot)$ on A_1, A_2, A respectively and three controls $(\alpha_1^\sharp(s), \alpha_2^\sharp(s), a^\sharp(s)) \in A_1 \times A_2 \times A$ such that* (*a*) *$\nu_1, \nu_2, \nu_{\mathcal{H}} \geq 0$, $\nu_1(s, A_1) + \nu_2(s, A_2) + \nu_{\mathcal{H}}(s, A) = 1$;*

(*b*) *up to extraction, $b_1^\varepsilon(X^\varepsilon(s), t-s, \alpha_1^\varepsilon) \to b_1(X(s), t-s, \alpha_1^\sharp(s)) \cdot \nu_i(s, A_1)$, and the same holds for $b_2, b_{\mathcal{H}}$ with measures ν_2, ν_H and controls $\alpha_2^\sharp, \alpha_{\mathcal{H}}^\sharp$;*

(*c*) *for $i = 1, 2$, $b_i(X(s), t-s, \alpha_i^\sharp(s)) \cdot e_N = 0$ ν_i-a.e. on $\{X(s) \in \mathcal{H}\}$.*

In particular, the dynamic obtained by using $(\alpha_0, \alpha_1^\sharp, \alpha_2^\sharp)$ is regular.

Proof — We use a slight modification of the procedure leading to relaxed control as follows. We write

$$b_1^\varepsilon\big(X^\varepsilon(s), t-s, \alpha_1^\varepsilon(s)\big)\mathbb{1}_{\{X^\varepsilon \in \Omega_1\}}(s) = \int_{A_1} b_1^\varepsilon\big(X^\varepsilon(s), t-s, \alpha\big)\, \nu_1^\varepsilon(s, \mathrm{d}\alpha)\,,$$

where $\nu_1^\varepsilon(s, \cdot)$ stands for the measure defined on A_1 by $\nu_1^\varepsilon(s, E) = \delta_{\alpha_1^\varepsilon}(E)\mathbb{1}_{\{X^\varepsilon \in \Omega_1\}}(s)$, for any Borelian set $E \subset A_1$. Similarly we define ν_2^ε and $\nu_{\mathcal{H}}^\varepsilon$ for the other terms. Notice that $\nu_{\mathcal{H}}^\varepsilon$ is a bit more complex measure since it concerns controls of the form $a = (\alpha_1, \alpha_2, \mu)$ on A, but it works as for ν_1^ε so we omit the details.

Note that, for any s, $\nu_1^\varepsilon(s, A_1) + \nu_2^\varepsilon(s, A_2) + \nu_{\mathcal{H}}^\varepsilon(s, A) = 1$ and therefore the measures $\nu_1^\varepsilon(s, \cdot), \nu_2^\varepsilon(s, \cdot), \nu_{\mathcal{H}}^\varepsilon(s, \cdot)$ are uniformly bounded in ε. Up to successive extractions of subsequences, they all converge weakly to some measures ν_1, ν_2, $\nu_{\mathcal{H}}$. Since the total mass is 1, we obtain in the limit $\nu_1(s, A_1) + \nu_2(s, A_2) + \nu_{\mathcal{H}}(s, A) = 1$.

Using that up to extraction X^ε converges uniformly on $[0, t]$, using the local uniform convergence of the b_1^ε, we get that

$$\int_{A_1} b_1^\varepsilon\big(X^\varepsilon(s), t-s, \alpha\big)\, \nu_1^\varepsilon(s, \mathrm{d}\alpha) \underset{\varepsilon \to 0}{\longrightarrow} \int_{A_1} b_1\big(X(s), t-s, \alpha\big)\, \nu_1(s, \mathrm{d}\alpha),$$

weakly in $L^\infty(0, T)$. Introducing $\pi_1(s) := \int_{A_1} \nu_1(s, \mathrm{d}\alpha)$ and using the convexity of A_1 together with a measurable selection argument (see [7, Theorem 8.1.3]), the last integral can be written as $b_1\big(X(s), \sigma(s), \alpha_1^\sharp(s)\big)\pi_1(s)$ for some control $\alpha_1^\sharp \in L^\infty(0, T; A_1)$. The same procedure for the other two terms provides the controls $\alpha_2^\sharp(\cdot)$, $a^\sharp(\cdot)$ and functions $\pi_2(\cdot)$, $\pi_{\mathcal{H}}(\cdot)$, which yields (*a*) and (*b*).

We now turn to property (*c*) that we prove for b_1, the proof being identical for b_2. Since $(X_N^\varepsilon)_+ := \max(X_N^\varepsilon, 0)$ is a sequence of Lipschitz continuous functions which converges uniformly to $(X_N)_+$ on $[0, t]$, up to an additional extraction of subsequence, we may assume that the derivatives converge weakly in L^∞ (weak–$*$ convergence). As a consequence, $\frac{d}{ds}\big[(X_N^\varepsilon)_+\big]\mathbb{1}_{\{X \in \mathcal{H}\}}$ converges weakly to $\frac{d}{ds}\big[(X_N)_+\big]\mathbb{1}_{\{X \in \mathcal{H}\}}$.

By Stampacchia's Theorem we have

$$\frac{d}{ds}\big[(X^\varepsilon_N)_+\big] = \dot X^\varepsilon_N(s)\,\mathbb{1}_{\{X^\varepsilon\in\Omega_1\}}(s) \quad \text{for almost all } s\in(0,t).$$

Therefore, the above convergence reads, in $L^\infty(0,T)$weak–$*$

$$\dot X^\varepsilon_N(s)\mathbb{1}_{\{X^\varepsilon\in\Omega_1\}}(s)\mathbb{1}_{\{X\in\mathcal{H}\}}(s) \longrightarrow \dot X_N(s)\mathbb{1}_{\{X\in\Omega_1\}}(s)\mathbb{1}_{\{X\in\mathcal{H}\}}(s) = 0\,.$$

Using the expression of $\dot X^\varepsilon(s)$, $\big(b^\varepsilon_1\big(X^\varepsilon(s),t-s,\alpha^\varepsilon_1(s)\big)\cdot e_N\big)\mathbb{1}_{\{X^\varepsilon\in\Omega_1\}}(s)\mathbb{1}_{\{X\in\mathcal{H}\}}(s)\to 0$ in $L^\infty(0,T)$ weak–$*$ which implies that

$$\Big(b_1\big(X(s),t-s,\alpha^\sharp_1(s)\big)\cdot e_N\Big)\,\pi_i(s) = 0 \ \text{ a.e. on } \{X(s)\in\mathcal{H}\}\,, \tag{6.22}$$

which yields property (c). This means that $b_i(X(s),t-s,\alpha^\sharp_i(s))$ is tangential on $\mathcal{H}$ so that combining them with some b_0 (which is tangential by definition), we get a regular dynamic on $\mathcal{H}$.

Q.E.D.

We now want to prove that the controls (α_1,α_2,a) yield regular strategies, not only the $(\alpha^\sharp_1,\alpha^\sharp_2,a^\sharp)$. In order to proceed we introduce the set of regular dynamics:

$$\forall (z,s)\in\mathcal{H}\times[0,t]\,,\quad K(z,s) := \big\{b_\mathcal{H}\big(z,s,a_*\big)\,, a_*\in A_0^{\mathrm{reg}}(z,s)\big\}\subset\mathbb{R}^N\,.$$

We notice that, for any $z\in\mathcal{H}$ and $s\in[0,T]$, $K(z,s)$ is closed and convex, and the mapping $(z,s)\mapsto K(z,s)$ is continuous on $\mathcal{H}$ for the Hausdorff distance. Then, for any $\eta>0$, we consider the subset of $[0,t]$ consisting of times s for which one has singular (η-enough) dynamics for the control $a(\cdot)$, namely

$$E^\eta_{\mathrm{sing}} := \Big\{s\in[0,t] : X(s)\in\mathcal{H} \text{ and } \operatorname{dist}\Big(b_\mathcal{H}\big(X(s),t-s,a(s)\big); K\big(X(s),t-s\big)\Big)\geq\eta\Big\}\,.$$

If $s\in E^\eta_{\mathrm{sing}}\neq\emptyset$, since $K(X(s),t-s)$ is closed and convex, there exists an hyperplane separating $b_\mathcal{H}\big(X(s),t-s,a(s)\big)$ from $K(X(s),t-s)$ and we can construct an affine function $\Psi_s:\mathbb{R}^N\to\mathbb{R}$ of the form $\Psi_s(z)=A(s)z+B(s)$ such that

$$\Psi_s\Big(b_\mathcal{H}\big(X(s),t-s,a(s)\big)\Big)\geq 1 \text{ if } s\in E^\eta_{\mathrm{sing}}\,,\quad \Psi_s\leq 0 \text{ on } K\big(X(s),t-s\big)\,.$$

In other words, Ψ_s "counts" the singular dynamics.

Since the mapping $s \mapsto b_{\mathcal{H}}\big(X(s), t-s, a(s)\big)$ is measurable and $s \mapsto K\big(X(s), t-s\big)$ is continuous, we can assume that $s \mapsto A(s), B(s)$ are measurable and bounded (because the distance $\eta > 0$ is fixed), which allows to define the quantity

$$I(\eta) := \begin{cases} \displaystyle\int_0^t \big(\Psi_s(\dot{X}(s)\big)\mathbb{1}_{E^\eta_{\text{sing}}}(s)\,\mathrm{d}s & \text{if } E^\eta_{\text{sing}} \neq \emptyset \\ 0 & \text{if } E^\eta_{\text{sing}} = \emptyset\,. \end{cases}$$

By definition, it is clear that $I(\eta) \geq |E^\eta_{\text{sing}}|$ (the Lebesgue measure of E^η_{sing}). The following result gives a converse estimate

Lemma 6.3.10 *For any $\eta > 0$, $I(\eta) \leq 0$.*

Proof — Let $\eta > 0$. If $E^\eta_{\text{sing}} = \emptyset$ there is nothing to do so let us assume that this is not the case, and take some $s \in E^\eta_{\text{sing}}$. Since Ψ_s is affine, using the weak convergence of $\dot{X}^\varepsilon$ we know that

$$I(\eta) = \lim_{\varepsilon \to 0} I^\varepsilon(\eta) := \int_0^t \big(\Psi_s(\dot{X}^\varepsilon(s)\big)\mathbb{1}_{E^\eta_{\text{sing}}}(s)\,\mathrm{d}s\,.$$

The strategy is to use Lemma 6.3.9 to pass to the limit and estimate $I^\varepsilon(\eta)$, knowing that at each level $\varepsilon > 0$, the dynamics are regular. In order to keep this information in the limit, dealing with the b^ε_i-terms is handled by property (c) of Lemma6.3.9. But the $b^\varepsilon_{\mathcal{H}}$-term is more delicate: we need first to fix a regular control independent of ε.

To do so, we start by noticing that for fixed $\varepsilon > 0$ and $s \in [0, t]$, for each $a^\varepsilon(s) \in A_0^{\text{reg}}(X^\varepsilon(s), t-s)$ there exists a $\tilde{a}^\varepsilon(s) \in A_0^{\text{reg}}(X(s), t-s)$ such that

$$b^\varepsilon_{\mathcal{H}}(X^\varepsilon(s), t-s, a^\varepsilon(s)) = b_{\mathcal{H}}(X(s), t-s, \tilde{a}^\varepsilon(s)) + o_\varepsilon(1)\,.$$

Indeed, this comes from a measurable selection argument and the fact that X^ε converges uniformly to X, while $b^\varepsilon_{\mathcal{H}}$ also converges locally uniformly (with respect to its first variable). So, rewriting the expansion of $\dot{X}^\varepsilon$ and using that Ψ_s is affine we get

$$\begin{aligned} I^\varepsilon(\eta) = &\int_0^t \Psi_s\Big(\sum_{i=1,2} b^\varepsilon_i\big(X^\varepsilon(s), t-s, \alpha^\varepsilon_i(s)\big)\mathbb{1}_{\{X^\varepsilon \in \Omega_i\}}(s)\Big)\mathbb{1}_{E^\eta_{\text{sing}}}(s)\,\mathrm{d}s \\ &+ \int_0^t A(s)\Big(b_{\mathcal{H}}\big(X(s), t-s, \tilde{a}^\varepsilon(s)\big)\,\mathbb{1}_{\{X^\varepsilon \in \mathcal{H}\}}(s)\Big)\mathbb{1}_{E^\eta_{\text{sing}}}(s)\,\mathrm{d}s + o_\varepsilon(1)\,. \end{aligned}$$

Moreover, by construction and using again a measurable selection argument (see Filippov's Lemma [7, Theorem 8.2.10]), there exists a control $a_\star(s) \in K(X(s), t-s)$ such that

$$A(s) b_{\mathcal{H}}(X(s), t-s, a_\star(s)) = \max_{a \in K(X(s), t-s)} A(s) b_{\mathcal{H}}(X(s), t-s, a).$$

Therefore,

$$I^\varepsilon(\eta) \leq \int_0^t \Psi_s \Big\{ \sum_{i=1,2} b_i^\varepsilon\big(X^\varepsilon(s), t-s, \alpha_i^\varepsilon(s)\big) \mathbb{1}_{\{X^\varepsilon \in \Omega_i\}}(s) \\ + b_{\mathcal{H}}\big(X(s), t-s, a_\star(s)\big)\, \mathbb{1}_{\{X^\varepsilon \in \mathcal{H}\}}(s) \Big\} \mathbb{1}_{E^\eta_{\text{sing}}}(s)\, \mathrm{d}s + o_\varepsilon(1)\,.$$

Now we pass to the weak limit, using Lemma 6.3.9 but with a constant $b_{\mathcal{H}}$ instead of $b^\varepsilon_{\mathcal{H}}$ and, more importantly, a constant control $a_\star$. In other words, the measure $\nu^\varepsilon_{\mathcal{H}}$ is actually independent of ε in this situation. We get some measures $\nu_1, \nu_2, \nu_{\mathcal{H}}$ and some controls $\alpha_1^\sharp, \alpha_2^\sharp$ and $a^\sharp = a_*$ here, for which

$$\lim_{\varepsilon\to 0} I^\varepsilon(\eta) \leq \int_0^t \Psi_s \Big\{ \sum_{i=1,2} b_i\big(X(s), t-s, \alpha_i^\sharp(s)\big)\nu_i(s, A_i) \\ + b_{\mathcal{H}}\big(X(s), t-s, a_\star(s)\big)\nu_H(s, A) \Big\} \mathbb{1}_{E^\eta_{\text{sing}}}(s)\, \mathrm{d}s\,.$$

Recall that by construction $b_{\mathcal{H}}(X(s), t-s, a_\star(s)) \in K(X(s), t-s)$ and that $\alpha_1^\sharp, \alpha_2^\sharp$ are regular controls. Therefore, since $\nu_1(s, A_1) + \nu_2(s, A_2) + \nu_{\mathcal{H}}(s, A) = 1$ and the set $K(X(s), t-s)$ is convex, we deduce that the convex combination satisfies

$$\Psi_s \Big\{ \sum_{i=1,2} b_i\big(X(s), t-s, \alpha_i^\sharp(s)\big)\nu_i(s, A_i) + b_{\mathcal{H}}\big(X(s), t-s, a_\star(s)\big)\nu_H(s, A) \Big\} \leq 0\,.$$

The conclusion is that $I(\eta) = \lim_{\varepsilon\to 0} I^\eta(\eta) \leq 0$ and the result is proved.

Q.E.D.

Proof of Proposition 6.3.7 — The first part (i) is done in Lemma 6.3.8. As for (ii), we proved above that for any $\eta > 0$, $|E^\eta_{\text{sing}}| \leq I(\eta) = 0$, so that set E^η_{sing} is of zero Lebesgue measure. Hence, using a countable union of negligeable sets we deduce that

$$\Big\{ s \in [0,t] : X(s) \in \mathcal{H} \text{ and } b_{\mathcal{H}}\big(X(s), t-s, a(s)\big) \notin K\big(X(s), t-s\big)\Big)\Big\}$$

is also of zero Lebesgue measure. This means that for almost any $s \in (0,t)$, the strategy obtained by choosing a as control is regular, which concludes the proof.

Q.E.D.

6.4 Adding a Specific Problem on the Interface

This section is devoted to explain the main adaptations and differences when we consider the more general problem

$$\begin{cases} u_t + H_1(x,t,u,Du) = 0 & \text{for } x \in \Omega_1\,,\\ u_t + H_2(x,t,u,Du) = 0 & \text{for } x \in \Omega_2\,,\\ u_t + H_0(x,t,u,D_T u) = 0 & \text{for } x \in \mathcal{H}\,,\\ u(x,0) = u_0(x) & \text{for } x \in \mathbb{R}^N\,. \end{cases} \tag{6.23}$$

Here, since H_0 is only defined on $\mathcal{H}$, the gradient $D_T u$ consists only on the tangential derivative of u: if $x = (x', x_N) \in \mathbb{R}^{N-1} \times \mathbb{R}$, $D_T u = D_{x'} u$ (or $(D_{x'}u, 0)$ depending on the convention we choose). In order to simplify some formula, we may write Du instead of $D_T u$ and therefore $H_0(x,t,u,Du)$ instead of $H_0(x,t,u,D_T u)$, keeping in mind that H_0 depends only on $p = Du$ through $p_T = D_T u$.

As we explained in Section 3.1, the conditions on $\mathcal{H}$ for those equations have to be understood in the relaxed (Ishii) sense, namely for (6.23)

$$\begin{cases} \max\Big(u_t + H_0(x,t,u,D_T u), u_t + H_1(x,t,u,Du), u_t + H_2(x,t,u,Du)\Big) \geq 0\,,\\ \min\Big(u_t + H_0(x,t,u,D_T u), u_t + H_1(x,t,u,Du), u_t + H_2(x,t,u,Du)\Big) \leq 0\,, \end{cases} \tag{6.24}$$

meaning that, for the supersolution [resp. subsolution] condition, at least one of the inequation has to hold.

In this section, we use the notation with H_0 as a sub/superscript in the mathematical objects to differentiate from the "non"-H_0 case since these are not exactly the same, in particular of course, the value functions differ whether H_0 is present or not.

We say here that the "standard assumptions in the co-dimension-1 case" are satisfied for (6.23) if $(\mathbf{H_{BA-CP}})$ holds for (b_i, c_i, l_i), $i = 0,1,2$ and $(\mathbf{NC}_{\mathcal{H}})$ holds for H_1 and H_2.

6.4.1 The Control Problem

The control problem is solved exactly as in the case of (6.1) that was considered above. We just need to add a specific control set A_0 and triples (b_0, c_0, l_0), defining $\mathbf{BCL}_0(x,t)$ when $x \in \mathcal{H}$ as for $\mathbf{BCL}_1$ and $\mathbf{BCL}_2$. Since the case $i = 0$ is specific because $\mathcal{H}$ can be identified with $\mathbb{R}^{N-1} \times \{0\}$, we set for all (x,t,α_0), $b_0(x,t,\alpha_0) = (b_0'(x,t,\alpha_0), 0)$ so that $b_0 \cdot p$ reduces to the scalar product of the first $(N-1)$ components.

Using this convention, we define now the new **BCL** as

$$\mathbf{BCL}^{H_0}(x,t) := \begin{cases} \mathbf{BCL}_1(x,t) & \text{if } x \in \Omega_1\,, \\ \mathbf{BCL}_2(x,t) & \text{if } x \in \Omega_2\,, \\ \overline{\text{co}}(\mathbf{BCL}_0, \mathbf{BCL}_1, \mathbf{BCL}_2)(x,t) & \text{if } x \in \mathcal{H}\,, \end{cases}$$

where the convex hull takes into account here the three sets $\mathbf{BCL}_i$ for $i = 0, 1, 2$ so that of course, on $\mathcal{H}$ we make a convex combination of all the (b_i, c_i, l_i), $i = 0, 1, 2$.

Lemma 6.4.1 *The set-valued map* $\mathbf{BCL}^{H_0}$ *satisfies* $(\mathbf{H_{BCL}})$.

The proof is an obvious adaptation of Lemma 6.1.1 so we skip it.

In order to describe the trajectories of the differential inclusion with $\mathbf{BCL}^{H_0}$, we have to enlarge the control space with A_0 (and introduce a new parameter μ_0 for the convex combination):

$$A^{H_0} := A_0 \times A_1 \times A_2 \times [0,1]^3\,, \quad \text{and} \quad \mathcal{A}^{H_0} := L^\infty(0,T; A^{H_0})$$

so that the extended control takes the form $a = (\alpha_0, \alpha_1, \alpha_2, \mu_0, \mu_1, \mu_2)$ and if $x \in \mathcal{H}$,

$$(b_{\mathcal{H}}, c_{\mathcal{H}}, l_{\mathcal{H}}) = \sum_{i=0}^{2} \mu_i (b_i, c_i, l_i)\,,$$

with $\mu_0 + \mu_1 + \mu_2 = 1$.

With this modification, solving the differential inclusion with $\mathbf{BCL}^{H_0}$ and the description of trajectories is similar to that in the **BCL**-case (see Lemma 6.1.2), except that the control has the form $a(\cdot) = (\alpha_0, \alpha_1, \alpha_2, \mu_0, \mu_1, \mu_2)(\cdot) \in \mathcal{A}^{H_0}$.

Then we define $\mathbf{U}^-_{H_0}$ by

$$\mathbf{U}^-_{H_0}(x,t) := \inf_{\mathcal{T}_{H_0}(x,t)} \left\{ \int_0^t l(X(s), t-s, a(s)) \exp(-D(s))\,\mathrm{d}s + u_0(X(t)) \exp(-D(t)) \right\}\,,$$

where $\mathcal{T}_{H_0}(x,t)$ is of course the space of trajectories associated with $\mathbf{BCL}^{H_0}$.

6.4.2 The Minimal Solution

As far as the value-function $\mathbf{U}^-_{H_0}$ is concerned, only easy adaptations are needed to handle H_0 and the related control problem. Of course we assume that H_0 also satisfies $(\mathbf{H_{Conv}})$, **(NC)**, **(TC)** and $(\mathbf{Mon}\text{-}u)$, as it is the case for H_1 and H_2.

Lemma 6.1.3 holds here with

$$H^{H_0}(x,t,u,p) := \sup_{(b,c,l)\in \mathbf{BCL}^{H_0}(x,t)} \big(-b\cdot p + cu - l\big)\,,$$

$$\mathbb{F}^{H_0}(x,t,u,(p_x,p_t)) := p_t + H^{H_0}(x,t,u,p)\,,$$

and of course we have to add H_0 in the max of the right-hand sides:

$$H^{H_0}(x,t,r,p) = \max\Big(H_0(x,t,r,p), H_1(x,t,r,p), H_2(x,t,u,p)\Big)\,,$$

$$\mathbb{F}^{H_0}(x,t,u,(p_x,p_t)) = \max\big(p_t + H_0(x,t,r,p), p_t + H_1(x,t,u,p), p_t + H_2(x,t,u,p)\big)\,.$$

Then, minimality of $\mathbf{U}^-_{H_0}$ follows exactly as in Proposition 6.1.4:

Proposition 6.4.2 *Assume that the "standard assumptions in the co-dimension-*1 *case" are satisfied for* (6.23)*. Then the value function* $\mathbf{U}^-_{H_0}$ *is an Ishii viscosity solutions of* (6.23)*. Moreover* $\mathbf{U}^-_{H_0}$ *is the minimal supersolution of* (6.23)*.*

Notice that a tangential dynamic $b\in \mathbf{B}^{H_0}_T(x,t)$ is now expressed as a convex combination

$$b = \mu_0 b_0 + \mu_1 b_1 + \mu_2 b_2 \tag{6.25}$$

for which $\mu_0+\mu_1+\mu_2 = 1$, $\mu_0,\mu_1,\mu_2\in[0,1]$ and $(\mu_1 b_1 + \mu_2 b_2)\cdot e_N = 0$, but here by definition, $b_0\cdot e_N = 0$.

Then, all the results of Section 6.1.3 apply, except that we need a little adaptation for Lemma 6.1.8 in order to take into account the b_0-contribution: *Proof of Lemma 6.1.8 in the* $\mathbf{BCL}^{H_0}$*-case —* The only modification consists in rewriting the convex combination as

$$\mu_0 b_0(x,t,\alpha_0) + (1-\mu_0)\left(\frac{\mu_1}{1-\mu_0} b_1(x,t,\alpha_1) + \frac{\mu_2}{1-\mu_0} b_2(x,t,\alpha_2)\right)\,,$$

and we apply the arguments of Lemma 6.1.8 to the convex combination

$$\frac{\mu_1}{1-\mu_0} b_1(x,t,\alpha_1) + \frac{\mu_2}{1-\mu_0} b_2(x,t,\alpha_2)\,.$$

Then, setting

$$\psi^{H_0}(y,s) := \mu_0 b_0(x,t,\alpha_0) + (1-\mu_0)\Big(\mu_1^\sharp(y,s)(b_1,c_1,l_1) + \mu_2^\sharp(y,s)(b_2,c_2,l_2)\Big)\,,$$

it is easy to check that the lemma holds for the $\mathbf{BCL}^{H_0}$-case.

Q.E.D.

Finally, the minimal solution $\mathbf{U}^-_{H_0}$ can also be characterized through $H^{H_0}_T$. The proof follows exactly the "non-H_0" case with obvious adaptations so that we omit it:

Theorem 6.4.3 *Assume that the "standard assumptions in the co-dimension-1 case" are satisfied for (6.23). Then* $\mathbf{U}^-_{H_0}$ *is the unique Ishii solution of (6.23) such that*

$$u_t + H_T^{H_0}(x,t,u,D_T u) \leq 0 \quad \text{on} \quad \mathcal{H} \times (0,T)\,.$$

6.4.3 The Maximal Solution

Surprisingly, for the maximal solution, the case of (6.23) is very different. And we can see it on the result for subsolutions, analogue to Lemma 6.3.5:

Lemma 6.4.4 *If* $u : \mathbb{R}^N \times (0,T) \to \mathbb{R}$ *is an usc subsolution of (6.1), then it satisfies*

$$u_t + \min(H_0(x,t,u,D_T u), H_T^{\mathrm{reg}}(x,t,u,D_T u) \leq 0 \quad \text{on } \mathcal{H} \times (0,T)\,. \tag{6.26}$$

We omit the proof since it is the same as that of Lemma 6.3.5 (taking into account the b_0-terms), but of course the conclusion is that the H_0-inequality necessarily holds if the H_T^{reg} does not, hence the min.

The important fact in Lemma 6.4.4 is that, in the without H_0, while (6.21) keeps the form of an HJB-inequality for a control problem, it is not the case anymore for (6.26) where the min looks more like an Isaacs' equation for a differential game. As we already mention it in the introduction of this part, this is the analogue for discontinuities of the phenomena which arises in exit time problems/Dirichlet problem where the maximal Ishii subsolution involves a "worse stopping time" on the boundary: we refer to [17] and [23] for details.

As an illustration, let us provide the form of the maximal solution of (6.23) in the particular case when for any $x \in \mathcal{H}$, $t \in (0,T)$, $r \in \mathbb{R}$ and $p_T \in \mathbb{R}^{N-1}$

$$H_0(x,t,r,p_T) \leq H_T^{\mathrm{reg}}(x,t,u,p_T)\;. \tag{6.27}$$

Proposition 6.4.5 *Assume that the "standard assumptions in the co-dimension-1 case" are satisfied and assume that (6.27) holds. Let* $V : \mathcal{H} \times (0,T) \to \mathbb{R}$ *be the unique solution of*

$$u_t + H_0(x,t,u,D_T u) = 0 \quad \text{on } \mathcal{H} \times (0,T)\,,$$

with the initial data $(u_0)_{|\mathcal{H}}$ *and let* $V_i : \Omega_i \times [0,T] \to \mathbb{R}$ *be the unique solutions of the problems*

$$u_t + H_i(x,t,u,Du) = 0 \quad \text{on } \Omega_i \times (0,T)\;,$$
$$u(x,t) = V(x,t) \quad \text{on } \mathcal{H} \times (0,T)\;,$$

$$u(x,0) = (u_0)_{|\overline{\Omega_i}} \quad on\ \overline{\Omega_i}\ .$$

Then the maximal (sub)solution of (6.23) *is given by*

$$\mathbf{U}^+_{H_0}(x,t) = \begin{cases} V_i(x,t) & \text{if } x \in \Omega_i \\ V(x,t) & \text{if } x \in \mathcal{H}\ . \end{cases}$$

Before giving the short proof of Proposition 6.4.5, we examine a simple example in dimension 1 showing the main features of this result. We take

$$\mathbf{BCL}_1(x,t) := \{(\alpha,0,0);\ |\alpha| \le 1\}\ ,$$

$$\mathbf{BCL}_2(x,t) := \{(\alpha,0,1);\ |\alpha| \le 1\}\ ,$$

and $\mathbf{BCL}_0(0,t) = \{(0,0,2)\}$. In which case

$$H_1(p) = |p|\ ,\ H_2(p) := |p| - 1\ ,\ H_T^{\mathrm{reg}} = 0\ ,\ H_0 = -2\ .$$

Hence (6.27) holds. It is easy to check that, if $u_0(x) = 0$ for all $x \in \mathbb{R}$

$$V(t) = 2t\ ,\ V_1(x,t) = 0\ ,\ V_2(x,t) = t \quad \text{for } x \in \mathbb{R},\ t \ge 0.$$

This example shows the followings: first, the value-function $\mathbf{U}^+_{H_0}$ is discontinuous although we have controllability/coercivity for the Hamiltonians H_1 and H_2; it is worth pointing out anyway that the global coercivity is lost since we use the Hamiltonian $\min(H_0, H_1, H_2)$ on $\mathcal{H}$ for the subsolutions instead of $\min(H_1, H_2)$.

Then, the values of $V(t)$ may seem strange since we use the maximal cost 2 but as we mention it above, this phenomena looks like the "worse stopping time" appearing in exit time problems. Finally, and this is even more surprising, the form of $\mathbf{U}^+_{H_0}$ shows that no information is transfered from Ω_1 to Ω_2: indeed, from the control point of view, starting from $x < 0$ where the cost is 1, it would seem natural to cross the border 0 to take advantage of the 0-cost in Ω_1 but this is not the case, even if $x < 0$ is close to 0. We have here state-constraint problems both in $\Omega_1 \times [0,T]$ and $\Omega_2 \times [0,T]$. This also means that the differential games features not only implies that one is obliged to take the maximal cost at $x = 0$ but also may prevent the trajectory to go from a less favourable region to a more favourable region.

Unfortunately we are unable to provide a general formula for $\mathbf{U}^+_{H_0}$, i.e. which would be valid for all cases without (6.27). Of course, trying to define $\mathbf{U}^+_{H_0}$ as in Proposition 6.4.5 but V being the solution of

$$u_t + \min\Big\{H_0(x,t,u,D_T u), H_T^{\mathrm{reg}}(x,t,u,D_T u)\Big\} = 0 \quad \text{on } \mathcal{H} \times (0,T)\,, \tag{6.28}$$

does not work as the following example shows it. In dimension 1, we take $H_1(p) = H_2(p) = |p|$, $H_0 > 0$ and $u_0(x) = -|x|$ in $\mathbb{R}$. Since $H_T^{\mathrm{reg}} = 0$, we have $H_0 > H_T^{\mathrm{reg}}$ and solving the above pde gives $V = 0$. Computing V_1 and V_2 as above gives $-|x| - t$ in both case. Hence V_1 and V_2 are just the restriction to $\Omega_1 \times [0,T]$ and $\Omega_2 \times [0,T]$ respectively of the solution of

$$u_t + |u_x| = 0 \quad \text{in } \mathbb{R} \times (0,T)\ ,$$

with the initial data u_0. Now defining $\mathbf{U}^+_{H_0}$ as in Proposition 6.4.5, we see that we do not have a subsolution: indeed the discontinuity of $\mathbf{U}^+_{H_0}$ at any point $(0,t)$ implies that $(0,t)$ is a maximum point of $\mathbf{U}^+_{H_0} - px$ for any $p \in \mathbb{R}$ and therefore we should have the inequality

$$\min(H_0, |p|, |p|) \leq 0\ ,$$

which is not the case if $|p| > 0$.

Remark 6.4.6 *Even if we were are able to provide a general formula for $\mathbf{U}^+_{H_0}$, we have some (again strange) informations on this maximal subsolution: first $\mathbf{U}^+_{H_0} \geq \mathbf{U}^+$ in $\mathbb{R}^N \times (0,T)$ since $\mathbf{U}^+$ is a subsolution of (6.23), a surprising result since it shows that adding H_0 on $\mathcal{H} \times (0,T)$ does not decrease the maximal subsolution as it could be thought from the control interpretation. On the other hand, Lemma 6.4.4 provides an upper estimates of $\mathbf{U}^+_{H_0}$ on $\mathcal{H} \times (0,T)$, namely the solution of (6.28).*

Proof of Proposition 6.4.5 — First, by our assumptions, V exists and is continuous: indeed, in order to obtain V we solve a standard Cauchy problem in $\mathbb{R}^{N-1} \times [0,T]$. Next by combining the argument of [17] (See also [23]) with the localization arguments of Section 3.2, V_1 and V_2 exists and are continuous in $\Omega_1 \times [0,T]$ and $\Omega_2 \times [0,T]$ respectively, with continuous extensions to $\overline{\Omega_1} \times [0,T]$ and $\overline{\Omega_2} \times [0,T]$. Moreover, the normal controllability implies that

$$V_1(x,t), V_2(x,t) \leq V(x,t) \quad \text{on } \mathcal{H} \times (0,T)\ .$$

Hence, defined in that way, $\mathbf{U}^+_{H_0}$ is upper semicontinuous (it may be discontinuous as we already see it above).

It is easy to check that $\mathbf{U}^+$ is a solution of (6.23). Indeed the subsolution properties on $\Omega_1 \times (0,T), \Omega_2 \times (0,T)$ are obvious. On $\mathcal{H} \times (0,T)$ they come from the properties of V since $\mathbf{U}^+_{H_0} = V$ on $\mathcal{H} \times (0,T)$; hence the H_0-inequality for V implies the subsolution inequality for $\mathbf{U}^+$.

For the supersolution ones, they comes from the properties of V_1, V_2 and V and the formulation of the Dirichlet problem since $(\mathbf{U}^+_{H_0})_* = \min(V_1, V_2, V) = \min(V_1, V_2)$ on

$\mathcal{H} \times (0,T)$. Indeed if ϕ is a smooth function in $\mathbb{R}^N \times (0,T)$ and if $(\bar{x},\bar{t}) \in \mathcal{H} \times (0,T)$ is a minimum point of $(\mathbf{U}^+_{H_0})_* - \phi$, we have several cases: if $(\mathbf{U}^+_{H_0})_*(\bar{x},\bar{t}) = V_1(\bar{x},\bar{t}) < V(\bar{x},\bar{t})$, then $(\bar{x},\bar{t})$ is a minimum point of $V_1 - \phi$ on $\overline{\Omega_1} \times (0,T)$ and, since V_1 is a solution of the Dirichlet problem in $\overline{\Omega_1} \times (0,T)$ with the Dirichlet data V, we have

$$\max\big(\phi_t(\bar{x},\bar{t}) + H_1(\bar{x},\bar{t},V_1(\bar{x},\bar{t}),D\phi(\bar{x},\bar{t})), V_1(\bar{x},\bar{t}) - V(\bar{x},\bar{t})\big) \geq 0\ ,$$

hence $\phi_t(\bar{x},\bar{t}) + H_1(\bar{x},\bar{t},V_1(\bar{x},\bar{t}),D\phi(\bar{x},\bar{t})) \geq 0$, which gives the answer we wish. The case when $(\mathbf{U}^+_{H_0})_*(\bar{x},\bar{t}) = V_2(\bar{x},\bar{t}) < V(\bar{x},\bar{t})$ is treated in a similar way. Finally if $(\mathbf{U}^+_{H_0})_*(\bar{x},\bar{t}) = V_1(\bar{x},\bar{t}) = V_2(\bar{x},\bar{t}) = V(\bar{x},\bar{t})$, we use that $(\bar{x},\bar{t})$ is a minimum point of $V - \phi$ on $\mathcal{H} \times (0,T)$ and therefore

$$\phi_t(\bar{x},\bar{t}) + H_0(\bar{x},\bar{t},V(\bar{x},\bar{t}),D\phi(\bar{x},\bar{t})) \geq 0\ ,$$

implying the viscosity supersolution inequality we wanted.

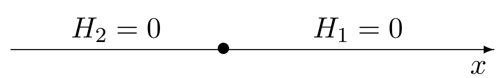
It remains to prove that any subsolution u of (6.23) is below $\mathbf{U}^+_{H_0}$. This comes from Lemma 6.4.4 which implies, using a standard comparison result on $\mathcal{H} \times [0,T]$ that $u(x,t) \leq V(x,t) = \mathbf{U}^+_{H_0}(x,t)$ on $\mathcal{H} \times [0,T]$.

Q.E.D.

Chapter 7

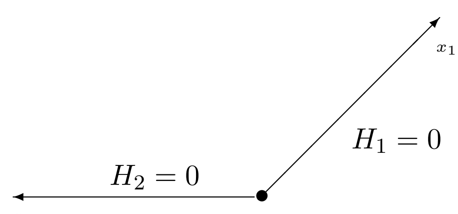
The "Network" point-of view

The "network approach" is a complete change of point of view which can easily be explained in dimension 1. If we consider an Hamilton-Jacobi Equation with a discontinuity at $x = 0$, we have in mind the picture



Here, in $\mathbb{R}$, Ishii's definition of viscosity solutions is natural and we have, at $x = 0$, the natural viscosity inequalities involving $\min(H_1, H_2)$ and $\max(H_1, H_2)$.

But, since the equations are different in the sets $\{x > 0\}$ and $\{x < 0\}$, we can see as well the picture as



where $J_1 = \{x > 0\}$ and $J_2 = \{x < 0\}$ become two different branches of a (simple) network and actually, it also becomes natural to introduce adapted coordinates on J_1, J_2 which are nothing but $x_1 = x$ on J_1 and $x_2 = -x$ on J_2.

Two important consequences of this different point of view are
(i) the "natural" test-functions are not the same as in the Ishii approach since they can be chosen differently in J_1 and J_2, with just a continuity assumption at $x = 0$

and
(ii) a "junction condition" is needed at $x = 0$ but the Ishii inequalities do not seem as natural as in the "flat case" and no condition appears as an obvious replacement for them.

7.1 Unflattening $\mathbb{R}^N$ or the Effects of a Change of Test-Functions

To go further, we come back on our original framework in $\mathbb{R}^N$ with $\Omega_1, \Omega_2, \mathcal{H}$ introduced in Section 3.1 where an analogous remark holds, just replacing J_1 by Ω_1, J_2 by Ω_2 and 0 by $\mathcal{H}$. We first define the space of "natural" test-functions

Definition 7.1.1 *We denote by* $\mathrm{PC}^1(\mathbb{R}^N \times [0,T])$ *the space of piecewise* C^1*-functions* $\psi \in C(\mathbb{R}^N \times [0,T])$ *such that there exist* $\psi_1 \in C^1(\bar{\Omega}_1 \times [0,T])$, $\psi_2 \in C^1(\bar{\Omega}_2 \times [0,T])$ *such that* $\psi = \psi_1$ *in* $\bar{\Omega}_1 \times [0,T]$ *and* $\psi = \psi_2$ *in* $\bar{\Omega}_2 \times [0,T]$.

An (obvious) important point in this definition is that $\psi = \psi_1 = \psi_2$ on $\mathcal{H} \times [0,T]$ and $D_{\mathcal{H}}\psi = D_{\mathcal{H}}\psi_1 = D_{\mathcal{H}}\psi_2$ on $\mathcal{H} \times [0,T]$, $\psi_t = (\psi_1)_t = (\psi_2)_t$ on $\mathcal{H} \times [0,T]$. We recall here that $D_{\mathcal{H}}$ is the tangential derivative.

This change of test-functions is a first step but it remains to examine the kind of "junction condition" we can impose on $\mathcal{H} \times [0,T]$, since, contrary to what happens for the Ishii definition, no obvious choice seems to stand out.

The first attempt could be to try the standard Ishii inequalities with this larger set of test-functions. On the simplest example where the equations are

$$\begin{cases} u_t + H_1(x,t,u,Du) = 0 & \text{in } \Omega_1 \times (0,T) , \\ u_t + H_2(x,t,u,Du) = 0 & \text{in } \Omega_2 \times (0,T) , \end{cases} \tag{HJ-gen}$$

and without additional Hamiltonian on $\mathcal{H}$, these conditions are

$$\begin{cases} \min(u_t + H_1(x,t,u,Du), u_t + H_2(x,t,u,Du)) \le 0 & \text{on } \mathcal{H} \times (0,T) , \\ \max(u_t + H_1(x,t,u,Du), u_t + H_2(x,t,u,Du)) \ge 0 & \text{on } \mathcal{H} \times (0,T) . \end{cases}$$

But it is easy to check that, with test-functions in $\mathrm{PC}^1(\mathbb{R}^N \times [0,T])$, there is no subsolutions if H_1, H_2 are both coercive. The argument is the following: if $u - \varphi$ has a maximum at some point $(0,t) \in \mathcal{H} \times (0,T)$, then $u - (\varphi + C|x_N|)$ also has a maximum at the same point and since $\varphi_C(x,t) := \varphi(x,t) + C|x_N|$ belongs to $\mathrm{PC}^1(\mathbb{R}^N \times [0,T])$

we can use it to test the inequalities. But since the Hamiltonians are coercive, taking $C > 0$ large enough yields an impossibility.

As a consequence of this simple remark, we see that some additional junction condition has to be added.

7.2 Flux-Limited Solutions for Control Problems and Quasi-Convex Hamiltonians

As it is clear from Chapter 6, in control problems, it is more natural to use some condition of the type

$$u_t + G(x, t, u, D_{\mathcal{H}} u) = 0 \quad \text{on } \mathcal{H} \times (0, T) \,, \tag{FL}$$

which is called a "flux-limiter" condition in the network litterature (cf. Imbert and Monneau [83, 84, 86]). Indeed, in control problems, one may have in mind a specific control on $\mathcal{H}$, i.e. a specific dynamic, discount and cost as in Section 6.1. Concrete modelizations and applications lead to a variety of different flux-limiter conditions at the boundary, expressed as specific functions G.

The connections between different types of Kirchhoff conditions and "flux-limiter" conditions (which can also be seen as the links between various type of definitions for junction conditions) is extensively studied in [83, 84, 86] and we give below some results which fall into the scope of this book.

But we first turn to the definition of "flux-limited sub and supersolutions" *in the case of control problems*, for which we introduce the following notations: if, for $x \in \overline{\Omega}_1$, $t \in [0, T]$, $r \in \mathbb{R}$, $p \in \mathbb{R}^N$, we have

$$H_1(x, t, r, p) := \sup_{\alpha_1 \in A_1} \left\{ -b_1(x, t, \alpha_1) \cdot p + c_1(x, t, \alpha_1) r - l_1(x, t, \alpha_1) \right\} \,, \tag{7.1}$$

$$H_1^-(x, t, r, p) := \sup_{\alpha_1 \in A_1 \,:\, b_1(x,t,\alpha_1)\cdot e_N \leq 0} \left\{ -b_1(x, t, \alpha_1) \cdot p + c_1(x, t, \alpha_1) r - l_1(x, t, \alpha_1) \right\} \,, \tag{7.2}$$

$$H_1^+(x, t, r, p) := \sup_{\alpha_1 \in A_1 \,:\, b_1(x,t,\alpha_1)\cdot e_N > 0} \left\{ -b_1(x, t, \alpha_1) \cdot p + c_1(x, t, \alpha_1) r - l_1(x, t, \alpha_1) \right\} \,, \tag{7.3}$$

and for $x \in \overline{\Omega}_2$, $t \in [0, T]$, $r \in \mathbb{R}$, $p \in \mathbb{R}^N$

$$H_2(x, t, r, p) := \sup_{\alpha_2 \in A_2} \left\{ -b_2(x, t, \alpha_2) \cdot p + c_2(x, t, \alpha_2) r - l_2(x, t, \alpha_2) \right\} \,, \tag{7.4}$$

$$H_2^-(x,t,r,p) := \sup_{\alpha_2\in A_2\,:\,b_2(x,t,\alpha_2)\cdot e_N<0} \{-b_2(x,t,\alpha_2)\cdot p + c_2(x,t,\alpha_2)r - l_2(x,t,\alpha_2)\}\ , \tag{7.5}$$

$$H_2^+(x,t,r,p) := \sup_{\alpha_2\in A_2\,:\,b_2(x,t,\alpha_2)\cdot e_N\geq 0} \{-b_2(x,t,\alpha_2)\cdot p + c_2(x,t,\alpha_2)r - l_2(x,t,\alpha_2)\}\ , \tag{7.6}$$

Finally, for the specific control problem on $\mathcal{H}$, we define for any $x \in \mathcal{H}$, $t \in [0,T]$, $r \in \mathbb{R}$, and $p_\mathcal{H} \in \mathbb{R}^{N-1}$

$$G(x,t,r,p_\mathcal{H}) := \sup_{\alpha_0\in A_0} \{-b_0(x,t,\alpha_0)\cdot p_\mathcal{H} + c_0(x,t,\alpha_0)r - l_0(x,t,\alpha_0)\}\,. \tag{7.7}$$

For $i = 1,2$, the functions b_i, c_i, l_i are at least continuous functions defined on $\bar\Omega_i \times [0,T] \times A_i$ and b_0, c_0, l_0 are also continuous functions defined on $\mathcal{H} \times [0,T] \times A_0$.

Notice first that the $+/-$ notation refers to the sign of $b \cdot e_N$ in the supremum, which implies that, for instance, H_1^- is nondecreasing with respect to p_N (the normal gradient variable) while H_1^+ is nonincreasing with respect to p_N.

The definition is the

Definition 7.2.1 (Flux-limited viscosity solutions for control problems) *An u.s.c., locally bounded function $u : \mathbb{R}^N \times (0,T) \to \mathbb{R}$ is a subsolution of* (HJ-Gen)-(FL) *if it is a classical viscosity subsolution of* (HJ-Gen) *and if, for any test-function $\psi \in \mathrm{PC}^1(\mathbb{R}^N \times [0,T])$ and any local maximum point $(x,t) \in \mathcal{H} \times (0,T)$ of $u - \psi$ in $\mathbb{R}^N \times (0,T)$, we have, at (x,t)*

$$\max\Big(\psi_t + G(x,t,u,D_\mathcal{H}\psi), \psi_t + H_1^+(x,t,u,D\psi_1), \psi_t + H_2^-(x,t,u,D\psi_2))\Big) \leq 0\,.$$

A l.s.c., locally bounded function $v : \mathbb{R}^N \times (0,T) \to \mathbb{R}$ is a supersolution of (HJ-Gen)-(FL) *if it is a classical viscosity supersolution of* (HJ-Gen) *and if, for any test-function $\psi \in \mathrm{PC}^1(\mathbb{R}^N \times [0,T])$ and any local minimum point $(x,t) \in \mathcal{H} \times (0,T)$ of $u - \psi$ in $\mathbb{R}^N \times (0,T)$, we have, at (x,t)*

$$\max\Big(\psi_t + G(x,t,v,D_\mathcal{H}\psi), \psi_t + H_1^+(x,t,v,D\psi_1), \psi_t + H_2^-(x,t,v,D\psi_2))\Big) \geq 0\,.$$

Several remarks have to be made on this definition which is very different from the classical ones: first we have a "max" both in the definition of supersolutions AND subsolutions; then we do not use the full Hamiltonians H_i but H_1^+ and H_2^-. These changes are justified when looking at the interpretation of the viscosity solutions inequalities in the optimal control framework. Indeed (*i*) the subsolution inequality means that any control is sub-optimal, *i.e.* if one tries to use a specific control, the result may not be optimal. But, of course, such a control has to be associated with an "admissible" trajectory: for example, if we are on $\mathcal{H}$, a b_1 pointing towards Ω_2 cannot be associated to a real trajectory, this is why we use H_1^+. And an analogous remark justifies H_2^-. Finally the "max" comes just from the fact that we test all sub-optimal controls.

(*ii*) Analogous remarks hold for the supersolution inequality, except that this inequality is related to the optimal trajectory, which has to be admissible anyway.

With these remarks, the reader may be led to the conclusion that an "universal" definition of solutions of (HJ-Gen) with the condition (FL) can hardly exist: if we look at control problems where the controller tries to maximize some profit, then the analogue of the H_1^+, H_2^- above seem still relevant because of their interpretation in terms of incoming dynamics but the max should be replaced by min. Therefore it seems that such particular definitions have to be used in each case since, again, the Kirchhoff condition does not seem natural in the control framework.

Remark 7.2.2 *Definition 7.2.1 provides the notion of "flux-limited viscosity solutions" for a problem with a co-dimension* 1 *discontinuity but it can be used in different frameworks, in particular in problems with boundary conditions: we refer to Guerand [76] for results on state constraints problems and [75] in the case of Neumann conditions where "effective boundary conditions and new comparison results are given, both works being in the case of quasi-convex Hamiltonians.*

7.3 Junction Viscosity Solutions, Kirchhoff Type Conditions

Even if flux-limited viscosity solutions have their advantages (and we will study this notion of solutions in the sequel), it may seem more natural to consider a definition of viscosity solution with a min / max condition on the junction involving H_1 and H_2 instead of their nondecreasing/nonincreasing parts.

In the next sections, we present the general notion of junction viscosity solutions, which is called "relaxed solution" in [83] but, because of the similarity to the classical notion of viscosity solutions, it seems to us that "junction viscosity solutions" is more appropriate. In Section 7.4 below, we prove that, in some sense, the notions of flux-limited solutions and junction viscosity solutions are equivalent – at least in the quasi-convex framework – in the case of (FL)-conditions, but also that Kirchhoff type conditions can be reduced to a (FL)-condition. These results are strongly inspired from [83] but we provide here simplified proofs.

7.3.1 General Junction Viscosity Solutions

Let us introduce the notion of G-(JVSub)/(JVSuper) (standing for G-junction viscosity sub/supersolution) as follows

Definition 7.3.1 *An u.s.c., locally bounded function $u : \mathbb{R}^N \times (0,T) \to \mathbb{R}$ is a G-*(JVSub) *of* (HJ-Gen) *if it is a classical viscosity subsolution of* (HJ-Gen) *and if, for any test-function $\psi \in \mathrm{PC}^1(\mathbb{R}^N \times [0,T])$ and any local maximum point $(x,t) \in \mathcal{H} \times (0,T)$ of $u - \psi$ in $\mathbb{R}^N \times (0,T)$,*

$$\min\Big(\psi_t + G(x,t,v,D_{\mathcal{H}}\psi), \psi_t + H_1(x,t,u,D\psi_1), \psi_t + H_2(x,t,u,D\psi_2))\Big) \leq 0\,. \quad (7.8)$$

A l.s.c., locally bounded function $v : \mathbb{R}^N \times (0,T) \to \mathbb{R}$ is a G-(JVSuper) *of* (HJ-Gen) *if it is a classical viscosity supersolution of* (HJ-Gen) *and if, for any test-function $\psi \in \mathrm{PC}^1(\mathbb{R}^N \times [0,T])$ and any local minimum point $(x,t) \in \mathcal{H} \times (0,T)$ of $u - \psi$ in $\mathbb{R}^N \times (0,T)$,*

$$\max\Big(\psi_t + G(x,t,v,D_{\mathcal{H}}\psi), \psi_t + H_1(x,t,v,D\psi_1), \psi_t + H_2(x,t,v,D\psi_2))\Big) \geq 0\,. \quad (7.9)$$

Of course, a G-(JVS) (i.e. a junction viscosity solution) is a function which is both G-(JVSub) and G-(JVSuper).

Before providing a result on the equivalence between the notion of flux-limited viscosity sub/supersolution and junction viscosity sub/supersolution (see section 7.4), we want to point out that one of the advantage of the notion of junction viscosity solution is that it can be applied to a wider class of junction conditions: typically the definition of sub/supersolutions for the Kirchhoff condition is nothing but the definition of junction condition but where the G-flux condition is replaced by the Kirchhoff condition.

7.3.2 The Kirchhoff Condition

A rather natural junction condition, used in various applications, is the *Kirchhoff condition*

$$\frac{\partial u}{\partial n_1} + \frac{\partial u}{\partial n_2} = 0 \quad \text{on } \mathcal{H} \times (0,T)\,, \quad \text{(KC)}$$

where, for $i = 1, 2$, $n_i(x)$ denotes the unit outward normal to $\partial\Omega_i$ at $x \in \partial\Omega_i$. Of course, this Kirchhoff condition has to be taken in the viscosity solutions sense which is here the sense of the following definition when (KC) is associated with the equations

Definition 7.3.2 (Viscosity solutions for the Kirchhoff condition) *An u.s.c., locally bounded function $u : \mathbb{R}^N \times (0,T) \to \mathbb{R}$ is a subsolution of* (HJ-Gen)-(KC) *if it is a classical viscosity subsolution of* (HJ-Gen) *and if, for any test-function $\psi \in \mathrm{PC}^1(\mathbb{R}^N \times [0,T])$ and any local maximum point $(x,t) \in \mathcal{H} \times (0,T)$ of $u - \psi$ in $\mathbb{R}^N \times (0,T)$, we have, at the point (x,t)*

$$\min\Big(\frac{\partial\psi_1}{\partial n_1} + \frac{\partial\psi_2}{\partial n_2}, \psi_t + H_1(x,t,u,D\psi_1), \psi_t + H_2(x,t,u,D\psi_2))\Big) \le 0\,.$$

A l.s.c., locally bounded function $v : \mathbb{R}^N \times (0,T) \to \mathbb{R}$ is a supersolution of (HJ-Gen)-(KC) *if it is a classical viscosity supersolution of* (HJ-Gen) *and if, for any test-function $\psi \in \mathrm{PC}^1(\mathbb{R}^N \times [0,T])$ and any local minimum point $(x,t) \in \mathcal{H} \times (0,T)$ of $u - \psi$ in $\mathbb{R}^N \times (0,T)$, we have, at the point (x,t)*

$$\max\Big(\frac{\partial\psi_1}{\partial n_1} + \frac{\partial\psi_2}{\partial n_2}, \psi_t + H_1(x,t,v,D\psi_1), \psi_t + H_2(x,t,v,D\psi_2))\Big) \ge 0\,.$$

This definition seems natural in the network framework since it keeps the basic features of the notion of viscosity solutions, just taking into account the "two branches" problem with the new set of test-functions. In particular, this definition uses the full Hamiltonians H_1 and H_2 (contrary to what will be the case below), and it thus usable for any Hamiltonians H_1 and H_2, without any convexity properties.

7.4 From One Notion of Solution to the Others

The aim of this section is to link Flux-limited solutions, junction viscosity solutions and the Kirchhoff condition. We prove equivalent formulations between (FL) and (JVS), and identifying the flux-limiter associated with the Kirchhoff condition.

7.4.1 Flux-Limited and Junction Viscosity Solutions

The equivalence of the two notions of solutions is given by the

Proposition 7.4.1 *Assume that the "standard assumptions in the co-dimension-1 case" hold. Then*

(i) an u.s.c., locally bounded function $u : \mathbb{R}^N \times (0,T) \to \mathbb{R}$ *is a flux-limited subsolution of* (HJ-Gen)-(FL) *with flux-limiter* G *if and only if it is a* G-(JVSub).
(ii) a l.s.c., locally bounded function $v : \mathbb{R}^N \times (0,T) \to \mathbb{R}$ *is a flux-limited supersolution of* (HJ-Gen)-(FL) *if it is a* G-(JVSuper).

Proof — In all this proof, ψ is always a generic test-function in $\mathrm{PC}^1(\mathbb{R}^N \times [0,T])$ and the maximum or minimum of $u-\psi$ in $\mathbb{R}^N \times (0,T)$ is always denoted by (x,t), which we assume to be located on $\mathcal{H} \times (0,T)$.

Subsolutions – we just sketch the proof here since this case is easy. If u is a flux-limited subsolution, it clearly satisfies (7.8). Indeed, if $u-\psi$ has a maximum at (x,t), then $\psi_t + G(x,t,v,D_{\mathcal{H}}\psi) \leq 0$ from the definition of flux-limited subsolutions. To prove the converse, we use the arguments of Proposition 3.4.1 to show that if $(x,t) \in \mathcal{H} \times (0,T)$

$$\begin{aligned} u(x,t) &= \limsup\{u(y,s)\ ; (y,s) \to (x,t),\ y \in \Omega_1\} \\ &= \limsup\{u(y,s)\ ; (y,s) \to (x,t),\ y \in \Omega_2\}\ . \end{aligned}$$

Using Proposition 7.4.3 at the end of this section allows to prove that, at $(x,t) \in \mathcal{H} \times (0,T)$, local maximum point of $u-\psi$ in $\mathbb{R}^N \times (0,T)$ then

$$\psi_t + H_1^+(x,t,u,D\psi_1) \leq 0 \quad , \quad \psi_t + H_2^-(x,t,u,D\psi_2)) \leq 0\ .$$

It remains to prove that $\psi_t + G(x,t,v,D_{\mathcal{H}}\psi) \leq 0$, which is done as follows: for any $C>0$, $u-(\psi + C|x_N|)$ has also a maximum at $(x,t) \in \mathcal{H} \times (0,T)$ but taking $C>0$ large enough in (7.8) yields that the min cannot be reached by the H_1/H_2-terms since both Hamiltonians are coercive. Thus necessarily, the non-positive min is given by the junction condition and the result follows.

Supersolutions – This case is a little bit more delicate. Of course, a flux-limited supersolution v satisfies (7.9) since $H_1 \geq H_1^+$ and $H_2 \geq H_2^-$. The main point is to prove that supersolutions of (7.9) are flux-limited supersolutions.

If $(x,t) \in \mathcal{H} \times (0,T)$ is a local maximum point of $u-\psi$, (7.9) holds and we wish to show that

$$\max\Big(\psi_t + G(x,t,v,D_{\mathcal{H}}\psi), \psi_t + H_1^+(x,t,v,D\psi_1), \psi_t + H_2^-(x,t,v,D\psi_2))\Big) \geq 0\ .$$

Assuming this is not the case, then necessarily $\psi_t + G(x,t,v,D_{\mathcal{H}}\psi) < 0$, $\psi_t + H_1^+(x,t,v,D\psi_1) < 0$ and $\psi_t + H_2^-(x,t,v,D\psi_2) < 0$, and (7.9) implies

$$\max\Big(\psi_t + H_1^-(x,t,v,D\psi_1), \psi_t + H_2^+(x,t,v,D\psi_2))\Big) \geq 0\ .$$

We assume for example that $\psi_t + H_1^-(x,t,v,D\psi_1) \geq 0$, the other case being treated similarly.

Refering the reader to Chapter 12 where the properties of H_1^+, H_2^- are described we see that, if $D\psi_i = p_T^i + p_N^i e_N$ for $i = 1,2$, where $p_T^i \in \mathcal{H}$ and $p_N^i \in \mathbb{R}$, then these inequalities imply for instance

$$-\psi_t(x,t) > H_1^+(x,t,v,p_T^1 + p_N^1 e_N)) \text{ and therefore } -\psi_t(x,t) > \min_s(H_1(x,t,v,p_T^1 + se_N))\,.$$

We deduce from the inequality $\psi_t + H_1^-(x,t,v,D\psi_1) \geq 0$ that $p_N^1 > m_1^+(x,t,v,p_T^1)$ where m_1^+ is the largest point of where $s \mapsto H_1(x,t,v,p_T^1 + se_N)$ reaches its minimum.

Now we have two cases: either $\psi_t + H_2^+(x,t,v,D\psi_2)) \geq 0$ and similarly, $p_N^2 < m_2^-(x,t,v,p_T^1)$, the least minimum point for H_2. In this first case, we set

$$\tilde\psi(x,t) := \begin{cases} \tilde\psi_1(x,t) = \psi_1(x,t) + (m_1^+(x,t,v,p_T^1) - p_N^1)x_N & \text{if } x_N > 0 \\ \tilde\psi_2(x,t) = \psi_2(x,t) + (m_2^-(x,t,v,p_T^2) - p_N^2)x_N & \text{if } x_N < 0\,. \end{cases}$$

This new test-function still belongs to $\mathrm{PC}^1(\mathbb{R}^N \times [0,T])$ and $v - \tilde\psi$ has still a minimum point at (x,t), therefore (7.9) holds with $\tilde\psi$. But, since by construction $D\tilde\psi_1(x,t) = m_1^+(x,t,v,p_T^1)$ while $D\tilde\psi_2(x,t) = m_2^-(x,t,v,p_T^2)$, it follows that for $i = 1,2$,

$$\tilde\psi_t + H_i(x,t,u,D\tilde\psi_i) = \tilde\psi_t + \min_s(H_i(x,t,v,p_T^i + se_N)) < 0\,.$$

Therefore $\tilde\psi_t + G(x,t,v,D_{\mathcal{H}}\tilde\psi) \geq 0$, which obviously implies $\psi_t + G(x,t,v,D_{\mathcal{H}}\psi) \geq 0$, so that the flux-limited condition holds.

If, on the contrary, $\psi_t + H_2^+(x,t,v,D\psi_2)) < 0$, then $\psi_t + H_2(x,t,v,D\psi_2)) < 0$ and the change of test-function reduces to

$$\tilde\psi(x,t) := \begin{cases} \tilde\psi_1(x,t) = \psi_1(x,t) + (m_1^+(x,t,v,p_T^1) - p_N^1)x_N & \text{if } x_N > 0 \\ \tilde\psi_2(x,t) = \psi_2(x,t) & \text{if } x_N < 0\,. \end{cases}$$

We conclude as in the first case.

Q.E.D.

Remark 7.4.2 *The above proof works both in the control framework (convex coercive Hamiltonians), but as the reader can see, it only requires* $(\mathbf{H_{QC}})$ *and the results of Section 12.2. Thus it applies also in the quasi-convex framework, under* $(\mathbf{H_{QC}})$*. But, whether the* (JVS) *formulation is a universal, applicable definition without any (quasi)convexity assumption is not clear to us.*

We conclude this section by a result which is more general than the one we use in the proof of Proposition 7.4.1 but which implies, in particular, that the flux-limited subsolution inequality can be reduced by dropping the H_1^+, H_2^- since these inequalities are automatic. But, of course, it can be applied in far more general situations.

Proposition 7.4.3 *Assume that u is an usc, locally bounded subsolution of*

$$u_t + H(x,t,u,Du) = 0 \quad in\ \Omega \times (0,T)\ ,$$

where H is a continuous function and Ω is a smooth domain of $\mathbb{R}^N$. If there exists $x \in \partial\Omega$, $t \in (0,T)$ and $r > 0$ such that
(i) for any $(y,s) \in \partial\Omega \cap B(x,r) \times (t-r,t+r)$

$$u(y,s) = \limsup\{u(z,\tau);\ (z,\tau) \to (y,t), z \in \Omega\}\ ,$$

(ii) The distance function d to $\partial\Omega$ is smooth in $\overline{\Omega} \cap B(x,r)$,
(iii) There exists a function $L : [\overline{\Omega} \times (0,T)] \cap [B(x,r) \times (t-r,t+r)] \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}$ such that $L \leq H$ on $[\overline{\Omega} \times (0,T)] \cap [B(x,r) \times (t-r,t+r)] \times \mathbb{R} \times \mathbb{R}^N$ and

$$\lambda \mapsto L(y,s,u,p+\lambda Dd(y))\ ,$$

is a decreasing function for any $(y,s,u,p) \in [\overline{\Omega} \times (0,T)] \cap [B(x,r) \times (t-r,t+r)] \times \mathbb{R} \times \mathbb{R}^N$.
Then u is a subsolution of

$$u_t + L(x,t,u,Du) = 0 \quad on\ [\partial\Omega \cap B(x,r)] \times (t-r,t+r)\ .$$

Moreover, if we can take $L = H$ the same result is valid for supersolutions.

Of course, this result immediately applies to $H = H_1$, $\Omega = \Omega_1$ and $L = H_1^+$ for any $(x,t) \in \mathcal{H} \times (0,T)$, provided that the subsolution satisfies (i) which is a consequence of Proposition 3.4.1 if we have suitable normal controlability and tangential continuity type assumptions.

This result for subsolutions consists at looking only at dynamics which are going inside Ω_1; for the supersolution one, we have to assume that ALL the dynamics are going inside [which is not actually the type of situations we are considering].

Proof — We consider a test-function ψ which is C^1 on $\overline{\Omega} \times (0,T)$ and we assume that $(y,s) \in \partial\Omega \cap B(x,r) \times (t-r,t+r)$ is a strict local maximum point of $u - \psi$. To prove the L-inequality, we consider the function

$$(z,\tau) \mapsto u(z,\tau) - \psi(z,\tau) - \frac{\alpha}{d(z)}\ ,$$

where $\alpha > 0$ is a parameter devoted to tend to 0.

Using (i), it is easy to show that, for α small enough, this function has a local maximum at $(\bar{z}, \bar{\tau})$ (depending on α but we drop this dependence for the sake of simplicity of notations) and moreover, as $\alpha \to 0$

$$(\bar{z}, \bar{\tau}) \to (y, s)\ ,\ u(\bar{z}, \bar{\tau}) \to u(y, s)\ .$$

Writing the viscosity subsolution inequality for u, we have

$$\psi_t(\bar{z}, \bar{\tau}) + H(\bar{z}, \bar{\tau}, u(\bar{z}, \bar{\tau}), D\psi(\bar{z}, \bar{\tau}) - \frac{\alpha}{[d(\bar{z})]^2} Dd(\bar{z})) \leq 0\ ,$$

which implies that the same inequality holds for L since $L \leq H$.

$$\psi_t(\bar{z}, \bar{\tau}) + L(\bar{z}, \bar{\tau}, u(\bar{z}, \bar{\tau}), D\psi(\bar{z}, \bar{\tau}) - \frac{\alpha}{[d(\bar{z})]^2} Dd(\bar{z})) \leq 0\ .$$

Finally we use the monotonicity property of L in the $Dd(y)$-direction which yields

$$\psi_t(\bar{z}, \bar{\tau}) + L(\bar{z}, \bar{\tau}, u(\bar{z}, \bar{\tau}), D\psi(\bar{z}, \bar{\tau})) \leq 0\ .$$

And the conclusion follows by letting α tends to 0, using the continuity of L.

For the supersolution property, we argue in an analogous way, looking at a minimum point and introducing a "$+\dfrac{\alpha}{d(z)}$" terms instead of the "$-\dfrac{\alpha}{d(z)}$" -one.

Q.E.D.

7.4.2 Kirchhoff Condition and Flux-Limiters

Here we compare the sub/supersolution of (HJ-Gen) associated with the Kirchhoff condition (KC) and with (FL)-conditions, mainly in the context of Chapter 6. We also consider the cases of more general Kirchhoff type conditions. To simplify, we drop the dependence of the Hamiltonians in u since this does not create much more difficulty in the proofs.

The results of this section are based on the analysis of various properties of Hamiltonians (in particular H_T^{reg}) which are provided in Appendix 12. Notice that this appendix is written in a slightly more general form, where the Hamiltonians depend on u for the sake of completeness, but the results, of course, apply here.

Our main result is

Proposition 7.4.4 *Assume that either the "standard assumptions in the co-dimension-1 case", or that* $(\mathbf{H_{BA-HJ}})$ *and* $(\mathbf{H_{QC}})$ *hold.*
(i) If u *is a subsolution for the Kirchhoff Condition then* u *is a flux-limited subsolution with* $G = H_T^{\rm reg}$*.*
(ii) If v *is a supersolution for the Kirchhoff Condition then* v *is a flux-limited supersolution with* $G = H_T^{\rm reg}$*.*

It is worth pointing out that this result holds both in the convex and non-convex case, provided that $(\mathbf{H_{QC}})$ is satisfied.

Proof — Of course, in both results, only the viscosity inequalities on $\mathcal{H}$ are different and therefore we concentrate on this case. We also point out that the proof is the same in the convex and non-convex case since it mainly uses $(\mathbf{H_{QC}})$ and Lemma 12.2.1. Therefore we concentrate only on the $(\mathbf{H_{QC}})$ case.

In order to prove (i), we first notice that the H_1^+, H_2^- inequalities clearly hold on $\mathcal{H}$ as a consequence of Proposition 7.4.3. Hence we have just to prove that, if $(\bar x, \bar t) \in \mathcal{H} \times (0,T)$ is a strict local maximum point of $u - \psi$ for some function $\psi = (\psi_1, \psi_2) \in \mathrm{PC}^1(\mathbb{R}^N \times [0,T])$ then

$$\psi_t(\bar x, \bar t) + H_T^{\rm reg}\big(\bar x, \bar t, D_{\mathcal{H}}\psi(\bar x, \bar t)\big) \leq 0 \ .$$

In particular, $(\bar x, \bar t)$ is a strict local maximum point of $((x',0),t) \mapsto u((x',0),t) - \psi((x',0),t)$ on $\mathcal{H}$ and we consider the function

$$(x,t) \mapsto u(x,t) - \psi((x',0),t) - \chi(x_N) - \frac{(x_N)^2}{\varepsilon^2} \ , \tag{7.10}$$

with, for some small $\kappa > 0$

$$\chi(y_N) := \begin{cases} (\lambda - \kappa) y_N & \text{if } y_N \geq 0, \\ (\lambda + \kappa) y_N & \text{if } y_N < 0, \end{cases}$$

where, referring to Lemma 12.2.1n, λ is a minimum point of the coercive, continuous function $s \mapsto \tilde H^{\rm reg}(\bar x, \bar t, D_{x'}\psi(\bar x, \bar t) + s e_N)$. As a consequence of this lemma,

$$H_T^{\rm reg}(\bar x, \bar t, D_{x'}\psi(\bar x, \bar t)) = H_1^-(\bar x, \bar t, D_{x'}\psi(\bar x, \bar t) + \lambda e_N) = H_2^+(\bar x, \bar t, D_{x'}\psi(\bar x, \bar t) + \lambda e_N) \ . \tag{7.11}$$

By standard arguments, the function defined in (7.10) has a maximum point $(x_\varepsilon, t_\varepsilon)$ near $(\bar x, \bar t)$ and $(x_\varepsilon, t_\varepsilon) \to (\bar x, \bar t)$ as ε tends to 0 since $(\bar x, \bar t)$ is a strict local maximum point of $(x,t) \mapsto u(x,t) - \psi((x',0),t)$ on $\mathcal{H}$.

Now we examine the quantity

$$Q_\varepsilon := H_1\Big(x_\varepsilon, t_\varepsilon, D_{x'}\psi((x'_\varepsilon, 0), t_\varepsilon) + (\lambda - \kappa)e_N + \frac{2(x_\varepsilon)_N}{\varepsilon^2})\Big)$$

which is defined only if $(x_\varepsilon)_N \geq 0$. Since $H_1 \geq H_1^-$ and H_1^- is increasing in the e_N-direction, it follows that

$$\begin{aligned} Q_\varepsilon &\geq H_1^-\Big(x_\varepsilon, t_\varepsilon, D_{x'}\psi((x'_\varepsilon, 0), t_\varepsilon) + (\lambda - \kappa)e_N + \frac{2(x_\varepsilon)_N}{\varepsilon^2})\Big) \\ &\geq H_1^-\big(x_\varepsilon, t_\varepsilon, D_{x'}\psi((x'_\varepsilon, 0), t_\varepsilon) + (\lambda - \kappa)e_N)\big) \\ &\geq H_1^-\big(\bar{x}, \bar{t}, D_{x'}\psi(\bar{x}, \bar{t}) + (\lambda - \kappa)e_N\big) + o_\varepsilon(1) \\ &= H_T^{\mathrm{reg}}(\bar{x}, \bar{t}, D_{x'}\psi(\bar{x}, \bar{t}) + \lambda e_N) + o_\varepsilon(1) + O(\kappa)\ . \end{aligned}$$

An analogous inequality holds if $(x_\varepsilon)_N \leq 0$ with H_2 and H_2^+.

From these two inequalities and the fact that the choice of $\chi(x_N)$ prevents the inequality "$\frac{\partial\psi_1}{\partial n_1} + \frac{\partial\psi_2}{\partial n_2} \leq 0$" to hold, we get the result.

We now turn to the proof of (ii). Let v be a supersolution for the Kirchhoff Condition: we have to prove that v is a flux-limited supersolution with $G = H_T^{\mathrm{reg}}$.

To do so, we consider a test-function $\psi = (\psi_1, \psi_2) \in \mathrm{PC}^1(\mathbb{R}^N \times [0, T])$ such that $v - \psi$ reaches a local strict minimum at $(\bar{x}, \bar{t}) \in \mathcal{H} \times (0, T)$. For $i = 1, 2$, we use the notations

$$a = \psi_t(\bar{x}, \bar{t})\ ,\ p' = D_{x'}\psi(\bar{x}, \bar{t})\ ,\ \lambda_i = \frac{\partial\psi_i}{\partial x_N}(\bar{x}, \bar{t})\ .$$

By the supersolution property of v, we have (dropping the dependence in $\bar{x}, \bar{t}$)

$$\max(-\lambda_1 + \lambda_2, a + H_1(p' + \lambda_1 e_N), a + H_2(p' + \lambda_2 e_N)) \geq 0\ ,$$

and we want to prove that

$$\max(a + H_T^{\mathrm{reg}}(p'), a + H_1^+(p' + \lambda_1 e_N), a + H_2^-(p' + \lambda_2 e_N)) \geq 0\ .$$

We argue by contradiction assuming that

$$\max(a + H_T^{\mathrm{reg}}(p'), a + H_1^+(p' + \lambda_1 e_N), a + H_2^-(p' + \lambda_2 e_N)) < 0\ .$$

Now we look at the sub-differential of v at $(\bar{x}, \bar{t})$ but restricted to each domain $\overline{\Omega}_1$ and $\overline{\Omega}_2$, i.e for example for $\overline{\Omega}_1$, we denote by $D^-_{\overline{\Omega}_1} v(\bar{x}, \bar{t})$ the set of (p_x, p_t) such that

$$v(y, t) \geq v(\bar{x}, \bar{t}) + p_x \cdot (y - \bar{x}) + p_t(t - \bar{t}) + o(|y - \bar{x}| + |t - \bar{t}|)\ ,$$

for any $y \in \overline{\Omega}_1$ and t close to $\bar t$. And an analogous definition is used for $D^-_{\overline{\Omega}_2} v(\bar x, \bar t)$. Such a sub-differentials are closed and convex sets and we have $((p', \lambda_i), a) \in D^-_{\overline{\Omega}_i} v(\bar x, \bar t)$. By these properties, the sets of λ such that $((p', \lambda), a) \in D^-_{\overline{\Omega}_i} v(\bar x, \bar t)$ are non-empty closed intervals of $\mathbb{R}$ and we can consider the supremum $\tilde\lambda_1$ of such λ in $D^-_{\overline{\Omega}_1} v(\bar x, \bar t)$ and the infimum $\tilde\lambda_2$ of such λ in $D^-_{\overline{\Omega}_2} v(\bar x, \bar t)$.

Since $((p', \tilde\lambda_i), a) \in D^-_{\overline{\Omega}_i} v(\bar x, \bar t)$, there exists a test-function $\tilde\psi = (\tilde\psi_1, \tilde\psi_2) \in \mathrm{PC}^1(\mathbb{R}^N \times [0,T])$ such that $v - \tilde\psi$ reaches a local strict minimum at $(\bar x, \bar t) \in \mathcal{H} \times (0,T)$ and with

$$a = \tilde\psi_t(\bar x, \bar t) \ , \ p' = D_{x'}\tilde\psi(\bar x, \bar t) \ , \ \tilde\lambda_i = \frac{\partial \psi_i}{\partial x_N}(\bar x, \bar t) \ .$$

We claim that

$$a + H_1(p' + \tilde\lambda_1 e_N) \geq 0 \quad \text{and} \quad a + H_2(p' + \tilde\lambda_2 e_N) \geq 0 \ . \tag{7.12}$$

Indeed we introduce the test-function $\tilde\psi_\delta$ where $\tilde\lambda_1$ is changed into $\tilde\lambda_1 + \delta$ for $0 < \delta \ll 1$ in the $\tilde\psi$-test-function, we see that $((p', \tilde\lambda_2 + \delta), a)$ cannot be in $D^-_{\overline{\Omega}_1} v(\bar x, \bar t)$ by the definition of $\tilde\lambda_1$ and therefore $(\bar x, \bar t)$ cannot be anymore a minimum point of $v - \tilde\psi_\delta$. If (x_δ, t_δ) is such a minimum point we have necessarily $(x_\delta)_N > 0$ and the H_1-viscosity inequality holds. But $(x_\delta, t_\delta) \to (\bar x, \bar t)$ as $\delta \to 0$ and we recover the H_1-viscosity inequality at $(\bar x, \bar t)$. Of course, an analogous proof can be done for H_2.

Therefore we have at the same time (7.12) and

$$\max(-\tilde\lambda_1 + \tilde\lambda_2, a + H_1(p' + \tilde\lambda_1 e_N), a + H_2(p' + \tilde\lambda_2 e_N)) \geq 0 \ ,$$

coming from the Kirchhoff condition. But, $\tilde\lambda_1 \geq \lambda_1$ and $\lambda \mapsto H_1^+(p' + \lambda_1 e_N)$ being decreasing, we have $a + H_1^+(p' + \tilde\lambda_1 e_N) < 0$. In the same way, $a + H_2^-(p' + \tilde\lambda_2 e_N)) < 0$. Hence

$$a + H_1^-(p' + \tilde\lambda_1 e_N) \geq 0 \quad \text{and} \quad a + H_2^+(p' + \tilde\lambda_2 e_N)) \geq 0 \ .$$

Taking into account Lemma 12.2.1 through (12.8) and the monotonicity properties of H_1^- and H_2^+, we deduce that necessarily $\tilde\lambda_2 < m_1 \leq m_2 < \tilde\lambda_1$, where m_1, m_2 are defined in Lemma 12.2.1.

But by the structure of the sub-differential, this means that, for any $\varepsilon > 0$,

$$((p', \tilde m_2 + \varepsilon), a) \in D^-_{\overline{\Omega}_1} v(\bar x, \bar t) \,, \qquad ((p', \tilde m_1 - \varepsilon), a) \in D^-_{\overline{\Omega}_2} v(\bar x, \bar t) \,,$$

which leads to

$$\max(-(m_2 + \varepsilon) + (m_1 - \varepsilon), a + H_1(p' + (m_2 + \varepsilon)e_N), a + H_2(p' + (m_1 - \varepsilon)e_N)) \geq 0 \ .$$

But we reach a contradiction since each term is strictly negative for ε small enough, because of the definition of m_1, m_2 and the fact that $a + H_T^{\rm reg}(p') < 0$.

Q.E.D.

We conclude this section by a caracterization of the solution associated to $H_T^{\rm reg}$ in the non-convex case.

Proposition 7.4.5 *Under the assumptions of Proposition 7.4.4, an u.s.c. function u is an Ishii subsolution of* (HJ-Gen) *if and only if it is a flux-limited subsolution associated to the flux-limiter $H_T^{\rm reg}$.*

Proof — We first show that an Ishii subsolution of (HJ-Gen) is a subsolution of (HJ-Gen)-(FL) for the flux limiter $H_T^{\rm reg}$. Of course, we just interested in the inequalities on $\mathcal{H}$.

Let u be an Ishii subsolution of (HJ-Gen)-(FL); by Proposition 7.4.3, we already know that the H_1^+ and H_2^- inequalities hold and therefore we have just to check the $H_T^{\rm reg}$-one. To do so, we pick a test-function $\psi : \mathbb{R}^{N-1} \times (0,T) \to \mathbb{R}$ and we assume that $x' \mapsto u((x',0),t) - \psi(x',t)$ has a strict, local maximum point at $(\bar{x},\bar{t}) = ((\bar{x}',0),t) \in \mathcal{H} \times (0,T)$. Then we consider the function

$$(x,t) = ((x', x_N), t) \mapsto u(x,t) - \psi(x',t) - \lambda x_N - \frac{x_N^2}{\varepsilon^2} \ ,$$

for $0 < \varepsilon \ll 1$, where $m_1 \leq \lambda \leq m_2$, m_1, m_2 being defined in Lemma 12.2.1 at the point $(\bar{x},\bar{t})$, and $p' = D_{x'}\psi(\bar{x},\bar{t})$. This function has a local maximum point at a point $(x_\varepsilon, t_\varepsilon)$ which converges to $(\bar{x},\bar{t})$.

If $(x_\varepsilon, t_\varepsilon) \in \Omega_1 \times (0,T)$, we have

$$\psi_t(x_\varepsilon, t_\varepsilon) + H_1(x_\varepsilon, t_\varepsilon, D_{x'}\psi(x_\varepsilon, t_\varepsilon) + \lambda e_N + \frac{2x_N}{\varepsilon^2} e_N) \leq 0 \ .$$

But $H_1 \geq H_1^+$ and using the monotonicity property of H_1^+ (which allows to drop the $\frac{2x_N}{\varepsilon^2}$-term), together with the continuity of H_1^+ and of the derivatives of ψ, we obtain

$$\psi_t(\bar{x},\bar{t}) + H_1^+(\bar{x},\bar{t}, D_{x'}\psi(\bar{x},\bar{t}) + \lambda e_N) \leq o_\varepsilon(1) \ ,$$

i.e.

$$\psi_t(\bar{x},\bar{t}) + H_T^{\rm reg}(\bar{x},\bar{t}, D_{x'}\psi(\bar{x},\bar{t}) + \lambda e_N) \leq o_\varepsilon(1) \ ,$$

and the conclusion follows by letting ε tend to 0. The two other cases $(x_\varepsilon, t_\varepsilon) \in \Omega_2 \times (0,T)$ and $(x_\varepsilon, t_\varepsilon) \in \mathcal{H} \times (0,T)$ can be treated similarly.

Conversely, we assume that u is a subsolution with the flux limiter H_T^{reg}, we have to show that it satisfies the right Ishii subsolution inequalites on $\mathcal{H}$. Let φ be a smooth function and $(\bar{x},\bar{t}) \in \mathcal{H} \times (0,T)$ be a maximum point of $u-\varphi$, we have to show

$$\min(a + H_1(\bar{x},\bar{t},p' + \lambda e_N), a + H_2(\bar{x},\bar{t},p' + \lambda e_N)) \leq 0 \ ,$$

with

$$a = \varphi_t(\bar{x},\bar{t}), \quad p' = D_{x'}\varphi(\bar{x},\bar{t}), \quad \lambda = \frac{\partial \varphi}{\partial x_N}(\bar{x},\bar{t}) \ .$$

But the flux-limited boundary condition says

$$a + H_1^+(\bar{x},\bar{t},p' + \lambda e_N) \leq 0, \quad a + H_2^-(\bar{x},\bar{t},p' + \lambda e_N) \leq 0 \ ,$$

(and $a + H_T^{\mathrm{reg}}(\bar{x},\bar{t},p' + \lambda e_N) \leq 0$), therefore we have to prove that

$$\text{either} \quad a + H_1^-(\bar{x},\bar{t},p' + \lambda e_N) \leq 0 \quad \text{or} \quad a + H_2^+(\bar{x},\bar{t},p' + \lambda e_N) \leq 0 \ ,$$

since $H_i = \max(H_i^+, H_i^-)$ for $i = 1,2$.

If $m_1 = m_1(\bar{x},\bar{t},p')$ and $m_2 = m_2(\bar{x},\bar{t},p')$ are given by Lemma 12.2.1, the result is obvious if $m_1 \leq \lambda \leq m_2$ while if $\lambda < m_1$, we have

$$a + H_1^-(\bar{x},\bar{t},p' + \lambda e_N) \leq a + H_1^-(\bar{x},\bar{t},p' + m_1 e_N) = a + H_T^{\mathrm{reg}}(\bar{x},\bar{t},p' + \lambda e_N) \leq 0 \ ,$$

and if $\lambda > m_2$, we have

$$a + H_2^+(\bar{x},\bar{t},p' + \lambda e_N) \leq a + H_2^+(\bar{x},\bar{t},p' + m_2 e_N) = a + H_T^{\mathrm{reg}}(\bar{x},\bar{t},p' + \lambda e_N) \leq 0 \ ,$$

so the proof is complete.

Q.E.D.

Chapter 8

Comparison Results for Flux-Limited Solutions and Applications

This part is devoted to prove comparison results for flux limited solutions; the original proofs given in [83, 84] were based on the rather technical construction of a "vertex function". We present here the simplified proof(s) of [12].

8.1 Comparison Results in the Convex Case

The main result of this section is the following.

Theorem 8.1.1 (Comparison principle) *Assume that we are in the "standard assumptions in the co-dimension-*1 *case". If* $u, v : \mathbb{R}^N \times (0,T) \to \mathbb{R}$ *are respectively an u.s.c. bounded flux-limited subsolution and a l.s.c. bounded flux-limited supersolution of* (HJ-Gen)-(FL) *and if* $u(x,0) \le v(x,0)$ *in* $\mathbb{R}^N$*, then* $u \le v$ *in* $\mathbb{R}^N \times (0,T)$.

We recall that the "standard assumptions in the co-dimension-1 case" mean that Assumption $(\mathbf{H_{BA-CP}})$ is satisfied by for (b_i, c_i, l_i) for $i = 0,1,2$ and that Assumption $(\mathbf{NC}_{\mathcal{H}})$ holds.

Proof — In order to simplify the proof, we provide it in full details only when the $c_i \equiv 0$ for $i = 0,1,2$; the general case only contains minor additional technical difficulties which can easily be handled after the classical changes $u \to \exp(-Kt)u$, $v \to \exp(-Kt)v$ which allow to reduce to the case when $c_i \ge 0$ for $i = 0,1,2$.

We follow Section 3.2 and we first check (**LOC1**)-evol: the function $\chi : \mathbb{R}^N \times (0,T) \to \mathbb{R}$ defined by

$$\bar\chi(x,t) := -Kt - (1+|x|^2)^{1/2} - \frac{1}{T-t} ,$$

is, for $K>0$ large enough, a strict subsolution of (HJ-Gen)-(FL) with $\bar\chi(x,t) \to -\infty$ when $|x| \to +\infty$ or $t \to T^-$. We replace u by either $u_\mu := u + (1-\mu)\bar\chi$ (a choice which does not use the convexity of the Hamiltonians) or $u_\mu := \mu u + (1-\mu)\bar\chi$ (a choice which uses the convexity of the Hamiltonians). Borrowing also the arguments of Section 3.2, (**LOC2**)-evol also holds and therefore we are lead to show that **(LCR)**-evol is valid in the case when u is an η-strict subsolution of (HJ-Gen)-(FL).

For a point $(\bar x, \bar t)$ where $\bar x \in \Omega_1$ or $\bar x \in \Omega_2$, the proof of **(LCR)**-evol in $\overline{Q^{\bar x,\bar t}_{r,h}}$ is standard, hence we have just to treat the case when $\bar x \in \mathcal{H}$. At this point, we make an other reduction in the proof: using Section 3.4, with $y = (t,x')$ and $z = x_N$, Assumptions **(TC)**,**(NC)** and **(Mon)** clearly hold in our case and therefore Theorem 3.4.2 applies. As a consequence, we can assume w.l.o.g that u is Lipschitz continuous with respect to all variables while it is C^1 and semi-convex in the (t,x')-variables; this last property implies that u_t and $D_{x'}u$ are continuous w.r.t. all variables.

We set $M := \max\limits_{\overline{Q^{x,t}_{r,h}}}(u-v)$ and we assume that $M>0$. If this maximum is achieved on $\partial_p Q^{x,t}_{r,h}$, the result is obvious so we may assume that it is achieved at $(\tilde x, \tilde t) \notin \partial_p Q^{x,t}_{r,h}$. Again, if $\tilde x \in \Omega_1$ or $\tilde x \in \Omega_2$, we easily obtain a contradiction and therefore we can assume that $\tilde x \in \mathcal{H}$.

We set $a = u_t(\tilde x, \tilde t)$, $p' = D_{x'}u(\tilde x, \tilde t)$ and we claim that we can solve the equations

$$a + H_1^-(\tilde x, \tilde t, \tilde p' + \lambda_1 e_N) = -\eta/2 \quad , \quad a + H_2^+(\tilde x, \tilde t, \tilde p' + \lambda_2 e_N)) = -\eta/2 ,$$

where we recall that $-\eta$ is the constant which measures the strict subsolution property of u.

In order to prove the existence of λ_1, we look at maximum points of the function

$$u(x,t) - \frac{|x-\tilde x|^2}{\varepsilon^2} - \frac{|t-\tilde t|^2}{\varepsilon^2} - \frac{\varepsilon}{x_N} ,$$

in $(\overline{Q^{x,t}_{r,h}}) \cap (\Omega_1 \times [0,T])$ and for $0 < \varepsilon \ll 1$. This function achieves its maximum at $(x_\varepsilon, t_\varepsilon)$ which converges to $(\tilde x, \tilde t)$ as $\varepsilon \to 0$ and by the semi-convexity of u in t and x', one has

$$u_t(x_\varepsilon, t_\varepsilon) + H_1(x_\varepsilon, t_\varepsilon, D_{x'}u(x_\varepsilon, t_\varepsilon) + \lambda_\varepsilon e_N) \le -\eta ,$$

for some $\lambda_\varepsilon \in \mathbb{R}$ which is bounded since u is Lipschitz continuous. Letting ε tend to 0 and using that $u_t(x_\varepsilon, t_\varepsilon) \to a$, $D_{x'}u(x_\varepsilon, t_\varepsilon) \to p'$ by the semi-convexity property of u, together with an extraction of subsequence for $(\lambda_\varepsilon)_\varepsilon$, yields the existence of $\bar{\lambda} \in \mathbb{R}$ such that

$$a + H_1(\tilde{x}, \tilde{t}, \tilde{p}' + \bar{\lambda} e_N) \leq -\eta\ .$$

Since $H_1^- \leq H_1$ we get $a + H_1^-(\tilde{x}, \tilde{t}, \tilde{p}' + \bar{\lambda} e_N) \leq -\eta$. Then we use the fact that $\lambda \mapsto a + H_1^-(\tilde{x}, \tilde{t}, \tilde{p}' + \lambda e_N)$ is continuous, nondecreasing in $\mathbb{R}$ and tends to $+\infty$ when $\lambda \to +\infty$ to get the existence of $\bar{\lambda} < \lambda_1$ solving the equation with $-\eta/2$. In this framework, λ_1 is necessarily unique since the convex function $\lambda \mapsto a + H_1^-(\tilde{x}, \tilde{t}, \tilde{p}' + \lambda e_N)$ only has flat parts at its minimum, while clearly λ_1 is not a minimum point for this function. The proof for λ_2 is analogous and we skip it.

In order to build the test-function, we set $h(t) := \lambda_1 t_+ - \lambda_2 t_-$ where $t_+ = \max(t, 0)$ and $t_- = \max(-t, 0)$, and

$$\chi(x_N, y_N) := h(x_N) - h(y_N) = \begin{cases} \lambda_1(x_N - y_N) & \text{if } x_N \geq 0\ ,\ y_N \geq 0\ , \\ \lambda_1 x_N - \lambda_2 y_N & \text{if } x_N \geq 0\ ,\ y_N < 0\ , \\ \lambda_2 x_N - \lambda_1 y_N & \text{if } x_N < 0\ ,\ y_N \geq 0\ , \\ \lambda_2(x_N - y_N) & \text{if } x_N < 0\ ,\ y_N < 0\ . \end{cases} \tag{8.1}$$

Now, for $0 < \varepsilon \ll 1$ we define a test function as follows

$$\psi_\varepsilon(x, t, y, s) := \frac{|x - y|^2}{\varepsilon^2} + \frac{|t - s|^2}{\varepsilon^2} + \chi(x_N, y_N) + |x - \tilde{x}|^2 + |t - \tilde{t}|^2\ .$$

In view of the definition of h, we see that for any $(x, t) \in \mathbb{R}^N \times [0, T]$ the function $\psi_\varepsilon(x, \cdot, t, \cdot) \in \mathrm{PC}^1(\mathbb{R}^N \times [0, T])$ and for any $(y, s) \in \mathbb{R}^N \times [0, T]$ the function $\psi_\varepsilon(\cdot, y, \cdot, s) \in \mathrm{PC}^1(\mathbb{R}^N \times [0, T])$.

We know look at the maximum points of

$$(x, t, y, s) \mapsto u(x, t) - v(y, t) - \psi_\varepsilon(x, t, y, s)\ ,$$

in $\left[\overline{Q_{r,h}^{x,t}}\right]^2$. By standard arguments, this function has maximum points $(x_\varepsilon, t_\varepsilon, y_\varepsilon, s_\varepsilon)$ such that $(x_\varepsilon, t_\varepsilon, y_\varepsilon, s_\varepsilon) \to (\tilde{x}, \tilde{t}, \tilde{x}, \tilde{t})$. Moreover, using the semi-convexity of u, we have

$$p'_\varepsilon = \frac{2(x'_\varepsilon - y'_\varepsilon)}{\varepsilon^2} \to p' \quad \text{and} \quad \frac{2(t_\varepsilon - s_\varepsilon)}{\varepsilon^2} \to a\ ,$$

and the Lipschitz continuity of u implies that

$$(p_\varepsilon)_N = \frac{2((x_\varepsilon)_N - (y_\varepsilon)_N)}{\varepsilon^2}\ ,$$

remains bounded.

We have to consider different cases depending on the position of x_ε and y_ε in $\mathbb{R}^N$. Of course, we have no difficulty for the cases $x_\varepsilon, y_\varepsilon \in \Omega_1$ or $x_\varepsilon, y_\varepsilon \in \Omega_2$, and even less because of the above very precise properties on the derivatives of the test-function; only the cases where x_ε, y_ε are in different domains or on $\mathcal{H}$ cause problem.

We have to consider three cases

1. $x_\varepsilon \in \Omega_1$, $y_\varepsilon \in \Omega_2 \cup \mathcal{H}$ or $x_\varepsilon \in \Omega_2$, $y_\varepsilon \in \Omega_1 \cup \mathcal{H}$.
2. $x_\varepsilon \in \mathcal{H}$, $y_\varepsilon \in \Omega_1$ or $x_\varepsilon \in \mathcal{H}$, $y_\varepsilon \in \Omega_2$.
3. $x_\varepsilon \in \mathcal{H}$, $y_\varepsilon \in \mathcal{H}$.

Case 1: If $x_\varepsilon \in \Omega^1$, $y_\varepsilon \in \Omega_2 \cup \mathcal{H}$, we use that u is a subsolution of $u_t + H_1(x,t,Du) \leq -\eta$ in $\Omega_1 \times (0,T)$: taking into account the above properties of the test-function, we have

$$a + o_\varepsilon(1) + H_1(x_\varepsilon, t_\varepsilon, p' + o(1) + \lambda_1 e_N + (p_\varepsilon)_N e_N) \leq -\eta \ . \tag{8.2}$$

which implies, using the fact that any term in H_1 remains in a compact subset

$$a + H_1^-(\tilde{x}, \tilde{t}, p' + \lambda_1 e_N + (p_\varepsilon)_N e_N) \leq -\eta + o_\varepsilon(1) \ .$$

But, since $(p_\varepsilon)_N \geq 0$, we have also, thanks to the monotonicity of H_1^- in the e_N-direction

$$a + H_1^-(\tilde{x}, \tilde{t}, p' + \lambda_1 e_N) \leq -\eta + o_\varepsilon(1) \ ,$$

which is a contradiction with the definition of λ_1.

The case $x_\varepsilon \in \Omega^2$, $y_\varepsilon \in \Omega_1 \cup \mathcal{H}$ can be treated in an analogous way.

Case 2: Since $x_\varepsilon \in \mathcal{H}$, we have the subsolution inequality

$$\max\Big(a + G(\tilde{x}, \tilde{t}, p') \ ; \ a + H_1^+(\tilde{x}, \tilde{t}, p' + \lambda_1 e_N + (p_\varepsilon)_N e_N) \ ; \\ a + H_2^-(\tilde{x}, \tilde{t}, p' + \lambda_1 e_N + (p_\varepsilon)_N e_N)\Big) \leq -\eta + o_\varepsilon(1) \ .$$

On the other hand, if $y_\varepsilon \in \Omega_1$, since v is a supersolution of $v_t + H_1(x,t,Dv) = 0$ in Ω_1 this implies

$$a + H_1(\tilde{x}, \tilde{t}, p' + \lambda_1 e_N + (p_\varepsilon)_N e_N) \geq o_\varepsilon(1) \tag{8.3}$$

Now the aim is to show that the same inequality holds for H_1^+ and to do so, we evaluate this quantity for H_1^-: taking into account the fact that $(p_\varepsilon)_N \leq 0$ and the monotonicity of H_1^- in the e_N-direction yields

$$a + H_1^-(\tilde{x}, \tilde{t}, p' + \lambda_1 e_N + (p_\varepsilon)_N e_N) \leq -\eta/2 + O_\varepsilon(1) < 0 \text{ if } \varepsilon \text{ is small enough} \ ,$$

and since $H_1 = \max(H_1^-, H_1^+)$, we actually have

$$a + H_1^+(\tilde{x}, \tilde{t}, p' + \lambda_1 e_N + (p_\varepsilon)_N e_N) \geq o_\varepsilon(1)$$

which gives a contradiction when compared with the subsolution property. And the same contradiction is obtained for the case $y_\varepsilon \in \Omega_2$ with λ_2 and H_2^+ having the role previously played by λ_1 and H_1^-.

Case 3: If $x_\varepsilon \in \mathcal{H}$, $y_\varepsilon \in \mathcal{H}$, we have viscosity sub and supersolution inequalities for the same Hamiltonian and the contradiction follows easily.

And the proof is complete.

Q.E.D.

8.2 Flux-Limited Solutions and Control Problems

In this section, we come back on the control problem of Section 6.1 which we address here from a different point of view.

In order to do that, we first have to define the admissible trajectories among all the solutions of the differential inclusion: We say that a solution $(X, T, D, L)(\cdot)$ of the differential inclusion starting from $(x, t, 0, 0)$ is an admissible trajectory if

1. there exists a global control $a = (\alpha_1, \alpha_2, \alpha_0)$ with $\alpha_i \in \mathcal{A}_i := L^\infty(0, \infty; A_i)$ for $i = 0, 1, 2$,
2. there exists a partition $\mathbb{I} = (\mathbb{I}_1, \mathbb{I}_2, \mathbb{I}_0)$ of $(0, +\infty)$, where $\mathbb{I}_1, \mathbb{I}_2, \mathbb{I}_0$ are measurable sets, such that $X(s) \in \overline{\Omega_i}$ for any $s \in \mathbb{I}_i$ if $i = 1, 2$ and $X(s) \in \mathcal{H}$ if $s \in \mathbb{I}_0$,
3. for almost every $0 \leq s \leq t$

$$(\dot{X}, \dot{D}, \dot{L})(s) = \sum_{i=0}^{2} (b_i, c_i, l_i)(X(s), t - s, \alpha_i(s)) \mathbb{1}_{\{\mathbb{I}_i\}}(s)\,. \tag{8.4}$$

In Equation (8.4), we have dropped $T(s)$ since we are in the $b^t \equiv -1$ case and therefore $T(s) = t - s$ for $s \leq t$. The set of all admissible trajectories $(X, \mathbb{I}, a)$ issued from a point $X(0) = x \in \mathbb{R}^N$ (at $T(s) = t$) is denoted by $\mathcal{T}_x$. Notice that, under the controllability assumption $(\mathbf{NC}_\mathcal{H})$, for any point $x \in \overline{\Omega_1}$, there exist trajectories starting from x, which stay in $\overline{\Omega_1}$, and the same remark holds for points in $\overline{\Omega_2}$. These trajectories are clearly admissible (with either $\mathbb{I}_1 \equiv 1$ or $\mathbb{I}_2 \equiv 1$) and therefore $\mathcal{T}_x$ is never void.

Remark 8.2.1 *It is worth pointing out that, in this approach, the partition* $\mathbb{I}_0, \mathbb{I}_1, \mathbb{I}_2$ *which we impose for admissible trajectories, implies that there is no mixing on* $\mathcal{H}$ *between the dynamics and costs in* Ω_1 *and* Ω_2*, contrarily to the approach of Section 6.1. A priori, on* $\mathcal{H}$*, we have an independent control problem or we can use either* (b_1, c_1, l_1) *or* (b_2, c_2, l_2)*, but no combination of* (b_1, c_1, l_1) *and* (b_2, c_2, l_2).

The value function is then defined as

$$\mathbf{U}_G^{\mathrm{FL}}(x,t) := \inf_{(X,\mathbb{I},a)\in\mathcal{T}_x} \int_0^t \Big\{ \sum_{i=0}^{2} l_i(X(s), t-s, \alpha_i(s)) \mathbb{1}_{\{\mathbb{I}_i\}}(s) \Big\} e^{-D(s)}\, \mathrm{d}s \; .$$

By standard arguments based on the Dynamic Programming Principle and the above comparison result, we have the

Theorem 8.2.2 *The value function* $\mathbf{U}_G^{\mathrm{FL}}$ *is the unique flux limited solution of* (HJ-Gen)-(FL).

Before considering the connections with the results of Section 6.1, we want to point out that among all these “flux-limited value-functions”, there is a particular one which corresponds to either no specific control on $\mathcal{H}$ (i.e. we just consider the trajectories such that $\mathbb{I}_0 \equiv \emptyset$) or, and this is of course equivalent, to a cost l_0 which is $+\infty$. This value function is denoted by $\mathbf{U}^{\mathrm{FL}}$.

The aim is to show that the value functions of regional control are flux-limited solutions.

Theorem 8.2.3 *Under the assumptions of Theorem 8.1.1 (comparison result), for any Hamiltonian* H_0 *we have*

- (i) $\mathbf{U}^- \leq \mathbf{U}^+ \leq \mathbf{U}^{\mathrm{FL}}$ *in* $\mathbb{R}^N \times [0,T]$.
- (ii) $\mathbf{U}^- = \mathbf{U}_G^{\mathrm{FL}}$ *in* $\mathbb{R}^N \times [0,T]$ *where* $G = H_T$ *and* $\mathbf{U}^-_{H_0} = \mathbf{U}_G^{\mathrm{FL}}$ *in* $\mathbb{R}^N \times [0,T]$ *where* $G = \max(H_T, H_0)$.
- (iii) $\mathbf{U}^+ = \mathbf{U}_G^{\mathrm{FL}}$ *in* $\mathbb{R}^N \times [0,T]$ *where* $G = H_T^{\mathrm{reg}}$.

This result shows that, by varying the flux-limiter G, we have access to the different value functions described in Section 6.1.

Proof — For (i), the inequalities can just be seen as a consequence of the definition of $\mathbf{U}^-, \mathbf{U}^+, \mathbf{U}^{\mathrm{FL}}$ remarking that we have a larger set of dynamics-costs for $\mathbf{U}^-$ and

$\mathbf{U}^+$ than for $\mathbf{U}^{\mathrm{FL}}$. From a more pde point of view, applying Proposition 7.4.3, it is easy to see that $\mathbf{U}^-, \mathbf{U}^+$ are flux-limited subsolutions of (HJ-gen)-(FL) since they are subsolutions of

$$u_t + H_1^+(x,t,u,Du) \leq 0 \quad \text{in } \Omega_1 \; ,$$
$$u_t + H_2^-(x,t,u,Du) \leq 0 \quad \text{in } \Omega_2 \; .$$

Then Theorem 8.1.1 allows us to conclude.

For (ii) and (iii), we have to prove respectively that $\mathbf{U}^-$ is a solution of (HJ-gen)-(FL) with $G = H_T$, $\mathbf{U}^-_{H_0}$ is a solution of (HJ-gen)-(FL) with $G = \max(H_T, H_0)$ and $\mathbf{U}^+$ with $G = H_T^{\mathrm{reg}}$. Then the equality is just a consequence of Theorem 8.1.1.

For $\mathbf{U}^-$, the subsolution property just comes from the above argument for the H_1^+, H_2^--inequalities and from Proposition 6.1.5 for the H_T-one. The supersolution inequality is a consequence of the proof of Lemma 5.4.1: alternative **A)** implies that one of the H_1^+, H_2^--inequalities hold while alternative **B)** implies that the H_T-one holds. The same is true for $\mathbf{U}^-_{H_0}$.

For $\mathbf{U}^+$, the subsolution property follows from the same arguments as for $\mathbf{U}^-$, both for the H_1^+, H_2^--inequalities and from Proposition 6.2.2 for the H_T^{reg}-one. The supersolution inequality is a consequence of Theorem 6.3.3: alternative **A)** implies that one of the H_1^+, H_2^--inequalities hold while alternative **B)** implies that the H_T^{reg}-one holds.

And the proof is complete.

Q.E.D.

Inequalities in Theorem 8.2.3-(i) can be strict: various examples are given in [10]. The following one in dimension 1 shows that we can have $\mathbf{U}^+ < \mathbf{U}^{\mathrm{FL}}$ in $\mathbb{R}$.

Example 8.2.4 *Let $\Omega_1 = (0, +\infty)$, $\Omega_2 = (-\infty, 0)$. We choose*

$$b_1(\alpha_1) = \alpha_1 \in [-1, 1] \; , \; l_1(\alpha_1) = \alpha_1 \; ,$$

$$b_2(\alpha_2) = \alpha_2 \in [-1, 1] \; , \; l_1(\alpha_2) = -\alpha_2 \; .$$

It is clear that the best strategy is to use $\alpha_1 = -1$ in Ω_1, $\alpha_2 = 1$ in Ω_2 and an easy computation gives

$$\mathbf{U}^+(x) = \int_0^{+\infty} - \exp(-t)dt = -1 \; ,$$

because we can use these strategies in Ω_1, Ω_2 but also at 0 since the combination

$$\frac{1}{2} b_1(\alpha_1) + \frac{1}{2} b_2(\alpha_2) = 0 \; ,$$

has a cost -1*. In other words, the "push-push" strategy at* 0 *allows to maintain the* -1 *cost.*

But for $\mathbf{U}^{\mathrm{FL}}$*, this "push-push" strategy at* 0 *is not allowed and, since the optimal trajectories are necessarily monotone, the best strategy when starting at* 0 *is to stay at* 0 *but here with a best cost which is* 0*. Hence* $\mathbf{U}^{\mathrm{FL}}(0) = 0 > \mathbf{U}^+(0)$ *and it is easy to show that* $\mathbf{U}^{\mathrm{FL}}(x) > \mathbf{U}^+(x)$ *for all* $x \in \mathbb{R}$*.*

Theorem 8.2.3 can be interpreted in several ways: first the key point is what kind of controlled trajectories we wish to allow on $\mathcal{H}$ and, depending on this choice, different formulations have to be used for the associated HJB problem. It could be thought that the flux-limited approach is more appropriate, in particular because of Theorem 8.1.1 which is used intensively in the above proof.

8.3 A Comparison Result in the Quasi-convex Case

We address the same question but without assuming the H_i's to be convex in p but satisfying only the following "quasi-convex" assumption (again we consider the case when the H_i's are independent of u in order to simplify)

$(\mathbf{H_{QC}})$ For $i = 1, 2$, $H_i = \max(H_i^+, H_i^-)$ where H_i^+, H_i^- are bounded from below, Lipschitz continuous functions such that, for any x in a neighborhood of $\mathcal{H}$, $t \in [0, T]$ and $p \in \mathbb{R}^N$

$\lambda \mapsto H_1^+(x, t, p + \lambda e_N)$ is decreasing, $\lambda \mapsto H_1^-(x, t, p + \lambda e_N)$ is increasing and tends to $+\infty$ as $\lambda \to +\infty$, locally uniformly w.r.t. x, t and p, and

$\lambda \mapsto H_2^+(x, t, p + \lambda e_N)$ is increasing, $\lambda \mapsto H_2^-(x, t, p + \lambda e_N)$ is decreasing and tends to $+\infty$ as $\lambda \to -\infty$, locally uniformly w.r.t. x, t and p. .

Theorem 8.3.1 (Comparison principle in the non-convex case) *If* $(\mathbf{H_{QC}})$ *holds and that the Hamiltonians* $H_i^\pm$ *and* G *satisfy* $(\mathbf{H_{BA-HJ}})$*. Then the result of Theorem 8.1.1 remains valid.*

Proof — We just sketch it since it follows very closely the proof of Theorem 8.1.1. The only difference is that Section 3.4 only allows to reduce to the case when the strict subsolution u is semi-convex in the (t, x')-variables but not C^1. This obliges us to first look at a maximum of

$$(x, t, y, s) \mapsto u(x, t) - v(y, s) - \frac{|x' - y'|^2}{\varepsilon^2} - \frac{|t - s|^2}{\varepsilon^2} \ ,$$

where $x = (x', x_N)$, $y = (y', x_N)$, which is, of course, an approximation of $\overline{\max_{Q^{x,t}_{r,h}}} (u - v)$.

If $(\tilde{x}, \tilde{t}, \tilde{y}, \tilde{s})$ is a maximum point of this function, we remark that the semi-convexity of u implies that u is differentiable w.r.t. x' and t at $(\tilde{x}, \tilde{t})$ and we have

$$a := \frac{2(\tilde{t} - \tilde{s})}{\varepsilon^2} = u_t(\tilde{x}, \tilde{t}) \quad \text{and} \quad p' := \frac{2(\tilde{x}' - \tilde{y}')}{\varepsilon^2} = D_{x'} u(\tilde{x}, \tilde{t}) \; .$$

Then we solve the λ_1, λ_2-equations with such a and p'; it is worth pointing out that λ_1 and λ_2 are not uniquely defined but this is not important in the proof.

Finally we consider the maxima of the function

$$(x, t, y, s) \mapsto u(x, t) - v(y, s) - \frac{|x' - y'|^2}{\varepsilon^2} - \frac{|t - s|^2}{\varepsilon^2} - \chi(x_N, y_N) - \frac{|x_N - y_N|^2}{\gamma^2} |x - \tilde{x}|^2 + |t - \tilde{t}|^2 \; ,$$

where $0 < \gamma \ll 1$ is a parameter devoted to tend to 0 first.

Using the normal controlability assumption, it is easy to show that

$$|(p_\varepsilon)_N| = \frac{2|(x_\varepsilon)_N - (y_\varepsilon)_N|}{\gamma^2} = O(|p'_\varepsilon| + 1) \; ,$$

and this allows to perform all the arguments of the proof of using the semi-convexity of u which again implies the differentiability of u w.r.t. t and x' and $u_t(x_\varepsilon, t_\varepsilon) \to a, D_{x'} u(x_\varepsilon, t_\varepsilon) \to p'$. The proof then follows as in the convex case.

Q.E.D.

8.4 Convergence of the vanishing viscosity approximation (I)

In classical viscosity solutions' theory, obtaining the convergence of the vanishing viscosity method is just a simple exercice which shows the power of the combination of the half-relaxed limit method with a strong comparison result.

But, in the present discontinuous framework, although classical viscosity solutions, (CVS) in short, still have good stability properties as described in Section 3.1, the lack of uniqueness makes this stability far less effective: the two half-relaxed limits are lying between the minimal one $\mathbf{U}^-$ and the maximal one $\mathbf{U}^+$ and one cannot really obtain the convergence in that way, except if $\mathbf{U}^+ = \mathbf{U}^-$.

The next idea is to turn to flux-limited solutions (FLS in short) for which we have a general comparison result but, in order to identify the limit of the vanishing viscosity method, a flux-limiter is required and, to the best of our knowledge, there is no obvious way to determine it. Actually we refer the interested reader to Section 11.3.2 for a discussion on more general discontinuities where the problem is still open.

We refer anyway to [83, 84] for general stability results for FLS and to Camilli, Marchi and Schieborn [38] for the first results on the convergence of the vanishing viscosity method.

Finally, the notion of junction viscosity solutions (JVS in short) certainly provides the simplest and most natural proof, in particular by using the Lions-Souganidis comparison result for JVS, since it holds without convexity assumptions on the Hamiltonians.

In this book, we give several different proofs of the following result: the first one inspired from [12] uses only the properties of $\mathbf{U}^+$ as flux-limited solution, the second one inspired from Imbert and Nguyen [86] uses the Kirchhoff condition and connections between JVS and FLS and the last one is the most general one for non-convex Hamiltonian we just mention above.

In the setting of Chapter II, we show now that the vanishing viscosity approximation converges towards the function $\mathbf{U}^+$ defined in the (CVS)-approach.

Theorem 8.4.1 (Vanishing viscosity limit) *Assume that we are in the "standard assumptions in the co-dimension-1 case". For any $\varepsilon > 0$, let u^ε be a viscosity solution of*

$$u_t^\varepsilon - \varepsilon \Delta u^\varepsilon + H(x,t,u^\varepsilon,Du^\varepsilon) = 0 \quad in \quad \mathbb{R}^N \times (0,T)\,, \tag{8.5}$$

$$u^\varepsilon(x,0) = u_0(x) \quad in \quad \mathbb{R}^N\,, \tag{8.6}$$

where $H = H_1$ in Ω_1 and H_2 in Ω_2, and u_0 is bounded continuous function in $\mathbb{R}^N$. If the u^ε are uniformly bounded in $\mathbb{R}^N \times (0,T)$ and C^1 in x_N in a neighborhood of $\mathcal{H}$, then, as $\varepsilon \to 0$, the sequence $(u^\varepsilon)_\varepsilon$ converges locally uniformly in $\mathbb{R}^N \times (0,T)$ to $\mathbf{U}^+$, the maximal Ishii subsolution of (6.1).

Remark 8.4.2 *A priori (8.5)-(8.6) is a uniformly parabolic problem and the regularity we assume on (u^ε) is reasonnable. Indeed the function u^ε is expected to be C^1 since it is also expected to be in $W^{2,r}_{loc}$ (for any $r > 1$). On the other hand, it is worth pointing out that, as long as $\varepsilon > 0$, it is not necessary to impose a condition on $\mathcal{H}$ because of the strong diffusion term.*

Contrary to the proof relying on the Lions-Souganidis approach, the arguments we use in this section strongly rely on the structure of the Hamiltonians and on the arguments of the comparison proof. It has the advantage anyway to identify the limit in terms of control problems. An other way to do it goes through the connections between the Kirchoff condition and Flux-limiters conditions (See Section 7.4.2).

Proof — We first recall that, by Theorem 6.3.6, $\mathbf{U}^+$ is the maximal subsolution (and Ishii solution) of (6.1) and we proved in Theorem 8.2.3 that it is the unique flux-limited solution of (HJ-Gen)-(FL) with $G = H_T^{\mathrm{reg}}$. We recall that the flux-limiter condition consists in complementing (HJ-Gen) with the condition

$$\max\Big(u_t+H_T^{\mathrm{reg}}(x,t,D_{\mathcal{H}}u), u_t+H_1^+(x,t,D_xu), u_t+H_2^-(x,t,D_xu)\Big) = 0 \quad \text{on } \mathcal{H}\times(0,T)\ ,$$

in the sense of Definition 7.2.1. Let us classically consider the half-relaxed limits (see Section 3.1 for a definition)

$$\underline{u}(x,t) := \liminf{}_* \, u^\varepsilon(x,t) \qquad \overline{u}(x,t) := \limsup{}^* \, u^\varepsilon(x,t)\ .$$

We observe that we only need to prove the following inequality

$$\mathbf{U}^+(x,t) \leq \underline{u}(x,t) \quad \text{ in } \mathbb{R}^N \times [0,T). \tag{8.7}$$

Indeed, by the maximality of $\mathbf{U}^+$ we have $\overline{u}(x,t) \leq \mathbf{U}^+(x,t)$ in $\mathbb{R}^N \times [0,T)$; moreover, by definition we have $\overline{u}(x,t) \geq \underline{u}(x,t)$ in $\mathbb{R}^N \times (0,T)$, therefore if we prove (8.7) we can conclude that $\mathbf{U}^+(x,t) \leq \underline{u}(x,t) \leq \overline{u}(x,t) \leq \mathbf{U}^+(x,t)$ which implies that $(u^\varepsilon)_\varepsilon$ converges locally uniformly to $\mathbf{U}^+$ in $\mathbb{R}^N \times [0,T)$.

In order to prove the inequality, $\mathbf{U}^+ \leq \underline{u}$ in $\mathbb{R}^N \times [0,T)$, we are going to make several reductions along the lines of Chapter 3 by changing $\mathbf{U}^+$ but we keep the notation $\mathbf{U}^+$ for the changed function for the sake of simplicity of notations. In the same way, we should argue on the interval $[0,T']$ for $0 < T' < T$ but we keep the notation T for T'.

First, thanks to the localization arguments of Chapter 3, we can assume that $\mathbf{U}^+$ is a strict subsolution such that $\mathbf{U}^+(x,t) \to -\infty$ as $|x| \to +\infty$, uniformly w.r.t. $t \in [0,T]$. Therefore there exists $(\bar{x},\bar{t}) \in \mathbb{R}^N \times [0,T]$ such that

$$M := \mathbf{U}^+(\bar{x},\bar{t}) - \underline{u}(\bar{x},\bar{t}) = \sup_{(x,t)\in\mathbb{R}^N\times[0,T]} \big(\mathbf{U}^+(x,t) - \underline{u}(x,t)\big)\ .$$

We assume by contradiction that $M > 0$ and of course this means that $\bar{t} > 0$. The cases when $\bar{x} \in \Omega_1$ or $\bar{x} \in \Omega_2$ can be treated by classical methods, hence we may assume that $\bar{x} \in \mathcal{H}$.

Next, by the regularization argument of Chapter 3 we can assume in addition that $\mathbf{U}^+$ is C^1 at least in the $t, x_1, \dots, x_{N-1}$ variables; moreover we can suppose that $(\bar{x},\bar{t})$ is a strict maximum point of $\mathbf{U}^+ - \underline{u}$.

Since $\mathbf{U}^+$ is C^1 in the (t,x')-variables, the flux-limited subsolution condition can be written as

$$(\mathbf{U}^+)_t(\bar{x},\bar{t}) + H_T^{\mathrm{reg}}(\bar{x},\bar{t},D_{x'}\mathbf{U}^+(\bar{x},\bar{t})) \leq -\eta\,,$$

where $\eta > 0$ measure the strict subsolution property. Therefore

$$H_T^{\mathrm{reg}}(\bar{x},\bar{t},D_{x'}\mathbf{U}^+(\bar{x},\bar{t})) \leq -(\mathbf{U}^+)_t(\bar{x},\bar{t}) - \eta\,,$$

and, borrowing ideas from Lemma 12.2.1 in the Appendix of this part (Chapter 12), there exist two solutions λ_1,λ_2, with $\lambda_2 < \lambda_1$, of the equation

$$\tilde{H}^{\mathrm{reg}}\Big(\bar{x},\bar{t},D_{x'}\mathbf{U}^+(\bar{x},\bar{t}) + \lambda e_N\Big) = -(\mathbf{U}^+)_t(\bar{x},\bar{t}) - \eta/2\,.$$

Note that, since $\bar{x},\bar{t}$ are fixed, $a = -(\mathbf{U}^+)_t(\bar{x},\bar{t})$ and $p' = D_{x'}\mathbf{U}^+(\bar{x})$ are also fixed, so that λ_1,λ_2 are constants (we mean: independent of the parameter $\varepsilon > 0$ that is to come below). We proceed now with the construction of the test-function: let $\chi(x_N,y_N)$ be defined as in (8.1) and

$$\psi_\varepsilon(x,y,t,s) := \frac{|t-s|^2}{\varepsilon^{1/2}} + \frac{|x'-y'|^2}{\varepsilon^{1/2}} + \chi(x,y) + \frac{|x_N-y_N|^2}{\varepsilon^{1/2}}\,.$$

Note that $\psi_\varepsilon(\cdot,y,\cdot,s),\psi_\varepsilon(x,\cdot,t,\cdot) \in \mathrm{PC}^1(\mathbb{R}^N\times[0,T])$.

Since $(\bar{x},\bar{t})$ is a strict global maximum point of $\mathbf{U}^+ - \underline{u}$ and since $\underline{u}(\bar{x},\bar{t}) = \liminf_* u^\varepsilon(\bar{x},\bar{t})$, the function $\mathbf{U}^+(x,t) - \underline{u}(y,s) - \psi_\varepsilon(x,y,t,s)$ has local maximum points $(x_\varepsilon,y_\varepsilon,t_\varepsilon,s_\varepsilon)$ which converge to $(\bar{x},\bar{x},\bar{t},\bar{t})$. For the sake of simplicity of notations, we drop the ε and just denote by (x,y,t,s) such a maximum point.

We now consider 5 different cases, depending on the position of (x,y,t,s).

CASE 1/2: $x_N > 0$ and $y_N \leq 0$ (or $x_N < 0$ and $y_N \geq 0$). We use the subsolution condition for $\mathbf{U}^+$ in Ω_1: recalling that $\mathbf{U}^+$ is regular in the (t,x')-variables, we first can write it as

$$(\mathbf{U}^+)_t(x,t) + H_1\left(x,t,D_{x'}\mathbf{U}^+(x,t) + \lambda_1 e_N + \frac{2(x_N-y_N)}{\varepsilon^{1/2}}\right) \leq -\eta\,,$$

where we have used the regularity of $\mathbf{U}^+$ to deduce that

$$(\mathbf{U}^+)_t(x,t) = \frac{2(t-s)}{\varepsilon^{1/2}} \quad \text{and} \quad D_{x'}\mathbf{U}^+(x,t) = \frac{2(x'-y')}{\varepsilon^{1/2}}\,. \tag{8.8}$$

Then, using further the regularity of $\mathbf{U}^+$ and recalling that $(\mathbf{U}^+)_t$ and $D_{x'}\mathbf{U}^+$ are continuous not only in t,x' but also x_N, we have $(\mathbf{U}^+)_t(x,t) = (\mathbf{U}^+)_t(\bar{x},\bar{t}) + o_\varepsilon(1)$, $D_{x'}\mathbf{U}^+(x,t) = D_{x'}\mathbf{U}^+(\bar{x},\bar{t}) + o_\varepsilon(1)$ and therefore

$$(\mathbf{U}^+)_t(\bar{x},\bar{t}) + H_1\left(x,t,D_{x'}\mathbf{U}^+(\bar{x},\bar{t}) + \lambda_1 e_N + \frac{2(x_N-y_N)}{\varepsilon^{1/2}}\right) \leq -\eta + o_\varepsilon(1)\,.$$

Next, using that $H_1^- \leq H_1$, H_1^- is non decreasing and $(x_N - y_N) > 0$ we get from the above property

$$\begin{aligned} H_1^-\left(x,t,D_{x'}\mathbf{U}^+(\bar{x},\bar{t})+\lambda_1 e_N\right) &\leq H_1^-\left(x,t,D_{x'}\mathbf{U}^+(\bar{x},\bar{t})+\lambda_1 e_N+\frac{2(x_N-y_N)}{\varepsilon^{1/2}}\right) \\ &\leq -(\mathbf{U}^+)_t(\bar{x},\bar{t})-\eta+o_\varepsilon(1)\,. \end{aligned}$$

From this inequality, since $D_{x'}\mathbf{U}^+(\bar{x},\bar{t})+\lambda_1 e_N$ remains bounded, using the continuity of H_1^- yields

$$H_1^-\left(\bar{x},\bar{t},D_{x'}\mathbf{U}^+(\bar{x},\bar{t})+\lambda_1 e_N\right) \leq -(\mathbf{U}^+)_t(\bar{x},\bar{t})-\eta+o_\varepsilon(1)\,.$$

On the other hand, the construction of λ_1 implies that

$$H_1^-\left(\bar{x},\bar{t},D_{x'}\mathbf{U}^+(\bar{x},\bar{t})+\lambda_1 e_N\right) = -(\mathbf{U}^+)_t(\bar{x},\bar{t})-\eta/2\ ,$$

therefore we obtain a contradiction for ε small enough.
The case $x_N < 0$ and $y_N \geq 0$ is completely similar, using H_2 instead of H_1.

CASE 3/4: $x_N = 0$ and $y_N > 0$ (or < 0). We use the supersolution viscosity inequality for u^ε at (y,t), using (8.8)

$$O(\varepsilon^{1/2})+(\mathbf{U}^+)_t(x,t)+H_1\Big(y,s,D_{x'}\mathbf{U}^+(x,t)+\lambda_1 e_N+\frac{2(x_N-y_N)}{\varepsilon^{1/2}}+o_\varepsilon(1)\Big) \geq 0\,. \quad (8.9)$$

We first want to show that we can replace H_1 by H_1^+ in this inequality.

Indeed, using the arguments of CASE 1/2 and the fact that $x_N - y_N = -y_N < 0$, we are led by the definition of λ_1 to

$$O(\varepsilon^{1/2})+(\mathbf{U}^+)_t(x,t)+H_1^-\Big(y,s,D_{x'}\mathbf{U}^+(x,t)+\lambda_1 e_N+\frac{2(x_N-y_N)}{\varepsilon^{1/2}}+o_\varepsilon(1)\Big) < 0\ .$$

We deduce that (8.9) holds true with H_1^+.

Moreover, by the subsolution condition of $\mathbf{U}^+$ on $\mathcal{H}$ we have

$$(\mathbf{U}^+)_t(x,t)+H_1^+\Big(x,D_{x'}\mathbf{U}^+(x,t)+\lambda_1 e_N+\frac{2(x_N-y_N)}{\varepsilon^{1/2}}+o_\varepsilon(1)\Big) \leq -\eta$$

therefore the conclusion follows by standard arguments putting together the two inequalities for H_1^+ and letting ε tend to zero. If $y_N < 0$, we can repeat the same argument using H_2^-.

CASE 5: $x_N = y_N = 0$. Let us remark that this case is not possible. We observe that, by assumption, u^ε is regular in the x_N-variable. Therefore the above maximum point property on $\mathbf{U}^+ - u^\varepsilon - \psi_\varepsilon$ implies that 0 is a minimum point of $z_N \mapsto u^\varepsilon((y',z_N),s)+\psi_\varepsilon(x,(y',z_N),t,s))$. But, by construction of the function χ, we have $\lambda_2 < \lambda_1$ and this is a contradiction.

Q.E.D.

8.5 Extension to second-order equations?

In this section, we consider second-order equations of the form

$$u_t + H_i(x,t,Du) - \mathrm{Tr}(a_i(x)D^2u) = 0 \quad \text{in } \Omega_i \times (0,T)\ ,$$

where a_i are continuous functions which are assumed to be on the standard form, i.e. $a_i = \sigma_i \cdot \sigma_i^T$, where σ_i^T is the transpose matrix of σ_i. We suppose that the σ_i's are bounded, Lipschitz continuous functions and in order that the definition of flux-limited solutions make sense, the following property has to be imposed

$$\sigma_i((x',0)) = 0 \quad \text{for } i=1,2 \text{ and for all } x' \in \mathbb{R}^{N-1}.$$

The main question is: can we prove a comparison result in this framework?

There are several difficulties that we list below

(i) in general, we cannot regularize the subsolution as we did above,

(ii) because of the second-order term, the normal controllability cannot be used efficiently outside $\mathcal{H}$,

(iii) a two-parameters proof as in the non-convex case is difficult to handle with the second-order term.

We take this opportunity to remark that the above proof has several common points with the comparison proof for nonlinear Neumann boundary conditions: in fact, it can be described as a "double Neumann" proof since H_1^- (almost) plays the role of a Neumann boundary condition for the equation in Ω_2 and conversely H_2^+ (almost) plays the role of a Neumann boundary condition for the equation in Ω_1. There is anyway a crucial additional difficulty: H_1^-, H_2^+ are NOT strictly monotone functions w.r.t. the normal gradient direction; therefore, a general "one-parameter proof", avoiding the use of $\gamma \ll \varepsilon$ may (perhaps) exist but it is probably rather technical and it may also require additional assumptions on the H_i's.

Instead, the following result gives some conditions under which the proof of Theorem 8.3.1 still works.

Theorem 8.5.1 (Comparison principle in the second-order case) *Under the assumptions of Theorem 8.3.1, the result of Theorem 8.1.1 is valid provided that the two following assumptions hold, for $i=1,2$, in a neighborhood of $\mathcal{H}$*
(i) $H_i(x,t,p) = H_{i,1}(x',t,p') + H_{i,2}(x_N,p_N)$,
(ii) $\sigma_i = \sigma_i(x_N)$ with $\sigma_i(0)=0$, σ_i being locally Lipschitz continuous and bounded.

It is worth pointing out that this result holds for non-convex Hamiltonians, but clearly with rather restrictive assumptions on the H_i and σ_i. We refer to Imbert and Nguyen [86] for general results for second-order equations in the case of *networks* where not only comparison results are obtained but the notions of FLS and JVS are discussed and applications are given.

Proof — The proof follows readily the proof of Theorem 8.3.1 and we just add the following comments
– The structure conditions we impose on the H_i's and σ_i's ensures that we can perform a regularization of the subsolution by sup-convolution in the spirit of Proposition 3.4.5: in particular, the Hamiltonians both satisfy Assumption **(TC-s)**. This is the first reason to impose (i) and (ii).
– Once this regularization is done, we still have to control the dependence in the derivatives in x_N (or all the terms involving the parameter γ): this is where the special dependence in x_N of the H_i's and σ_i's plays a role.
– In all the steps where the properties of λ_1, λ_2 are crucial, the second-order term is small since $|\sigma_i(x_N)| = O(|x_N|)$ and therefore $|a_i(x_N)| = O(x_N^2)$ which can be combined with the facts that

$$\frac{|(x_\varepsilon)_N - (y_\varepsilon)_N|^2}{\gamma^2} \to 0 \quad \text{as } \gamma \to 0 \,,$$

and the second-order derivatives are a $O(\gamma^{-2})$.

Q.E.D.

Remark 8.5.2 *Anticipating the main result of Section 7.4.2 showing that the Kirchhoff boundary conditions is equivalent to a flux-limited boundary condition with $G = H_T^{\mathrm{reg}}$ under the assumptions of Theorems 8.1.1 or 8.3.1, these two results also provide the comparison for the (KC)-condition. The proof(s) would apply readily if we were able to show that we can choose $\lambda_1 > \lambda_2$ in the test-function (the function χ) but this is not obvious at this point and this property will be clarified in Section 7.4.2.*

Chapter 9

A Comparison Result for Junction Viscosity Solutions and Applications

In this chapter we expose the Lions-Souganidis approach of the comparison proof for Flux-Limited solutions and apply it to the case of Kirchoff's condition, as well as second-order equations before going back on the vanishing viscosity method.

9.1 Preliminary lemmas

We begin with some simple lemmas in dimension 1.

Lemma 9.1.1 *If $u : [0, r] \to \mathbb{R}$ is a Lipschitz continuous subsolution of*

$$H(u_x) = 0 \quad \textit{in } (0, r) \ ,$$

where H is a continuous function and if

$$\underline{p} = \liminf_{x \to 0} \left[\frac{u(x) - u(0)}{x} \right] < \overline{p} = \limsup_{x \to 0} \left[\frac{u(x) - u(0)}{x} \right] \ ,$$

then $H(p) \le 0$ for all $p \in [\underline{p}, \overline{p}]$.

Remark 9.1.2 *In Lemma 9.1.1, the subsolution is assumed to be Lipschitz continuous and this is consistent with the fact that we consider equations with coercive*

Hamiltonians (or satisfying **(NC)***). This assumption ensures that we have bounded* $\underline{p}$ *and* $\overline{p}$*, but this is not really necessary as the proof will show. Without this assumption, we can still prove at least that* $H(p) \leq 0$ *for all* $p \in]\underline{p}, \overline{p}[$ *if* $\underline{p} < +\infty$*. The importance of this remark is more for supersolutions: we use below an analogous result for them and it is less natural to assume them to be Lipschitz continuous.*

Proof — Let $(x_k)_k$ be a sequence of points of $(0, r)$ such that

$$x_k \to 0 \quad , \quad \frac{u(x_k) - u(0)}{x_k} \to \underline{p} \; .$$

We pick $\underline{p} < p < \overline{p}$ and consider the function $\psi(y) = u(y) - u(0) - py$ on the interval $[0, x_k]$. Since $\psi(0) = 0$, $\psi(x_k) < 0$ and $\limsup_{x\to 0} \left[\frac{\psi(x)}{x}\right] = \overline{p} - p > 0$, there exists $\tilde{x}_k \in (0, x_k)$ which is a maximum point of ψ and the subsolution property implies $H(p) \leq 0$, which is what we wanted to prove. By the continuity of H, this property holds also true for $\underline{p}$ and $\overline{p}$.

Q.E.D.

Lemma 9.1.3 *The result of Lemma 9.1.1 remains valid if* $\underline{p} = \overline{p}$*.*

Proof — For $\varepsilon > 0$ close to 0, we consider $v(x) = u(x) + \varepsilon x \sin(\log(x))$. The function $x \mapsto x \sin(\log(x))$ is Lipschitz continuous and therefore, $H(v_x) \leq o_\varepsilon(1)$. Moreover

$$\frac{v(x) - v(0)}{x} = \frac{u(x) - u(0)}{x} + \varepsilon \sin(\log(x)) \; ,$$

and therefore

$$\liminf_{x\to 0} \left[\frac{v(x) - v(0)}{x}\right] = \overline{p} - \varepsilon < \limsup_{x\to 0} \left[\frac{v(x) - v(0)}{x}\right] = \overline{p} + \varepsilon \; .$$

Since $\overline{p} - \varepsilon < \overline{p} < \overline{p} + \varepsilon$, Lemma 9.1.1 implies $H(\overline{p}) \leq o_\varepsilon(1)$ and the conclusion follows by letting ε tend to 0.

Q.E.D.

Remark 9.1.4 *Of course, analogous results hold for supersolutions: if* v *is a supersolution of* $H(v_x) \geq 0$ *in* $(0, r)$*, it suffices to use that u=-v(x) is a subsolution of* $-H(-u_x) \leq 0$ *in* $(0, r)$*.*

We conclude this section by the following lemma which connects the 1-d and multi-dimensional situations. Here we consider a set

$$Q := \big\{(y,x) : y \in \mathcal{V}\,,\ x \in [0,\delta[\big\} \subset \mathbb{R}^{p+1}$$

where $\mathcal{V}$ is a neighborhood of 0 in $\mathbb{R}^p$, and $\delta > 0$. We denote by $D^+_Q w$ and $D^-_Q w$ the super and sub-differentials of w with respect to both variables (y,x). If w is differentiable with respect to y at $(0,0)$, it can be expected that the sub/super-differentials of w in both variables are given by

$$(D_y w(0,0), D^-_x w(0,0)) \quad \text{and} \quad (D_y w(0,0), D^+_x w(0,0))\,.$$

The following result gives a more precise formulation.

Lemma 9.1.5 *Let $w : Q \to \mathbb{R}$ be a function which is continuous at $(0,0)$, such that the functions $y \mapsto w(y,x)$ are uniformly Lipschitz continuous in $\mathcal{V}$ with respect to $x \in [0,\delta[$ and that $y \mapsto w(y,0)$ is differentiable at 0.*

(a) *Superdifferential case*

We assume moreover that w is upper-semicontinuous in Q and that, for any $x \in [0,\delta[$, the function $y \mapsto w(y,x)$ is semi-convex in $\mathcal{V}$. If

$$\overline{p} = \limsup_{x\to 0} \left[\frac{w(0,x) - w(0,0)}{x}\right]$$

exists and is finite, then for any $p \geq \overline{p}$,

$$(D_y w(0,0), p) \in D^+_Q w(0,0)\,.$$

(b) *Subdifferential case*

We assume moreover that w is lower-semicontinuous in Q and that, for any $x \in [0,\delta[$, the function $y \mapsto w(y,x)$ is semi-concave in $\mathcal{V}$. If

$$\underline{q} = \liminf_{x\to 0} \left[\frac{w(0,x) - w(0,0)}{x}\right]$$

exists and is finite, then for any $q \leq \underline{q}$,

$$(D_y w(0,0), q) \in D^-_Q w(0,0)\,.$$

The interest of this lemma is clear: under suitable assumptions, we can connect 1-d and multi-d sub or super-differentials. This will be a key step for applying Lemma 9.1.1 to multi-d problems.

Proof — We only do the proof in case (a), the other case working with obvious adaptations. Take $p \geq \bar{p}$ and set

$$\bar{w}(y,x) := w(y,x) - w(0,0) - D_y w(0,0) \cdot y - px\,.$$

We want to show that $\bar{w}(y,x) \leq o(|y|) + o(x)$. To do so, we use the decomposition

$$\bar{w}(y,x) = [\bar{w}(y,x) - \bar{w}(0,x)] + \bar{w}(0,x)\ .$$

By the definition of $\bar{p}$ and $p \geq \bar{p}$, $\bar{w}(0,x) = w(0,x) - w(0,0) - px \leq o(x)$. Therefore it remains to estimate the bracket.

To do so, we introduce a regularization by convolution of $\bar{w}$ in the y-variable. Let $(\rho_\varepsilon)_\varepsilon$ be a resolution of identity in $\mathbb{R}^p$, *i.e.* a sequence of positive, C^∞-functions on $\mathbb{R}^p$ with compact support in $B_\infty(0,\varepsilon)$ such that $\int_{\mathbb{R}^p} \rho_\varepsilon(z)dz = 1$. Then we set

$$\bar{w}_\varepsilon(y,x) := \int_{\mathbb{R}^p} \bar{w}(y+z,x)\rho_\varepsilon(z)\,dz\ .$$

For any $x \in [0,\delta[$, the functions $y \mapsto \bar{w}_\varepsilon(y,x)$ are C^1 in a neighborhood of 0 and therefore

$$\bar{w}_\varepsilon(y,x) - \bar{w}_\varepsilon(0,x) = \int_0^1 D_y\bar{w}_\varepsilon(sy,x)\cdot y\,ds\ .$$

Now we examine $D_y\bar{w}_\varepsilon(sy,x)$. Since, for any $x \in [0,\delta[$, $y \mapsto \bar{w}(y,x)$ is Lipschitz continuous, we have

$$D_y\bar{w}_\varepsilon(sy,x) = \int_{\mathbb{R}^p} D_y\bar{w}(sy+z,x)\rho_\varepsilon(z)\,dz\ .$$

Moreover, by the semi-convexity assumption and the continuity at $(0,0)$, we have $D_y\bar{w}(y,x) = D_y w(y,x) - D_y w(0,0) \to 0$ when $(y,x) \to (0,0)$. This implies that $D_y\bar{w}_\varepsilon(sy,x) = o_x(1) + o_y(1) + o_\varepsilon(1)$ as $(y,x,\varepsilon) \to (0,0,0)$, uniformly with respect to $s \in [0,1]$. Therefore $\bar{w}_\varepsilon(y,x) - \bar{w}_\varepsilon(0,x) = |y|(o_x(1) + o_y(1) + o_\varepsilon(1))$. Letting ε tend to 0, we end up with

$$\bar{w}(y,x) - \bar{w}(0,x) = o(|y|) + |y|o_x(1) = o(|y|)\,,$$

which yields the desired property.

Q.E.D.

9.2 Back on the Kirchhoff condition

Now, with the notations of Section 7.1, we come back to the study of the problem

$$u_t + H_1(x,t,u,Du) = 0 \quad \text{in } \Omega_1 \times (0,T)\ ,$$

$$u_t + H_2(x,t,u,Du) = 0 \quad \text{in } \Omega_2 \times (0,T)\ ,$$

with the *Kirchhoff condition*

$$\frac{\partial u}{\partial n_1} + \frac{\partial u}{\partial n_2} = 0 \quad \text{on } \mathcal{H} \times (0,T)\ ,$$

where, for $i=1,2$ $n_i(x)$ denotes the unit outward to $\partial\Omega_i$ at $x \in \partial\Omega_i$. We recall that this Kirchhoff condition has to be taken in the viscosity solutions sense, namely

$$\min\Big(u_t + H_1(x,t,u,Du), u_t + H_2(x,t,u,Du), \frac{\partial u}{\partial n_1} + \frac{\partial u}{\partial n_2}\Big) \leq 0 \quad \text{on } \mathcal{H}\ \times (0,T)\,,$$

for the subsolution condition and

$$\max\Big(u_t + H_1(x,t,u,Du), u_t + H_2(x,t,u,Du), \frac{\partial u}{\partial n_1} + \frac{\partial u}{\partial n_2}\Big) \geq 0 \quad \text{on } \mathcal{H}\,,$$

for the supersolution condition, using test-functions in $\mathrm{PC}^1(\mathbb{R}^N \times [0,T])$.

Thanks to Section 3.2, we are not going to prove a full comparison result but only a local comparison result. This result relies on the regularization process introduced in Section 3.4 with the variables $y = (t,x')$ where $x' = (x_1, x_2, \cdots, x_{N-1})$, $z = x_N$ and, denoting $p_x = (p_{x'}, p_{x_N})$,

$$F\Big(((t,x'),x_N),u,((p_t,p_{x'}),p_{x_N})\Big) = \begin{cases} p_t + H_1(x,t,u,p_x) & \text{if } x_N > 0 \\ p_t + H_2(x,t,u,p_x) & \text{if } x_N < 0 \end{cases}$$

On the interface, F is given (differently for the sub and supersolution) the Kirchhoff condition.

In the following result, we concentrate on **(LCR)**-evol in a neighborhood of points $x \in \mathcal{H}$ since otherwise it is a standard result.

Theorem 9.2.1 *Assume that, for any $(x,t) \in \mathcal{H} \times (0,T]$, there exists $r > 0$ such that H_1 and H_2 satisfy* **(TC)** *and* **(NC)** *with $y = (t,x')$ and $z = x_N$ and* **(Mon)** *with $\lambda_R = 0$ in $B_\infty((x,t),r)$. Then* **(LCR)***-evol holds for any subsolution u and supersolution v if either v satisfies (3.16) or* **(TC-s)** *holds.*

Of course, the main interest of this result is to allow to prove a **(GCR)** which is valid for non convex Hamiltonians H_1 and H_2; in addition, it is easy to see that the proof we give below (and which is almost exactly the Lions-Souganidis one) can provide a comparison result for different types of "junction conditions" on $\mathcal{H}$ and also for more general networks problems.

However this proof does not provide THE answer to all the problems of discontinuities in HJ Equations because, on one hand, it uses the fact that the discontinuity is of codimension 1 – at least under this form – and, on the other hand, this is precisely the junction condition which is related to the kind of application we have in mind as we saw in the previous sections: minimizing or maximizing in control problems changes the junction condition, so there is no hope to have a unified theory. Of course, differential games should lead to even more complex situations.

Proof — If u is a subsolution and v is supersolution of (HJ-Gen)-(KC) in $Q^{x,t}_{r,h}$ we wish to prove that there exists $r > 0$, $0 < h < t$ such that, if $\max_{\overline{Q^{x,t}_{r,h}}}(u - v) > 0$, then

$$\max_{\overline{Q^{x,t}_{r,h}}}(u - v) \leq \max_{\partial_p Q^{x,t}_{r,h}}(u - v)\,.$$

The proof of this result is based on the arguments of Section 3.4, and more precisely on Theorem 3.4.2 and 3.4.5: by using Theorem 3.4.2, we can assume without loss of generality that u is a Lipschitz continuous, η-strict subsolution of the equation and that u is semi-convex in x' and t. Using similar arguments, it is also possible to assume that v is semi-concave in x' and t but only under the conditions of Theorem 3.4.5. We point out that these reductions allows to have a supersolution v which is Lipschitz continuous in x' and t, uniformly in x_N, but which can still be discontinuous in x_N.

Next we consider a point point $(\bar{x}, \bar{t})$ where $\max_{\overline{Q^{x,t}_{r,h}}}(u - v) > 0$ is achieved. Of course, we can assume that $\bar{t} > 0$ and $(\bar{x}, \bar{t}) \notin \partial_p Q^{x,t}_{r,h}$. It is also clear that we can assume without loss of generality that $\bar{x} \in \mathcal{H}$ (otherwise only the H_1 or H_2 equation plays a role and we are in the case of a standard proof).

A key consequence of the semi-convexity of u and of the semi-concavity of v in the variables x', t is that u, v are differentiable in x' and t at the maximum point $(\bar{x}, \bar{t})$ (since semi-convex functions are differentiable at maximum points) and we have

$$D_{x'}u(\bar{x}, \bar{t}) = D_{x'}v(\bar{x}, \bar{t}) \quad \text{and} \quad u_t(\bar{x}, \bar{t}) = v_t(\bar{x}, \bar{t})\,.$$

Moreover, if we denote by (p', p_N, p_t) any element in the superdifferential of u at (y, s) close to $(\bar{x}, \bar{t})$, then $(p', p_t) \to (D_{x'}u(\bar{x}, \bar{t}), u_t(\bar{x}, \bar{t}))$ as (y, s) tends to $(\bar{x}, \bar{t})$. For the supersolution v, the same property may not be true for the elements of the subdifferentials since v can be discontinuous at $(\bar{x}, \bar{t})$. To turn around this difficulty we introduce

$$\tilde{v}((x', x_N), t) := \min\left(v((x', x_N), t), v((x', 0), t) + K|x_N|\right) ,$$

where $K > 0$. If we choose K large enough, the function $\tilde{v}$ is a supersolution of the equation for $x_N \neq 0$ as the minimum of two supersolutions and is continuous at $(\bar{x}, \bar{t})$. As a consequence of this continuity property, $\tilde{v}$ being still semi-concave in (x', t) as the minimum of semi-concave functions in (x', t), for any element (p', p_N, p_t) in the subdifferential of v at (y, s) close to $(\bar{x}, \bar{t})$, then $(p', p_t) \to (D_{x'}v(\bar{x}, \bar{t}), v_t(\bar{x}, \bar{t}))$ as (y, s) tends to $(\bar{x}, \bar{t})$.

These properties of u and $\tilde{v}$ allow us to argue only in the x_N variable since, taking into account the regularity of H_1, H_2, we have

$$\tilde{H}_1(u_{x_N}) \leq -\eta < 0 \leq \tilde{H}_1(\tilde{v}_{x_N}) \quad \text{for } x_N > 0 ,$$

$$\tilde{H}_2(u_{x_N}) \leq -\eta < 0 \leq \tilde{H}_2(\tilde{v}_{x_N}) \quad \text{for } x_N < 0 ,$$

where for $i = 1, 2$,

$$\tilde{H}_i(p_N) = u_t(\bar{x}, \bar{t}) + H_i(\bar{x}, \bar{t}, u(\bar{x}, \bar{t}), (D_{x'}u(\bar{x}, \bar{t}), p_N)) + o(1) ,$$

the $o(1)$ tending to 0 as $\bar{r} \to 0$ if we consider the equations in $B((\bar{x}, \bar{t}), \bar{r})$. It is worth pointing out that for the H_i-equations for v, we have used the fact that both $r \mapsto H_i(x, t, r, p)$ are increasing and that $u(\bar{x}, \bar{t}) > v(\bar{x}, \bar{t})$.

In order to proceed, we compute the superdifferentials for u in the two directions $x_N > 0$ and $x_N < 0$. We recall that since the test-functions are different in Ω_1 and Ω_2, these superdifferentials are different. For $x_N > 0$, we have $D_1^+ u(0) = [\overline{p}_1, +\infty)$ where $\overline{p}_1$ is defined as the $\overline{p}$ in Lemma 9.1.1, but we are referring here to Ω_1. For $x_N < 0$, we have $D_2^+ u(0) = [-\infty, -\overline{p}_2)$ where $\overline{p}_2$ is defined as the $\overline{p}$ in Lemma 9.1.1 but for $u(-x_N)$, in Ω_2.

Using the definition of viscosity subsolution together with Lemma 9.1.1 and 9.1.5, we obtain, since $n_1 = -e_N$ and $n_2 = e_N$

$$\min(-p_1 + p_2, \tilde{H}_1(p_1) + \eta, \tilde{H}_2(p_2) + \eta) \leq 0 ,$$

for any $p_1 \geq \overline{p}_1$ and $p_2 \leq -\overline{p}_2$; moreover

$$\tilde{H}_1(p_1) + \eta \leq 0 \quad \text{if } p_1 \in [\underline{p}_1, \overline{p}_1] ,$$
$$\tilde{H}_2(p_2) + \eta \leq 0 \quad \text{if } p_2 \in [-\overline{p}_2, -\underline{p}_2] .$$

For the supersolution v, we argue through $\tilde{v}$ but the aim is really to identify the subdifferential of v at $(\bar{x}, \bar{t})$. We first notice that, $(\bar{x}, \bar{t})$ being a maximum point of $u-v$, then $u(x,t)-v(x,t) \leq u(\bar{x},\bar{t})-v(\bar{x},\bar{t})$ for any (x,t) and the Lipschitz continuity of u implies

$$-C|(x,t)-(\bar{x},\bar{t})| \leq u(x,t)-u(\bar{x},\bar{t}) \leq v(x,t)-v(\bar{x},\bar{t}) \ ,$$

for C large enough, and in particular in the x_N-direction

$$-C|x_N| \leq v((\bar{x}', x_N), \bar{t}) - v(\bar{x},\bar{t}) \ .$$

Hence the subdifferentials of v at $(\bar{x}, \bar{t})$ in both directions (namely $D_1^- v(0)$ and $D_2^- v(0)$) are non empty.

Applying Lemma 9.1.1 to $\tilde{v}$, we obtain that

$$\begin{aligned} \tilde{H}_1(q_1) &\geq 0 \quad \text{if } q_1 \in [\underline{q}_1, \overline{q}_1] \ , \\ \tilde{H}_2(q_2) &\geq 0 \quad \text{if } q_2 \in [-\overline{q}_2, -\underline{q}_2] \ . \end{aligned}$$

and we also have $D_1^- \tilde{v}(0) = (-\infty, \underline{q}_1)$, $D_2^- \tilde{v}(0) = [-\underline{q}_2, +\infty)$.

On the other hand, using Lemma 9.1.5, $(D_{x'} v(\bar{x},\bar{t}), q_1, v_t(\bar{x},\bar{t})) \in D^-_{\overline{\Omega}_1} \tilde{v}(\bar{x},\bar{t})$ [(1)] for any $q_1 \leq \underline{q}_1$.

In order to connect the sub-differentials of v and $\tilde{v}$, we use the following classical result whose proof is an exercise left to the reader.

Lemma 9.2.2 *Let $w_1, w_2 : A \subset \mathbb{R}^p \to \mathbb{R}$ be two lsc functions such that $w_1(z_0) = w_2(z_0)$ for some $z_0 \in A$. Then*

$$D_A^- \min(w_1, w_2)(z_0) \subset D_A^- w_1(z_0) \cap D_A^- w_2(z_0) \, .$$

Applying the result with $A := \overline{\Omega}_1 \times [0,T]$, $z_0 := (\bar{x}, \bar{t})$, $w_1(x,t) = v(x,t)$ and $w_2(x,t) = v((x',0),t) + K|x_N|$, we deduce that $(D_{x'} v(\bar{x},\bar{t}), q_1, v_t(\bar{x},\bar{t})) \in D^-_{\overline{\Omega}_1} v(\bar{x},\bar{t})$. Of course, the same arguments can be used for $D^-_{\overline{\Omega}_2} v(\bar{x},\bar{t})$.

Hence, we have

$$\max(-q_1 + q_2, \tilde{H}_1(q_1), \tilde{H}_2(q_2)) \geq 0 \ ,$$

for any $q_1 \leq \underline{q}_1$ and $q_2 \geq -\underline{q}_2$.

In order to conclude, we examine the different possibilities.

[(1)] We recall that $D^-_{\overline{\Omega}_1} \tilde{v}(\bar{x},\bar{t})$ denotes the subdifferential *related to* $\overline{\Omega}_1$ of the function $\tilde{v}$ at $(\bar{x},\bar{t})$

Case 1 : Either $[\underline{p}_1, \overline{p}_1] \cap [\underline{q}_1, \overline{q}_1] \neq \emptyset$ or $[-\overline{p}_2, -\underline{p}_2] \cap [-\overline{q}_2, -\underline{q}_2] \neq \emptyset$: this means that there exists p such that we have

$$\text{either } \tilde{H}_1(p) + \eta \leq 0 \leq \tilde{H}_1(p)\,, \qquad \text{or } \tilde{H}_2(p) + \eta \leq 0 \leq \tilde{H}_2(p)\;,$$

and in each case we reach a contradiction.

Case 2 : Otherwise, since 0 is a maximum point of $u - v$, we have necessarily $\overline{p}_1 \leq \overline{q}_1$ and therefore $\underline{p}_1 \leq \overline{p}_1 < \underline{q}_1 \leq \overline{q}_1$. Considering the function $p \mapsto \tilde{H}_1(p)$ which is less that $-\eta$ in $[\underline{p}_1, \overline{p}_1]$ and positive in $[\underline{q}_1, \overline{q}_1]$, we see that there exists $\overline{p}_1 < r_1 < \underline{q}_1$ such that $\tilde{H}_1(r_1) = -\eta/2$.

We can show in the same way that $-\overline{q}_2 \leq -\underline{q}_2 < -\overline{p}_2 \leq -\underline{p}_2$ and there exists $-\underline{q}_2 < r_2 < -\overline{p}_2$ such that $\tilde{H}_2(r_2) = -\eta/2$. Then, choosing $\delta > 0$ small enough and $p_1 = r_1 - \delta$, $p_2 = r_2 + \delta$, we have $p_1 \geq \overline{p}_1$ and $p_2 \leq -\overline{p}_2$. Therefore the viscosity inequalities give

$$\min(-p_1 + p_2, \tilde{H}_1(p_1) + \eta, \tilde{H}_2(p_2) + \eta) \leq 0\;,$$

but with the choice of δ, $\tilde{H}_1(p_1) + \eta > 0$, $\tilde{H}_2(p_2) + \eta > 0$, which implies $-p_1 + p_2 \leq 0$.

Similarly, choosing $q_1 = r_1 + \delta$ and $q_2 = r_2 - \delta$ and using $\tilde{H}_1(q_1) < 0$, $\tilde{H}_2(p_2) < 0$, we get $-q_1 + q_2 \geq 0$.

So, at the same time $-r_1 + r_2 + 2\delta \leq 0$ and $-r_1 + r_2 - 2\delta \geq 0$ and we also reach a contradiction in this case.

Hence the proof is complete[(2)].

Q.E.D.

9.3 Extension to second-order problems

The same approach allows to deal with second-order problems, with similar structure assumptions on the Hamiltonians:

Theorem 9.3.1 (Comparison principle in the second-order case (LS-version)) *Under the assumptions of Theorem 8.5.1, except* $(\mathbf{H_{QC}})$*, the result of Theorem 8.5.1 is valid provided that the two following assumptions hold, for* $i = 1, 2$*, in a neighborhood of* $\mathcal{H}$

[(2)] The authors wish to thank Peter Morfe for pointing out several unclear points in this proof which led us to several improvements, in particular the statements of Lemma 9.1.5 and 9.2.2.

(i) $H_i(x,t,p) = H_{i,1}(x',t,p') + H_{i,2}(x_N,p_N)$,
(ii) $\sigma_i = \sigma_i(x_N)$ *with* $\sigma_i(0) = 0$, σ_i *being locally Lipschitz continuous and bounded.*

Proof — We just give a very brief sketch based on two remarks

– Assumptions $(i)-(ii)$ above are used in this context to make a regularization of both the sub and the supersolution in the variables x' and t, allowing to assume without loss of generality that the subsolution is semi-convex in (x',t) while the supersolution is semi-concave in these variables.

– Lemma 9.1.1 remains valid, even with the second-order term, allowing to get exactly the same viscosity inequalities for $x_N = 0$.

Q.E.D.

9.4 Convergence of the Vanishing Viscosity Approximation (II): the Convex Case

In this section, we use the above results for obtaining the convergence of the vanishing viscosity method. The following result is formulated in the same way as Theorem 8.4.1 but it can be proved under slightly more general assumption (typically the assumptions of Theorem 8.1.1 or Theorem 8.3.1), at the cost of a slightly less precise result; we provide such result in the next section.

Theorem 9.4.1 *Assume that we are in the "standard assumptions in the co-dimension-1 case". For any* $\varepsilon > 0$*, let* u^ε *be a viscosity solution of*

$$u_t^\varepsilon - \varepsilon\Delta u^\varepsilon + H(x,t,u^\varepsilon,Du^\varepsilon) = 0 \quad in \quad \mathbb{R}^N \times (0,T)\,, \tag{9.1}$$

$$u^\varepsilon(x,0) = u_0(x) \quad in \quad \mathbb{R}^N\,, \tag{9.2}$$

where $H = H_1$ *in* Ω_1 *and* H_2 *in* Ω_2*, and* u_0 *is bounded continuous function in* $\mathbb{R}^N$*. If the* u^ε *are uniformly bounded in* $\mathbb{R}^N \times (0,T)$ *and* C^1 *in* x_N *in a neighborhood of* $\mathcal{H}$*, then, as* $\varepsilon \to 0$*, the sequence* $(u^\varepsilon)_\varepsilon$ *converges locally uniformly in* $\mathbb{R}^N \times (0,T)$ *to a continuous function* u *which is at the same time*

(i) the maximal Ishii's subsolution of (6.1)*,*
(ii) the unique solution of the Kirchhoff problem,
(iii) the unique solution associated to the flux-limiter H_T^{reg}*.*

Proof — We begin the proof by remarking that Section 7.4.2 provides the equivalence of properties (i), (ii) and (iii).

The strategy of the proof is (almost) standard since we use the half-relaxed limits method but we are going to use Section 7.4.2 to go through Kirchhoff's condition. To do so, we use the

Lemma 9.4.2 *The half-relaxed limits* $\overline{u} = \limsup^* u^\varepsilon$ *and* $\underline{u} = \liminf_* u^\varepsilon$ *are respectively sub and supersolution of the Kirchhoff problem.*

This lemma is not standard: it is not an usual stability result for viscosity solutions since we have to use test-function which are not smooth across $\mathcal{H}$. But if the lemma is proved then the result just follows from Theorem 8.1.1, using first Proposition 7.4.4 to connect the Kirchhoff condition with the flux-limiter condition H_T^{reg}.

Q.E.D.

Proof of Lemma 9.4.2 — We prove the result for $\overline{u}$, the one for $\underline{u}$ being analogous. Let $\phi \in \mathrm{PC}^1(\mathbb{R}^N \times [0,T])$ be a test-function and let $(\bar{x}, \bar{t})$ be a strict local maximum point of $\overline{u} - \phi$. The only difficulty is when $\bar{x} \in \mathcal{H}$ and therefore we concentrate on this case.

By standard arguments, $u^\varepsilon - \phi$ has a local maximum point at $(x_\varepsilon, t_\varepsilon)$ and $(x_\varepsilon, t_\varepsilon) \to (\bar{x}, \bar{t})$ as $\varepsilon \to 0$. Now, if there exists a subsequence $(x_{\varepsilon'}, t_{\varepsilon'})$ with $x_{\varepsilon'} \notin \mathcal{H}$, the classical arguments can be applied and passing to the limit (along another subsequence) in the inequality

$$\phi_t(x_\varepsilon, t_\varepsilon) - \varepsilon \Delta \phi(x_\varepsilon, t_\varepsilon) + H(x_\varepsilon, t_\varepsilon, u^\varepsilon(x_\varepsilon, t_\varepsilon), D\phi(x_\varepsilon, t_\varepsilon)) \leq 0$$

yields the result.

The main difficulty is when $x_\varepsilon \in \mathcal{H}$ for all ε small enough since ϕ is not smooth at $(x_\varepsilon, t_\varepsilon)$. Here we have to consider

$$\frac{\partial \phi}{\partial x_N}(((\bar{x})', 0+), \bar{t}) = \lim_{x_N \to 0, x_N > 0} \frac{\partial \phi}{\partial x_N}(((\bar{x})', x_N), \bar{t})$$

and

$$\frac{\partial \phi}{\partial x_N}(((\bar{x})', 0-), \bar{t}) = \lim_{x_N \to 0, x_N < 0} \frac{\partial \phi}{\partial x_N}(((\bar{x})', x_N), \bar{t}) \ .$$

If $-\frac{\partial \phi}{\partial x_N}(((\bar{x})', 0+), \bar{t}) + \frac{\partial \phi}{\partial x_N}(((\bar{x})', 0-), \bar{t}) \leq 0$, then the Kirchoff subsolution condition is satisfied and the result holds.

Otherwise, the maximum point property at $(x_\varepsilon, t_\varepsilon)$ implies, since u_ε is smooth, that

$$\frac{\partial u_\varepsilon}{\partial x_N}(((x_\varepsilon)', 0), \bar{t}) \leq \frac{\partial \phi}{\partial x_N}(((x_\varepsilon)', 0+), \bar{t}) \ ,$$

and
$$\frac{\partial u_\varepsilon}{\partial x_N}((((x_\varepsilon)',0),\bar{t}) \geq \frac{\partial \phi}{\partial x_N}((((x_\varepsilon)',0-),\bar{t}) \; .$$

Therefore
$$-\frac{\partial \phi}{\partial x_N}((((x_\varepsilon)',0+),\bar{t}) + \frac{\partial \phi}{\partial x_N}((((x_\varepsilon)',0-),\bar{t}) \leq 0 \; ,$$

which is a contradiction for ε small enough since both partial derivatives are continuous in x'.

Q.E.D.

9.5 Convergence of the Vanishing Viscosity Approximation (III): the General (non-convex) Case

In this section, we use the Lions-Souganidis comparison result to show that the vanishing viscosity approximation converges to the unique solution of the Kirchhoff problem; this gives an other version of Theorem 9.4.1 in a non-convex setting.

Theorem 9.5.1 *Assume that, for any $\varepsilon > 0$, u^ε is a viscosity solution of*

$$u_t^\varepsilon - \varepsilon \Delta u^\varepsilon + H(x,t,Du^\varepsilon) = 0 \quad \text{in} \quad \mathbb{R}^N \times (0,T) \, , \tag{9.3}$$

$$u^\varepsilon(x,0) = u_0(x) \quad \text{in} \quad \mathbb{R}^N \, , \tag{9.4}$$

where $H = H_1$ in Ω_1 and H_2 in Ω_2, and u_0 is bounded continuous function in $\mathbb{R}^N$. Under the assumptions of Theorem 9.2.1 and if the sequence (u^ε) is uniformly bounded in $\mathbb{R}^N \times (0,T)$, C^1 in x_N in a neighborhood of $\mathcal{H}$, then, as $\varepsilon \to 0$, the sequence $(u^\varepsilon)_\varepsilon$ converges locally uniformly to the unique solution of the Kirchhoff problem in $\mathbb{R}^N \times (0,T)$.

Proof — The proof follows exactly the proof of Theorem 9.4.1, just using Theorem 9.2.1 instead of Theorem 8.1.1, and without switching to a flux-limiter formulation.

Q.E.D.

Remark 9.5.2 *This third result on the convergence of the vanishing viscosity approximation may appear as being more general since it covers the case of non-convex*

Hamiltonians. But we point out that Theorem 9.2.1 requires (3.16) *or* **(TC-s)** *which limit its range of applications. A third possibility (that we did not explore) would be to use the framework of Theorem 8.3.1 which has the defect of re-introducing the quasi-convexity but with more general assumptions on the Hamiltonians otherwise. Of course such strategy would require an analogue of Proposition 7.4.4 for non-convex (but quasi-convex) Hamiltonians: we refer the reader to Imbert and Monneau [84] or Imbert and Nguyen [86] for such results.*

Chapter 10

Application: KPP-type problems with discontinuities (I)

10.1 Introduction: KPP Equations and front propagations

In this section, we are interested in Kolmogorov-Petrovsky-Piskunov [93] type equations (KPP in short), whose simplest form is

$$u_t - \frac{1}{2}\Delta u = cu(1-u) \quad \text{in } \mathbb{R}^N \times (0,+\infty)\ , \tag{10.1}$$

where c is a nonnegative constant.

Such reaction-diffusion equation appears in several different models in Physics (combustion for example) and Biology (typically for the evolution of population) and, in all these applications, one of the main interest comes front the large time behavior of the solutions which is mainly described in terms of front propagations. One of the main ingredients to understand this behavior is the study of the existence of *travelling waves solutions*, i.e. solutions which can be written as

$$u(x,t) := q(x \cdot e - \alpha t)\ ,$$

where $q : \mathbb{R} \to [0,1]$ is a smooth enough function, $e \in \mathbb{R}^N$ is such that $|e| = 1$, and $\alpha \in \mathbb{R}$. The travelling wave connects the instable equilibrium $u = 0 = q(-\infty)$ with the stable one $u = 1 = q(+\infty)$.

The connection between these travelling waves solutions and front propagation phenomenon is clear: the existence of such a solution implies that the hyperplanes

$x \cdot e = constant$ propagate with a normal velocity α. And clearly, understanding the propagation of such flat fronts is a key step towards dealing with more complicated fronts.

The case of KPP Equations is complicated in terms of travelling waves: while for other nonlinearities (for example cubic non linearities like $f(u) = (u - \mu)(1 - u^2))$ there exists a unique characteristic velocity, KPP Equations admit a critical velocity $\alpha^* > 0$ such that travelling waves solutions exist for all $\alpha \geq \alpha_*$. And it is well-known that the large time behavior of the solutions (in particular the choice(s) of the velocity) depends on the behavior at infinity of the initial data. Actually this large time behavior can be rather complicated since it can be explained by the "mixing" of several different travelling waves as explained in Hamel and Nadirashvili [78].

We are going to concentrate here on the case where the minimal velocity α_* is selected. In this case it is known that $\alpha_* = \sqrt{2c}$ and that the large time behavior of the solutions of the KPP Equation is described by a front propagating with a $\sqrt{2c}$ normal velocity, where the front separates the regions where u is close to 0 and to 1.

In order to prove this result, Freidlin [67] introduced a scaling in space and time $(x, t) \to (\frac{x}{\varepsilon}, \frac{t}{\varepsilon})$ which has the double advantage to preserve the velocities and to allow to observe in finite times the large time behavior of the solution by examining the behavior of the scaled solution as $\varepsilon \to 0$. Hence one has to study the behavior when $\varepsilon \to 0$ of

$$u_\varepsilon(x, t) = u\left(\frac{x}{\varepsilon}, \frac{t}{\varepsilon}\right) ,$$

which solves the singular perturbation problem

$$(u_\varepsilon)_t - \frac{\varepsilon}{2}\Delta u_\varepsilon = \frac{c}{\varepsilon}u_\varepsilon(1 - u_\varepsilon) \quad \text{in } \mathbb{R}^N \times (0, +\infty) .$$

We complement this pde with the initial data

$$u_\varepsilon(x, 0) = g(x) \quad \text{in } \mathbb{R}^N ,$$

where $g : \mathbb{R}^N \to \mathbb{R}$ is a compactly supported continuous function satisfying $0 \leq g(x) \leq 1$ in $\mathbb{R}^N$.

The reader might be surprised by this unscaled initial data but, in this approach, the role of g is just to initialize the position of the front, given here by the boundary of the support of g, $\Gamma_0 := \partial \operatorname{supp}(g)$.

In this context, the following properties can be proved:

$$u_\varepsilon(x, t) = \exp\left(-\frac{I(x, t) + o(1)}{\varepsilon}\right) ,$$

where I is the unique viscosity solution of the variational inequality

$$\min\left(I_t + \frac{1}{2}|DI|^2 + c, I\right) = 0 \quad \text{in } \mathbb{R}^N \times (0, +\infty) ,$$

with

$$I(x,0) = \begin{cases} 0 & \text{if } x \in G_0 \\ +\infty & \text{if } x \in \mathbb{R}^N \setminus G_0. \end{cases}$$

Moreover $I = \max(J, 0)$ where J is the unique viscosity solution of

$$J_t + \frac{1}{2}|DJ|^2 + c = 0 \quad \text{in } \mathbb{R}^N \times (0, +\infty) .$$

The importance of this second part of the result is to allow for an easy computation of J, and therefore I, through the Oleinik-Lax formula

$$J(x,t) = \frac{[d(x, G_0)]^2}{t} - ct ,$$

where $G_0 = \text{supp}(g)$. Hence $u_\varepsilon(x,t) \to 0$ in the domain $\{I > 0\} = \{J > 0\} = \{d(x, G_0) > \sqrt{2c}t\}$ and it can be shown that $u_\varepsilon(x,t) \to 1$ in the interior of the set $\{I = 0\} = \{J \leq 0\} = \{d(x, G_0) \leq \sqrt{2c}\,t\}$. Therefore the propagating front is $\Gamma_t = \{d(x, G_0) = \sqrt{2c}\,t\}$ which means a propagation with normal velocity $\sqrt{2c}$ as predicted by the travelling waves.

Such kind of results, in the more general cases of x, t dependent velocities $c(x,t)$, diffusion and drift terms, were obtained by Freidlin [67] using probabilistic Large Deviation type methods and later pdes' proofs, based on viscosity solutions' arguments, were introduced by Evans and Souganidis [55, 54]. They were then developed not only for KPP Equations but for other reaction-diffusion equations by Barles, Evans and Souganidis [14], Barles, Bronsard and Souganidis [13], Barles, Georgelin and Souganidis [15]. Later, these front propagation problems were considered in connections with the "levet-set approach": one of the first articles in this direction was the one by Evans, Soner and Souganidis [53] (see also Barles, Soner and Souganidis [20]). The most general results in this direction are obtained through the "geometrical approach" of Barles and Souganidis [28]. A complete overview of all these developments can be found in the CIME course of Souganidis [118] where a more complete list of references is given.

Of course, the aim of the section is to extend the results for KPP Equations to the case of discontinuous diffusions, drifts and reaction terms.

Before doing so, we come back to the main steps of the above mentioned result

1. Introduce the change of variable $I_\varepsilon := -\varepsilon \log(u_\varepsilon)$ and show that I_ε is uniformly locally bounded.

2. Pass to the limit by using the half-relaxed limit method in the equation satisfied by I_ε.

3. Prove a strong comparison result for the variational inequality which allows to prove that $I_\varepsilon \to I$ locally uniformly in $\mathbb{R}^N \times (0, +\infty)$.

4. Show that $I = \max(J, 0)$, when this is true (see just below).

All these steps are classical, except perhaps the last one which is related to the *Freidlin condition*: J is given by a formula of representation given by the associated control problem and Freidlin's condition holds if the optimal trajectories for points (x,t) such that $J(x,t) > 0$ remain in the domain $\{J > 0\}$.

It is worth pointing out that this condition is not always satisfied, but keep in mind that this fourth step is only used to give a simplest form to the result.

10.2 A simple discontinuous example

In order to introduce discontinuities in the KPP Equation, but also to point out an interesting feature of the fronts associated to this equation, let us consider a 1-d example borrowed from Freidlin's book [67] in which we have

$$u_t - \frac{1}{2}\Delta u = c(x)u(1-u) \quad \text{in } \mathbb{R} \times (0, +\infty) \ ,$$

where $c(x) = c_1$ if $x < 1$ and $c(x) = c_2$ if $x \geq 1$. We also assume that $G_0 = (-\infty, 0)$, i.e. the front is located at $x = 0$ initially.

For the control formulation for the function J, we follow the approach of Part II: for $x \in \Omega_1 := \{x < 1\}$, we set

$$\mathbf{BCL}_1(x,t) := \Big\{ \Big(v_1\,,\, 0\,,\, -c_1\Big);\ v_1 \in \mathbb{R}^N \Big\} .$$

and for $x \in \Omega_2 := \{x > 1\}$, we set

$$\mathbf{BCL}_2(x,t) := \Big\{ \Big(v_1\,,\, 0\,,\, -c_2\Big);\ v_2 \in \mathbb{R}^N \Big\} .$$

Therefore the cost $-c$ is discontinuous at $x = 1$.

The following formula allows to compute explicitly function J

$$J(x,t) = \inf\left\{\int_0^t \left(\frac{|\dot y(s)|^2}{2} - c(y(s))\right) ds\ ;\ y(0)=x,\ y(t)\le 0\right\}\ .$$

Notice that, a priori, we should have been careful with this formal formula since the function c is discontinuous at $x=1$ but, at this point of the book, it should be clear for the reader that the present situation is quite easy to handle. Indeed, if the trajectory stays on the line $x=1$, we are just going to choose $c(y(s)) = \max(c_1,c_2)$, which yields the minimal possible cost, and with simply a tangential dynamic.

From now on, we assume for our purpose that $c_2 > c_1$. And we adress the question: when does the front, starting from $x=0$ reach the value 1? If we just consider the domain $x<1$, the answer should be $t_1 = \left(\sqrt{2c_1}\right)^{-1}$ since the velocity of the front is $\sqrt{2c_1}$ in this domain.

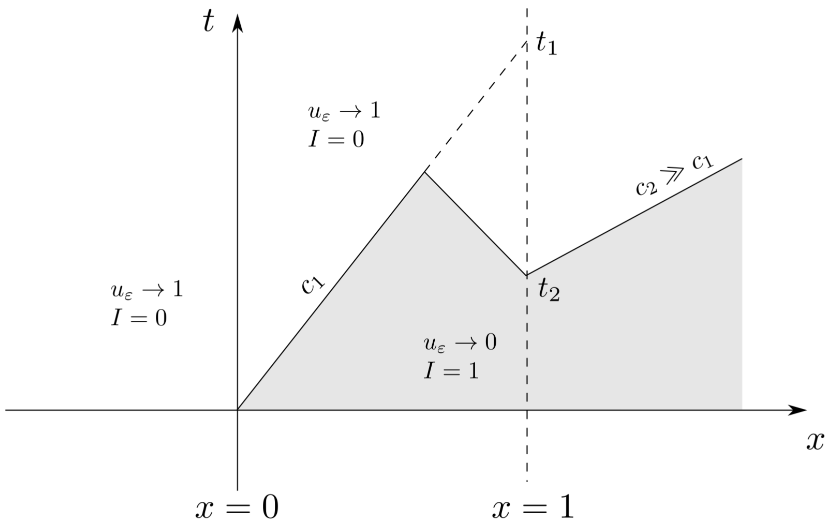
But we may also examine $J(1,t)$ and compute the smallest t for which it is 0 (which indeed corresponds to $u=1$). It is clear that an optimal trajectory should stay at $x=1$ on an interval $[0,h]$ and then a straight line to reach $x=0$. Therefore

$$J(1,t) = \min_{0\le h\le t}\left(-c_2 h + \frac{1}{2(t-h)} - c_1(t-h)\right)\ .$$

An easy computation gives

$$J(1,t) = \begin{cases} \dfrac{1}{2t} - c_1 t & \text{if } t \le \dfrac{1}{\sqrt{2(c_2-c_1)}}\ , \\ \sqrt{2}\sqrt{c_2-c_1} - c_2 t & \text{otherwise}\end{cases}\ ,$$

and the front reaches 1 either at time $t_1 = \left(\sqrt{2c_1}\right)^{-1}$ or $t_2 = \sqrt{2}\dfrac{\sqrt{c_2-c_1}}{c_2}$, depending which one is the smallest one. Hence, an easy computation shows that, if $c_2 > 2c_1$, we have $t_2 < t_1$ and, for the front, a picture like



We observe at time t_2 a strange phenomenon: a new front is created at $x = 1$, ahead of the front travelling in Ω_1 with velocity $\sqrt{2c_1}$. This kind of phenomenon can arise even if $c(x)$ is continuous but the computations are easier to describe in the discontinuous setting. We also point out that Freidlin's condition holds true in this example.

In the next sections, we first provide results for general KPP Equations in the framework of Part II, i.e. in the case when we have discontinuities on an hyperplane. Then we consider some extensions to more general type of discontinuities which uses some particular features of the KPP Equations.

10.3 KPP Equations with discontinuities on an hyperplane

With the notations of Part II, we consider the problem

$$(u_\varepsilon)_t - \frac{\varepsilon}{2}\mathrm{Tr}(a(x)D^2 u_\varepsilon) - b(x)\cdot Du_\varepsilon = \frac{1}{\varepsilon}f(x,u_\varepsilon) \quad \text{in } \mathbb{R}^N \times (0,+\infty)\,, \tag{10.2}$$

where, in Ω_i, $a = a^{(i)}$, $b = b^{(i)}$, $f = f^{(i)}$ for $i = 1,2$, where $a^{(i)}, b^{(i)}, f^{(i)}$ are bounded Lipschitz continuous functions taking values respectively in $\mathcal{S}^N$, $\mathbb{R}^N$ and $\mathbb{R}$. We assume that the following additional properties hold

(Uniform ellipticity) there exists $\nu > 0$ such that

$$a^{(i)}(x)p \cdot p \geq \nu |p|^2 \quad \text{for any } x, p \in \mathbb{R}^N \; . \tag{10.3}$$

(KPP-nonlinearity) For $i = 1, 2$ and for any $x \in \Omega_i$: $u \mapsto f^{(i)}(x, u)$ is differentiable at 0 and for any $u \in [0, 1]$

$$\begin{cases} f^{(i)}(x,0) = f^{(i)}(x,1) = 0, \quad f^{(i)}(x,u) > 0 \quad \text{if } 0 < u < 1 \\ c^{(i)}(x) = \dfrac{\partial f^{(i)}}{\partial u}(x,0) = \sup\limits_{0<u<1} \left(\dfrac{f^{(i)}(x,u)}{u} \right), \end{cases} \tag{10.4}$$

with $c^{(i)}$ being bounded Lipschitz continuous on $\overline{\Omega_i}$.

Of course, the prototypal example of $f^{(i)}$ is $f^{(i)}(x,u) = c^{(i)}(x)u(1-u)$ which is not a globally Lipschitz continuous function of u but since all the solutions u_ε will take values in $[0, 1]$, this is not a problem.

Next we complement (10.2) with the initial data

$$u_\varepsilon(x, 0) = g(x) \quad \text{in } \mathbb{R}^N \; , \tag{10.5}$$

where $g : \mathbb{R}^N \to \mathbb{R}$ is a compactly supported continuous function such that $0 \leq g(x) \leq 1$ in $\mathbb{R}^N$. As above we denote by G_0 the support of g which is assumed to be a non-empty compact subset of $\mathbb{R}^N$ with

$$\overline{\mathrm{Int}(G_0)} = G_0 \; .$$

In order to formulate the result, we introduce the Hamiltonians defined for $i = 1, 2$ by

$$H_i(x,p) := \frac{1}{2} a^{(i)}(x)p \cdot p - b^{(i)}(x) \cdot p + c^{(i)}(x) \; .$$

As we already noticed in the previous subsection, keep in mind that in the control viewpoint, the cost is $l^{(i)} = -c^{(i)}$.

Theorem 10.3.1 *As $\varepsilon \to 0$, we have*

$$-\varepsilon \log(u_\varepsilon) \to I \quad \textit{locally uniformly in } \mathbb{R}^N \times (0, +\infty) \; ,$$

where I can be seen as either the unique solution of

$$\begin{cases} \min(I_t + H_i(x, DI), I) = 0 & \textit{in } \Omega_i \times (0, +\infty) \; , \\ I(x,0) = \begin{cases} 0 & \textit{if } x \in G_0, \\ +\infty & \textit{otherwise} \; , \end{cases} \end{cases} \tag{10.6}$$

associated to the Kirchhoff condition

$$\frac{\partial I}{\partial n_1} + \frac{\partial I}{\partial n_2} = 0 \quad \text{on } \mathcal{H} \times (0, +\infty) \ , \tag{10.7}$$

or equivalently, the maximal Ishii solution of variational inequality (10.6) *in* $\mathbb{R}^N \times (0, +\infty)$*. Moreover we have*

$$u_\varepsilon(x,t) \to \begin{cases} 0 & \text{in } \{I > 0\}, \\ 1 & \text{in the interior of the set } \{I = 0\}. \end{cases}$$

Finally if Freidlin's condition holds then $I = \max(J, 0)$ *where* J *is either the unique solution of*

$$\begin{cases} J_t + H_i(x, DJ) = 0 & \text{in } \Omega_i \times (0, +\infty) \ , \\ J(x,0) = \begin{cases} 0 & \text{if } x \in G_0, \\ +\infty & \text{otherwise} \ , \end{cases} \end{cases} \tag{10.8}$$

associated to the Kirchhoff condition, or equivalently the maximal Ishii solution of (10.8) *in* $\mathbb{R}^N \times (0, +\infty)$*. Function* J *is given by the following representation formula*

$$J(x,t) = \inf \left\{ \int_0^t l(y(s), \dot{y}(s)) ds; \ y(0) = x, \ y(t) \in G_0, \ y \in H^1(0,T) \right\} \ ,$$

where $l(y(s), \dot{y}(s)) = \dfrac{1}{2}[a^{(i)}(y(s))]^{-1}(\dot{y}(s) - b^{(i)}(y(s))) \cdot (\dot{y}(s) - b^{(i)}(y(s))) - c^{(i)}(y(s))$ *if* $y(s) \in \Omega_i$ *and with the regular control procedure on* $\mathcal{H} \times (0, +\infty)$*.*

We can summarize this result by saying that the "usual" KPP-result holds true provided that the "action functional" J is suitably defined, taking only regular controls on $\mathcal{H} \times (0, +\infty)$, using the links between the maximal Ishii viscosity solution, flux limted solutions and junction viscosity solutions for the Kirchhoff condition.

Proof — The proof relies on classical arguments which remains valid because of the results of Theorem 10.4.2 given in the appendix of this section.

The aim is make the change of variable

$$I_\varepsilon(x,t) = -\varepsilon \log(u_\varepsilon(x,t))$$

and to show that $I_\varepsilon \to I$ locally uniformly in $\mathbb{R}^N \times (0, +\infty)$. But, in order to do so, we first need local uniform bounds on I_ε.

We first notice that, by the Maximum Principle, we have

$$0 \le u_\varepsilon(x,t) \le 1 \quad \text{in } \mathbb{R}^N \times (0, +\infty) \ ,$$

and therefore $I_\varepsilon(x,t) \geq 0$ in $\mathbb{R}^N \times (0,+\infty)$. In addition, I_ε is well-defined because $u_\varepsilon(x,t) > 0$ in $\mathbb{R}^N \times (0,+\infty)$ by the Strong Maximum Principle.

Getting an upper bound on I_ε is done by using the trick introduced in [17, 18] (the reader can look in those references for the details which follow): we set

$$I_\varepsilon^A(x,t) = -\varepsilon \log\left(u_\varepsilon(x,t) + \exp(-A/\varepsilon)\right) \ ,$$

where $A \gg 1$. Then $o(1) \leq I_\varepsilon^A(x,t) \leq A$, and it is easy to show that

$$\limsup{}^* \ I_\varepsilon^A = \min(\limsup{}^* \ I_\varepsilon, A) \ .$$

Therefore controlling I_ε^A uniformly in A provides the same control on I_ε.

Next, using that $f^{(i)}(x,u_\varepsilon) \geq 0$ in $\mathbb{R}^N \times (0,+\infty)$, the function I_ε^A satisfies

$$(I_\varepsilon^A)_t - \frac{\varepsilon}{2}\mathrm{Tr}(a^{(i)}(x)D^2 I_\varepsilon^A) + \frac{1}{2}a^{(i)}(x)DI_\varepsilon^A \cdot DI_\varepsilon^A - b^{(i)}(x)\cdot DI_\varepsilon^A \leq 0 \quad \text{in } \Omega_i \times (0,+\infty) \ ,$$

and the ellipticity assumption together with a Cauchy-Schwartz inequality on the $b^{(i)}$-term leads to

$$(I_\varepsilon^A)_t - \frac{\varepsilon}{2}\mathrm{Tr}(a^{(i)}(x)D^2 I_\varepsilon^A) + \frac{1}{2}\nu|DI_\varepsilon^A|^2 \leq k(\nu) \quad \text{in } \Omega_i \times (0,+\infty) \ ,$$

for some constant $k(\nu)$ large enough, depending only on $\|b^{(i)}\|_\infty$ and ν.

Passing to the limit through the half-relaxed limit method and setting $\bar{I}_A = \limsup^* \ I_\varepsilon^A$, we get the following inequality for $\bar{I}_A$:

$$\begin{cases} (\bar{I}_A)_t + \frac{1}{2}\nu|D\bar{I}_A|^2 \leq k(\nu) & \text{in } \mathbb{R}^N \times (0,+\infty) \ , \\ \bar{I}_A(x,0) = \begin{cases} 0 & \text{if } x \in G_0, \\ A & \text{otherwise} \ . \end{cases} & \end{cases}$$

The Oleinik-Lax formula then implies

$$\bar{I}_A(x,t) \leq \frac{[d(x,G_0)]^2}{2\nu t} + k(\nu)t \quad \text{in } \mathbb{R}^N \times (0,+\infty) \ ,$$

which is the desired uniform bound.

Therefore we can perform the I_ε change of function and we obtain

$$(I_\varepsilon)_t - \frac{\varepsilon}{2}\mathrm{Tr}(a^{(i)}(x)D^2 I_\varepsilon) + \frac{1}{2}a^{(i)}(x)DI_\varepsilon \cdot DI_\varepsilon - b^{(i)}(x)\cdot DI_\varepsilon \leq -\frac{f^{(i)}(x,u_\varepsilon)}{u_\varepsilon} \quad \text{in } \Omega_i \times (0,+\infty) \ ,$$

where we have kept the notation u_ε in the right-hand side to emphasize the role of the quantity $f^{(i)}(x,u_\varepsilon)/u_\varepsilon$. Indeed we have both

$$-\frac{f^{(i)}(x,u_\varepsilon)}{u_\varepsilon} \geq -c^{(i)}(x) \quad \text{for any } x \ ,$$

and

$$-\frac{f^{(i)}(x,u_\varepsilon)}{u_\varepsilon} \to -c^{(i)}(x) \quad \text{if } u_\varepsilon(x,t) \to 0 \ ,$$

and this last case occurs if $I_\varepsilon(x,t)$ tends to a strictly positive quantity.

Using these properties, Theorem 9.4.1 implies that $\overline{I} = \limsup^* I_\varepsilon$ and $\underline{I} = \liminf_* I_\varepsilon$ are respectively sub and supersolutions of the variational inequality (10.6) associated with Kirchhoff condition on $\mathcal{H}$.

In order to conclude, we have just to use Theorem 10.4.2: with the notations of this result, we have

$$\overline{I}(x,t) \le I^+(x,t) \le \underline{I}(x,t) \quad \text{in } \mathbb{R}^N \times (0,+\infty) \ ,$$

and, I^+ being continuous, this implies that $I_\varepsilon \to I^+$ locally uniformly in $\mathbb{R}^N \times (0,+\infty)$.

The proof is complete since the other results can be obtained exactly as in the standard KPP case.

Q.E.D.

10.4 Appendix: the variational inequality

In this section, we study the control/game problems related to the functions I and J arising in the statement of Theorem 10.3.1, together with the properties of the associated Bellman equation or variational inequality.

To do so, we follow the approach of Part II: for $x \in \Omega_i$, we set

$$\mathbf{BCL}_i(x,t) := \left\{ \left(v_i \, , \, 0 \, , \, \frac{1}{2}[a^{(i)}(x)]^{-1}(v_i - b^{(i)}(x)) \cdot (v_i - b^{(i)}(x)) - c^{(i)}(x)\right); \ v_i \in \mathbb{R}^N \right\}.$$

This definition can be seen as using v as a control, hence authorizing any possible dynamic $v \in \mathbb{R}^N$ at any point (x,t), but with a cost $l(x,v) = l_i(x,v) = \frac{1}{2}[a^{(i)}(x)]^{-1}(v - b^{(i)}(x)) \cdot (v - b^{(i)}(x)) - c^{(i)}(x)$ if $x \in \Omega_i$.

Of course, we are in an unbounded control framework but this does not create any major additional difficulty.

It remains to define the dynamic/cost on $\mathcal{H}$ (regular or not) and for $x \in \mathcal{H}$, we have: $(v,0,l) \in \mathbf{BCL}_T(x,t)$ if $v \in \mathcal{H}$, $v = \alpha v_1 + (1-\alpha) v_2$ and

$$l = \alpha l_1(x,v_1) + (1-\alpha) l_2(x,v_2) \ ,$$

where as above, $l_i(x,v_i) = \frac{1}{2}[a^{(i)}(x)]^{-1}(v_i - b^{(i)}(x)) \cdot (v_i - b^{(i)}(x)) - c^{(i)}(x)$. The set $\mathbf{BCL}_T^{\mathrm{reg}}(x,t)$ is defined in the same way, adding the condition $v_1 \cdot e_N \le 0$, $v_2 \cdot e_N \ge 0$.

If $I_0 \in C_b(\mathbb{R}^N)$, we introduce

$$J^-(x,t) = \inf_{\mathcal{T}_{x,t}} \left\{ \int_0^t l(X(s),\dot{X}(s))ds + I_0(X(T)) \right\} \ ,$$

$$J^+(x,t) = \inf_{\mathcal{T}^{\rm reg}_{x,t}} \left\{ \int_0^t l(X(s),\dot{X}(s))ds + I_0(X(T)) \right\} \ ,$$

where, in these formulations, we have replaced $v_i(s)$ $(i = 1, 2)$ or $v(s)$ by $\dot{X}(s)$.

In the same way, we introduce

$$I^-(x,t) = \inf_{\mathcal{T}_{x,t},\theta} \left\{ \int_0^{t\wedge\theta} l(X(s),\dot{X}(s))ds + \mathbb{1}_{t<\theta} I_0(X(T)) \right\} \ ,$$

$$I^+(x,t) = \inf_{\mathcal{T}^{\rm reg}_{x,t},\theta} \left\{ \int_0^{t\wedge\theta} l(X(s),\dot{X}(s))ds + \mathbb{1}_{t<\theta} I_0(X(T)) \right\} \ .$$

Following the methods of Part II, it is easy to show the following result

Theorem 10.4.1

(*i*) *The value-functions J^- and J^+ are continuous and respectively the minimal Ishii supersolution (and solution) and maximal Ishii subsolution (and solution) of the equation*

$$J_t + H(x, DJ) = 0 \quad in\ \mathbb{R}^N \times (0,+\infty) \ , \tag{10.9}$$

where $H = H_i$ in $\Omega_i \times (0,+\infty)$ with the initial data

$$J(x,0) = I_0(x) \quad in\ \mathbb{R}^N \ .$$

(*ii*) **(SCR)** *holds for the flux-limited problems for Equation (10.9) with flux-limiters H_T and $H_T^{\rm reg}$ on $\mathcal{H}$; J^- is the unique flux-limited solution associated to the flux-limiter H_T and J^+ is the unique flux-limited solution associated to the flux-limiter $H_T^{\rm reg}$. J^+ is also the unique solution associated to the Kirchhoff condition on $\mathcal{H}$.*

(*iii*) *The value-functions I^- and I^+ are continuous and respectively the minimal Ishii supersolution (and solution) and maximal Ishii subsolution (and solution) of the equation*

$$\min(I_t + H(x, DI), I) = 0 \quad in\ \mathbb{R}^N \times (0,+\infty) \ , \tag{10.10}$$

where $H = H_i$ in $\Omega_i \times (0,+\infty)$ with the initial data

$$I(x,0) = I_0(x) \quad in\ \mathbb{R}^N \ .$$

(*iv*) **(SCR)** *holds for the flux-limited problems for the variational inequality* (10.10) *with flux-limiters* H_T *and* H_T^{reg}; I^- *is the unique flux-limited solution associated to the flux-limiter* H_T *and* I^+ *is the unique flux-limited solution associated to the flux-limiter* H_T^{reg}. I^+ *is also the unique solution associated to the Kirchhoff condition on* $\mathcal{H}$.

In order to treat the KPP problem, we have to extend this result to the case of discontinuous I_0, with possibly infinite values. Of course, stricto sensu, a **(SCR)** cannot hold in this case. Indeed, if u and v are respectively a sub and supersolution of either (10.9) or (10.10) with initial data I_0, the inequalities at time $t = 0$ are

$$u(x,0) \leq I_0^*(x) \quad \text{and} \quad v(x,0) \geq (I_0)_*(x) \quad \text{in } \mathbb{R}^N \ ,$$

and it is false in general that $u(x,0) \leq v(x,0)$ in $\mathbb{R}^N$. Therefore we have to extend the meaning of **(SCR)** by saying that a **(SCR)** holds in this context if we have

$$u(x,t) \leq v(x,t) \quad \text{in } \mathbb{R}^N \times (0,+\infty) \ ,$$

hence for all $t > 0$.

With this modified definition, we can formulate a simple result which is exactly what we need (we do not try to reach the full generality here):

Theorem 10.4.2 *Assume that* $\overline{\mathrm{Int}(G_0)} = G_0$, *then the results of Theorem 10.4.1 remain true if* $I_0(x) = A\mathbb{1}_{G_0}$ *for some* $A > 0$, *and even if* $A = +\infty$.

Proof — We begin with the case when $A < +\infty$ and we provide the full proof only in the I-case, the J-one being obtained by similar and even simpler arguments.

Step 1: *Approximation of the data.*

In order to prove the analogue of (*iii*), we can approximate I_0 by above and below by sequences $((I_0)^A)_A$ and $((I_0)_A)_A$ of bounded continuous initial data such that

$$(I_0)^A \downarrow I_0^* \quad \text{and} \quad (I_0)_A \uparrow (I_0)_* \ .$$

We denote by $(I^A)^\pm$ and $(I_A)^\pm$ the minimal and maximal solutions given by Theorem 10.4.1 with these intial data.

If u, v are respectively a subsolution and a supersolution of the variational inequality with initial data I_0, they are respectively subsolution with $(I_0)^A$ and supersolution with $(I_0)_A$. Therefore

$$u \leq (I^A)^+ \quad \text{and} \quad (I_A)^- \leq v \quad \text{in } \mathbb{R}^N \times (0,+\infty) \ .$$

It remains to pass to the limit in the variational formulas for $(I^A)^+$ and $(I_A)^-$. This step is easy for $(I^A)^-$ by the stability of solutions of differential inclusion (one has just to be careful of the fact that we obtain $(I_0)_*$ in the formula at the limit).

For $(I^A)^+$, things are more delicate since we have to deal with regular trajectories. But here, we can take advantage of the inequality we wish to show and first argue with a FIXED trajectory (here also one has to be careful because we obtain $(I_0)^*$ in the formula at the limit).

Step 2: *Both functions $(I^A)^+$ and $(I^A)^-$ are continuous.*

In order to prove the claim, we can use the approach of the authors in [26], showing that $I = (I^A)^-$ or $(I^A)^+$ both satify

$$-\eta(t) \leq I_t(x,t) \leq C \ .$$

for some positive function η which may tend to $+\infty$ when $t \to 0$ and for some constant C. This inequality is obtained by using the arguments of [26]: we just use a sup-convolution in time

$$\sup_{0\leq s\leq t} \left(I(x,s) - \eta(s)(t-s)\right) ,$$

and combine it with a comparison result for flux-limited solutions (with the suitable flux-limiter for $(I^A)^-$ and $(I^A)^+$).

This argument shows that $(I^A)^-$ and $(I^A)^+$ are Lipschitz continuous in x (for $t > 0$) where they are strictly positive. Indeed, if $I > 0$, variational inequality (10.10) implies that $H(x, DI) = -I_t \leq \eta(t)$, and the coercivity of H implies a bound on DI. Then, it is a simple exercice to extend it to all points in $\mathbb{R}^N \times (0, +\infty)$, whether $I > 0$ or $I = 0$.

Step 3: *Strong Comparison Result.*

For the proofs of the **(SCR)**, we still consider $(I_0)^A$, $(I_0)_A$ but the **(SCR)** for either H_T, $H_T^{\rm reg}$ or the Kirchhoff condition. In the case of $H_T^{\rm reg}$, for example, we obtain

$$u \leq (I^A)^+ \quad \text{and} \quad (I_A)^+ \leq v \quad \text{in } \mathbb{R}^N \times (0, +\infty) \ .$$

To conclude in this case, we have to use Proposition 6.3.7 to pass to the limit by extracting a sequence of trajectories which converges to a regular trajectorie. The case of H_T is simpler.

Step 4: *Passing to the limit to treat the case $A = \infty$.*

In the case where $A = +\infty$, we first notice that all solutions associated with initial data like $I_0(x) = A\mathbb{1}_{G_0}$, and slightly enlarging or slightly reducing the set G_0

are uniformly locally bounded with respect to A (this can be obtained by choosing appropriate trajectories such as straight lines). And the limiting function are

$$I^-(x,t) = \inf_{\mathcal{T}_{x,t,\theta}} \left\{ \int_0^{t\wedge\theta} l(X(s),\dot{X}(s))ds;\ X(T)\in G_0 \right\} ,$$

$$I^+(x,t) = \inf_{\mathcal{T}^{\mathrm{reg}}_{x,t},\theta} \left\{ \int_0^{t\wedge\theta} l(X(s),\dot{X}(s))ds;\ X(T)\in G_0 \right\} .$$

Now, if u is a subsolution then, for all A and $C = \max_i(||c_i||_\infty)$, $\min(u, A - Ct)$ is also a subsolution associated to the initial data $A1\!\!1_{G_0}$. Indeed, sonce the Hamiltonians are convex, the infimum of two subsolutions remains a subsolution. We then use the first result to conclude. We can use a similar argument for the supersolution, using this time a comparison with $(I_A)^\pm$, depending of the result we want.

Q.E.D.

Chapter 11

More remarks

11.1 Interested in an Emblematic Example?

The aim of this section is to give an overview of the results of Part II to the unwise, reckless and foolhardy (partial) reader of this book who wishes

(*i*) to have an idea of what can be done in the emblematic particular case when we are in a framework which is the HJ analogue of 1-d scalar conservation laws with a discontinuous flux ...

(*ii*) ... without reading the rest of this book!

Thus, the reader will find here some redundancy concerning definitions, results, ideas... with respect to the previous sections. We try to keep it simple and refer to those previous sections for more precise results and proofs.

Here we look at the problem

$$u_t + H(x, u_x) = 0 \quad \text{in } \mathbb{R} \times (0, T) \ , \tag{11.1}$$

where the Hamiltonian H is given by

$$H(x,p) = \begin{cases} H_1(p) & \text{if } x > 0, \\ H_2(p) & \text{if } x < 0. \end{cases}$$

In this definition of H, H_1, H_2 are continuous functions which are coercive, i.e.

$$H_1(p), H_2(p) \to +\infty \quad \text{as } |p| \to +\infty \ ,$$

and we will look at two main cases : the “Lipschitz case” where both H_i’s are supposed to be Lipschitz continuous in $\mathbb{R}$ and the “convex case” where the H_i’s are supposed to be convex (and non necessarily Lipschitz continuous, even if this case is not completely covered by the results of Part II [(1)]).

In the “Lipschitz case”, a natural sub-case is the one when the H_i (i=1,2) are *quasi-convex*, i.e. when they are built as the maximum of an increasing and a decreasing function. For this reason, we will write

$$H_1 = \max(H_1^+, H_1^-) \quad \text{and} \quad H_2 = \max(H_2^+, H_2^-)\ ,$$

where (this is strange but the reader has to keep in mind that *characteristics* play a role in these problems...) H_1^+, H_2^+ are the decreasing parts of H_1, H_2 respectively and H_1^-, H_2^- their increasing parts.

Equation (11.1) has to be complemented by an initial datum

$$u(x,0) = u_0(x) \quad \text{in } \mathbb{R}\ , \tag{11.2}$$

where u_0 is assumed to be bounded and continuous in $\mathbb{R}$.

Of course, the first key question is: what kind of condition has to be imposed at $x = 0$ where the Hamiltonians is discontinuous?

For a reader who is familiar with the notion of viscosity solutions, if no other conditions comes from the problem one is interested in, the immediate answer is: apply the *classical definition of viscosity solutions* (CVS) introduced by H. Ishii, i.e.

$$\min(u_t + H_1(u_x), u_t + H_2(u_x)) \leq 0\ ,$$
$$\max(u_t + H_1(u_x), u_t + H_2(u_x)) \geq 0\ .$$

We recall that these sub and supersolutions properties have to be tested with test-functions which are C^1 in $\mathbb{R} \times (0,T)$[(2)].

Unfortunately (or fortunately?), this classical notion of solutions is not adapted for treating the case of problems in which we have an additional “transfer condition” at $x = 0$. To be convinced by this claim, it suffices to look at Kirchhoff’s condition

$$-u_x(0^+,t) + u_x(0^-,t) = 0 \quad \text{on} \quad \{0\} \times (0,T)\,, \tag{11.3}$$

for which testing with test-functions which are C^1 on $\mathbb{R} \times (0,T)$ is meaningless.

For the Kirchhoff condition but also for more general conditions like

$$G(u_t, -u_x(0^+,t), u_x(0^-,t)) = 0 \quad \text{on} \quad \{0\} \times (0,T)\,, \tag{11.4}$$

$G(a,b,c)$ is a continuous function which is increasing in a and b[(3)], one has to use

(1) but we trust the reader to be able to fill up the gaps!

(2) we do not detail these properties here and refer the reader to Section 3.1

(3) Precise assumptions will be given later on.

a notion of *"Network viscosity solutions"* for which one has to test the viscosity properties with continuous, "piecewise C^1"-test-functions. More precisely $\phi \in C(\mathbb{R} \times (0,T))$ is a suitable test-function if there exists two functions ϕ_1, ϕ_2 which are C^1 in $\mathbb{R} \times (0,T)$ such that

$$\phi(x,t) = \begin{cases} \phi_1(x,t) & \text{if } x > 0, \\ \phi_2(x,t) & \text{if } x < 0, \end{cases}$$

with $\phi_1(0,t) = \phi_2(0,t)$ for any $t \in (0,T)$. In order to define "Network viscosity solutions" (and without entering into details), we add that, for viscosity properties which have to hold at a point $(0,t)$ we use the derivatives of ϕ_1 for the H_1-term or all type of derivatives coming from the $x > 0$-domain and the derivatives of ϕ_2 for the H_2-term or all type of derivatives coming from the $x < 0$-domain.

Unfortunately one may use the notion of "Network viscosity solutions" with conditions at $x = 0$ in (at least) two slightly different ways.

The "flux-limiter" notion of solutions of Imbert-Monneau which is valid in the quasi-convex case, i.e. in a more general framework than the "convex case".

A general "flux-limiter" condition at $x = 0$ is

$$u_t + A = 0 \quad \text{on} \quad \{0\} \times (0,T)\,, \tag{11.5}$$

where A is a real constant and the notion of flux-limiter is written with the following viscosity inequalities for $x = 0$

$$\max(u_t + H_1^+(u_x), u_T + H_2^-(u_x), u_t + A) \leq 0 \quad \text{on} \quad \{0\} \times (0,T)\,,$$

$$\max(u_t + H_1^+(u_x), u_T + H_2^-(u_x), u_t + A) \geq 0 \quad \text{on} \quad \{0\} \times (0,T)\,.$$

Why using only H_1^+ and H_2^-? The most (vague and) convincing answer is probably through the *characteristics*: we use inequalities which test caracteristics which are entering the right domain ($[0,+\infty)$ for H_1 and $(-\infty, 0]$ for H_2)[(4)]. We respectively call these conditions the sub and supersolution flux-limiter conditions.

In the above definition, we can replace the $u_t + A$-term by a more general $\chi(u_t)$-term where the function $\tau \mapsto \chi(\tau)$ is strictly increasing.

The notion of "junction (viscosity) solutions" which is closer to the Ishii formulation since the inequalities for $x = 0$ read

$$\min(u_t + H_1(u_x), u_t + H_2(u_x), G(u_t, -u_x(0^+,t), u_x(0^-,t))) \leq 0\ ,$$

$$\max(u_t + H_1(u_x), u_t + H_2(u_x), G(u_t, -u_x(0^+,t), u_x(0^-,t))) \geq 0\ .$$

[(4)] For people working in control, replace "characteristics" by "dynamics"

Notice a key difference between these notions of solutions: while the "flux-limiter" one uses a "max-max" inequalities, the "junction" one uses the classical "min-max" inequalities.

To understand both the kind of results one can prove and the role and connections of these notions of solutions, a natural entrance door is the *vanishing viscosity method* for the simplest problem (11.1)-(11.2). The result is the

Theorem 11.1.1 *Assume that, for any $\varepsilon > 0$, u^ε is a viscosity solution of*

$$u_t^\varepsilon - \varepsilon u_{xx}^\varepsilon + H(x, u_x^\varepsilon) = 0 \quad in \quad \mathbb{R} \times (0,T)\,, \tag{11.6}$$

$$u^\varepsilon(x,0) = u_0(x) \quad in \quad \mathbb{R}\,, \tag{11.7}$$

If the u^ε are uniformly bounded in $\mathbb{R} \times [0,T)$ and C^1 in x in a neighborhood of $x = 0$ for $t > 0$, then, as $\varepsilon \to 0$, the sequence $(u^\varepsilon)_\varepsilon$ converges locally uniformly to the unique "junction' solution" of the Kirchhoff problem (11.1)-(11.2)-(11.3).

The formal proof of the first part of this result is straightforward: u^ε being C^1 in x in a neighborhood of $x = 0$ for $t > 0$, one has

$$-u_x^\varepsilon(0^+, t) + u_x^\varepsilon(0^-, t) = 0 \quad \text{on} \quad \{0\} \times (0,T)\,,$$

and it suffices to pass to the limit using the good stability properties of viscosity solutions.

We provide three different proofs of this results in Part II (!): the most general one is obtained via the Lions-Souganidis' arguments which are very close to the above formal proof. Using an almost classical stability argument for viscosity solutions, the half-relaxed limits of u^ε are "junction sub and supersolution" of the Kirchhoff problem, i.e.

$$\min(u_t + H_1(u_x), u_t + H_2(u_x), -u_x(0^+, t) + u_x(0^-, t)) \le 0\,,$$

$$\max(u_t + H_1(u_x), u_t + H_2(u_x), -u_x(0^+, t) + u_x(0^-, t)) \ge 0\,,$$

which are as similar as it could be to the classical Ishii formulation, despite of the different spaces of test-functions. We refer to Section 7.1 for a more precise definition of the Kirchhoff condition. It is worth pointing out that the notion of "junction solution" is not only necessary to define properly the Kirchhoff condition but it also plays a key role in the proof via the stability result.

Therefore the convergence of the vanishing viscosity method is not an issue. But two further questions can be addressed

– Is it possible to caracterize the unique "junction solution" of (11.1)-(11.2)-(11.3) in terms of classical viscosity solutions (CVS)?

– In the "convex case", is it possible to write down an explicit formula for solutions of the Kirchhoff problem? (a la Oleinik-Lax). In other words, is there an underlying control problem which gives a control formula for this solution?

Our second result gives the answer to these questions, of course in the "convex case". We point out that the results are unavoidably a little bit vague to avoid a too long statement (but precise results can be found in Chapter 6).

Theorem 11.1.2 *In the "convex case",*
(i) (CVS) *of* (11.1)-(11.2) *(with the natural Ishii conditions at* $x = 0$*) are not unique in general. There is a minimal* (CVS) *(denoted by* $\mathbf{U}^-$*) and a maximal CVS (denoted by* $\mathbf{U}^+$*) which are both given explicitly as value functions of suitable control problems.*
(ii) *If* m_1 *is the largest minimum point of* H_1 *and* m_2 *the least minimum of* H_2*, a sufficient condition in order to have* $\mathbf{U}^- = \mathbf{U}^+$ *is* $m_2 \geq m_1$*.*
(iii) *The solution of the Kirchhoff problem is* $\mathbf{U}^+$*. Hence the vanishing viscosity method converges to the maximal* (CVS)*.*

This result shows the weakness of (CVS) for equations with discontinuities: although they are very stable because of the half- relaxed limits method, they are not unique in this framework and this is, of course, more than a problem. Result (ii) is a last desesperate attempt to maintain uniqueness in a rather general case but it seems to be a little bit anecdotic...

Result (iii) is a first bridge between the notions of (CVS) and "junction solution" and it is proved using in a key way the notion of "flux-limiter solutions". It opens the way to the next question which can be formulated in several different ways (but which all concern the relations between different notions of solutions)

– *For control problems*, two particular value-functions appear in Theorem 11.1.2: $\mathbf{U}^-$ and $\mathbf{U}^+$. Both may be interesting for some particular application but, clearly, the caracterisation as (CVS) is not appropriate. Is there a way to caracterize them in an other way?

– *From the pde point of view*, Result (iii) gives a connection between the "junction solution" for the Kirchhoff condition and a value-function of a control problem: is it possible to do it for more general conditions (11.4) and in a rather explicit way?

The answer is provided in the following result which relies on the notion of "flux-limiter solutions".

Theorem 11.1.3
(i) In the quasi-convex case

– *For any A, there exists a unique "flux-limiter solution" of* (11.1)-(11.2)-(11.5). *Moreover we have a comparison result for the flux-limiter problem.*

– *If G satisfies : there exists $\alpha, \beta \geq 0$ with $\alpha + \beta > 0$ and for any $a_1 \geq a_2$, $b_1 \geq b_2$, $c_1 \geq c_2$*

$$G(a_1, b_1, c_1) - G(a_2, b_2, c_2) \geq \alpha(a_1 - a_2) + \beta(b_1 - b_2) + \beta(c_1 - c_2) ,$$

then any sub-solution or supersolution of (11.1)-(11.2)-(11.4)[5] *is a flux-limiter solution with*

$$\chi(a) = \max_{p_1,p_2} \left(\min \left(a + H_1^-(p_1), a + H_2^+(p_2), G(a, -p_1, p_2)\right)\right) .$$

(ii) In the convex case

– *The value function $\mathbf{U}^-$ is associated to the flux-limiter*

$$A^- = \min_s (\max(H_1(s), H_2(s))) .$$

– *The value function $\mathbf{U}^+$ is associated to the flux-limiter*

$$A^+ = \min_s (\max(H_1^-(s), H_2^+(s))) .$$

The second part of this result shows that value-functions of control problems can be caracterized as a "flux-limiter solution" of (11.1)-(11.2) with the right flux-limiter conditions at $x = 0$. Contrarily to (CVS), we have a uniqueness result but, as the vanishing viscosity method shows it, stability becomes a problem since one has to identify the right flux-limiter for the limiting problem.

Remark 11.1.4 *The case of more general junctions' conditions at $x = 0$ like* (11.1)-(11.2)-(11.4) *can be treated by the Lions-Souganidis approach: in particular, we have a comparison result for* (11.1)-(11.2)-(11.4) *in the case of general Hamiltonians H_1, H_2 without assuming them to be quasi-convex. Of course, the monotonicity properties of G are necessary not only for having such a comparison result but even for the notion of "junction solution" to make sense.*

[5] defined as in the case of the Kirchhoff conditions

11.2 The Case of Quasi-Convex Hamiltonians

The aim of this section is to revisit the results of this part in the case of quasi-convex Hamiltonians and to answer the question: what is still true in this case up to a reformulation of the statements? Some results are already given in this part like Remark 7.4.2, Theorem 8.3.1 and Proposition 7.4.4: we gather and complement them here.

To do so, we consider Problem (6.1) in the case when Assumption ($\mathbf{H_{QC}}$) holds and the Hamiltonians $H_i^{\pm}$ and H_0 satisfy ($\mathbf{H_{BA-HJ}}$). We assume that these assumptions are satisfied throughout this section.

Of course, we are going to consider all possible notions of solutions: classical viscosity solutions, flux-limited solutions and junction solutions.

Theorem 11.2.1 *Under the above assumptions, we have*

(i) The notions of flux-limited and junction solutions are equivalent; they both satisfy a comparison principle[(6)].

(ii) Classical viscosity solutions are not unique in general but there exist a minimal solution $\mathbf{U}^-$ *and a maximal solution* $\mathbf{U}^+$ *which are flux-limited with the flux-limiters* $\max(H_T, H_0)$ *and* H_T^{reg} *respectively where* H_T *and* H_T^{reg} *are given by* (12.7) *and* (12.8).

(iii) A junction sub or supersolution with the generalized Kirchhoff condition

$$G(x,t,u_t,D_T u,-\frac{\partial u}{\partial x_N}(x',0+),\frac{\partial u}{\partial x_N}(x',0-))=0 \quad \text{on } \mathcal{H}\times(0,T)\ ,$$

wheref G *satisfies : there exists* $\alpha,\beta\geq 0$ *with* $\alpha+\beta>0$ *and for any* $x\in\mathcal{H}$, $p'\in\mathcal{H}$, $t\in(0,T)$, $a_1\geq a_2$, $b_1\geq b_2$, $c_1\geq c_2$

$$G(x,t,a_1,p',b_1,c_1)-G(x,t,a_2,p',b_2,c_2)\geq \alpha(a_1-a_2)+\beta(b_1-b_2)+\beta(c_1-c_2)\ ,$$

is also a flux-limited sub or supersolution with the flux-limiter $A(x,t,a,p')$ *is given by*

$$\min_{s_1,s_2}\left(\max\left(a+H_1^-(x,t,p'+s_1e_N),a+H_2^+(x,t,p'+s_2e_N),G(x,t,a,p',-s_1,s_2)\right)\right)\ ,$$

where a *stands for the* u_t*-derivative.*

[(6)] with different proofs.

(*iv*) *The vanishing viscosity method converges to* $\mathbf{U}^+$*, the maximal classical viscosity solution of Problem (6.1).*

Proof — We just give a brief sketch of the proof since most of the arguments are already provided in differents proofs we already gave. For example, the proof of (i) is nothing but the proof of Proposition 7.4.1 where just the properties of the $H_i^\pm$ play a role, while the proof of (iv) is the general one of Theorem 9.5.1. Concerning (iii), we already point out that the arguments of the proof of Proposition 7.4.4 remain valid without the convexity assumptions; in order to treat the more general G-case, the result of Lemma 12.3.1 is needed.

Therefore only the proof of (ii) requires some details. We first remark that Proposition 7.4.5 (which can be slightly generalized to provide the appropriate result) gives the answer for $\mathbf{U}^+$ and $H_T^{\rm reg}$: indeed, since Ishii and flux-limited subsolutions associated to the flux-limiter $H_T^{\rm reg}$ are the same, then the maximal Ishii subsolution is also a flux-limited solution associated to the flux-limiter $H_T^{\rm reg}$.

The subsolution properties for $\mathbf{U}^-$ follows along the same lines; but supersolutions properties are so easy to obtain and we are going to show how to prove that $\mathbf{U}^-$ is a flux-limited supersolution with the flux-limiter H_T. To do so, we consider $(\bar x,\bar t)\in \mathcal{H}\times(0,T)$ and we introduce the subdifferential relatively to each domain Ω_i: we say that (p_x,p_t) is in $D^-_{\Omega_i}\mathbf{U}^-(\bar x,\bar t)$ if and only if we have, for any $(x,t)\in\overline{\Omega_i}\times(0,T)$, close to $(\bar x,\bar t)$

$$\mathbf{U}^-(x,t)\geq \mathbf{U}^-(\bar x,\bar t)+p_t(t-\bar t)+p\cdot(x-\bar x)+o(|t-\bar t|+|x-\bar x|)\,.$$

Concerning p_x, we write it below as $(p_x',(p_x)_N)$ where $p_x'\in\mathbb{R}^{N-1}$ is the tangential component and $(p_x)_N$ corresponds to the normal (generalized) derivative. We recall classical results in the following lemma.

Lemma 11.2.2 (Sub-differentials on $\mathcal{H}$)
(i) For $i=1,2$, $(p_x,p_t)\in D^-_{\Omega_i}\mathbf{U}^-(\bar x,\bar t)$ if and only if there exists a C^1-function φ_i such that $(\bar x,\bar t)$ is a (strict) local minimum point of $\mathbf{U}^- -\varphi_i$ on $\overline{\Omega_i}\times(0,T)$ and with $D_x\varphi_i(\bar x,\bar t)=p_x$, $(\varphi_i)_t(\bar x,\bar t)=p_t$.
(ii) Conversely, if $(\bar x,\bar t)$ a minimum point of $\mathbf{U}^- -\varphi$ with $\varphi=(\varphi_1,\varphi_2)\in \mathrm{PC}^1(\mathbb{R}^N\times[0,T])$ then $(D_x\varphi_i(\bar x,\bar t),(\varphi_i)_t(\bar x,\bar t)\in D^-_{\Omega_i}\mathbf{U}^-(\bar x,\bar t)$ for $i=1,2$.
(iii) If $D^-_{\Omega_1}\mathbf{U}^-(\bar x,\bar t)$ [resp. $D^-_{\Omega_2}\mathbf{U}^-(\bar x,\bar t)$] is non-empty and if $(p^1_x,p^1_t)\in D^-_{\Omega_1}\mathbf{U}^-(\bar x,\bar t)$ [resp. $(p^2_x,p^2_t)\in D^-_{\Omega_2}\mathbf{U}^-(\bar x,\bar t)$], then there exists $(\overline{p}^1_x,\overline{p}^1_t)\in D^-_{\Omega_1}\mathbf{U}^-(\bar x,\bar t)$ such that $(\overline{p}^1_x)'=(p^1_x)'$, $\overline{p}^1_t=p^1_t$ and $(\overline{p}^1_x)_N\geq(p^1_x)_N$. In other words, $(((p^1_x)',p_N),p^1_t)\in D^-_{\Omega_1}\mathbf{U}^-(\bar x,\bar t)$ if and only if $p_N\leq(\overline{p}^1_x)_N$ [resp. there exists $(\overline{p}^2_x,\overline{p}^2_t)\in D^-_{\Omega_2}\mathbf{U}^-(\bar x,\bar t)$ such that $(\overline{p}^2_x)'=(p^2_x)'$, $\overline{p}^2_t=p^2_t$ and $(\overline{p}^2_x)_N\leq(p^2_x)_N$. In other words, $(((p^2_x)',p_N),p^2_t)\in D^-_{\Omega_2}\mathbf{U}^-(\bar x,\bar t)$ if and only if $p_N\geq(\overline{p}^2_x)_N$.]

From Lemma 11.2.2, it is therefore obvious that we can argue only on these subdifferential and the "tangential derivatives" are equal which means that we have to consider $(p_x^1, p_t^1) \in D^-_{\Omega_1}\mathbf{U}^-(\bar{x}, \bar{t})$ and $(p_x^2, p_t^2) \in D^-_{\Omega_2}\mathbf{U}^-(\bar{x}, \bar{t})$ only when $p_t^1 = p_t^2$ and p_x^1, p_x^2 differ only by their normal component to $\mathcal{H}$ since this is the case for test-functions.

For these reasons, we are going to argue as if we were in dimension 1, dropping the tangential part of p_x^1, p_x^2 and to simplify matter we will also drop the dependence in all the variables in the Hamiltonians except on $p_t = p_t^1 = p_t^2$ and p_x^1, p_x^2 which now correspond to the normal derivatives only.

We have to show that, for any $(p_x^1, p_t) \in D^-_{\Omega_1}\mathbf{U}^-(\bar{x}, \bar{t})$ and $(p_x^2, p_t) \in D^-_{\Omega_2}\mathbf{U}^-(\bar{x}, \bar{t})$, we have

$$\max(p_t + H_1^+(p_x^1), p_t + H_2^-(p_x^2), p_t + \max(H_0, H_T)) \geq 0 \ .$$

We argue by contradiction assuming that there exists such $(p_x^1, p_t), (p_x^2, p_t)$ for which

$$\max(p_t + H_1^+(p_x^1), p_t + H_2^-(p_x^2), p_t + \max(H_0, H_T)) < 0 \ .$$

This implies in particular that $p_t + H_T < 0$.

Next we use Lemma 11.2.2-(i) and we claim that

$$p_t + H_1(\overline{p}_x^1) \geq 0 \text{ and } p_t + H_2(\overline{p}_x^2) \geq 0 \ .$$

Indeed, for example for $\overline{p}_x^1$, by Lemma 11.2.2-(i), there exists φ_1 such that $(\bar{x}, \bar{t})$ is a strict local minimum point of $\mathbf{U}^- - \varphi_1$ and $D_x\varphi_1(\bar{x}, \bar{t}) = \overline{p}_x^1$, $(\varphi_1)_t(\bar{x}, \bar{t}) = p_t$, but, for any $\varepsilon > 0$, this is not the case for the function $\mathbf{U}^-(x,t) - \varphi_1(x,t) - \varepsilon x_N$ since $(p_x^1 + \varepsilon, p_t) \notin D^-_{\Omega_1}\mathbf{U}^-(\bar{x}, \bar{t})$. Therefore this function has a minimum point in Ω_1 which converges as $\varepsilon \to 0$ to $(\bar{x}, \bar{t})$. An analogous argument shows the result for $\overline{p}_x^2$.

From all the above informations we deduce that, H_1^+ being decreasing and $\overline{p}_x^1 \geq p_x^1$, we necessarely have $p_t + H_1^-(\overline{p}_x^1) \geq 0$ and for any $p \geq \overline{p}_x^1$, $p_t + H_1^-(p) \geq 0$. In particular, $\max(H_1(p), H_2(p)) \geq H_1(p) \geq H_1^-(p) \geq -p_t > H_T$. In the same way, for any $q \leq \overline{p}_x^2$, $\max(H_1(q), H_2(q)) \geq H_2(q) \geq H_1^-(q) \geq -p_t > H_T$. And if r is a minimum point of the function $\max(H_1, H_2)$ we have $\overline{p}_x^2 < r < \overline{p}_x^1$.

From this inequality, we see that $(r, p_t) \in D^-_{\Omega_1}\mathbf{U}^-(\bar{x}, \bar{t})$ and $(r, p_t) \in D^-_{\Omega_2}\mathbf{U}^-(\bar{x}, \bar{t})$. Hence (r, p_t) is in the $\mathbb{R}^N$ subdifferential of $\mathbf{U}^-$ at $(\bar{x}, \bar{t})$ and the (Ishii) viscosity inequality yields

$$\max(p_t + H_1(r), p_t + H_2(r), p_t + H_0) \geq 0 \ ,$$

but $H_1(r) = H_2(r) = H_T$ and this inequality is a contradiction.

Q.E.D.

11.3 Summary, Comments and Questions

As the title indicates, the aim of this chapter is to summarize and comment the results we have provided in the co-dimension 1 case.

11.3.1 On the different notions of solution

Let us examine the three approaches we have described.

The first one, using *Ishii's notion of viscosity solutions*, has the advantage to be very stable and universal in the sense that it can be formulated for any type of Hamiltonians, convex or not. But Chapter 6 shows that it has poor uniqueness properties in the present situation. In the simple case of the optimal control framework we have considered, with a discontinuity on an hyperplane $\mathcal{H}$ and a specific control on $\mathcal{H}$, we are able to identify the minimal solution ($\mathbf{U}^-$) and the maximal solution ($\mathbf{U}^+$): if $\mathbf{U}^-$ is a natural value-function providing the minimal cost over all possible controls, $\mathbf{U}^+$ completely ignores some controls and in particular all the specific control on $\mathcal{H}$.

Why can $\mathbf{U}^+$ be an Ishii viscosity solution of the Bellman Equations anyway? The answer is that the Ishii subsolution condition on $\mathcal{H}$ is not strong enough in order to force the subsolutions to see all the particularities of the control problem on $\mathcal{H}$. This generates unwanted (or not?) subsolutions. We point out that, as all the proofs of Chapter 6 show, there is a complete disymmetry between the sub and supersolutions properties in this control setting: this fact is natural and well-known due to the form of the problem but it is accentuated in the discontinuous framework.

This lack of uniqueness properties for Ishii's viscosity solutions leads to consider different notions of solutions but, in some interesting applications, one may recover this uniqueness since $\mathbf{U}^- = \mathbf{U}^+$. We point out Lemma 12.2.2 below which provides a condition under which $H_T = H_T^{\mathrm{reg}}$ and therefore $\mathbf{U}^- = \mathbf{U}^+$. This condition is formulated directly on the Hamiltonians and can sometimes be easy to check (see for example, Section 16.3).

In the *Network Approach*, one can either use the notion of *flux-limited solutions* or the notion of *junction viscosity solutions*. The first one is particularly well-adapted to control problems and has the great advantage to reinforce the subsolutions conditions on $\mathcal{H}$ and, through the flux-limiter, to allow to consider various control problems at the same time by just varying this flux-limiter. The value-functions $\mathbf{U}^-$ and $\mathbf{U}^+$ are reinterpreted in this framework as value-functions associated to particular flux-limiters.

But we are very far from the universality of the definition of viscosity solutions since this "max-max" definition in the case of convex Hamiltonians has to be replaced by a "min-min" one in the case of concave ones, and it has no analogue for general ones. On the other hand, this notion of solution is less flexible in terms of stability properties compared to Ishii solutions.

The notion of *junction viscosity solution* tries to recover all the good properties of Ishii solutions for general Hamiltonians: it is valid for any kind of "viscosity solutions compatible" junction conditions, it is stable and the Lions-Souganidis proof (even if there are some limitations in Theorem 9.2.1) is the only one which is valid for general Hamiltonians with Kirchhoff's boundary conditions. Though this approach is not as well-adapted to control problems as the flux-limiter one, it gives however a common formulation for for problems when the controller wants to minimize some cost (which leads to convex Hamiltonians) or maximize it (which leads to concave Hamiltonians).

The Kirchhoff boundary condition is one of the most natural "junction condition" in the networks theory but a priori, it has no connection with control problems. However, as it is shown by Proposition 7.4.4 together with Theorem 8.2.3, this boundary condition is associated $\mathbf{U}^+$. The explanation is maybe in the next paragraph.

In fact, the main interest of the approach by junction solution, using the Lions-Souganidis comparison result, is to provide the convergence of the vanishing viscosity method in the most general framework, without using some convexity or quasi-convexity assumption on the Hamiltonians. In the convex setting, we have several proofs of the convergence to $\mathbf{U}^+$ which shows that it is the most stable value-function if we add a stochastic noise on the dynamic.

In the next parts, we examine stratified solutions in $\mathbb{R}^N$ or in general domains, *i.e.* essentially the generalization of $\mathbf{U}^-$ which we aim at caracterizing as the unique solution of a suitable problem with the right viscosity inequalities. And we will emphasize the (even more important) roles of the subsolution inequalities, normal controlability, tangential continuity...etc. But we will not consider questions related to $\mathbf{U}^+$ and the vanishing viscosity method, even if some of these questions are really puzzling.

11.3.2 Towards more general discontinuities: a bunch of open problems.

A very basic and minimal summary of Part II can be expressed as follows: for Problem (6.1), we are able to provide an explicit control formula for the minimal supersolution (and solution) $\mathbf{U}^-$, and also an explicit control formula for the maximal (and solution) $\mathbf{U}^+$; moreover, $\mathbf{U}^+$ is the limit of the vanishing viscosity method.

A natural question is: is it possible to extend such results to more general type of discontinuities?

In the case of $\mathbf{U}^-$, which may be perhaps considered as being the more natural solution from the control point of view, the answer is yes and this is not so surprising since, by Corollary 4.2.8, we know in a very general framework that $\mathbf{U}^-$ is the minimal viscosity supersolution of the Bellman Equations, therefore we already have a lot of informations on $\mathbf{U}^-$.

In the next parts, we provide a rather complete study of stratified solutions in $\mathbb{R}^N$ and then in general domains, which are the natural generalization of $\mathbf{U}^-$ in the case when the codimension-1 discontinuity is replaced by discontinuities on Whitney's stratifications. As in Section 6.1, we characterize the stratified solution $\mathbf{U}^-$ as the unique solution of a suitable problem with suitable viscosity inequalities. The methods which are used to study Ishii solutions, relying partly on control arguments and partly on pde ones, can be extended to this more general setting and we will emphasize the (even more important) roles of the subsolution inequalities, normal controlability, tangential continuity...etc.

But the case of the maximal subsolution (and solution) $\mathbf{U}^+$ is more tricky and several questions can be asked, in particular

(*i*) Can one provide an explicit control formula for $\mathbf{U}^+$?

(*ii*) Is it still true that the vanishing viscosity method converges to $\mathbf{U}^+$?

Before describing the difficulties which appear even for rather simple configurations and in order to be more specific and to fix ideas, we consider two interesting examples: the first one is the case when we still have two domains but the interface is not smooth, typically Figure 11.1 below.

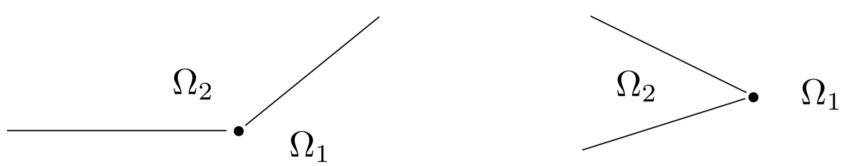


Figure 11.1: Two domains with a non-smooth interface

A second very puzzling example is the "cross-case" where $\mathbb{R}^2$ is decomposed into its four main quadrants, see Figure 11.2 below. And of course, one may also have in mind "triple-junction configurations" in between these two cases.

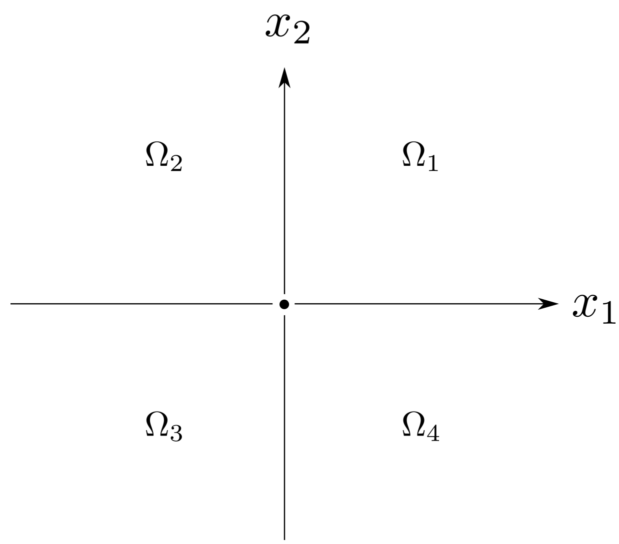


Figure 11.2: The cross-case

The importance of the above questions is due to the numerous applications and we can mention for example front propagations phenomenas or Large Deviations type results: in both case, one has to identify the limit of the vanishing viscosity method and an "action functional" which exactly means to answer the above questions if the diffusions and/or drift involved in these problems are discontinuous.

We refer for example to Souganidis [119] and references therein for the viscosity solutions' approach of front propagations in reactions diffusion equations (like KPP (Kolmogorov-Petrovskii-Piskunov) type equations) and to Bouin [33] and references therein for front propagation in kinetic equations. For the viscosity solutions' approach of Large Deviations problems, we refer to [18] (see also [23]).

Now we turn to the above questions which are largely open even in the two simple cases described above. We first remark that most of the results of this part, in particular those obtained by pde methods, use in a crucial way the codimension-1 feature of the problem, via the normal direction which determines which are the inward and outward dynamics to the Ω_i's but also the $H_i^{\pm}$, and therefore the key H_T^{reg} Hamiltonian.

concerning Question (i), in terms of control, the additional difficulty is to identify the "regular strategies" which allow to stay at the new discontinuity point (0 in the cross-case) and then to show that using only these "regular strategies", $\mathbf{U}^+$ is an Ishii solution of the problem. For Question (ii), the proofs which are given above use either $\mathbf{U}^+$ (and therefore require an answer to Question (i)) or the codimension-1 feature of the problem via the Kirchhoff condition.

For all these reasons, even in the very simple configurations we propose above, we DO NOT know the right answer... but we hope that some readers will be able to find it!

In order to show the difficulty, we provide a "simple" result in the cross-case in $\mathbb{R}^2$, which DOES NOT give the result we wish but which uses the natural ingredients which should be useful to get it.

We are going to consider the problem

$$u_t + H_i(Du) = 0 \quad \text{in } \Omega_i \times (0,T)\ , \text{ for } i = 1,2,3,4,$$

where the Hamiltonian H_i are given by

$$H_i(p) = \sup_{\alpha_i \in A_i} \{-b_i(\alpha_i)\cdot p - l_i(\alpha_i)\}\ .$$

where A_i are compact metric spaces. We are in a very simplified framework since we do not intend to provide general results, so we also assume that the Hamiltonians H_i are coercive, and even that there exists $\delta > 0$ such that

$$B(0,\delta) \subset \{b_i(\alpha_i);\quad \alpha_i \in A_i\} \quad \text{for any } i = 1,2,3,4\ .$$

This is natural as a normal controllability assumption.

Of course, these equations in each Ω_i have to be complemented by the Ishii conditions on the two axes: except for $x = 0$, we are in the framework described in this part since we face a co-dimension 1 discontinuity. Therefore we concentrate on the case $x = 0$ where, in order to identify $\mathbf{U}^+$, we have to identify the "H_T^{reg}", i.e. the "regular strategies" which allow to remain at $x = 0$.

In order to do so, we introduce the set $\mathcal{A}$ of controls $(\alpha_1,\alpha_2,\alpha_3,\alpha_4)$ such that, on one hand, $b_i(\alpha_i) \in D_i$ for $i = 1,2,3,4$ where

$$D_i = \{b_i(\alpha_i);\ b_i(\alpha_i)\cdot x \leq 0 \text{ for all } x \in \Omega_i\}\ ,$$

and, on the other hand, there exists a convex combination of the $b_i(\alpha_i)$ such that $\sum_{i=0}^{4} \mu_i b_i(\alpha_i) = 0$. Such a convex combination may not be unique and we denote by Δ the set of all such convex combinations.

Finally we set

$$H_T^{\mathrm{reg-cross}} := \sup_{\mathcal{A}} \Big\{ \inf_{\Delta} \Big(-\sum_{i=0}^{4} \mu_i l_i(\alpha_i) \Big) \Big\}\ .$$

Notice that here, since we consider a zero-dimensional set, the Hamiltonian $H_T^{\mathrm{reg-cross}}$ reduces to a real number. We have the

Lemma 11.3.1 *If $u : \mathbb{R}^2 \times (0,T) \to \mathbb{R}$ is an Ishii subsolution of the above problem then*

$$u_t + H_T^{\mathrm{reg-cross}} \leq 0 \quad \textit{on } \{0\} \times (0,T)\ .$$

Proof — Let ϕ be a C^1 function on $(0,T)$ and $\bar{t}$ be a strict local maximum point of $u(0,t) - \phi(t)$. We have to show that $\phi_t(\bar{t}) + H_T^{reg-cross} \leq 0$.

To do so, we consider $(\alpha_i)_i \in \mathcal{A}$ and, for $\delta > 0$ small, we consider the affine functions

$$\psi_i(p) = \phi_t(\bar{t}) - b_i(\alpha_i) \cdot p - l_i(\alpha_i) - \delta\ .$$

Applying Farkas' Lemma, there are two possibilities; the first one is: there exists $\bar{p}$ such that $\psi_i(\bar{p}) \geq 0$ for all i. In that case, we consider the function $(x,t) \mapsto u(x,t) - \psi(t) - \bar{p} \cdot x - \frac{|x|^2}{\varepsilon}$ for $0 < \varepsilon \ll 1$.

Since $\bar{t}$ is a strict local maximum point of $u(0,t) - \phi(t)$, this function has a local maximum point at $(x_\varepsilon, t_\varepsilon)$ and $(x_\varepsilon, t_\varepsilon) \to (0, \bar{t})$ as $\varepsilon \to 0$. Wherever the point x_ε is, we have an inequality of the type

$$\phi_t(t_\varepsilon) + H_i(\bar{p} + \frac{2x}{\varepsilon}) \leq 0\ .$$

But if such H_i inequality holds, this means that we are on $\overline{\Omega_i}$ and in particular

$$\phi_t(t_\varepsilon) - b_i(\alpha_i) \cdot (\bar{p} + \frac{2x}{\varepsilon}) - l_i(\alpha_i) \leq 0\ .$$

Recalling that $b_i(\alpha_i) \in D_i$, this implies

$$\phi_t(t_\varepsilon) - b_i(\alpha_i) \cdot \bar{p} - l_i(\alpha_i) \leq 0\ .$$

For ε small enough, this inequality is a contradiction with $\psi_i(\bar{p}) \geq 0$ and therefore this first case cannot hold.

Therefore, we are always in the second case: there exists a convex combination of the ψ_i, namely $\sum_{i=0}^{4} \mu_i \psi_i$ which gives a negative number. In that case, it is clear that we have

$$\sum_{i=0}^{4} \mu_i b_i(\alpha_i) = 0 \quad \text{and} \quad \phi_t(\bar{t}) - \sum_{i=0}^{4} \mu_i l_i(\alpha_i) - \delta \leq 0\ .$$

This implies that

$$\phi_t(\bar{t}) + \inf_{\Delta} \Big(- \sum_{i=0}^{4} \mu_i l_i(\alpha_i) \Big) - \delta \leq 0\ ,$$

and since this is true for any $(\alpha_i)_i \in \mathcal{A}$ and for any $\delta > 0$, we have the result.

Q.E.D.

The interest of this proof is to show the two kinds of arguments which seem useful to obtain an inequality for the subsolutions at 0: (i) to find the suitable set Δ of "regular strategies" which allow to stay fixed at 0; (ii) to have suitable properties on the b_i's which allow to deal with the $2x/\varepsilon$-term in the Hamiltonians, in other words we have to define suitable "outgoing strategies".

Again this result is not satisfactory and we do not think that it leads to the desired result in the cross case.

Chapter 12

Appendix to Part II

12.1 On equivalent definitions H_T and $H_T^{\rm reg}$

Let us first recall some definitions:

$$H_1(x,t,r,p) := \sup_{\alpha_1 \in A_1} \{-b_1(x,t,\alpha_1)\cdot p + c_1(x,t,\alpha_1)r - l_1(x,t,\alpha_1)\}\ , \tag{12.1}$$

$$H_2(x,t,r,p) := \sup_{\alpha_2 \in A_2} \{-b_2(x,t,\alpha_2)\cdot p + c_2(x,t,\alpha_2)r - l_2(x,t,\alpha_2)\}\ , \tag{12.2}$$

Of course, here H_i is defined for $x \in \bar\Omega_i$, $t \in (0,T)$, $r \in \mathbb{R}$ and $p \in \mathbb{R}^N$. We also introduced in Section 7.1 the Hamiltonians H_1^- and H_2^+ where the sup are taken over the (b,c,l) such that $b_1(x,t,\alpha_1)\cdot e_N \le 0$, and $b_2(x,t,\alpha_2)\cdot e_N \ge 0$ respectively.

We recall also that we defined H_T and $H_T^{\rm reg}$ in Section 6.1.3, using the subsets $\mathbf{BCL}_T(x,t), \mathbf{BCL}_T^{\rm reg}(x,t)$: for $x \in \mathcal{H}$, $t \in (0,T)$, $r \in \mathbb{R}$, $p \in \mathbb{R}^N$

$$H_T(x,t,r,p) := \sup_{(b,c,l)\in \mathbf{BCL}_T(x,t)} \big\{ -b\cdot p + cu - l \big\}\,, \tag{12.3}$$

The second Hamiltonian is defined similarly but by considering only *regular* tangential dynamics b:

$$H_T^{\rm reg}(x,t,r,p) := \sup_{\mathbf{BCL}_T^{\rm reg}(x,t)} \big\{ -b\cdot p + cu - l \big\}\,. \tag{12.4}$$

The first result of this appendix is to connect two equivalent definition of those tangential Hamiltonians. To do so, let us introduce for $x \in \mathcal{H}$, $t \in (0,T)$, $r \in \mathbb{R}$, $p \in \mathbb{R}^N$, we set

$$\tilde H(x,t,r,p) := \max(H_1(x,t,r,p), H_2(x,t,r,p)) \tag{12.5}$$

$$\tilde H^{\rm reg}(x,t,r,p) := \max(H_1^-(x,t,r,p), H_2^+(x,t,r,p))\ . \tag{12.6}$$

Lemma 12.1.1 *If H_1, H_2 are given by (12.1)-(12.2) and $H_T, H_T^{\rm reg}$ by (12.3)-(12.4), we have*

$$H_T(x,t,r,p') = \min_{s\in\mathbb{R}} \tilde{H}(x,t,r,p'+se_N)\,, \tag{12.7}$$

$$H_T^{\rm reg}(x,t,r,p') = \min_{s\in\mathbb{R}} \tilde{H}^{\rm reg}(x,t,r,p'+se_N)\,. \tag{12.8}$$

Proof — We only provide the full proof in the case of H_T, the one for $H_T^{\rm reg}$ follows from the same arguments, just changing the sets of (b_1,c_1,l_1), (b_2,c_2,l_2) we consider.

We introduce the function $\varphi:\mathbb{R}\to\mathbb{R}$ defined by

$$\varphi(s) := \max(H_1(x,t,r,p'+se_N), H_2(x,t,r,p'+se_N))\ .$$

This function is convex, continuous and coercive since both H_1, H_2 have these properties and therefore there exists $\bar{s}\in\mathbb{R}$ such that $\varphi(\bar{s}) = \min_{s\in\mathbb{R}} \varphi(s)$. As a consequence, $0\in\partial\varphi(\bar{s})$, the convex subdifferential of φ.

We apply a classical result on the subdifferentials of convex functions defined as suprema of convex (or C^1) functions (cf [111]): here

$$\varphi(s) = \sup\left\{-b_1\cdot(p'+se_N)+c_1r-l_1; -b_2\cdot(p'+se_N)+c_2r-l_2\right\}\ ,$$

where the supremum is taken over all $(b_1,c_1,l_1)\in \mathbf{BCL}_1(x,t)$ and $(b_2,c_2,l_2)\in \mathbf{BCL}_2(x,t)$.

The functions $s\mapsto -b_i\cdot(p'+se_N)+c_ir-l_i$ for $i=1,2$ and $(b_i,c_i,l_i)\in\mathbf{BCL}_i(x,t)$ are all C^1 and $\partial\varphi(\bar{s})$ is the convex hull of their gradients for all the (b_i,c_i,l_i) such that $\varphi(\bar{s}) = -b_i\cdot(p'+se_N)+c_ir-l_i$. Since $\mathbf{BCL}_1(x,t), \mathbf{BCL}_2(x,t)$ are convex, this means that one of the following cases holds:

(a) either the above supremum is only achieved at a unique (b_i,c_i,l_i) but then φ is differentiable at $\bar{s}$ and $0=\partial\varphi(\bar{s}) = -b_i\cdot e_N$;

(b) or there exists $(b_1,c_1,l_1)\in\mathbf{BCL}_1(x,t)$, $(b_2,c_2,l_2)\in\mathbf{BCL}_2(x,t)$ and $\mu\in[0,1]$ such that

$$\begin{cases}\varphi(\bar{s}) = -b_1\cdot(p'+\bar{s}e_N)+c_1r-l_1 = -b_2\cdot(p'+\bar{s}e_N)+c_2r-l_2\\ 0=\mu(-b_1\cdot e_N)+(1-\mu)(-b_2\cdot e_N) \quad \text{i.e. } (\mu b_1+(1-\mu)b_2)\cdot e_N = 0\ .\end{cases}$$

In case (b), we deduce that

$$\begin{aligned}\varphi(\bar{s}) &= \mu(-b_1\cdot(p'+\bar{s}e_N)+c_1r-l_1)+(1-\mu)(-b_2\cdot(p'+\bar{s}e_N)+c_2r-l_2)\\ &= -(\mu b_1+(1-\mu)b_2)\cdot p' + (\mu c_1+(1-\mu)c_2)r - (\mu l_1+(1-\mu)l_2)\\ &\leq \quad H_T(x,t,r,p')\ .\end{aligned}$$

But on the other hand, for any $(\tilde{b}_1, \tilde{c}_1, \tilde{l}_1) \in \mathbf{BCL}_1(x,t)$, $(\tilde{b}_2, \tilde{c}_2, \tilde{l}_2) \in \mathbf{BCL}_2(x,t)$ such that $(\tilde{\mu}\tilde{b}_1 + (1-\tilde{\mu})\tilde{b}_2)\cdot e_N = 0$ for some $\tilde{\mu} \in [0,1]$, the definition of φ implies that

$$\begin{aligned}\varphi(\bar{s}) &\geq \tilde{\mu}(-\tilde{b}_1\cdot(p'+\bar{s}e_N) + \tilde{c}_1 r - \tilde{l}_1) + (1-\tilde{\mu})(-\tilde{b}_2\cdot(p'+\bar{s}e_N) + \tilde{c}_2 r - \tilde{l}_2)\\ &= -(\tilde{\mu}\tilde{b}_1 + (1-\tilde{\mu})\tilde{b}_2)\cdot p' + (\mu\tilde{c}_1 + (1-\mu)\tilde{c}_2)r - (\mu\tilde{l}_1 + (1-\mu)\tilde{l}_2),\end{aligned}$$

which, taking the supremum on all such $(\tilde{b}_1, \tilde{c}_1, \tilde{l}_1)$, $(\tilde{b}_2, \tilde{c}_2, \tilde{l}_2)$ and $\tilde{\mu}$, gives $\varphi(\bar{s}) \geq H_T(x,t,r,p')$. So, equality holds, which gives the result.

Dealing with case (a) follows from the same arguments as in case (b), with $\mu = 0$ or 1. So the Lemma is proved.

Q.E.D.

12.2 Properties in the quasi-convex setting

We now turn to a more general setting: we do not assume here that $H_i(x,t,r,p)$, $i = 1,2$ are control-type Hamiltonians given by (12.1) and (12.2), only that they satisfy some quasi-convex property. Hypothesis $(\mathbf{H_{QC}})$ was introduced in Section 8.3 and we formulate it here with the additional variable r since it works exactly the same:

$(\mathbf{H_{QC}})$ For $i = 1,2$, $H_i = \max(H_i^+, H_i^-)$ where

- (i) H_i^+, H_i^- are bounded from below, Lipschitz continuous functions such that, for any x in a neighborhood of $\mathcal{H}$, $t \in [0,T]$, $r \in \mathbb{R}$ and $p \in \mathbb{R}^N$
- (ii) $\lambda \mapsto H_1^+(x,t,r,p+\lambda e_N)$ is decreasing, $\lambda \mapsto H_1^-(x,t,r,p+\lambda e_N)$ is increasing and tends to $+\infty$ as $\lambda \to +\infty$, locally uniformly w.r.t. x, t, r and p, and
- (iii) $\lambda \mapsto H_2^+(x,t,r,p+\lambda e_N)$ is increasing, $\lambda \mapsto H_2^-(x,t,r,p+\lambda e_N)$ is decreasing and tends to $+\infty$ as $\lambda \to -\infty$, locally uniformly w.r.t. x, t, r and p.

So, in this section, when speaking about H_T and H_T^{reg}, we understand them as defined by (12.5) and (12.6).

Lemma 12.2.1 *If H_1, H_2 satisfy $(\mathbf{H_{QC}})$ and if H_T^{reg} is defined by (12.8), then there exists $m_1 = m_1(x,t,r,p')$ and $m_2 = m_2(x,t,r,p')$ such that $m_1 \leq m_2$ and*

$$\tilde{H}^{\mathrm{reg}}(x,t,r,p'+se_N) = \begin{cases} H_2^+(x,t,r,p'+se_N) > H_1^-(x,t,r,p'+se_N) & \text{if } s \leq m_1\\ H_1^-(x,t,r,p'+se_N) = H_2^+(x,t,r,p'+se_N) & \text{if } m_1 \leq s \leq m_2\\ H_1^-(x,t,r,p'+se_N) > H_2^+(x,t,r,p'+se_N) & \text{if } s \geq m_2\end{cases}$$

In particular, $H_1^-(x,t,r,p'+se_N) = H_2^+(x,t,r,p'+se_N) = H_T^{\rm reg}(x,t,r,p')$ *if* $m_1 \leq s \leq m_2$ *and if* $\lambda > H_T^{\rm reg}(x,t,r,p')$*, the equation* $\tilde H^{\rm reg}(x,t,r,p'+se_N) = \lambda$ *has exactly two solutions* $s_1 < m_1 \leq m_2 < s_2$ *with*

$$H_2^+(x,t,r,p'+s_1e_N) = \lambda \quad \textit{and} \quad H_1^-(x,t,r,p'+s_2e_N) = \lambda\ .$$

Proof — We introduce the function $\varphi : \mathbb{R} \to \mathbb{R}$ defined by

$$\varphi(s) := H_1^-(x,t,r,p'+se_N) - H_2^+(x,t,r,p'+se_N)\ .$$

Because of $(\mathbf{H_{QC}})$, the function φ is increasing and $\varphi(s) \to +\infty$ as $s \to +\infty$ while $\varphi(s) \to -\infty$ as $s \to -\infty$. Therefore, there exists $m_1 \leq m_2$ such that $\varphi(s) < 0$ if $s < m_1$, $\varphi(s) = 0$ if $m_1 \leq s \leq m_1$ and $\varphi(s) > 0$ if $s > m_2$. This argument together with the monotonicity properties of H_1^- and H_2^+ immediately gives the result.

Q.E.D.

Lemma 12.2.2 *We denote by* $m_1^+(x,t,r,p')$ *the largest minimum point of the function* $s \mapsto H_1(x,t,r,p'+se_N)$ *and* $m_2^-(x,t,r,p')$ *the least minimum of the function* $s \mapsto H_2(x,t,r,p'+se_N)$*. If* $m_2^-(x,t,r,p') \geq m_1^+(x,t,r,p')$ *for any* (x,t,r,p') *then* $H_T = H_T^{\rm reg}$ *on* $\mathcal{H} \times [0,T] \times \mathbb{R}^{N-1}$*.*

We point out the importance of this lemma: indeed, in the case when we have to deal only with H_1, H_2 and there is no Hamiltonian G on $\mathcal{H}$ (the case of $\mathbf{U}^{\rm FL}$), then this result gives a very easy sufficient condition to check in order to have $\mathbf{U}^- = \mathbf{U}^+$, i.e. the uniqueness of the Ishii solution. We recall that in the quasi-convex setting, H_T and $H_T^{\rm reg}$ are defined through (12.5) and (12.6).

Proof — We first remark that $H_T \geq H_T^{\rm reg}$ since, by definition, $\tilde H \geq \tilde H^{\rm reg}$.

To prove the converse inequality, we first remark that, by definition of H_1^-, H_2^+, $H_1^-(x,t,r,p'+se_N)$ is given by

$$\begin{cases} H_1^-(x,t,r,p'+m_1^+(x,t,r,p')e_N) = \min_s \big(H_1^-(x,t,r,p'+se_N)\big) & \text{if } s \leq m_1^+(x,t,r,p') \\ H_1(x,t,r,p'+se_N) > \min_s \big(H_1^-(x,t,r,p'+se_N)\big) & \text{if } s > m_1^+(x,t,r,p') \end{cases} \tag{12.9}$$

while $H_2^+(x,t,r,p'+se_N)$ is given by

$$\begin{cases} H_2^+(x,t,r,p'+m_1^+(x,t,r,p')e_N) = \min_s \big(H_2^+(x,t,r,p'+se_N)\big) & \text{if } s \geq m_2^-(x,t,r,p') \\ H_2(x,t,r,p'+se_N) > \min_s \big(H_2^+(x,t,r,p'+se_N)\big) & \text{if } s < m_2^-(x,t,r,p') \end{cases} \tag{12.10}$$

On the other hand, Lemma 12.2.1 implies that $H_T^{\rm reg}(x,t,r,p') = H_1^-(x,t,r,p'+se_N) = H_2^+(x,t,r,p'+se_N)$ if $m_1 \leq s \leq m_2$; indeed, we recall that $H_1^-(x,t,r,p'+se_N)$ is increasing in s while $H_2^+(x,t,r,p'+se_N)$ is decreasing in s.

For such a real s, if $s \leq m_1^+(x,t,r,p')$, by (12.9), then

$$H_1^-(x,t,r,p'+se_N) = H_1^-(x,t,r,p'+m_1^+(x,t,r,p')e_N) = \min_s(H_1(x,t,r,p'+se_N)) \; .$$

On the contrary, by (12.10), we have

$$H_2^+(x,t,r,p'+se_N) = H_2(x,t,r,p'+se_N) \; ,$$

because $m_2^-(x,t,r,p') \geq m_1^+(x,t,r,p')$. Now we look at the situation at the point $m_1^+(x,t,r,p')e_N$): using that $m_1^+(x,t,r,p') \leq m_2^-(x,t,r,p')$, we have

$$\begin{aligned} H_2(x,t,r,p'+m_1^+(x,t,r,p')e_N) &= H_2^+(x,t,r,p'+m_1^+(x,t,r,p')e_N) \; , \\ &\leq H_2^+(x,t,r,p'+se_N) \text{ since } H_2^+ \text{ is decreasing} \\ &= H_1^-(x,t,r,p'+m_1^+(x,t,r,p')e_N) \\ &= H_1(x,t,r,p'+m_1^+(x,t,r,p')e_N) \; . \end{aligned}$$

From there, we first deduce that $\tilde{H}(x,t,r,p'+m_1^+(x,t,r,p')e_N) = H_1(x,t,r,p'+m_1^+(x,t,r,p')e_N)$ and since we know that $H_1^-(x,t,r,p'+m_1^+(x,t,r,p')e_N) = H_T^{\rm reg}(x,t,r,p')$, we finally deduce that $H_T(x,t,r,p') \leq H_T^{\rm reg}(x,t,r,p')$ by the definition of H_T, which the desired inequality.

The proof is exactly the same if $s \geq m_2^-(x,t,r,p')$, exchanging the role of H_1^-, H_2^+.

It remains to study the case when $m_1^+(x,t,r,p') \leq s \leq m_2^-(x,t,r,p')$. But, in this case, this double inequality on s implies

$$H_1^-(x,t,r,p'+se_N) = H_1(x,t,r,p'+se_N) \quad \text{and} \quad H_2^+(x,t,r,p'+se_N) = H_2(x,t,r,p'+se_N) \; ,$$

and therefore $H_T^{\rm reg}(x,t,r,p') = \tilde{H}(x,t,r,p'+se_N)$ which yields the conclusion.

Q.E.D.

12.3 Treating more general Kirchhoff type conditions

In this section we state a general lemma which is used to deal with more general Kirchoff conditions on the interface.

Lemma 12.3.1 *Assume that f,g* $: \mathbb{R} \to \mathbb{R}$ *and* $h : \mathbb{R}^2 \to \mathbb{R}$ *are contunous functions such that*

(i) f *is an increasing function with* $f(t) \to +\infty$ *as* $t \to +\infty$*,*

(ii) g *is a decreasing function with* $g(t) \to +\infty$ *as* $t \to -\infty$*,*

(iii) there exists $\alpha > 0$ *such that, for any* $t_2 \le t_1$ *and* $s_2 \le s_1$*, we have*

$$h(t_2, s_2) - h(t_1, s_1) \le \alpha(t_2 - t_1) + \alpha(s_2 - s_1)\ .$$

If $\psi : \mathbb{R}^2 \to \mathbb{R}$ *is the function defined by*

$$\psi(t, s) := \max(f(t), g(s), h(s, t))\ ,$$

then ψ *is a coercive continuous function in* $\mathbb{R}^2$*, there exists* $(\bar{t}, \bar{s})$ *such that*

$$\psi(\bar{t}, \bar{s}) = \min_{t,s} (\psi(t, s))\ , \tag{12.11}$$

and we have

$$f(\bar{t}) = g(\bar{s}) = h(\bar{t}, \bar{s})\ . \tag{12.12}$$

Moreover if a point $(\tilde{t}, \tilde{s}) \in \mathbb{R}^2$ *satisfies* (12.12) *then* $(\tilde{t}, \tilde{s})$ *is a minimum point of* ψ*.*

Proof — Using the three properties we impose on f, g, h, it is easy to prove that ψ is actually continuous and coercive: we point out that the assumption on h implies that $h(t, s)$ is a strictly decreasing function of t and a strictly increasing function of s with $h(t, s) \to +\infty$ if $t \to -\infty$, s remaining bounded or if $s \to +\infty$, t remaining bounded. Therefore such a minimum point $(\bar{t}, \bar{s})$ exists.

We have to show that (12.12) holds and to do so, we may assume without loss of generality that f is strictly increasing and g is strictly decreasing. Indeed, this is done by replacing $f(t)$ by $f(t) + \varepsilon t$ and $g(t)$ by $g(s) - \varepsilon s$ and remarking that the minimum points remain in a fixed compact subset of $\mathbb{R}^2$.

If $m = \min_{t,s} (\psi(t, s))$, we first notice that $h(\bar{t}, \bar{s}) = m$. Otherwise $h(\bar{t}, \bar{s}) < m$ and it is clear enough that, for $\delta > 0$ small enough, then

$$\psi(\bar{t} - \delta, \bar{s} + \delta) < \psi(\bar{t}, \bar{s})\ ,$$

a contradiction.

In the same way, if $f(\bar{t}) < m$, using the properties of h, there exists $\delta, \delta' > 0$ small enough such that $h(\bar{t} + \delta, \bar{s} + \delta') < m$, $g(\bar{s} + \delta') < m$ and $\psi(\bar{t} + \delta, \bar{s} + \delta') < \psi(\bar{t}, \bar{s})$, again a contradiction.

A similar proof allowing to conclude that $g(\bar{s}) = m$, (12.12) holds. Notice that if we have replaced $f(t)$ by $f(t) + \varepsilon t$ and $g(t)$ by $g(s) - \varepsilon s$, we can let ε tend to 0 and keep this property for at least one minimum point.

Now we consider a point $(\tilde{t}, \tilde{s}) \in \mathbb{R}^2$ which satisfies (12.12) and we pick any point $(t, s) \in \mathbb{R}^2$. We examine the different possible cases, taking into account the particular form of ψ and the monotonicity properties of f, g, h, using that, of course, $\psi(\tilde{t}, \tilde{s}) = f(\tilde{t}) = g(\tilde{s}) = h(\tilde{t}, \tilde{s})$:

- If $t \geq \tilde{t}$, $\psi(t, s) \geq f(t) \geq f(\tilde{t}) = \psi(\tilde{t}, \tilde{s})$.
- If $s \leq \tilde{s}$, the same conclusion holds by using that g is decreasing.
- If $t \leq \tilde{t}$ and $s \geq \tilde{s}$, then $\psi(t, s) \geq h(t, s) \geq h(\tilde{t}, \tilde{s}) = \psi(\tilde{t}, \tilde{s})$.

And the conclusion follows since we have obtained that ψ reaches its minimum at $(\tilde{t}, \tilde{s})$.

Q.E.D.

Part III

General Discontinuities: Stratified Problems

Chapter 13

Stratified Solutions: definition and comparison

13.1 Introduction and definitions

In this section, we consider Hamilton-Jacobi-Bellman Equations with more general discontinuities, namely discontinuities of any co-dimensions but with the restriction that these discontinuities form a "Whitney's stratification" (cf. Section 3.3).

To study such equations, we introduce the notion of "stratified solutions" for Equation (4.4) where $\mathbb{F}$ is given by (4.3), that is,

$$\mathbb{F}(x,t,U,DU) = 0 \quad \text{in} \quad \mathbb{R}^N \times [0,T]\,, \tag{13.1}$$

where $DU = (D_x U, D_t U)$ and

$$\mathbb{F}(x,t,r,p) := \sup_{(b,c,l)\in \mathbf{BCL}(x,t)} \big\{ -b\cdot p + cr - l \big\}\,. \tag{13.2}$$

Of course, this can be done in a suitable framework which is the one described in the section "Good Framework for HJ Equations with Discontinuities".

We assume that we have a stratification $\mathbb{M} = (\mathbf{M}^k)_{k=0\ldots(N+1)}$ on $\mathbb{R}^N \times [0,T]$ which may depend on t and that we see as the restriction on $\mathbb{R}^N \times [0,T]$ of a regular stratification on $\mathbb{R}^N \times \mathbb{R}$. Our (first) main assumption is that we have "good framework for HJ Equations with discontinuities" for Equation (13.1) in $\mathcal{O} = \mathbb{R}^N \times (0,T)$ (and of course, associated to the stratification $\mathbb{M}$): this implies in particular that $\mathbb{F}(x,t,u,P)$ is a continuous function, except perhaps on $\mathbf{M}^k$ for $k = 0,...,(N+1)$. This first and main assumption is complemented by an other one concerned with $t = 0$ since the initial data may be determined through an other equation. To do so, we also assume that we have "good framework for HJ Equations with discontinuities" for the equation $\mathbb{F}_{init} = 0$ in $\mathbb{R}^N$ associated to the stratification $\mathbb{M}_0 = (\mathbf{M}_0^k)_{k=0...(N+1)}$ where $\mathbf{M}_0^k = \mathbf{M}^k \cap \{t = 0\}$.

To state a definition, we introduce the Hamiltonians H^k defined in the following way: if $(x,t) \in \mathbf{M}^k$, $u \in \mathbb{R}$ and $p \in T_{(x,t)}\mathbf{M}^k$, we set

$$\mathbb{F}^k(x,t,u,p) := \sup_{\substack{(b,c,l)\in\mathbf{BCL}(x,t)\\ b\in T_{(x,t)}\mathbf{M}^k}} \left\{ -b\cdot p + cu - l \right\}, \tag{13.3}$$

and in the same way

$$\mathbb{F}^k_{init}(x,u,p_x) := \sup_{\substack{((b^x,0),c,l)\in\mathbf{BCL}(x,0)\\ b\in T_x\mathbf{M}_0^k}} \left\{ -b^x\cdot p_x + cu - l \right\}. \tag{13.4}$$

In these two definitions, we have used (we hope without ambiguity!) two different definitions of $\mathbf{M}^k$.

Definition 13.1.1 (Stratified sub and supersolutions)
(i) A locally bounded, lsc function $v : \mathbb{R}^N \times [0,T[\to \mathbb{R}$ is a stratified supersolution of Equation (13.1) iff it is an Ishii supersolution of this equation.
(ii) A locally bounded, usc function $u : \mathbb{R}^N \times [0,T[\to \mathbb{R}$ is a stratified subsolution of Equation (13.1) iff (a) it is an Ishii subsolution of this equation and (b) for any $k = 0,...,(N+1)$ it is a subsolution of

$$\mathbb{F}^k(x,t,u,(D_t u, D_x u)) \leq 0 \quad \text{on } \mathbf{M}^k,\ \text{for } t > 0,$$

and

$$(\mathbb{F}_{init})_*(x,u,D_x u) \leq 0 \quad \text{in } \mathbb{R}^N,\ \text{for } t = 0.$$
$$\mathbb{F}^k_{init}(x,u,D_x u) \leq 0 \quad \text{on } \mathbf{M}_0^k,\ \text{for } t = 0.$$

In addition, we will say that u is an η-strict stratified subsolution if the ≤ 0-inequalities are replaced by a $\leq -\eta < 0$-inequality where $\eta > 0$ is independent of x and t.

As it is clear in the definition, the concept of "stratified solution" just consists in super-imposing subsolutions' inequalities on each sets of discontinuity $\mathbf{M}^k$, including the time $t = 0$. Taking into account the situation described in Chapter 6 for a codimension 1 discontinuity, this is a natural way to prevent the system to ignore what happens on these discontinuities (but other assumptions like the normal controllability will play a role for that, too!). These subsolution conditions are real "$\mathbf{M}^k$" inequalities, i.e. they are obtained by looking at maximum points of $u - \varphi$ on $\mathbf{M}^k$ where φ is a test-function which is smooth on $\mathbf{M}^k$.

Before providing the comparison result for stratified sub and supersolutions of Equation (13.1), we come back on the assumptions for a "Good Framework for HJ Equations with Discontinuities", and in particular on sufficient (but also natural) conditions in terms of **BCL** for **(TC)** & **(NC)** to hold. We do it for the stratification in $\mathbb{R}^N \times (0,T)$ but we could argue in an analogous way for $t = 0$.

Since these assumptions are local, we can state them in a ball $B((x,t),r)$ centered at $(x,t) \in \mathbf{M}^k$ with a small radius $r > 0$ and we can assume that, in $B((x,t),r)$, $\mathbb{M}$ is an (AFS) with $\mathbf{M}^k = (x,t) + V_k$, where V_k is a k-dimensional vector space in $\mathbb{R}^{N+1}$ and $B((x,t),r)$ intersects only $\mathbf{M}^k, \mathbf{M}^{k+1}, \cdots, \mathbf{M}^{N+1}$. We denote by $V_k^\perp$ the orthogonal space to V_k and by $P^\perp$ the orthogonal projector on $V_k^\perp$.

In this framework, **(TC)** & **(NC)** are satisfied if, for any $0 \leq k \leq N+1$ and for any $(x,t) \in \mathbf{M}^k$, there exists contants $C_1, \delta > 0$ and a modulus $m : [0,+\infty) \to \mathbb{R}^+$ such that

(TC-BCL) For any $j \geq k$, if $(y_1,t_1),(y_2,t_2) \in \mathbf{M}^j \cap B(x,r)$ with $(y_1,t_1)-(y_2,t_2) \in V_k$,

$$\begin{cases} \mathrm{dist}_{\mathrm{H}} \left(\mathbf{B}(y_1,t_1), \mathbf{B}(y_2,t_1)\right) \leq C_1(|y_1 - y_2| + |t_1 - t_2|)\,, \\ \mathrm{dist}_{\mathrm{H}} \left(\mathbf{BCL}(y_1,t_1), \mathbf{BCL}(y_2,t_2)\right) \leq m\big(|y_1 - y_2| + |t_1 - t_2|\big)\,, \end{cases}$$

where $\mathrm{dist}_{\mathrm{H}}$ denotes the Hausdorff distance.

(NC-BCL) There exists $\delta = \delta(x,t) > 0$, such that, for any $(y,s) \in B((x,t),r) \setminus \mathbf{M}^k$, one has

$$B(0,\delta) \cap V_k^\perp \subset P^\perp \left(\mathbf{B}(y,t)\right)\,.$$

Of course, the case $k = 0$ is particular since $V_k = \{0\}$: here we impose a complete controllability of the system in a neighborhood of $x \in \mathbf{M}^0$ since the condition reduces to $B(0,\delta) \subset \mathbf{B}(y,t)$ because $V_k^\perp = \mathbb{R}^N$.

This normal controllability assumption plays a key role in all our analysis: first, in the proof of Theorem 4.2.9 below, to obtain the viscosity subsolution inequalities for the value function, in the comparison proof to allow the regularization (in a suitable sense) of the subsolutions and, last but not least, for the stability result.

We point out an easy consequence of these assumptions which will be used later on for obtaining the $\mathbb{F}^k$-inequality. With the same notations as above we set, for

$(y,s) \in B((x,t),r) \cap \mathbf{M}^k$

$$\mathbf{BCL}^k(y,s) := \{(b,c,l) \in \mathbf{BCL}(y,s);\ b \in T_{(y,s)}\mathbf{M}^k = V_k\}\ .$$

We have the

Lemma 13.1.2 *For any $(y,s) \in B((x,t),r) \cap \mathbf{M}^k$, $\mathbf{BCL}^k(y,s) \neq \emptyset$ and, for any $(b,c,l) \in \mathbf{BCL}^k(x,t)$ and $\eta > 0$, $\mathbf{BCL}^k(y,s) \cap B((b,c,l),\eta) \neq \emptyset$ if (y,s) is close enough to (x,t).*

The first part of the result is a direct consequence of **(NC-BCL)** since, by uppersemicontinuity and convexity, $0 \in \mathbf{B}(y,t)$, while the second part comes from **(TC-BCL)**.

13.2 The Comparison Result

The result is the

Theorem 13.2.1 *In the framework of "good framework for HJ Equations with discontinuities" which is described above, the comparison result between bounded stratified sub and supersolutions holds for Equation (13.1).*

Proof — Essentially the proof follows the main steps as the proof of Theorem 6.1.9 where it is shown that $\mathbf{U}^-$ is the unique solution of the Bellman Equation with the H_T-complemented inequality, which turns out to be an $\mathbf{M}^N$-inequality in the stratified setting. The only difference is that we have to use the most sophisticated form of Theorem 5.3.1.

Before describing these main steps, we introduce some notations and perform some reductions.

Let $u, v : \mathbb{R}^N \times [0,T[\to \mathbb{R}$ be respectively a bounded u.s.c. stratified subsolution and a bounded l.s.c. stratified supersolution of Equation (13.1)). Our aim is to show that $u \leq v$ in $\mathbb{R}^N \times [0,T[$[(1)].

This inequality is proved via two different comparison results: first, one has to show that $u(x,0) \leq v(x,0)$ in $\mathbb{R}^N$, which means to prove a comparison result for the stationary equation associated to the Hamiltonian $\mathbb{F}_{init}$, and then to show that $u \leq v$ in $\mathbb{R}^N \times]0,T[$, i.e. a comparison for the evolution problem.

(1) The reason why we do not include T in the comparison will be clarified later on.

The global strategy to obtain the comparison is the same in these two cases and the changes to pass from one to the other are minor. Therefore we are going to provide the full proof only in the evolution case, admitting that $u(x,0) \leq v(x,0)$ in $\mathbb{R}^N$. Actually, proving this property at $t=0$ only requires the additional argument of Kruzkov's change of variable

$$\tilde{u}(x) = -\exp(-\alpha u(x,0)) \quad \text{and} \quad \tilde{v}(x) = -\exp(-\alpha v(x,0)) \ ,$$

for $\alpha > 0$ small enough. With this change, one easily shows that all the $\mathbb{F}^k_{init}(x,r,p_x)$ become strictly increasing in r; we leave this easy checking (based on $(\mathbf{H_{BCL}})$-(iv)) to the reader. Then, the rest of the proof is done exactly as we proceed below for the evolution problem.

The second reduction consists in using the by-now classical change

$$\bar{u}(x,t) = \exp(-Kt)u(x,t) \quad \text{and} \quad \bar{v}(x,t) = \exp(-Kt)v(x,t) \ ,$$

which allows to reduce to the case when $c \geq 0$ for any $(b,c,l) \in \mathbf{BCL}(x,t)$ and for any $(x,t) \in \mathbb{R}^N \times]0,T[$.

Then the comparison proof in $\mathbb{R}^N \times]0,T[$ is done in five steps.

Step 1: *Reduction to a local comparison result* **(LCR)**-*evol* – Using the assumptions on the **BCL**, one easily proves that

$$\psi(x) := -\delta(1+|x|^2)^{1/2} - \delta^{-1}t$$

is a (smooth) stratified subsolution (and even η-strict subsolution for some $\eta > 0$); therefore changing $\bar{u}(x,t)$ into

$$\bar{u}_\mu(x,t) = \mu\bar{u}(x,t) + (1-\mu)\psi(x,t) \ ,$$

we are left to the case when we have to compare a $(1-\mu)\eta$-strict subsolution $\bar{u}_\mu$ and a supersolution v such that $\bar{u}_\mu(x,t) - v(x,t) \to -\infty$ when $|x| \to +\infty$. In other words, ((**LOC1**)) is satisfied. And so is ((**LOC2**)) by considering $\bar{u}_\mu(x,t) - \delta'(|x-\bar{x}|^2 + |t-\bar{t}|^2)$ where $(\bar{x},\bar{t})$ is the point where we wish to check ((**LOC2**)) and $\delta' > 0$ a small enough constant.

Thanks to Section 3.2, we can just prove local comparison results and, to do so, for the sake of simplicity of notations, we just denote by u a strict stratified subsolution and v a stratified supersolution.

Step 2: *Local comparison and argument by induction* – In order to prove **(LCR)**-evol, we are going to argue by induction but, since we have to use Theorem 5.3.1, we have to show, at the same time, a local comparison result non only for Equation (13.1) but also for equations of the type $\max(\mathbb{F}(x,t,w,Dw), w-\psi)=0$ where ψ is a continuous function. In fact, with the assumptions we use, there is no difference when proving **(LCR)**-evol for these two (slightly different) equations but, in order to be rigourous, we have to consider the "obstacle" one, which reduces to the $\mathbb{F}$–one if we choose $\psi(x)=K$ where the constant K is larger than $\max(||u||_\infty, ||v||_\infty)$.

For the sake of simplicity, we use below the generic expression ψ–Equation for the equation $\max(\mathbb{F}(x,t,w,Dw), w-\psi)=0$ and we will always assume that ψ is a continuous function, at least in a neighborhood of the domain we consider.

So, we are reduced now to show the following property, for any $(\bar{x},\bar{t})\in\mathbb{R}^N\times(0,T)$:

$\mathbf{LCR}^\psi(\bar{x},\bar{t})$: *There exists $r=r(\bar{x},\bar{t})>0$ and $h=h(\bar{x},\bar{t})\in(0,\bar{t})$ such that, if u and v are respectively a strict stratified subsolution*[2] *and a stratified supersolution of some ψ–Equation in $Q^{\bar{x},\bar{t}}_{r,h}$ and if $\max_{\overline{Q^{\bar{x},\bar{t}}_{r,h}}}(u-v)>0$, then*

$$\max_{\overline{Q^{\bar{x},\bar{t}}_{r,h}}}(u-v)\leq \max_{\partial_p Q^{\bar{x},\bar{t}}_{r,h}}(u-v)\,,$$

where we recall that $\partial_p Q^{\bar{x},\bar{t}}_{r,h}$ stands for the parabolic boundary of $Q^{\bar{x},\bar{t}}_{r,h}$, namely here $\partial B(\bar{x},r)\times[\bar{t}-h,\bar{t}]\cup\overline{B(\bar{x},r)}\times\{\bar{t}-h\}$.

It is clear that $\mathbf{LCR}^\psi(\bar{x},\bar{t})$ holds in $\mathbf{M}^{N+1}$ since $\mathbb{F}^{N+1}$ and all the ψ–Equations satisfy all the property ensuring a (standard) comparison result in the open set $\mathbf{M}^{N+1}$; therefore $\mathbf{LCR}^\psi(\bar{x},\bar{t})$ is satisfied for r and h small enough (see Section 3.2.4).

In order that it holds for $(\bar{x},\bar{t})$ in any $\mathbf{M}^k$, we use a (backward) induction on k and more precisely, we introduce the property

$$\mathbf{P}(k){:=}\Big\{\ \mathbf{LCR}^\psi(\bar{x},\bar{t}) \textit{ holds for any } (\bar{x},\bar{t})\in\mathbf{M}^k\cup\mathbf{M}^{k+1}\cup\cdots\cup\mathbf{M}^{N+1}\Big\}\,.$$

Since $\mathbf{P}(N+1)$ is true, the core of the proof consists in showing that $\mathbf{P}(k+1)$ implies $\mathbf{P}(k)$ for $0\leq k\leq N$. To do so, we assume that $(\bar{x},\bar{t})\in\mathbf{M}^k$ and want to prove that $\mathbf{LCR}^\psi(\bar{x},\bar{t})$ holds provided $\mathbf{P}(k+1)$ is satisfied.

Step 3: *Regularization of the subsolution* – In order to apply the ideas of Section 3.4.1, we use the definition of a regular stratification which allows us to assume that we are in the case of a flat stratification, in a neighborhood of $\bar{x}=0$, $\bar{t}>0$. We can

[2] According to the type of obstacle ψ we have to use in the proof of Theorem 5.3.1, we can assume w.l.o.g. that $u\leq\psi-\delta$ for some $\delta>0$ in $\overline{Q^{\bar{x},\bar{t}}_{r,h}}$ and therefore a strict subsolution of $\mathbb{F}=0$ or of the ψ-Equation have essentially the same meanings.

also assume that $\mathbf{M}^k$ is the k-dimensional manifold parametrized by $(t, x_1, \cdots, x_{k-1})$, given by the equations $x_k = x_{k+2} =, \cdots, = x_N = 0$. This reduction is based on a $W^{2,\infty}$-change of variable in x which is done *only for the regularization step* and then we come back to the initial framework by the inverse of the change.

In the new setting, we keep the notations $\mathbb{F}$, $\mathbb{F}^l$ (for all l) and u. We just point out here that the t-variable is always part of the tangent variables which explain some restriction in the assumption concerning the behavior of the $\mathbb{F}^l$ in t (cf. **(TC)**). Before proceeding, we emphasize the fact that, since r and h may depend on $(\bar{x}, \bar{t})$, we can handle without any difficulty the localization to reduce to the case of a flat stratification.

The next important remark is that, because of the assumptions **(NC)**, **(TC)**, Proposition 3.4.1 holds for the subsolution u and for all the points $(x,t) \in \mathbf{M}^0 \cup \mathbf{M}^1 \cup \cdots \mathbf{M}^N$. So, for any $(x,t) \in \mathbf{M}^k$, $k < N$,

$$u(x,t) = \limsup\{u(y,s),\ (y,s) \to (x,t),\ (y,s) \notin \mathbf{M}^k\}\ , \tag{13.5}$$

and for the special case where $(x,t) \in \mathbf{M}^N$,

$$\begin{aligned} u(x,t) &= \limsup\{u(y,s),\ (y,s) \to (x,t),\ (y,s) \in M_+^{(x,t)}\} \\ &= \limsup\{u(y),\ y \to x,\ y \in M_-^{(x,t)}\}, \end{aligned} \tag{13.6}$$

where, for $r, h > 0$ small enough, $M_+^{(x,t)}, M_-^{(x,t)} \subset \mathbf{M}^{N+1} \cap Q_{r,h}^{x,t}$ are the locally disjoint connected components of $\big((\mathbb{R}^N \times [0,T]) \setminus \mathbf{M}^N\big) \cap Q_{r,h}^{x,t}$.

If $(x,t) = (0,\bar{t}) \in \mathbf{M}^l$ for some $l \geq k$, we may assume that $Q_{r,h}^{x,t}$ only contains points of $\mathbf{M}^k, \mathbf{M}^{k+1}, \cdots, \mathbf{M}^{N+1}$. So, we regularize the strict subsolution in $Q_{r,h}^{x,t}$ by applying the idea of Section 3.4.1,using the variables $y = (t, x_1, \cdots, x_{k-1})$, $z = (x_k, x_{k+2}, \cdots, x_N)$ and

$$G((y,z),u,p) := \max(\mathbb{F}_*(x,t,r,p), \mathbb{F}^l(x,t,r,p), u - \psi)\,.$$

We both use an approximation by a sup-convolution and usual convolution with a smoothing kernel as in Proposition 3.4.4.

Assumptions **(NC)**, **(TC)** and **(Mon)** hold as a consequence of either $(\mathbf{H_{BCL}})$ or the additional assumptions coming from the "good framework for HJ Equations with discontinuities", therefore, by Proposition 3.4.2 and 3.4.4, we can assume that we have obtained a sequence of strict stratified subsolutions which are C^1 in the variables $y = (t, x_1, \cdots, x_{k-1})$.

Applying back the change of variables, and using that the above procedure gives a strict stratified subsolution in a neighborhood of $(x,t) = (0,\bar{t})$, there exists $r, h > 0$,

$t' > \bar{t}$ and a sequence $(u_\varepsilon)_\varepsilon$ of subsolutions of the stratified problem in $Q_{r,h}^{\bar{x},t'}$, which are in $C^0(\overline{Q_{r,h}^{\bar{x},t'}}) \cap C^1(\mathbf{M}^k \cap \overline{Q_{r,h}^{\bar{x},t'}})$ which are all $(\eta/2)$-strict subsolutions of Equation (4.4) in $Q_{r,h}^{\bar{x},t'}$. And because of Remark 3.2.5, we can assume as well that the u_ε's are strict subsolutions on $Q_{r,h}^{\bar{x},\bar{t}}$.

Step 4: *Properties of the regularized subsolution* – Step 3 has two consequences

- (a) for any $\varepsilon > 0$ small enough, $\mathbb{F}^k(x, t, u_\varepsilon, Du_\varepsilon) \leq -\eta/2 < 0$ on $\mathbf{M}^k \cap \overline{Q_{r,h}^{\bar{x},\bar{t}}}$ in a classical sense;
- (b) since u_ε is an $(\eta/2)$-strict subsolution of the ψ-Equation in $\mathcal{O} := Q_{r,h}^{\bar{x},\bar{t}} \setminus \mathbf{M}^k$ and since **(LCR)** holds there because $\mathbf{P}(k+1)$ holds, we use the subdynamic programming principle for subsolutions (cf. Theorem 5.3.1) which implies that each u_ε satisfies an $(\eta/2)$-strict dynamic programming principle in $\mathcal{O}$.

These two properties allow us to have **(LCR)**-evol in $Q_{r,h}^{\bar{x},\bar{t}}$ in the final step.

Step 5: *Performing the local comparison* – From the previous step we know that for each $\varepsilon > 0$, u_ε satisfies the hypotheses of Lemma 5.4.1 and we deduce from this lemma that

$$\forall (y,s) \in Q_{r,h}^{\bar{x},\bar{t}}, \quad (u_\varepsilon - v)(y,s) < \max_{\overline{Q_{r,h}^{\bar{x},\bar{t}}}} (u_\varepsilon - v)\,.$$

Using that $u = \limsup^* u_\varepsilon$, this yields a local comparison result (with inequality in the large sense) between u and v as $\varepsilon \to 0$.

Therefore we have shown that $\mathbf{P}(k+1)$ implies $\mathbf{P}(k)$, which ends the proof.

Q.E.D.

Remark 13.2.2 *As it is clear in the above proof, the special structure of* $\mathbb{M}$ *does not play any role and time-dependent stratifications do not differ so much from time-independent ones. We remark anyway that a difference is hidden in the normal controllability assumption is that we cannot have a normal direction of the form* $(0_{\mathbb{R}^N}, +/-1)$ *for* $\mathbf{M}^k$ *and this, for any* k*.*

Chapter 14

Stratified Solutions and Optimal Control Problems

14.1 The value-function as a stratified solution

In Section 4.2, we have already shown that the value-function U defined by (4.2) in Section 4.2.3 is an Ishii supersolution of $\mathbb{F} = 0$ and therefore it is a stratified supersolution. It remains to prove the subsolution's properties and, to do so, the behaviour of the dynamic is going to play a key role via Assumptions **(TC-BCL)** and **(NC-BCL)**.

Theorem 14.1.1 (Subsolution's Properties) *Under Assumptions* $(\mathbf{H_{BCL}})$*,* **(TC-BCL)** *and* **(NC-BCL)***, the value-function U satisfies*

(i) For any $k = 0..(N-1)$, $U^* = (U|_{\mathbf{M}^k})^*$ on $\mathbf{M}^k$;

(ii) for any $k = 0..(N-1)$, U is a subsolution of

$$\mathbb{F}^k(x, t, U, DU) = 0 \quad \text{on } \mathbf{M}^k \, .$$

In this result, we again point out – even if it is obvious– that (ii) is a viscosity inequality for an equation *restricted to* $\mathbf{M}^k$, namely it means that if ϕ is a smooth function on $\mathbf{M}^k \times (0,T)$ (or equivalently on $\mathbb{R}^N \times (0,T)$ by extension) and if $(x,t) \in \mathbf{M}^k \times (0,T)$ is a local maximum point of $U^* - \phi$ on $\mathbf{M}^k \times (0,T)$, then

$$\mathbb{F}^k(x, t, U^*(x,t), D\phi(x,t)) \leq 0 \ \ .$$

This is why point (i) is an important fact since it allows to restrict everything (including the computation of the usc envelope of U) to $\mathbf{M}^k$.

Proof — Since all the results are local, we can assume without loss of generality that we are in the case of an (AFS) (a complete proof being obtained via a simple change of variable).

We consider $(x,t)\in \mathbf{M}^k$ and a sequence $(x_\varepsilon,t_\varepsilon)\to(x,t)$ such that

$$U^*(x,t)=\lim_\varepsilon U(x_\varepsilon,t_\varepsilon)\ .$$

We have to show that we can assume that $(x_\varepsilon,t_\varepsilon)\in\mathbf{M}^k$. In all the sequel, we assume that $\varepsilon\ll 1$ in order that all the points remains in $B((x,t),r)$, the ball given by **(NC-BCL)**.

We assume that, on the contrary, $(x_\varepsilon,t_\varepsilon)\notin\mathbf{M}^k$ and we show how to build a sequence of points $(\bar x_\varepsilon,\bar t_\varepsilon)_\varepsilon$ with $(\bar x_\varepsilon,\bar t_\varepsilon)\in\mathbf{M}^k$ for any ε and with $U^*(x,t)=\lim_\varepsilon U(\bar x_\varepsilon,\bar t_\varepsilon)$.

By Theorem 4.2.4, we have

$$U(x_\varepsilon,t_\varepsilon)\le\int_0^\theta l\big(X(s),T(s)\big)\exp(-D(s))ds+U\big(X(\theta),T(\theta)\big)\exp(-D(\theta))\big)\ ,$$

for any solution (X,T,D,L) of the differential inclusion starting from $(x_\varepsilon,t_\varepsilon,0,0)$. Let $(\tilde x_\varepsilon,\tilde t_\varepsilon)$ be the projection of $(x_\varepsilon,t_\varepsilon)$ on $\mathbf{M}^k$; we have $n_\varepsilon:=(\tilde x_\varepsilon,\tilde t_\varepsilon)-(x_\varepsilon,t_\varepsilon)\in V_k^\perp$ and, using **(NC-BCL)**, for any $(y,s)\in B((x,t),r)$, there exists $b\in\mathbf{B}(y,s)$ such that, if $b=b_\top+b_\perp$ with $b_\top\in V_k$, $b_\perp\in V_k^\perp$, then $b_\perp:=\delta/2.n_\varepsilon|n_\varepsilon|^{-1}$.

Choosing such a dynamic b (with any constant discount-cost (c,l)), it is clear that $(X(s),T(s))\in B((x,t),r)$ for s small enough (independent of ε) and for $s_\varepsilon=2(|\tilde x_\varepsilon-x_\varepsilon|+|\tilde t_\varepsilon-t_\varepsilon|)/\delta$, we have $(\bar x_\varepsilon,\bar t_\varepsilon)=(X(s_\varepsilon),T(s_\varepsilon)=(\tilde x_e+y_\varepsilon,\tilde t_e+\tau_\varepsilon)$ where $(y_\varepsilon,\tau_\varepsilon)\in V_k$, $|(y_\varepsilon,\tau_\varepsilon)|=O(|\tilde x_\varepsilon-x_\varepsilon|+|\tilde t_\varepsilon-t_\varepsilon|)$. Therefore $(\bar x_\varepsilon,\bar t_\varepsilon)\in\mathbf{M}^k$ by Lemma 3.3.2 and we have using the Dynamic Programming Principle above with $\theta=s_\varepsilon$

$$U(x_\varepsilon,t_\varepsilon)\le O(s_\varepsilon)+U\big(X(s_\varepsilon),T(s_\varepsilon)\big)\exp(-D(\theta))=O(s_\varepsilon)+U\big(\bar x_\varepsilon,\bar t_\varepsilon\big)(1+O(s_\varepsilon))\,.$$

Finally since $s_\varepsilon\to0$ as $\varepsilon\to0$, we deduce that

$$\limsup_\varepsilon U\big(\bar x_\varepsilon,\bar t_\varepsilon\big)\ge\limsup_\varepsilon\ U(x_\varepsilon,t_\varepsilon)=U^*(x,t)\ ,$$

which shows (i) since $\big(\bar x_\varepsilon,\bar t_\varepsilon\big)\in\mathbf{M}^k$.

To prove (ii), we assume now that $(x_\varepsilon,t_\varepsilon)\in\mathbf{M}^k$ and we use again Theorem 4.2.4 which implies

$$U(x_\varepsilon,t_\varepsilon)\le\int_0^\theta l\big(X(s),T(s)\big)\exp(-D(s))ds+U\big(X(\theta),T(\theta)\big)\exp(-D(\theta))\big)\ ,\quad(14.1)$$

for any solution (X, T, D, L) of the differential inclusion starting from $(x_\varepsilon, t_\varepsilon, 0, 0)$.

But now we can use the result of Lemma 13.1.2: for any $(b, c, l) \in \mathbf{BCL}^k(x, t)$ and $\eta > 0$, $\mathbf{BCL}^k(y, s) \cap B((b, c, l), \eta) \neq \emptyset$ if (y, s) is close enough to (x, t). Solving locally the differential inclusion with $\mathbf{BCL}^k(y, s) \cap B((b, c, l), \eta)$ instead of $\mathbf{BCL}$ and using the associated solution in (14.1) allows to obtain the viscosity inequality for (b, c, l) as in the standard case.

Since this is true for any $(b, c, l) \in \mathbf{BCL}^k(x, t)$, the result is complete.

Q.E.D.

An immediate consequence of Theorem 14.1.1 is

Corollary 14.1.2 *Under the assumptions of Theorem 14.1.1, the value-function U is continuous in $\mathbb{R}^{N+1}$ and is the unique stratified solution of the Bellman Equation.*

14.2 Some Key Examples (a.k.a. does my problem enter into this framework?)

In this section, we consider a different point of view for stratified problems and we also give examples when the assumptions are satisfied. To do so, we assume that we are given a general regular stratification $(\mathbf{M}^k)_k$ of $\mathbb{R}^N$.

14.2.1 A control-oriented general example

Each manifold $\mathbf{M}^k$ is written as the union of its connected components $\mathbf{M}^{k,j}$

$$\mathbf{M}^k = \bigcup_{j=1}^{J(k)} \mathbf{M}^{k,j} ,$$

where $J(k) \in \mathbb{N} \cup \{+\infty\}$ and on each $\mathbf{M}^{k,j}$, we are given a space of control $A_{k,j}$ and functions $(b^{k,j}, c^{k,j}, l^{k,j})$ representing the dynamic, discount factor and cost for a control problem on $\mathbf{M}^{k,j}$. For the sake of simplicity, we assume that all these function are defined in $\mathbb{R}^N \times [0, T] \times A_{k,j}$ with the condition $b^{k,j}(x, t, \alpha_{k,j}) \in T_x\mathbf{M}^k$ for any $(x, t) \in \mathbf{M}^{k,j}$ and $\alpha_k \in A_k$ in order that the dynamic preserves $\mathbf{M}^{k,j}$ at least for a short time. If $(x, t) \in \mathbf{M}^{k,j}$, we can introduce the associated Hamiltonian

$$\tilde{H}^{k,j}(x, t, r, p) := \sup_{\alpha_{k,j} \in A_{k,j}} \left\{ -b^{k,j}(x, t, \alpha_{k,j}) \cdot p + c^{k,j}(x, t, \alpha_{k,j}) r - l^{k,j}(x, t, \alpha_{k,j}) \right\} ,$$

which is defined for $r \in \mathbb{R}$ and a priori only for $p \in T_x\mathbf{M}^k$ but we can as usual extend this definition for $p \in \mathbb{R}^N \times \mathbb{R}$.

If $(x,t) \in \mathbb{R}^N \times (0,T)$, we set $L(x,t) := \{(k,j);\ (x,t) \in \overline{\mathbf{M}^{k,j}}\}$, we have

$$\mathbf{BCL}(x,t) = \overline{\mathrm{Conv}} \left\{ \bigcup_{(k,j)\in L(x,t)} \{(b^{k,j}, c^{k,j}, l^{k,j})(x,t,\alpha_{k,j}),\ \alpha_{k,j} \in A_{k,j}\} \right\} .$$

And

$$\mathbb{F}(x,t,r,p) = \sup_{\substack{(k,j)\in L(x,t),\\ \alpha_{k,j}\in A_{k,j}}} \left\{ -b^{k,j}(x,t,\alpha_{k,j}) \cdot p + c^{k,j}(x,t,\alpha_{k,j})r - l^{k,j}(x,t,\alpha_{k,j}) \right\} .$$

In order to have Assumption **(TC)** to be satisfied, it is enough that each $(b^{k,j}, c^{k,j}, l^{k,j})$ satisfies $(\mathbf{H}_{BACP})$ and for **(NC)**, we have to assume that if $(x,t) \in \mathbf{M}^{\bar{k}}$, then the set

$$\overline{\mathrm{Conv}} \left(\bigcup_{\substack{(k,j)\in L(x,t),\\ k>\bar{k}}} \{(b^{k,j}, c^{k,j}, l^{k,j})(x,t,\alpha_{k,j}),\ \alpha_{k,j} \in A_{k,j}\} \right) ,$$

satisfies **(NC-BCL)** (instead of **B**).

14.2.2 A pde-oriented general example

Unfortunately this pde-oriented example will not be completely formulated in terms of pde and Hamiltonians, the difficulty being analogous to defining H_T in Part II. To simplify, we treat the case when the stratification does not depend on times, i.e. $\mathbf{M}^{k+1} = \tilde{\mathbf{M}}^k \times (0,T)$ for all $0 \le k \le N$, where $(\tilde{\mathbf{M}}^k)_k$ is a stratification of $\mathbb{R}^N$.

We start from $\mathbf{M}^{N+1}$ which we write as the union of its connected components

$$\mathbf{M}^{N+1} = \bigcup_{j=1}^{J(N+1)} \tilde{\mathbf{M}}^{N,j} \times (0,T) .$$

We consider the case when

$$\mathbb{F}^{N+1}(x,t,r,(p_x,p_t)) = p_t + \tilde{H}^{N,j}(x,t,r,p_x) \quad \text{in } \tilde{\mathbf{M}}^{N,j} \times (0,T) ,$$

for all j where the Hamiltonians $\tilde{H}^{N,j}$ are defined by

$$\tilde{H}^{N,j}(x,t,r,p) = \sup_{\alpha_{N,j}\in A_{N,j}} \left\{ -b^{N,j}(x,t,\alpha_{N,j}) \cdot p + c^{N,j}(x,t,\alpha_{N,j})r - l^{N,j}(x,t,\alpha_{N,j}) \right\} ,$$

where the control sets $A_{N,j}$ are compact metric spaces. A simple but natural situation is when all these Hamiltonians can be extended as continuous in $\mathbb{R}^N \times [0,T]$ functions satisfying $(\mathbf{H}_{BA-HJ})$. These Hamiltonians are the analogues of H_1, H_2 in Part II.

It remains to define $\mathbb{F}$ and $\mathbb{F}^{k+1}$ on all $\tilde{\mathbf{M}}^k \times (0,T)$ for $k < N$ and this has to be done by induction. For $k = N-1$, if

$$\mathbf{M}^N = \bigcup_{j=1}^{J(N)} \tilde{\mathbf{M}}^{N-1,j} \times (0,T) ,$$

we can assume that, on each $\tilde{\mathbf{M}}^{N-1,j} \times (0,T)$, we have an Hamiltonian $\tilde{H}^{N-1,j}$ and we have, for any $(x,t) \in \tilde{\mathbf{M}}^{N-1,j} \times (0,T)$

$$\mathbb{F}(x,t,r,(p_x,p_t)) = \max_{l \in L(x,t)} \Big(p_t + \tilde{H}^{N,l}(x,t,r,p_x), p_t + \tilde{H}^{N-1,j}(x,t,r,p_x) \Big) ,$$

with $L(x,t) := \{l;\ (x,t) \in \overline{\tilde{\mathbf{M}}^{N,l}} \times (0,T)\}$. On the other hand, $\mathbb{F}^N$ may be decomposed into two parts: the analogue of the H_T-one in Part II coming from $\mathbb{F}^{N+1}$ and the specific $\tilde{H}^{N-1,j}$-one reflecting a particular control problem on $\tilde{\mathbf{M}}^{N-1,j} \times (0,T)$. This means

$$\mathbb{F}^N(x,t,r,(p_x,p_t)) = \max \Big(\mathbb{F}_T^{N+1}(x,t,r,(p_x,p_t)), p_t + \tilde{H}^{N-1,j}(x,t,r,p_x) \Big) ,$$

where $\mathbb{F}_T^{N+1}(x,t,r,(p_x,p_t))$ is built in the following way: as in the previous section, we set

$$\overline{\mathrm{Conv}} \left(\bigcup_{l \in L(x,t)} \{(b^{N,j}, c^{N,j}, l^{N,j})(x,t,\alpha_{N,j}),\ \alpha_{N,j} \in A_{N,j}\} \right) ,$$

and, for $(x,t) \in \tilde{\mathbf{M}}^{N-1,j} \times (0,T)$ we denote by $\mathbf{BCL}_T^{N-1}(x,t)$ the subset of (b,c,l) in this closed convex envelope such that $b \in T_x\tilde{\mathbf{M}}^{N-1,j}$. Then

$$\mathbb{F}_T^{N+1}(x,t,r,(p_x,p_t)) = p_t + \sup_{\mathbf{BCL}_T^{N-1}(x,t)} \{-b \cdot p_x + cr - l\} .$$

For any k, the construction is analogous. For any connected component of $\tilde{\mathbf{M}}^{k,j} \times (0,T)$ of $\tilde{\mathbf{M}}^k \times (0,T)$, $\mathbb{F}$ and $\mathbb{F}^{k+1}$ are constructed in the same way by using, for $\mathbb{F}$, a maximum of the $\mathbb{F}^{k+2}, \mathbb{F}^{k+3}, \cdots, \mathbb{F}^{N+1}$ nearby and of $p_t + \tilde{H}^{k,j}(x,t,r,p_x)$ where $\tilde{H}^{k,j}$ is a specific Hamiltonian on $\mathbf{M}^{k,j} \times (0,T)$, while for $\mathbb{F}^{k+1}$, one has to built a tangential Hamiltonian $\mathbb{F}_T^{k+2}$ and take the maximum with $p_t + \tilde{H}^{k,j}(x,t,r,p_x)$. The construction of $\mathbb{F}_T^{k+2}$ is the same as in the previous section and is based on computing the element of $\mathbf{BCL}(x,t)$ for $(x,t) \in \tilde{\mathbf{M}}^{k,j} \times (0,T)$ coming from $\tilde{\mathbf{M}}^{\bar{k},l} \times (0,T)$ for $\bar{k} > k$ and for the nearby connected components of the $\tilde{\mathbf{M}}^{\bar{k}} \times (0,T)$.

Chapter 15

Stability results

Stability results are of course a fundamental feature of viscosity solutions. But in the case of stratified media the situation is more complex for two reasons: (i) the non-uniqueness of solutions; (ii) the possibility of moving/creating/deleting some parts of the stratification.

In order to deal with these difficulties, we proceed in two steps: we first provide a stability result for the *minimal solution* in the case where the structure of the stratification is constant. Then we extend this stability result when the stratification itself converges to some final stratification (in a specific sense).

15.1 Stability under constant structure of the stratification

A stability result for a stratified problem requires two ingredients; first a suitable notion of convergence for regular stratifications and then some assumptions on the convergence of the Hamiltonians.

It is clear enough that the first point is a key one and, in [25], the following definition is given for the convergence of regular stratifications

Definition 15.1.1 *We say that a sequence $(\mathbb{M}_\varepsilon)_\varepsilon$ of regular stratification of $\mathbb{R}^N$ converges to a regular stratification $\mathbb{M}$ if, for each $x \in \mathbb{R}^N$, there exists $r > 0$, an AFS $\mathbb{M}^\star = \mathbb{M}^\star(x,r)$ in $\mathbb{R}^N$ and, for any $\varepsilon > 0$, changes of coordinates $\Psi^x_\varepsilon, \Psi^x$ as in Definition 3.3.4 such that $\Psi^x_\varepsilon(x) = \Psi^x(x)$ and*

(i) $\Psi^x_\varepsilon(\mathbf{M}^k_\varepsilon \cap B(x,r)) = \mathbb{M}^\star \cap \Psi^x_\varepsilon(B(x,r))$, $\Psi^x(\mathbf{M}^k \cap B(x,r)) = \mathbb{M}^\star \cap \Psi^x(B(x,r))$.

(ii) the changes of coordinates Ψ^x_ε converge in $C^1(B(x,r))$ to Ψ^x and their inverses $(\Psi^x_\varepsilon)^{-1}$ defined on $\Psi^x(B(x,r))$ also converge in C^1 to $(\Psi^x)^{-1}$.

This definition essentially means that a sequence $(\mathbb{M}_\varepsilon)_\varepsilon$ of stratification converges to $\mathbb{M}$ if the $\mathbb{M}_\varepsilon$ are, locally, just *smooth*, little deformations of $\mathbb{M}$. This excludes a lot of interesting cases and, in particular, the following one in $\mathbb{R}^3$, see Figure 15.1: we define $\mathbb{M}$ by

$$\mathbf{M}^1 := \{(0,0,x_3),\ x_3 \in \mathbb{R}\}\ ,\ \mathbf{M}^2 := \{(x_1,|x_1|,x_3),\ x_1 \in \mathbb{R}\setminus\{0\},\ x_3 \in \mathbb{R}\}\,,$$

and $\mathbf{M}^0 = \emptyset$, $\mathbf{M}^3 = \mathbb{R}^3 \setminus (\mathbf{M}^1 \cup \mathbf{M}^2)$. Defining $\mathbb{M}_\varepsilon$ through

$$\mathbf{M}^2_\varepsilon := \{(x_1,(x_1^2+\varepsilon^2)^{1/2}-\varepsilon,x_3),\ x_1 \in \mathbb{R}\setminus\{0\},\ x_3 \in \mathbb{R}\}\,,$$

and with $\mathbf{M}^0_\varepsilon = \mathbf{M}^0$, $\mathbf{M}^1_\varepsilon = \mathbf{M}^1$ and $\mathbf{M}^3_\varepsilon = \mathbb{R}^3 \setminus (\mathbf{M}^1 \cup \mathbf{M}^2_\varepsilon)$, we see that we do not have the expected convergence with the above definition. Indeed, the dashed axis on Figure 15.1 which should converge to the x_3-axis of the limiting stratification does not exist in the approximating stratifications.

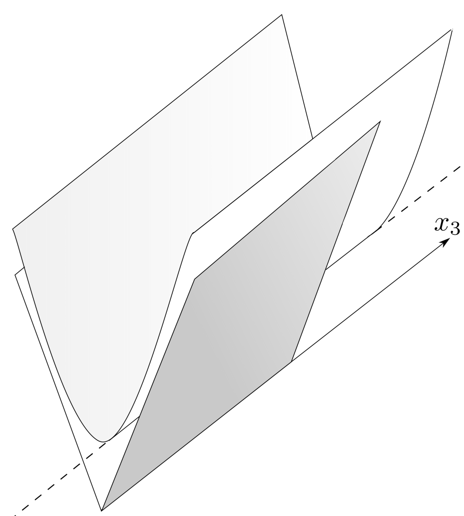


Figure 15.1: The "book" approximation

In the same way, if we set, $\mathbf{M}^0_\varepsilon = \emptyset$,

$$\mathbf{M}^1_\varepsilon := \{(\varepsilon,0,x_3),\ x_3 \in \mathbb{R}\} \cup \{(-\varepsilon,0,x_3),\ x_3 \in \mathbb{R}\}\ ,$$

$$\mathbf{M}^2_\varepsilon := \{(x_1+\varepsilon,x_1-\varepsilon,x_3),\ x_1>0,\ x_3 \in \mathbb{R}\} \cup \{(x_1-\varepsilon,x_1+\varepsilon,x_3),\ x_1<0,\ x_3 \in \mathbb{R}\}\,,$$

and $\mathbf{M}^3_\varepsilon = \mathbb{R}^3 \backslash (\mathbf{M}^1 \cup \mathbf{M}^2_\varepsilon)$, the $\mathbb{M}_\varepsilon$ do not converge in the sense of the above definition. This second example is more tricky since the limiting $\mathbf{M}^1$ is obtained by merging the two connected components of the $\mathbf{M}^1_\varepsilon$, a case which is clearly excluded by [25].

15.1.1 A better version of the stability result

The aim of this section is thus to provide a notion of convergence of stratification which corrects the defects above, and an associated notion of convergence for the Hamiltonians in order to have a first stability result in the case when we have a sequence of problems on stratified domains which have locally the same structure: by this sentence we mean that no new part of the stratification will appear (no creation of new discontinuities for the equation) and no part will be removed (no elimination of discontinuities). We address these questions later in this chapter.

To do so, we concentrate on the equation in $\mathbb{R}^N \times (0,T)$, the case $t = 0$ being treated analogously. In order to formulate the stability result, a notion of convergence of stratifications of [25] is changed into the more general following definition.

Definition 15.1.2 *We say that a sequence $(\mathbb{M}_\varepsilon)_\varepsilon$ of regular stratification of $\mathbb{R}^N \times (0,T)$ converges to a regular stratification $\mathbb{M}$ if*

(i) for any $k = 0, \cdots, N+1$, $\mathbf{M}^k_\varepsilon \to \mathbf{M}^k$ for the Hausdorff distance,

(ii) for any $k = 1, \cdots, N+1$, for any $(x,t) \in \mathbf{M}^k$, there exists $r > 0$ and, for any $\varepsilon > 0$, C^1-changes of coordinates $\Psi^{x,t}_\varepsilon : B((x,t),r) \to \mathbb{R}^N \times (0,T)$ such that
(i) $\Psi^{x,t}_\varepsilon(\mathbf{M}^k \cap B((x,t),r)) = \mathbf{M}^k_\varepsilon \cap B((x,t),r))$.
(ii) the changes of coordinates $\Psi^{x,t}_\varepsilon$ and their inverses $(\Psi^{x,t}_\varepsilon)^{-1}$ converge in C^1 to identity in a neighborhood of (x,t).

We denote this convergence by $\mathbb{M}_\varepsilon \xrightarrow{RS} \mathbb{M}$ where RS stands for Regular Stratification.

In this definition, contrarily to the preceeding one, we have, for any k, a C^1-convergence $\mathbf{M}^k_\varepsilon$ to $\mathbf{M}^k$ through the convergence of the $\Psi^{x,t}_\varepsilon$ but NOT for the whole stratification. Again we recall that no new part of the stratification (with a dimension $l < k$) can be created in this passage to the limit and no part of the stratification can really disappear (except with the merging of the above example).

Then we also consider, for each $\varepsilon > 0$, the associated Hamilton-Jacobi-Bellman problem in the stratified domain $\mathbb{M}_\varepsilon$. The meaning of sub and supersolutions is the one that is introduced in Definition 13.1.1, with the family of Hamiltonians $\mathbb{F}_\varepsilon$ and $(\mathbb{F}^k_\varepsilon)$ that are constructed from $\mathbb{M}_\varepsilon$ and some family $\mathbf{BCL}_\varepsilon$. In order to simplify we write $(\text{HJB-S})_\varepsilon$ for the equation associated to $\mathbb{F}_\varepsilon$ and $(\mathbb{F}^k_\varepsilon)$.

In order to formulate the following stability result, we have to define limiting Hamiltonians for the $\mathbb{F}^k_\varepsilon(x,t,r,p)$ which are defined only if $p \in T_{(x,t)}\mathbf{M}^k$. The definition of the RS-convergence gives us the right way to do it. If $(x,t) \in \mathbf{M}^k$, we set

$$\liminf{}_* \mathbb{F}^k_\varepsilon(x,t,r,p) = \liminf_{\substack{(x_\varepsilon,t_\varepsilon)\in\mathbf{M}^k_\varepsilon\to(x,t),\ r_\varepsilon\to r \\ p_\varepsilon\in T_{(x_\varepsilon,t_\varepsilon)}\mathbf{M}^k_\varepsilon\to p,\ \varepsilon\to 0}} \mathbb{F}^k(x_\varepsilon,t_\varepsilon,r_\varepsilon,p_\varepsilon)\ .$$

Notice that this definition is consistent with Definition 15.1.2 since if $p_\varepsilon \in T_{(x_\varepsilon,t_\varepsilon)}\mathbf{M}^k_\varepsilon \to p$ then $p \in T_{(x,t)}\mathbf{M}^k$.

Theorem 15.1.3 *Assume that $(\mathbb{M}_\varepsilon)_\varepsilon$ is a sequence of regular stratifications in $\mathbb{R}^N \times (0,T)$ such that $\mathbb{M}_\varepsilon \xrightarrow{RS} \mathbb{M}$, then the following holds*

(*i*) *if, for all $\varepsilon > 0$, v_ε is a lsc supersolution of (HJB-S)$_\varepsilon$, then $\underline{v} = \liminf_* v_\varepsilon$ is a lsc supersolution of (HJB-S), the HJB problem associated with $\mathbb{F} = \limsup^* \mathbb{F}_\varepsilon$.*

(*ii*) *If, for $\varepsilon > 0$, u_ε is an usc subsolution of (HJB-S)$_\varepsilon$ and if the Hamiltonians $(\mathbb{F}^k_\varepsilon)_{k=0..N}$ satisfy* **(NC)** *and* **(TC)** *with uniform constants and on uniform neighborhood of $\mathbb{M}$, then $\bar{u} = \limsup^* u_\varepsilon$ is a subsolution of (HJB-S) with $\mathbb{G}^k = \liminf_* \mathbb{F}^k_\varepsilon$ for any $k = 0..N$.*

In the statement of Theorem 15.1.3, we have used the notation $\mathbb{G}^k$ for $\liminf_* \mathbb{F}^k_\varepsilon$ because it is not clear a priori that we have a stratified problem, i.e. that there exists **BCL** and $\mathbb{F}$ is given by (4.3) such that $\mathbb{G}^k = \mathbb{F}^k$ given by (13.3).

Proof — Result (i) is standard since only the $\mathbb{F}_\varepsilon/\mathbb{F}$-inequalities are involved and therefore (i) is nothing but the standard stability result for discontinuous viscosity solutions with discontinuous Hamiltonians, see [88].

For (ii), contrarily to [25], we remain on the real stratification, without reducing to the case of a fixed AFS. If $(x_0,t_0) \in \mathbf{M}^k$ is a *strict* local maximum point of $\bar{u} - \phi$ on $\mathbf{M}^k$, where ϕ is a C^1 function in $\mathbb{R}^N \times (0,T)$, we consider the functions

$$u_\varepsilon(x,t) - \phi(x,t) - L\psi_\varepsilon(x,t)\ ,$$

where $\psi_\varepsilon(x,t) = \mathrm{dist}((\Psi^{x,t}_\varepsilon)^{-1}(x,t), \mathbf{M}^k)$, $\mathrm{dist}(\cdot, \mathbf{M}^k)$ denoting the distance to $\mathbf{M}^k$ which is smooth in a neighborhood of $\mathbf{M}^k$, except on $\mathbf{M}^k$.

For ε small enough, this function has a maximum point $(x_\varepsilon,t_\varepsilon)$ near (x_0,t_0). Of course, choosing a small enough neighborhood of (x_0,t_0) [in order to have no point of $\mathbf{M}^l$ for $l < k$] and ε small enough, we know that $(x_\varepsilon,t_\varepsilon) \in \mathbf{M}^l_\varepsilon$ for some $l \geq k$.

If $(x_\varepsilon, t_\varepsilon) \in \mathbf{M}^l_\varepsilon$ for $l > k$, we have (because u_ε is an usc subsolution of $(\text{HJB-S})_\varepsilon$ and since ψ_ε is differentiable outside $\mathbf{M}^k$)

$$\mathbb{F}^l_\varepsilon\Big(x_\varepsilon, t_\varepsilon, u_\varepsilon(x_\varepsilon, t_\varepsilon), D\phi(x_\varepsilon, t_\varepsilon) + LD\psi_\varepsilon(x_\varepsilon, t_\varepsilon)\Big) \leq 0 \;.$$

Next we remark that, on the one hand, $D\big[\operatorname{dist}((x,t), \mathbf{M}^k)\big]$ is orthogonal to $\mathbf{M}^k$ (or more precisely to its tangent space) and on the other hand $\big|D\big[\operatorname{dist}((x,t), \mathbf{M}^k)\big]\big| = 1$ where the distance function is differentiable; therefore by Definition 15.1.2 and the convergence of $(\Psi^{x,t}_\varepsilon)^{-1}$ to identity in C^1, $D\psi_\varepsilon(x_\varepsilon, t_\varepsilon)$ is a transverse vector to $\mathbf{M}^k_\varepsilon$. Moreover, using notations as if we were in the flat case, it is easy to see that

$$|[D\psi_\varepsilon(x_\varepsilon, t_\varepsilon)]_\perp| \geq \kappa > 0 \,,$$

for some $\kappa \in (0,1)$ which does not depend neither on ε nor on l. Here we have strongly used that the distance to $\mathbf{M}^k$ is smooth if we are not on $\mathbf{M}^k$.

Hence, using **(NC)** which holds in an uniform neighborhood of $\mathbf{M}^k_\varepsilon$ by assumptions, we deduce that the $\mathbb{F}^l_\varepsilon$-inequality cannot hold if we have chosen L large enough, and of course L can be chosen independently of ε.

Therefore $(x_\varepsilon, t_\varepsilon) \in \mathbf{M}^k_\varepsilon$ and $(x_\varepsilon, t_\varepsilon)$ is a local maximum point of $u_\varepsilon(x,t) - \phi(x,t)$ on $\mathbf{M}^k_\varepsilon$ (we can drop the distance term since we look at the function only on $\mathbf{M}^k_\varepsilon$ where $\psi_\varepsilon \equiv 0$ by definition of $\Psi^{x,t}_\varepsilon$). Hence

$$\mathbb{F}^k_\varepsilon\Big(x_\varepsilon, t_\varepsilon, u_\varepsilon(x_\varepsilon, t_\varepsilon), D\phi(x_\varepsilon, t_\varepsilon)\Big) \leq 0 \;.$$

But using that $\bar{u} = \limsup^* u_\varepsilon$ and that (x_0, t_0) is a strict local maximum point of $\bar{u} - \phi$ on $\mathbf{M}^k$, classical arguments imply that $(x_\varepsilon, t_\varepsilon) \to (x_0, t_0)$ and the conclusion of the proof follows as in the standard case.

Q.E.D.

15.1.2 Sufficient conditions for stability

We conclude this section with some sufficient conditions on **BCL** for the stability of solutions.

Lemma 15.1.4 *For any $\varepsilon > 0$, let $\mathbf{BCL}_\varepsilon$ satisfying* $(\mathbf{H_{BCL}})$, **(TC-BCL)** *and* **(NC-BCL)** *on $\mathbb{M}_\varepsilon$ with constants independent of ε and assume that $\mathbb{M}_\varepsilon \xrightarrow{RS} \mathbb{M}$ where $\mathbb{M}$ is* *a regular stratification. If, for any $(x,t) \in \mathbb{R}^N \times (0,T)$, we have*

$$\mathbf{BCL}(x,t) = \limsup{}^* \, \mathbf{BCL}_\varepsilon(x,t) = \bigcap_{r>0} \bigcap_{\varepsilon>0} \left(\bigcup_{\substack{|(y,s)-(x,t)|\leq r \\ \tilde\varepsilon \leq \varepsilon}} \mathbf{BCL}_{\tilde\varepsilon}(y,s) \right) ,$$

then $\mathbb{F} = \limsup^* \ \mathbb{F}_\varepsilon$ and, for every $k \in \{0, ..., N\}$, $\liminf_* \ \mathbb{F}^k_\varepsilon \geq \mathbb{F}^k$.

Proof — Since we can assume w.l.o.g. that we are in a flat (and static) situation, let us first notice that the Hamiltonians $\mathbb{F}^k_\varepsilon$ are all defined on the same set. Then the convergence of $\mathbf{BCL}_\varepsilon$ implies that $(\mathbf{BCL}_\varepsilon)|_k$ (the restriction to $\mathbf{M}^k \times [0,T]$) converges locally uniformly to $\mathbf{BCL}|_k$. It follows directly that

$$\mathbb{F}^k(x,r,p) := \sup_{\substack{(b,c,l)\in\mathbf{BCL}_\varepsilon(x,t)\\ b\in T_x\mathbf{M}^k}} \big\{ -b\cdot p + cr - l \big\} \longrightarrow \sup_{\substack{(b,l)\in\mathbf{BCL}(x,t)\\ b\in T_x\mathbf{M}^k}} \big\{ -b\cdot p + cr - l \big\} = \mathbb{F}^k(x,r,p) \, .$$

Q.E.D.

Corollary 15.1.5 *For any $\varepsilon > 0$, let $\mathbf{BCL}_\varepsilon$ satisfy $(\mathbf{H_{BCL}})$ with constants independent of ε, and consider an associated regular stratification $(\mathbb{M}_\varepsilon, \Psi_\varepsilon)$. We assume that $\mathbf{BCL}_\varepsilon \to \mathbf{BCL}$ in the sense of Haussdorf distance and that $\mathbb{M}_\varepsilon \xrightarrow{RS} \mathbb{M}$. Let U_ε be the unique solution of (HJB-S)$_\varepsilon$. Then*

$$U_\varepsilon \to U \quad \text{locally uniformly in } \mathbb{R}^N \times [0,\infty) \, ,$$

where U is the unique solution of the limit problem (HJB-S).

Proof — The proof is immediate: by the convergence of $\mathbf{BCL}_\varepsilon$ and $\mathbb{M}_\varepsilon$, after a suitable change of variables we are reduced to considering the case of a constant local AFS, $\mathbb{M}$. Then we apply Lemma 15.1.4 which implies that the $(\tilde{\mathbb{F}}^k_\varepsilon)_k$ converge to the $(\tilde{\mathbb{F}}^k)_k$. We invoke Theorem 15.1.3 which says that the half-relaxed limits of the U_ε are sub and supersolutions of the limit problem, (HJB-S). And finally, the comparison result implies that all the sequence converges to U.

Q.E.D.

15.2 Stability under structural modifications

In the previous section, we have provided a stability result in the case where there is no modification of the structure of the stratification. On the contrary, in this section, we consider cases where this structure can be changed by the appearance of new discontinuity sets or the disappearance of existing ones. Since the stability result of the first part will be, anyway, the keystone of this result we have to show how to introduce a new part of $\mathbf{M}^k$ or remove an existing one in order to manage these changes of stratifications. Again we only treat the case of $\mathbb{R}^N \times (0,T)$.

15.2.1 Introducing new parts of the stratification

The result is the

Proposition 15.2.1 *Let $(\mathbf{M}^k, \mathbb{F}^k)_k$ be a standard stratified problem and $u : \mathbb{R}^N \times (0,T) \to \mathbb{R}$ an usc subsolution of this problem. If $\mathcal{M}$ is a m-dimensional submanifold of $\mathbf{M}^l$ for some $m < l$ and if the normal controlability assumption is satisfied in a neighborhood of $\mathcal{M}$, then*

$$\mathbb{F}^{\mathcal{M}}(x,t,u,Du) \leq 0\,,$$

where for $x \in \mathcal{M}$, $t \in (0,T)$, $z \in \mathbb{R}$, $p = (p_x, p_t) \in \mathbb{R}^N \times \mathbb{R}$

$$\mathbb{F}^{\mathcal{M}}(x,t,z,p) := \sup_{\substack{(b,c,l)\in \mathbf{BCL}(x,t)\\ b\in T_{(x,t)}\mathcal{M}}} \big\{ -b\cdot p + cz - l \big\}\,.$$

This result means that we can create an artificial part of (here) $\mathbf{M}^m$ inside $\mathbf{M}^l$ since $\mathcal{M}$ can be seen as some new part of $\mathbf{M}^m$.

Proof — Since the result is local, we can assume without loss of generality that $\mathbf{M}^l = \mathbb{R}^l$ and that $\mathcal{M}$ is an affine subspace of $\mathbb{R}^l$. If $\phi : \mathbb{R}^N \times [0,T] \to \mathbb{R}$ is a smooth function and $(\bar x, \bar t) \in \mathcal{M}$ is a strict, local maximum point of $u - \phi$ on $\mathcal{M}$, we have to show that

$$\mathbb{F}^{\mathcal{M}}(\bar x, \bar t, u(\bar x, \bar t), D\phi(\bar x, \bar t)) \leq 0\,.$$

To do so, for $0 < \varepsilon \ll 1$, we consider the function defined on $\mathbf{M}^l = \mathbb{R}^l$

$$(x,t) \mapsto u(x,t) - \phi(x,t) - \frac{[d(x,t)]^2}{\varepsilon}\,,$$

where $d(x,t) = d((x,t),\mathcal{M})$ is the distance function to $\mathcal{M}$ which is C^1 outside $\mathcal{M}$ but not on $\mathcal{M}$. On the contrary, $(x,t) \mapsto [d(x,t)]^2$ is C^1 even on $\mathcal{M}$

By standard arguments, this function has a maximum point at $(x_\varepsilon, t_\varepsilon)$ and we have

$$(x_\varepsilon, t_\varepsilon) \to (\bar{x}, \bar{t}) \quad u(x_\varepsilon, t_\varepsilon) \to u(\bar{x}, \bar{t}) \quad \text{and} \quad \frac{[d(x_\varepsilon, t_\varepsilon)]^2}{\varepsilon} \to 0 \quad \text{as } \varepsilon \to 0 \,.$$

Since u is a subsolution of the stratified problem, we have

$$\mathbb{F}^l(x_\varepsilon, t_\varepsilon, u(x_\varepsilon, t_\varepsilon), D\phi(x_\varepsilon, t_\varepsilon) + \frac{2d(x_\varepsilon, t_\varepsilon)Dd(x_\varepsilon, t_\varepsilon)}{\varepsilon}) \leq 0 \,.$$

In order to deduce the result from this inequality, we use the tangential continuity (here the full continuity since our argument is restricted to $\mathbf{M}^l = \mathbb{R}^l$): if $(y_\varepsilon, s_\varepsilon)$ is the unique projection of $(x_\varepsilon, t_\varepsilon)$ on $\mathcal{M}$, we have $|y_\varepsilon - x_\varepsilon| + |t_\varepsilon - s_\varepsilon| = d(x_\varepsilon, t_\varepsilon)$ and using the tangential continuity we have

$$\mathbb{F}^l(y_\varepsilon, s_\varepsilon, u(x_\varepsilon, t_\varepsilon), D\phi(x_\varepsilon, t_\varepsilon) + \frac{2d(x_\varepsilon, t_\varepsilon)Dd(x_\varepsilon, t_\varepsilon)}{\varepsilon}) \leq o_\varepsilon(1) \,.$$

On the other hand, if $b^1 \in T_{(y_\varepsilon, s_\varepsilon)}\mathcal{M}$, we have $b^1 \cdot Dd(x_\varepsilon, t_\varepsilon) = 0$ because $(y_\varepsilon, s_\varepsilon)$ is the unique projection of $(x_\varepsilon, t_\varepsilon)$ on $\mathcal{M}$. Therefore

$$\mathbb{F}^{\mathcal{M}}(y_\varepsilon, s_\varepsilon, u(x_\varepsilon, t_\varepsilon), D\phi(x_\varepsilon, t_\varepsilon)) \leq o_\varepsilon(1) \,.$$

In order to conclude, we have just to use the tangential continuity on $\mathbf{M}^l = \mathbb{R}^l$ combined with the normal controllability: if $(b, c, l) \in \mathbf{BCL}(\bar{x}, \bar{t})$ with $b \in T_{(\bar{x}, \bar{t})}\mathcal{M}$, there exists $(b^1_\varepsilon, c^1_\varepsilon, l^1_\varepsilon) \in \mathbf{BCL}(y_\varepsilon, s_\varepsilon)$ with $b^1_\varepsilon \in T_{(y_\varepsilon, s_\varepsilon)}\mathcal{M}$ and such that $(b^1_\varepsilon, c^1_\varepsilon, l^1_\varepsilon) \to (b, c, l)$ as $\varepsilon \to 0$. Using this property, the result is obtained by letting ε tend to 0.

Q.E.D.

15.2.2 Eliminable parts of the stratification

In this section, the aim is to remove "artificial" parts of the stratification, i.e. parts on which there is no real discontinuity and the viscosity inequalities are just a consequence of those existing in lower codimensions manifolds. Our result is the

Proposition 15.2.2 *Let* $(\mathbf{M}^k, \mathbb{F}^k)_k$ *be a standard stratified problem and* $u : \mathbb{R}^N \times (0, T) \to \mathbb{R}$ *an usc subsolution of this problem. If* $\mathcal{M} \subset \mathbf{M}^k$ *is a submanifold such that*
(i) $\mathcal{M} \subset \overline{\mathbf{M}^l}$ *for some* $l > k$*,*
(ii) $\mathcal{M} \cup \mathbf{M}^l$ *is a* l*-dimensional submanifold of* $\mathbb{R}^N$ *and*

(iii) **BCL** *satisfies the tangential continuity assumption on* $(\mathcal{M} \cup \mathbf{M}^l) \times (0,T)$*, then* u *is a subsolution of*

$$\tilde{\mathbb{F}}^l(x,t,u,Du) \leq 0 \quad \text{on } (\mathcal{M} \cup \mathbf{M}^l) \times (0,T) \;,$$

where, for $x \in \mathcal{M} \cup \mathbf{M}^l$*,* $t \in (0,T)$*,* $z \in \mathbb{R}$*,* $p = (p_x, p_t) \in \mathbb{R}^N \times \mathbb{R}$

$$\tilde{\mathbb{F}}^l(x,t,z,p) := \sup_{\substack{(b,c,l) \in \mathbf{BCL}(x,t) \\ b \in T_{(x,t)}(\mathcal{M} \cup \mathbf{M}^l)}} \big\{ -b \cdot p + cz - l \big\} \,.$$

In other words, this proposition means that $\mathbf{M}^l$ can be replaced by $\mathcal{M} \cup \mathbf{M}^l$: the higher co-dimension discontinuity manifold $\mathcal{M}$ can be removed and integrated to $\mathbf{M}^l$. A typical case we have in mind is similar to the one we consider at the beginning of the stability chapter: if $\mathbb{R}^3$, we define $\mathbb{M}$ by

$$\mathbf{M}^1 := \{(0,0,x_3),\, x_3 \in \mathbb{R}\} \;,\; \mathbf{M}^2 := \{(x_1, x_1^2, x_3),\, x_1 \in \mathbb{R} \setminus \{0\},\, x_3 \in \mathbb{R}\} \,,$$

and $\mathbf{M}^0 = \emptyset$, $\mathbf{M}^3 = \mathbb{R}^3 \setminus (\mathbf{M}^1 \cup \mathbf{M}^2)$, we may have in mind to remove $\mathbf{M}^1$ and to see if we can replace $\mathbf{M}^2$ by $\{(x_1, x_1^2, x_3),\, x_1 \in \mathbb{R}\setminus,\, x_3 \in \mathbb{R}\}$.

Proof — Again we can assume without loss of generality that $\mathbf{M}^l = \mathbb{R}^l$ and that $\mathcal{M}$ is an affine subspace of $\mathbb{R}^l$. If $\phi : \mathbb{R}^N \times [0,T] \to \mathbb{R}$ is a smooth function and $(\bar{x}, \bar{t}) \in \mathcal{M}$ is a strict, local maximum point of $u - \phi$ on $(\mathcal{M} \cup \mathbf{M}^l)$, we have to show that

$$\tilde{\mathbb{F}}^l(\bar{x}, \bar{t}, u(\bar{x}, \bar{t}), D\phi(\bar{x}, \bar{t})) \leq 0 \;.$$

Here the difficulty is that the set $(b,c,l) \in \mathbf{BCL}(x,t)$ with $b \in T_{(x,t)}(\mathcal{M} \cup \mathbf{M}^l)$ is larger than the set for which $b \in T_{(x,t)}\mathcal{M}$.

If $b \in T_{(x,t)}\mathcal{M}$, then the desired inequality is nothing but a consequence of the $\mathbb{F}^k$-inequality on $\mathcal{M}$ and therefore we can assume without loss of generality that $b \notin T_{(x,t)}\mathcal{M}$ and we write

$$b = b^\top + b^\perp \quad \text{with } b^\top \in T_{(x,t)}\mathcal{M},\ b^\perp \text{ in its orthogonal space.}$$

Then we consider $D = \{(x,t) \in \mathbf{M}^l = \mathbb{R}^l;\ (x - \bar{x}, t - \bar{t}) \cdot b^\perp > 0\}$ and, for $0 < \varepsilon \ll 1$, we consider on D the function

$$(x,t) \mapsto u(x,t) - \phi(x,t) - \frac{\varepsilon}{(x - \bar{x}, t - \bar{t}) \cdot b^\perp} \;.$$

We first remark that the normal controllability assumption on $\mathbf{M}^k$ (and therefore on $\mathcal{M}$) implies that

$$u(\bar{x}, \bar{t}) = \limsup_{\substack{(x,t) \to (\bar{x}, \bar{t}) \\ (x,t) \in D \times [0,T]}} u(x,t) \;,$$

and because of this property, standard arguments show that this function has a maximum point at $(x_\varepsilon, t_\varepsilon) \in D \times [0,T]$ with

$$(x_\varepsilon, t_\varepsilon) \to (\bar{x}, \bar{t}) \quad u(x_\varepsilon, t_\varepsilon) \to u(\bar{x}, \bar{t}) \quad \text{and} \quad \frac{\varepsilon}{(x_\varepsilon - \bar{x}, t_\varepsilon - \bar{t}) \cdot b^\perp} \to 0 \quad \text{as } \varepsilon \to 0 \, .$$

Using the tangential continuity, there exists $(b^1_\varepsilon, c^1_\varepsilon, l^1_\varepsilon) \in \mathbf{BCL}(x_\varepsilon, t_\varepsilon)$ with $b^1_\varepsilon \in T_{(x_\varepsilon,t_\varepsilon)}\mathbf{M}^l$ and such that $(b^1_\varepsilon, c^1_\varepsilon, l^1_\varepsilon) \to (b, c, l)$ as $\varepsilon \to 0$ and the $\mathbb{F}^l$-inequality for such triplet yields

$$-b^1_\varepsilon \cdot \left(D\phi(x_\varepsilon, t_\varepsilon) - \frac{\varepsilon b^\perp}{((x_\varepsilon - \bar{x}, t_\varepsilon - \bar{t}) \cdot b^\perp)^2} \right) + c^1_\varepsilon u(x_\varepsilon, t_\varepsilon) - l^1_\varepsilon \leq 0 \, .$$

But $-b^1_\varepsilon \cdot -b^\perp \to |b^\perp|^2$ as $\varepsilon \to 0$ and therefore the corresponding term is positive for ε small enough; therefore

$$-b^1_\varepsilon \cdot D\phi(x_\varepsilon, t_\varepsilon) + c^1_\varepsilon u(x_\varepsilon, t_\varepsilon) - l^1_\varepsilon \leq 0 \, ,$$

and the conclusion follows by letting ε tends to 0.

Q.E.D.

15.2.3 Sub/Super-stratifications and a more general stability result

The two preceeding sections lead us to introduce the following definition

Definition 15.2.3 *Let $\mathbb{M} = (\mathbf{M}^k, \mathbb{F}^k)_k$, $\tilde{\mathbb{M}} = (\tilde{\mathbf{M}}^k, \tilde{\mathbb{F}^k})_k$ be standard stratified problems associated with the same* **BCL** *set.*
(i) $\tilde{\mathbb{M}}$ is said to be a super-stratification of $\mathbb{M}$ if it can be deduced from $\mathbb{M}$ by applying a finite (or countable) number of time Proposition 15.2.1.
(ii) $\tilde{\mathbb{M}}$ is said to be a sub-stratification of $\mathbb{M}$ if it can be deduced from $\mathbb{M}$ by applying a finite (or countable) number of time Proposition 15.2.2.

Before commenting these definitions, we use them to extend the notion of convergence of stratification.

Definition 15.2.4 *A sequence of stratifications $(\mathbb{M}_\varepsilon)_\varepsilon$ is said to converge to a stratification $\mathbb{M}$ if there exists a sequence $(\tilde{\mathbb{M}}_\varepsilon)_\varepsilon$ of stratification and a stratification $\tilde{\mathbb{M}}$ such that*

(*i*) *for any* ε, $\tilde{\mathbb{M}}_\varepsilon$ *is a super-stratification of* $\mathbb{M}_\varepsilon$,
(*ii*) $\tilde{\mathbb{M}}_\varepsilon \xrightarrow{RS} \tilde{\mathbb{M}}$,
(*iii*) $\mathbb{M}$ *is a sub-stratification of* $\tilde{\mathbb{M}}$.

Using this new notion, we have the

Theorem 15.2.5 *The stability results of Theorem 15.1.3 remains valid if the sequence of stratified problem* $(\mathbb{M}_\varepsilon)_\varepsilon$ *converges to a stratified problem* $\mathbb{M}$ *in the sense of Definition 15.2.4.*

Theorem 15.2.5 makes precise a very simple and natural idea: of course, the conditions imposed by Theorem 15.1.3 on the convergence of stratified problems are very restrictive and do not cover (for example) the convergence of problems without discontinuities (like, for instance, Fillipov's approximation) to a problem with discontinuities. To correct this defect, it suffices to introduce suitable "artificial" elements of stratification, using Proposition 15.2.1 (thus creating a super-stratification) then to use Theorem 15.1.3 and, at the end, we can drop some useless part of the obtained stratification using the elimination result of Proposition 15.2.2. Of course, all these operations require suitable tangential continuity or normal controllability assumptions which are partially hidden under the various definitions we give.

Chapter 16

Applications

16.1 Where the stratified formulation is needed

The following problem was studied by Giga and Hamamuki [72] as a model of $2-d$ nucleation in crystal growth phenomena. In [72], the equations were written with concave Hamiltonians but we re-formulate them with convex ones to be in the framework of this book, and in $\mathbb{R}^N$ instead of $\mathbb{R}^2$ since there is no additional difficulty.

The simplest equation takes the form

$$u_t + |D_x u| = I(x) \quad \text{in } \mathbb{R}^N \times (0,T) \tag{16.1}$$

where the function $I : \mathbb{R}^N \to \mathbb{R}$ is given by

$$I(x) = \begin{cases} 1 & \text{if } x \neq 0, \\ 0 & \text{if } x = 0. \end{cases}$$

This equation is associated with the initial data

$$u(x,0) = u_0(x) \quad \text{in } \mathbb{R}^N, \tag{16.2}$$

where $u_0 : \mathbb{R}^N \to \mathbb{R}$ is a bounded continuous function.

Of course, the key difficulty in this problem comes from the discontinuity of I: in terms of classical viscosity solutions' theory, Ishii's definition gives a subsolution condition which is

$$u_t + |Du| \leq I^*(x) = 1 \quad \text{in } \mathbb{R}^N \times (0,T)\ ,$$

and the fact that $I(0) = 0$ completely disappears in this condition.

On the other hand, and formally for the time being, the classical control interpretation of (16.1) is that the system can evolve at any velocity b^x with $|b^x| \leq 1$ and the cost is 1 outside 0 and 0 at 0. If we choose, $u_0 = 0$ then the natural value function is $U(x,t) = \min(|x|, t)$ in $\mathbb{R}^N \times [0,T]$ (adopting the strategy to go as quickly as possible to 0 and then staying there). On the other hand, one easily checks that $u(x,t) = t$ is a subsolution.

As a consequence, $u(x,t) > U(x,t)$ if $|x| < t$ although U should be the good solution; therefore we can expect no comparison result in this framework. But it is also clear that u is an "unnatural" subsolution, due to the fact that Ishii's definition erases the value 0 of I at $x = 0$ which is undoubtedly an important information!

Now we turn to the stratified formulation (which could certainly be simplified in this context): if $t > 0$, we have

$$\mathbf{BCL}(x,t) = \mathbf{BCL}(x) = \{((b^x, -1), 0, I(x));\ |b^x| \leq 1\} ,$$

while if $t = 0$, $\mathbf{BCL}(x,0)$ is the convex hull of $\mathbf{BCL}(x) \cup \{((0,0), 1, u_0(x))\}$.

The stratification of $\mathbb{R}^N \times (0,T)$ just contains $\mathbf{M}^1 = \{0\} \times (0,T)$ and $\mathbf{M}^{N+1} = (\mathbb{R}^N \setminus \{0\}) \times (0,T)$ and we have, for $t > 0$

$$\mathbb{F}^{N+1}(x,t,p) = p_t + |p_x| - 1 ,$$

since $I(x) = 1$ in $\mathbf{M}^{N+1}$, and

$$\mathbb{F}^1(x,t,p) = p_t ,$$

since for $\mathbb{F}^1$, we have to consider only $b^x = 0$ because this Hamiltonian considers the trajectories which stay on $\mathbf{M}^1$. And for $t = 0$, we just have the classical initial condition.

Therefore a subsolution of the problem is an usc function $u : \mathbb{R}^N \times [0,T] \to \mathbb{R}$ which satisfies

$$u_t + |D_x u| \leq 1 \quad \text{in } \mathbb{R}^N \times (0,T) , \tag{16.3}$$

$$u_t \leq 0 \quad \text{on } \mathbf{M}^1 , \tag{16.4}$$

this last subsolution inequality being understood as a 1-d inequality which is obtained by looking at maxima of $u(0,t) - \phi(t)$ for smooth functions ϕ, while the first one is just the classical Ishii's subsolution definition.

A supersolution of the problem is a lsc function $v : \mathbb{R}^N \times [0,T] \to \mathbb{R}$ which satisfies

$$v_t + |D_x v| \geq I(x) \quad \text{in } \mathbb{R}^N \times (0,T) . \tag{16.5}$$

It can be seen on this formulation that it consists in super-imposing the right subsolution inequalities on $\mathbf{M}^1$, while the supersolutions' conditions are nothing but the classical Ishii's conditions. Finally it is easy to see that the $\mathbb{F}_{init}$- conditions reduce to

$$u(x,0) \leq u_0(x) \leq v(x,0) \quad \text{in } \mathbb{R}^N \ . \tag{16.6}$$

In this framework, we can prove the

Theorem 16.1.1 *The following results hold*
(i) We have a comparison result between stratified sub and supersolutions of (16.1)-(16.2)*, i.e. sub and supersolutions which satisfy* (16.3)-(16.4) *and* (16.5) *respectively, with* (16.6)*.*
(ii) There exists a unique stratified solution of (16.1)-(16.2) *which is given by*

$$U(x,t) = \inf\left\{\int_0^t I(X(s))ds + u_0(X(t));\ X(0)=x,\quad |\dot{X}(s)| \leq 1\right\} \ .$$

(iii) This solution is the minimal Ishii's viscosity solution.
(iv) Finally, if $(I_k)_k$ is a sequence of continuous functions such that

$$\liminf_k {}_* \ I_k(x) = I(x) \quad \textit{and} \quad \limsup_k {}^* \ I_k(x) = I^*(x) = 1 \ ,$$

then the unique (classical) viscosity solutions u_k associated to I_k converges locally uniformly to U.

Proof — The proof just consists in applying the result of Chapters 13, 14 and 15, and therefore in checking the normal controlability and tangential regularity assumptions which is obvious here : the comparison result (i) is just a (very) particular case of Theorem 13.2.1, (ii) is obtained by examining carefully the value function of the stratified problem.

For (iii), it is enough to remark that any Ishii's supersolution is a supersolution of the stratified problem, as it was done in Chapter 14.

Finally (iv) is a straightforward adaptation of Chapter 15 : indeed, there exists a sequence $x_k \to 0$ such that $I_k(x_k) \to 0$ and using the stratification $\mathbf{M}^1_k = \{x_k\} \times (0,T)$ and $\mathbf{M}^N_k = \big(\mathbb{R}^N \times (0,T)\big) \setminus \mathbf{M}^1_k$, Proposition 15.2.1 shows that

$$(u_k)_t \leq I_k(x_k) \quad \text{in } \mathbf{M}^1_k \ ,$$

easily leading to the result through the stability result (Corollary 15.1.5) and part (i) of Theorem 16.1.1.

Q.E.D.

Remark 16.1.2 *In [72], Giga and Hamamuki have tested several notions of solutions for (16.1)-(16.2) and remarked that most of them were not completely adapted: for the notion of D or $\bar{D}$-solutions, they tried to impose for the points in $\mathbf{M}^1$ an Ishii's subsolution inequality with $I(x)$ (and not $I^*(x)$) but this was a $\mathbb{R}^N \times (0,T)$- inequality (and not a $\mathbf{M}^1$-one). Although imposing a stronger subsolution condition on $\mathbf{M}^1$ was going in the right direction, this inequality was too strong compared to (16.4), at least the $\bar{D}$-ones, and they found that the problem has no $\bar{D}$-solution in general. They ended up considering enveloppe solutions, i.e. using Result (iii) of Theorem 16.1.1.*

Of course, the simplest case we study above can be generalized in several ways, even if we wish to stay in a similar context: it is clear enough that the case when I vanishes at several points instead of one can be treated exactly in the same way, just changing $\mathbf{M}^1$. A more intriguing case which is considered in [72] is when there exists a closed subset $\mathcal{S}$ of $\mathbb{R}^N$ such that

$$I(x) = \begin{cases} 1 & \text{if } x \notin \mathcal{S}, \\ 0 & \text{if } x \in \mathcal{S}. \end{cases}$$

Giga and Hamamuki aim at treating the case of general closed subset $\mathcal{S}$ of $\mathbb{R}^N$. Unfortunately[(1)], in our framework, we cannot treat the case of any closed subset of $\mathbb{R}^N$. A natural framework is the following: there exists a stratification $\tilde{\mathbf{M}} = (\tilde{\mathbf{M}}^k)_k$ of $\mathbb{R}$ such that

$$\tilde{\mathbf{M}}^N = \mathcal{S}^c \cup Int(\mathcal{S}) \ ,$$

where $Int(\mathcal{S})$ denotes the interior of $\mathcal{S}$,and

$$\partial\mathcal{S} = \tilde{\mathbf{M}}^{N-1} \cup \tilde{\mathbf{M}}^{N-2} \cdots \cup \tilde{\mathbf{M}}^0 \ .$$

Once this hypothesis holds, then we set $\mathbf{M}^k = \tilde{\mathbf{M}}^{k-1} \times (0,T)$ for $1 \leq k \leq N+1$.

Clearly this assumption on $\mathcal{S}$ implies that $\partial\mathcal{S}$ has some regularity properties but, at least, it allows to use all the stratification arguments and therefore all the above results can be extended.

16.2 Where the stratified formulation may (unexpectedly) help

In [27], motivated by a model of solid combustion in heterogeneous media, Roquejoffre and the first author studied the time-asymptotic behaviour of flame fronts evolving with a periodic space-dependent normal velocity using the "level-sets approach".

(1) but maybe we are missing something...

We recall that the "level-sets approach" consists in identifying a moving front with the 0-level-set of a solution of a "geometric type" equation (for which one has a unique viscosity solution). Based on an idea appearing in Barles [21] for constant normal velocity, the "level-set approach" was first used for numerical computations by Osher and Sethian [106] who did these computations for more general normal velocities (in particular curvature dependent ones). Then Evans and Spruck [56], Chen, Giga and Goto [41] developed the theoretical basis.

In [27], the model was leading to an Eikonal Equation

$$u_t + R(x)|Du| = 0 \quad \text{in } \mathbb{R}^N \times (0, +\infty) , \tag{16.7}$$

where, in the most standard case, $R : \mathbb{R}^N \to \mathbb{R}$ is a positive, Lipschitz continuous function.

The key idea in the "level-set approach" is to assume that the initial front Γ_0 is the boundary of, say, a smooth domain Ω_0 (such smoothness is not necessary but we begin with this case to fix idea) and to choose the initial data

$$u(x, 0) = u_0(x) \quad \text{in } \mathbb{R}^N , \tag{16.8}$$

such that $u_0 \in C(\mathbb{R}^N)$ with $\Omega_0 = \{x : \; u_0(x) < 0\}$, $\Gamma_0 = \{x : \; u_0(x) = 0\}$ and $\mathbb{R}^N \setminus (\Omega_0 \cup \Gamma_0) = \{x : \; u_0(x) > 0\}$. Under this assumption, it can be proved that the sets

$$\Omega_t = \{x : \; u(x,t) < 0\},\ \Gamma_t = \{x : \; u(x,t) = 0\} \ \text{ and } \ \mathbb{R}^N \backslash (\Omega_t \cup \Gamma_t) = \{x : \; u(x,t) > 0\} ,$$

are independent of the choice of u_0 satisfying the above conditions, but they depend only on Ω_0 and Γ_0.

This result can be used for (16.7) when R is a positive, Lipschitz continuous function since the classical existence and uniqueness theory applies and allows to define $t \mapsto \Gamma_t$ as the level-set evolution of Γ_0 with normal velocity R. In addition, the solution u is given by the control formula

$$u(x,t) = \inf \{u_0(\gamma(t));\; \gamma(0) = x,\; |\dot\gamma(s)| \leq R(\gamma(s))\} \tag{16.9}$$

where γ is taken among all piecewise C^1 curves.

In [27], results on this propagation are given and in particular on the asymptotic velocity but only in the case of Lipschitz continuous functions R. However an interesting case (which is the purpose of an entire (but formal) section in [27]) was when R is given in $\mathbb{R}^2$ by

$$R(x) = R(x_1, x_2) = \begin{cases} M & \text{if } x_1 \in \mathbb{Z} \\ m & \text{otherwise,} \end{cases}$$

where m, M are positive constants. The interesting case is when $m \ll M$ for which we have "lines with maximal speed".

Here we can extend the $\mathbb{R}^2$-framework in an $\mathbb{R}^N$-one by setting

$$R(x) = R(x', x_N) = R(x') = \begin{cases} M & \text{if } x' \in \mathbb{Z}^{N-1} \\ m & \text{otherwise,} \end{cases}$$

where, as usual $x = (x', x_N)$ with $x' \in \mathbb{R}^{N-1}$.

We can address the problem for such discontinuous R within the stratified framework: we have the stratification $\mathbb{R}^N \times (0, +\infty) = \mathbf{M}^2 \cup \mathbf{M}^{N+1}$ where

$$\mathbf{M}^2 = (\mathbb{Z}^{N-1} \times \mathbb{R}) \times (0, +\infty) \;,$$

and $\mathbf{M}^{N+1}$ is its complementary set in $\mathbb{R}^N \times (0, +\infty)$. For $\mathbf{BCL}$, we have $\mathbf{BCL}(x,t) = \mathbf{BCL}(x) = \{((mv, -1), 0, 0);\ v \in \mathbb{R}^N,\ |v| \le 1\}$ if $x \in \mathbf{M}^{N+1}$ and for $x \in \mathbf{M}^2$, $\mathbf{BCL}(x,t) = \mathbf{BCL}(x) = \{((Mv, -1), 0, 0);\ v \in \mathbb{R}^N,\ |v| \le 1\}$. Notice that, since $M > m$, $\mathbf{BCL}$ is actually uppersemi-continuous on $\mathbf{M}^2$. Therefore a stratified subsolution $u : \mathbb{R}^N \times (0, +\infty)$ of (16.7) is an usc function with satisfies

$$u_t + m|Du| \le 0 \quad \text{in } \mathbf{M}^{N+1} \times (0, +\infty) \;, \tag{16.10}$$

$$u_t + M|Du| \le 0 \quad \text{in } \mathbf{M}^2 \times (0, +\infty) \;, \tag{16.11}$$

while a stratified supersolution $v : \mathbb{R}^N \times (0, +\infty)$ of (16.7) is an lsc function with satisfies

$$v_t + R(x)|Dv| \ge 0 \quad \text{in } \mathbb{R}^N \times (0, +\infty) \;. \tag{16.12}$$

Using results of Section 13.2, one can easily prove the

Theorem 16.2.1 *For any $u_0 \in C(\mathbb{R}^N)$, the problem (16.7)-(16.8) has a unique stratified solution given by (16.9). Moreover we have a comparison result for this problem.*

We leave the proof to the reader since it comes from a simple checking of the assumptions required in Section 13.2.

The next question concerns the asymptotic velocity when $t \to +\infty$. A classical method first consists in looking at initial datas of the form $u_0(x) = p \cdot x$ for some $p \in \mathbb{R}^N$ in order to obtain the velocity when the normal direction is p. Then the scaling $(x,t) \to (\frac{t}{\varepsilon}, \frac{x}{\varepsilon})$ –which preserves velocities– allows to see on finite times what happens for large time. It leads to study the equation satisfied by the function u_ε defined by $u_\varepsilon(t,x) := \varepsilon u(\frac{x}{\varepsilon}, \frac{t}{\varepsilon})$, namely

$$(u_\varepsilon)_t + R(\frac{x}{\varepsilon})|Du_\varepsilon| = 0 \quad \text{in } (0, +\infty) \times \mathbb{R}^N \;. \tag{16.13}$$

We also notice that the initial data is unchanged by the scaling, i.e. $u_\varepsilon(x,0) = p\cdot x$.

We can formulate the result in the following simple form

Theorem 16.2.2 *As $\varepsilon \to 0$, $u_\varepsilon(x,t) \to p\cdot x - t\bar H(p)$ where $\bar H(p) = \max(M|p_N|, m|p|)$ if $p = (p_1, p_2, \cdots, p_N)$.*

This theorem implies that, if $|p| = 1$, $\bar H(p)$ is the velocity of the front in the direction p.

Proof — We first remark that, by classical comparison results and since $m \le R(x) \le M$ in $\mathbb{R}^N$, we have

$$p\cdot x - Mt \le u_\varepsilon(x,t) \le p\cdot x - mt \quad \text{in } \mathbb{R}^N \times (0,+\infty)\ .$$

Therefore u_ε is locally uniformly bounded.

We give several arguments to treat the convergence of u_ε, following the method of Lions, Papanicolaou and Varadhan [98] together with the perturbed test-function method of Evans [57, 58] as in the article of Briani, Tchou and the two authors [24].

To do so, we first prove the

Lemma 16.2.3 *For any $p \in \mathbb{R}^N$, there exists a unique constant $\bar H(p)$ such that the equation*

$$R(x)|p + D_x w| = \bar H(p) \quad \text{in } \mathbb{R}^N\ . \tag{16.14}$$

has a bounded, Lipschitz continuous stratified solution $w = w(x,p)$. Moreover we have $\bar H(p) = \max(M|p_N|, m|p|)$.

Because of the very simple form of the initial data for u_ε and even more, the simple form of the limit of the u_ε's, there is a very quick proof to conclude. Indeed the function $\chi_\varepsilon(x,t) := p\cdot x - t\bar H(p) - \varepsilon w\left(\dfrac{x}{\varepsilon}, p\right)$ is a solution of (16.13) and moreover

$$\chi_\varepsilon(x,0) - \varepsilon||w(\cdot,p)||_\infty \le u_\varepsilon(x,0) \le \chi_\varepsilon(x,0) + \varepsilon||w(\cdot,p)||_\infty\ .$$

Therefore by comparison

$$\chi_\varepsilon(x,t) - \varepsilon||w(\cdot,p)||_\infty \le u_\varepsilon(x,t) \le \chi_\varepsilon(x,t) + \varepsilon||w(\cdot,p)||_\infty\ ,$$

which gives the result.

We provide more general arguments, which allow to take care of the case of more general initial datas and limits after the proof of Lemma 16.2.3.

Proof — This lemma is classical and so is its proof, except that, in our case, R is discontinuous. For $0 < \alpha \ll 1$, we consider the equation

$$R(x)|p + D_x w^\alpha| + \alpha w^\alpha = 0 \quad \text{in } \mathbb{R}^N . \tag{16.15}$$

Borrowing arguments in Section 13.2, it is easy to prove that (16.15) has a unique stratified solution. Moreover, w^α depends only on x' since R depends only on x' and is $\mathbb{Z}^{N-1}$-periodic since R is $\mathbb{Z}^{N-1}$-periodic: indeed, $w^\alpha(x', x_N)$ and $w^\alpha(x', x_N + h)$ are solutions of the same equation for any $h \in \mathbb{R}$ and therefore they are equal. Moreover, we have (again because of the comparison results) $-M|p| \leq \alpha w^\alpha(x) \leq -m|p|$ in $\mathbb{R}^N$ since $-M|p|/\alpha$ and $-m|p|/\alpha$ are respectively sub and supersolution of (16.15). Finally, the w^α are equi-Lipschitz continuous since αw^α is uniformly bounded and the term $R(x)|p + q|$ is coercive in q, uniformly in x. We point out that, in all this paragraph, we have used extensively the comparison result for stratified solutions of (16.15).

We can apply Ascoli's Theorem to the sequence $(w^\alpha(\cdot) - w^\alpha(0))_\alpha$ which is equi-Lipschitz continuous and equi-bounded by the periodicity of the w^α's: we can extract a subsequence $(w^{\alpha'}(\cdot) - w^{\alpha'}(0))_{\alpha'}$ which converges uniformly in $\mathbb{R}^N$ (by periodicity) to a periodic, Lipschitz continuous function w, and such that $\alpha' w^{\alpha'}(0)$ converges to a constant $-\lambda$. By the stability result for stratified solutions, w is a stratified solution of

$$R(x)|p + D_x w| = \lambda \quad \text{in } \mathbb{R}^N .$$

To prove that λ is unique, we assume by contradiction that there exists a bounded stratified solution w' of

$$R(x)|p + D_x w'| = \lambda' \quad \text{in } \mathbb{R}^N ,$$

for some different constant λ'. But the functions $(x, t) \mapsto w(x) - \lambda t$ and $(x, t) \mapsto w'(x) - \lambda' t$ are stratified solutions of the same equation and therefore, for any $t > 0$

$$||(w(x) - \lambda t) - (w'(x) - \lambda' t)||_\infty \leq ||w(x) - w'(x)||_\infty ,$$

an inequality which can hold only if $\lambda = \lambda'$.

It remains to show that $\lambda = \bar{H}(p)$ is given by the formula we claim. By the Dynamic Programming Principle, we have, for any $T > 0$

$$w(x) = \inf \left\{ \int_0^T (b(s) \cdot p + \bar{H}(p)) ds + w(X(s));\ X(0) = x,\ \dot{X}(s) = b(s) \in B(X(s)) \right\} ,$$

where $\mathbf{B}(y) = MB(0, 1)$ if $y \in \mathbf{M}^2$ and $\mathbf{B}(y) = mB(0, 1)$ if $y \in \mathbf{M}^{N+1}$ (we trust the reader to be able to translate in this setting the framework of Chapters 13 and 14 - , without any difficulty, even if we have dropped the b^t-term since $b^t \equiv -1$).

In order to compute the infimum in the above formula, we have several choice for $b(s)$: at any point of $\mathbb{R}^N$, we can choose $|b(s)| = m$ and clearly the minimal cost is $b(s)\cdot p = -m|p|$; if $X(s) \in \mathbf{M}^2$, we can choose $b(s) = +/-Me_N$ to stay on $\mathbf{M}^2$ and clearly the minimal cost is $b(s)\cdot p = -M|p_N|$. The optimal choice, at least if $X(s)\in\mathbf{M}^2$ is $\min(-m|p|, -M|p_N|) = -\max(m|p|, M|p_N|)$.

If the maximum is $m|p|$ and $p \neq 0$ (the case $p = 0$ is obvious and $\bar{H}(0) = 0$), we choose $b(s) = -mp/|p|$ and since w is bounded we have

$$\int_0^T (b(s)\cdot p + \bar{H}(p))ds = \int_0^T (-m|p| + \bar{H}(p))ds = T(-m|p| + \bar{H}(p)) \text{ is bounded },$$

and therefore $\bar{H}(p) = m|p|$.

If the maximum is $M|p_N|$, we can choose $x \in \mathbf{M}^2$ and $b(s) = -\mathrm{sign}(p_N)Me_N$ and clearly the minimal cost is $b(s)\cdot p = -M|p_N|$ and we conclude in the same way that $\bar{H}(p) = M|p_N|$.

Q.E.D.

Now we provide a more general proof of the convergence of the u_ε's based on the perturbed test-function method of Evans [57, 58] as in the article of Briani, Tchou and the two authors [24].

To do so, we introduce the half-relaxed limits $\overline{u} = \limsup^* u_\varepsilon$ and $\underline{u} = \liminf_* u_\varepsilon$ which are well-defined since the u_ε's are locally uniformly bounded. The key step is the

Lemma 16.2.4 *The functions $\overline{u}$ and $\underline{u}$ are respectively (classical) viscosity sub and supersolution of*

$$u_t + \bar{H}(Du) = 0 \quad \text{in } \mathbb{R}^N \times (0,+\infty)\ , \tag{16.16}$$

$$u(x,0) = p\cdot x \quad \text{in } \mathbb{R}^N\ . \tag{16.17}$$

If the lemma is proved, the result follows in an easy way since we have a comparison result for (16.16)-(16.17) for which $p\cdot x - t\bar{H}(p)$ is a solution, implying that

$$\overline{u}(x,t) \le p\cdot x - t\bar{H}(p) \le \underline{u}(x,t) \quad \text{in } \mathbb{R}^N \times [0,+\infty)\ ,$$

and the result.

Proof — We provide the proof only for $\overline{u}$, the one for $\underline{u}$ being analogous.

Let $\phi : \mathbb{R}^N \times (0,+\infty) \to \mathbb{R}$ be a smooth test-function and let $(\bar{x},\bar{t})$ be a strict local maximum point of $\overline{u} - \phi$: since we may assume without loss of generality that

$(\overline{u}-\phi)(\bar{x},\bar{t})=0$, this means that there exists $r,h>0$ such that $(\overline{u}-\phi)(x,t)\leq 0$ in $Q^{\bar{x},\bar{t}}_{r,h}$. Moreover there exists some $\delta>0$ such that $(\overline{u}-\phi)(x,t)\leq -2\delta$ on $\partial_p Q^{\bar{x},\bar{t}}_{r,h}$.

We want to show that $\phi_t(\bar{x},\bar{t})+\bar{H}(D\phi(\bar{x},\bar{t}))\leq 0$ and to do so, we argue by contradiction assuming that $\phi_t(\bar{x},\bar{t})+\bar{H}(D\phi(\bar{x},\bar{t}))>0$.

The next step consists in considering the function $\phi_\varepsilon(x,t):=\phi(x,t)+\varepsilon w(\dfrac{x}{\varepsilon},D\phi(\bar{x},\bar{t}))$ and to look at $(\phi_\varepsilon)_t(x,t)+R(\dfrac{x}{\varepsilon})|D\phi_\varepsilon(x,t)|$ in $Q^{\bar{x},\bar{t}}_{r,h}$. Formally, using the equation satisfied by $w(\cdot,D\phi(\bar{x},\bar{t}))$, we have

$$\begin{aligned}(\phi_\varepsilon)_t(x,t)+R(\frac{x}{\varepsilon})|D\phi_\varepsilon(x,t)| =&\phi_t(x,t)+R(\frac{x}{\varepsilon})|D\phi(x,t)+D_xw(\frac{x}{\varepsilon},D\phi(\bar{x},\bar{t}))|\\ =&\phi_t(\bar{x},\bar{t})+R(\frac{x}{\varepsilon})|D\phi(\bar{x},\bar{t})+D_xw(\frac{x}{\varepsilon},D\phi(\bar{x},\bar{t}))|+O(r)+O(h)\\ =&\phi_t(\bar{x},\bar{t})+\bar{H}(D\phi(\bar{x},\bar{t}))+O(r)+O(h),\end{aligned}$$

the terms $O(r),O(h)$ coming from the replacement of $D\phi(x,t)$ by $D\phi(\bar{x},\bar{t})$. Therefore, taking perhaps r,h,δ smaller we have

$$(\phi_\varepsilon)_t(x,t)+R(\frac{x}{\varepsilon})|D\phi_\varepsilon(x,t)|\geq\delta>0\quad\text{in } Q^{\bar{x},\bar{t}}_{r,h}.$$

This formal computation can be justified by looking carefully at the stratification formulation and such checking does not present any difficulty.

Moreover, on the compact set $\partial_p Q^{\bar{x},\bar{t}}_{r,h}$, we have $\phi(x,t)\geq\overline{u}+2\delta$ and therefore, because of the definition of $\overline{u}$, we have for ε small enough $\phi_\varepsilon(x,t)\geq u_\varepsilon+\delta$ on $\partial_p Q^{\bar{x},\bar{t}}_{r,h}$.

By comparison, we conclude that $\phi_\varepsilon(x,t)\geq u_\varepsilon+\delta$ in $Q^{\bar{x},\bar{t}}_{r,h}$ for any ε but taking the $\limsup^*$ this gives $\phi(x,t)\geq\overline{u}+\delta$ in $Q^{\bar{x},\bar{t}}_{r,h}$, a contradiction since $(\overline{u}-\phi)(\bar{x},\bar{t})=0$.

Q.E.D.

Q.E.D.

Remark 16.2.5 *The above arguments shows that the classical proofs for the homogenization of Hamilton-Jacobi Equations extend easily to the discontinuous case provided that one uses the right stratified formulation.*

16.3 KPP-type problems with discontinuities (II): remarks on more general discontinuities

In the proof of Theorem 10.3.1, even if we hide it carefully inside the proof of Theorem 10.4.2, we use in an essential way the various notions of solutions which are described in Part II, namely flux- limited solutions and junction solutions together with results concerning their links.

This heavy sophisticated machinery is a weakness if we want to address the case of more general discontinuities for which we are not able to provide such a precise analysis. Therefore, it is natural to consider what we can do in those more general cases.

In which cases can we conclude by using only the standard notion of Ishii's viscosity solution?

In the framework of Chapter 10, i.e. with only a discontinuity of codimension 1 on an hyperplan, the answer is straightforward and this can be seen from two slightly different points of view: on one hand, in order to conclude, it is enough that the functions I^+ and I^- appearing in Theorem 10.4.1 and Theorem 10.4.2 are equal, and so are J^+ and J^-. Lemma 12.2.2 gives conditions under which this happens.

On the other hand, and this is a more general point of view, we can also look for conditions under which Ishii's viscosity subsolutions are stratified subsolutions (since the supersolutions are the same). The conclusion then follows from the comparison result for stratified solutions. Since it is easy to see that, on the hyperplane, the $\mathbb{F}^N$-inequality on $\mathcal{H} \times (0,T)$ is the H_T-one, Lemma 12.2.2 still gives the answer.

In order to exploit this result and to go further, let us just consider the case when the $b^{(i)}$ are equal to 0 for $i = 1, 2$.

Under this assumption, we have

$$H_i(x,p) := \frac{1}{2} a^{(i)}(x)p \cdot p + c^{(i)}(x) \ ,$$

and the computation of $m_1(x,p'), m_2(x,p')$ is easy

$$m_i(x,p') = \frac{a^{(i)}(x)p' \cdot e_N}{a^{(i)}(x)e_N \cdot e_N} \ .$$

The property $m_2(x,p') \geq m_1(x,p')$ for any (x,p') which is required in Lemma 12.2.2 in order to have $H_T = H_T^{\mathrm{reg}}$ reads

$$\frac{a^{(2)}(x)p' \cdot e_N}{a^{(2)}(x)e_N \cdot e_N} \leq \frac{a^{(1)}(x)p' \cdot e_N}{a^{(1)}(x)e_N \cdot e_N} \ .$$

Using that this has to hold for any $p' \in \mathcal{H}$, we are led to

$$\frac{a^{(2)}(x)e_N}{a^{(2)}(x)e_N \cdot e_N} - \frac{a^{(1)}(x)e_N}{a^{(1)}(x)e_N \cdot e_N} = 0 \ , \tag{16.18}$$

because the inequality implies that this vector is colinear to e_N while its scalar product with e_N is 0. Under this condition, Theorem 10.3.1 can be proved using only the basic notion of viscosity solutions.

Can we go further?

The answer is yes, as can be seen in the following example in the "cross case": consider Equations (10.2) which holds in $Q_i \subset \mathbb{R}^2$ where the Q_i's are the four quadrants in $\mathbb{R}^2$, namely

$$Q_1 = \{x_1 > 0, x_2 > 0\}\ , Q_2 = \{x_1 < 0, x_2 > 0\}\ ,\ Q_3 = -Q_1\ ,\ Q_4 = -Q_2\ .$$

We assume that the $b^{(i)}$'s are also 0 and that, for $i = 1, 2, 3, 4$, $a^{(i)}(x) = \lambda^{(i)}(x)Id$ in Q_i for some bounded, Lipschitz continuous function $\lambda^{(i)}$. We assume also the existence of some constant $\nu > 0$ such that $\lambda^{(i)}(x) \geq \nu$ in Q_i for any i.

Under natural assumptions, the asymptotics of u_ε can easily be obtained in this framework: indeed

(i) $\underline{I}$ is an Ishii viscosity supersolution of the variational inequality in $\mathbb{R}^2 \times (0, +\infty)$,

(ii) $\overline{I}$ turns out to be a "stratified subsolution" of the variational inequality in $\mathbb{R}^2 \times (0, +\infty)$. Indeed, on the axes (except 0), i.e. on $\mathbf{M}^2$, the above analysis shows that $H_T = H_T^{\mathrm{reg}}$ inequality holds for $\overline{I}$ and therefore the $\mathbb{F}^2$-one holds too. At $x = 0$ for $t > 0$, i.e. on $\mathbf{M}^1$, we clearly have

$$\min(\overline{I}_t + \max_i(c^{(i)}(x), \overline{I}) \leq 0\ ,$$

because all the inequalities $\min(\overline{I}_t + c^{(i)}(x), \overline{I}) \leq 0$ hold by passage to the limit (stability) from the Q_i domain.

Hence, $\underline{I}$ and $\overline{I}$ are respectively stratified super and subsolution of the variational inequality and we can conclude since the comparison result for stratified solutions easily extend to this framework.

We refer to the next section for a (slightly) more general result allowing to show that, under suitable conditions, an Ishii viscosity subsolution is a stratified subsolution. This result allows to treat the following kind of KPP problems: we assume that $\mathbb{M} = (\mathbf{M}^k)_{k=0..N}$ is a stratification of $\mathbb{R}^N$ and that in the framework of Chapter 10, the u_ε are solutions of Equation 10.2 where

1. $a^{(i)}(x) = \lambda^{(i)}(x)Id$ in Ω_i where the Ω_i are the connected components of $\mathbf{M}^N$. We assume that the functions $\lambda^{(i)}$ are uniformly bounded and Lipschitz continuous functions and there exists a constant $\nu > 0$ such that $\lambda^{(i)}(x) \geq \nu$ in Ω_i for any i.

2. $b = 0$ in $\mathbb{R}^N$.

3. $f = f^{(i)}$ in Ω_i where the $f^{(i)}$ are KPP-nonlinearities, the $c^{(i)}$ being uniformly bounded and Lipschitz continuous on $\overline{\Omega_i}$.

Under these conditions, and if the initial data g is as in Chapter 10, the result is the

Proposition 16.3.1 *As $\varepsilon \to 0$, we have*

$$-\varepsilon \log(u_\varepsilon) \to I \quad \textit{locally uniformly in } \mathbb{R}^N \times (0,+\infty)\ ,$$

where I is the unique stratified solution of the equation with

$$\left\{ \begin{array}{ll} \min(I_t + H_i(x,DI), I) = 0 & \textit{in } \Omega_i \times (0,+\infty)\ , \\ I(x,0) = \left\{ \begin{array}{ll} 0 & \textit{if } x \in G_0, \\ +\infty & \textit{otherwise}\ . \end{array} \right. & \end{array} \right. \tag{16.19}$$

Moreover we have

$$u_\varepsilon(x,t) \to \left\{ \begin{array}{ll} 0 & \textit{in } \{I > 0\}, \\ 1 & \textit{in the interior of the set } \{I = 0\}. \end{array} \right.$$

Finally if Freidlin's condition holds then $I = \max(J,0)$ where J is the unique stratified solution of

$$\left\{ \begin{array}{ll} J_t + H_i(x,DJ) = 0 & \textit{in } \Omega_i \times (0,+\infty)\ , \\ J(x,0) = \left\{ \begin{array}{ll} 0 & \textit{if } x \in G_0, \\ +\infty & \textit{otherwise}\ . \end{array} \right. & \end{array} \right. \tag{16.20}$$

Function J is given by the following representation formula

$$J(x,t) = \inf \left\{ \int_0^t l(y(s),\dot y(s))ds;\ y(0) = x,\ y(t) \in G_0,\ y \in H^1(0,T) \right\}\ ,$$

where $l(y(s),\dot y(s)) = \dfrac{1}{2}[\lambda^{(i)}(y(s))]^{-1}|\dot y(s)|^2 - c^{(i)}(y(s))$ if $y(s) \in \Omega_i$.

Several remarks on this results
(i) We have left this result with a slightly imprecise statement, giving the equations only in $\mathbf{M}^N \times (0,T)$ and defining l only in $\mathbf{M}^N \times (0,T)$. The next section will (at least partially) show why this is enough.
(ii) As above in the "cross case", the proof that $\overline{I}$ is a stratified subsolution comes from the arguments given the next section.
(iii) The first part of this result holds for example in the counter-example in dimension 1 given in Chapter 10, the only point is that Freidlin's condition is not satisfied.

16.4 When do Ishii's viscosity solutions and stratified solutions coincide?

The aim of this section is to consider the question suggested by the title of the section: under which conditions can it be proved that a classical Ishii viscosity subsolution is a stratified subsolution? Of course, this question is meaningful only for subsolutions since the supersolutions are the same.

In the case of codimension-1 discontinuities (see Part II), this question consists in looking for conditions which ensure that $\mathbf{U}^+ \equiv \mathbf{U}^-$ and a partial (but rather general) answer is given by Lemma 12.2.2. The reader can check on examples that this lemma is not always of a simple use as it can be seen on Chapter 10 by adding drifts terms. Since we are looking for simple conditions which can easily be checked for more general types of discontinuities, the ones we propose in this section are unavoidably rather restrictive but they cover anyway some interesting cases as we will illustrate below by several examples.

In the following proposition, we assume that u is an u.s.c. classical viscosity subsolution of the equation

$$\mathbb{F}(y, u, Du) = 0 \quad \text{in } \mathcal{O} \subset \mathbb{R}^N ,$$

where $\mathcal{O}$ is an open set and

$$\mathbb{F}(y, r, p) = \sup_{(b,c,l)\in\mathbf{BCL}(y)} \big\{ -b\cdot p + cr - l \big\}$$

for some set-valued map $\mathbf{BCL} : \mathcal{O} \to \mathcal{P}(\mathbb{R}^{N+2})$. We assume that we are in the "good framework for HJ-Equations with discontinuities". Our aim is then to prove that u satisfies some tangential inequalities $\mathbb{F}^k \le 0$ for any $0 \le k < N$.

Let us begin with some simplifications: since all arguments that follow are purely local arguments, we can reduce the discussion to the case of a flat stratification. In this setting, $\mathbf{M}^k$ is a k-dimensional vector space, hence we identify $T_y\mathbf{M}^k \equiv \mathbf{M}^k \equiv \mathbb{R}^k$. We also assume that $\mathcal{O} = \mathbb{R}^M$, again because we use only local arguments.

Next, we recall the structure of the tangential hamiltonians: for any $y \in \mathbf{M}^k$, $r \in \mathbb{R}$ and $p \in T_y\mathbf{M}^k$

$$\mathbb{F}^k(y, r, p) := \sup_{\substack{(b,c,l)\in\mathbf{BCL}(y)\\ b\in T_y\mathbf{M}^k}} \big\{ -b\cdot p + cr - l \big\} .$$

Under the flat stratification assumption, the definition of $\mathbb{F}^k$ can be easily extended to any affine space which is parallel to $\mathbf{M}^k$, typically $e + \mathbf{M}^k \subset \mathbb{R}^N$ for some $e \in \mathbb{R}^N$;

more precisely, for any $y+e \in e+\mathbf{M}^k \subset \mathbb{R}^N$, $r \in \mathbb{R}$ and $p \in \mathbf{M}^k$ we set

$$\mathbb{F}^k(y+e,r,p) = \sup_{\substack{(b,c,l)\in \mathbf{BCL}(y+e)\\ b\in\mathbf{M}^k}} \left\{ -b\cdot p + cr - l \right\},$$

where we recall that $T_{y+e}\mathbf{M}^k = T_y\mathbf{M}^k = \mathbb{R}^k$, see Fig. 16.1 below. We also denote by $(\mathbf{M}^k)^\perp$ the orthogonal space to $\mathbf{M}^k$ in $\mathbb{R}^N$. Notice that for simplicity if $p \in \mathbf{M}^k = \mathbb{R}^k$ we still write p for the same vector, considered as a vector of $\mathbb{R}^M$.

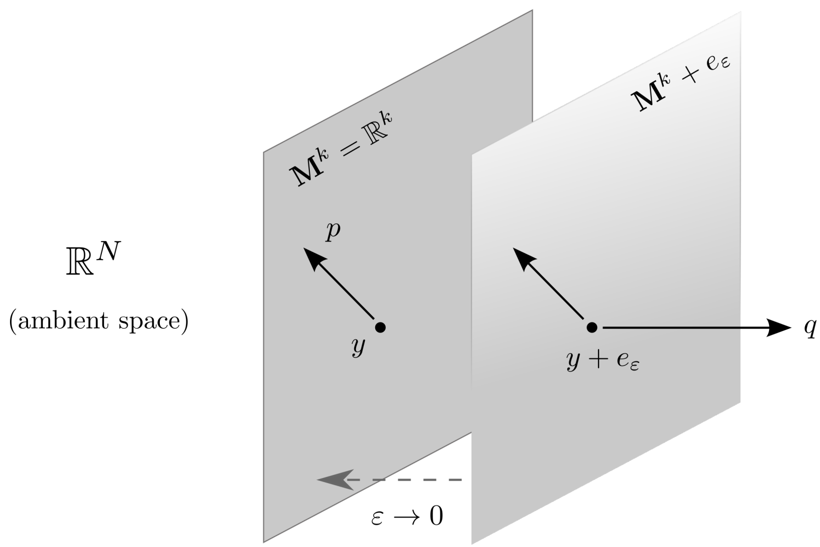


Figure 16.1:

We have the

Proposition 16.4.1 *Under the above conditions, we assume moreover that for any $y \in \mathbf{M}^k$, $r \in \mathbb{R}$, $p \in \mathbf{M}^k$, there exists a sequence $(e_\varepsilon)_\varepsilon$ of elements of $e_\varepsilon \in \mathbb{R}^N$ converging to 0 such that $y+e_\varepsilon \in e_\varepsilon + \mathbf{M}^k \subset \mathbb{R}^N$ and*

$$\mathbb{F}^k(y+e_\varepsilon,r,p) \leq \mathbb{F}(y+e_\varepsilon,r,p+q) + o_\varepsilon(1) \quad \text{for any } q \in (\mathbf{M}^k)^\perp,$$
$$\mathbb{F}^k(y,r,p) \leq \limsup_{\varepsilon\to 0} \mathbb{F}^k(y+e_\varepsilon,r,p).$$

Then u is stratified subsolution of $\mathbb{F}^k = 0$ in $\mathbf{M}^k$.

Proof — First, we can assume without loss of generality that u is Lipschitz continuous by the standard regularization process in the $\mathbf{M}^k$-direction. Then if φ is a smooth test-function and if $u-\varphi$ has a strict local maximum point at y (relatively to $\mathbf{M}^k$), we have to prove that $\mathbb{F}^k(y,u(y),D\varphi(y)) \leq 0$.

To do so, we consider the sequence $(e_\varepsilon)_\varepsilon$ associated to y, $r=u(y)$ and $p=D\varphi(y)$. Since $z\mapsto u(z+e_\varepsilon)$ converges locally uniformly to $z\mapsto u(z)$ on $\mathbf{M}^k$ by the Lipschitz continuity of u, the function $z\mapsto u(z+e_\varepsilon)-\varphi(z)$ has a local maximum point at y_ε (again relatively to $\mathbf{M}^k$) and y_ε converges to y as $\varepsilon\to 0$.

We introducing the function $z\mapsto u(z+e_\varepsilon)-\varphi(z)-\dfrac{[d(z,\mathbf{M}^k)]^2}{\alpha}-|z-y_\varepsilon|^2$ for $0<\alpha\ll 1$. For α small enough, this function has a maximum point at $y_{\varepsilon,\alpha}$ converging to y_ε as $\alpha\to 0$. And we have by the subsolution inequality in $\mathbf{M}^N$

$$\mathbb{F}(y_{\varepsilon,\alpha}+e_\varepsilon, u(y_{\varepsilon,\alpha}+e_\varepsilon), D\varphi(y_{\varepsilon,\alpha})+q_{\varepsilon,\alpha}+2(y_{\varepsilon,\alpha}-y_\varepsilon)) \leq 0 \; ,$$

where

$$q_{\varepsilon,\alpha}=\frac{2d(z,\mathbf{M}^k)Dd(z,\mathbf{M}^k)}{\alpha}\in(\mathbf{M}^k)^\perp \; .$$

Letting α tend to 0, using the Lipschitz continuity of u (which implies in particular that $q_{\varepsilon,\alpha}$ remains bounded), we conclude that there exists $q_\varepsilon\in(\mathbf{M}^k)^\perp$ such that

$$\mathbb{F}(y_\varepsilon+e_\varepsilon, u(y_\varepsilon+e_\varepsilon), D\varphi(y_\varepsilon)+q_\varepsilon) \leq 0 \; .$$

But the assumption on $\mathbb{F}$ together with its Lipschitz continuity in r and p, implies

$$\mathbb{F}^k(y_\varepsilon+e_\varepsilon, u(y), D\varphi(y)) \leq o_\varepsilon(1) \; .$$

The conclusion follows by letting $\varepsilon\to 0$.

Q.E.D.

Example 16.4.2 *We consider the equation*

$$u_t + a(x)|Du| = g(x) \quad \text{in } \mathbb{R}^2\times(0,T) \; ,$$

where $a=a_i$ and $g=g_i$ in Ω_i where the Ω_i are in Figure 11.2 and the functions a_i, g_i are continuous functions. Of course, we assume that $a_i(x)\geq 0$ for any $x\in\Omega_i$ $(1\leq i\leq 4)$.

Let us first consider $\mathbf{M}^2$ and the part $\{x_1=0, x_2>0\}$. Here

$$\mathbb{F}^2(x,t,(p_x,p_t)) = p_t + \sup\left\{-(\theta a_1 v_1 + (1-\theta)a_2 v_2)\cdot p_x - (\theta g_1 + (1-\theta)g_2)\right\} \; ,$$

where the supremum is taken on all $|v_1|,|v_2|\leq 1$ and all $0\leq\theta\leq 1$ such that $(\theta a_1 v_1 + (1-\theta)a_2 v_2)\cdot e_1 = 0$.

It is obvious that $\mathbb{F}^2$ *can be computed by choosing* v_1, v_2 *such that* $v_1 \cdot e_1 = v_2 \cdot e_1 = 0$, *i.e. by taking dynamics which are in the direction of* $\mathbf{M}^2$ *and writing*

$$-(\theta a_1 v_1 + (1-\theta) a_2 v_2).p_x - (\theta g_1 + (1-\theta) g_2) = \theta(-a_1 v_1 \cdot p_x - g_1) + (1-\theta)(-a_2 v_2 \cdot p_x - g_2) \ ,$$

it is clear that

$$\mathbb{F}^2(x, t, (p_x, p_t)) = \max(p_t + a_1|(p_x)_2| - g_1, p_t + a_2|(p_x)_2| - g_2) \ ,$$

where $(p_x)_2$ *is the second component of* p_x, *i.e. the tangential part of the gradient in space.*

Examining the condition to be checked for Proposition 16.4.1, we easily see that, choosing below the "±" depending if the max is achieved for the index 1 *or* 2, *we have*

$$\begin{aligned}\mathbb{F}^2(x \pm \varepsilon e_1, t, (p_x, p_t)) &= \max_i (p_t + a_i(x \pm \varepsilon e_1)|(p_x)_2| - g_i(x \pm \varepsilon e_1)) \\ &\leq p_t + a(x \pm \varepsilon e_1)|p_x| - g(x \pm \varepsilon e_1) \\ &= \mathbb{F}(x \pm \varepsilon e_1, t, (p_x, p_t)) \ ,\end{aligned}$$

just because $|(p_x)_2| \leq |p_x|$. *And of course the same property holds for the three other parts of* $\mathbf{M}^2$.

For $\mathbf{M}^1 := \{(0,0)\} \times (0,T)$, *the checking is even simpler since it use the fact that* $|p_x| \geq 0$ *and by considering* $(0,0) \pm e_1 \pm e_2$, *one can easily check that the* $\mathbb{F}^1$ *condition, i.e.* $p_t - \min(g_i) \leq 0$, *is satisfied.*

Remark 16.4.3 *As the above example shows, the result of Proposition 16.4.1 is not very sophisticated but it has the advantage to be very simple to apply.*

16.5 Other (undrafted) Applications

The first one we have in mind concerns **homogenization** in a chessbord type configuration: this problem is treated in Forcadel and Rao [62] but we think that the results of Chapter 13, 14 and 15 lead to more general results with simpler proofs, using also some ideas taken from the co-dimension 1 case in Barles, Briani, Chasseigne and Tchou [24].

We also take the opportunity of this section on applications which are not treated in this book to mention applications of Hamilton-Jacobi Equations on networks to traffic problems in Imbert, Monneau and Zidani [85], Forcadel and Salazar [61], Forcadel, Salazar, Wilfredo and Zaydan [63].

Chapter 17

Some Extensions

We start this section by recalling the main ideas of a comparison proof for stratified solutions

(i) We localize, i.e. we reduce the proof of a GCR to the proof of a LCR.

(ii) In order to show that the LCR holds, we first regularize the subsolution by a partial sup-convolution procedure using the tangential continuity and the normal controllability and then (still tangentially) with a standard convolution with a smoothing kernel.

(iii) After Step (ii) the subsolution is Lipschitz continuous w.r.t. all variables and C^1 w.r.t. the tangent variable and we use the "magical lemma" (Lemma 5.4.1) to conclude.

If we analyze these 3 steps, it is easy to see that Section 3.2 and the examples which are treated there shows in a rather clear (and still very incomplete) way that the localisation can be made via various arguments and is not really a limiting step (even if we agree that there are more complicated situations were this might become a problem). In the same way, Step (iii) is not really a limiting step, especially the way we use it in the proof by induction.

Hence we quickly realize that, in the generalizations, we wish to present in this chapter the main issue comes from Step (ii) and more precisely from the first part, i.e. the tangential sup-convolution procedure. This is why we are going to mainly insist on this point.

17.1 More general dependence in time

A (very?) restrictive assumption or, at least, an unusual assumption we have used so far concerns the dependence in time of the Hamiltonians and on the dynamics of the control problems. In general, it is well-known that a simple continuity assumption is a sufficient requirement.

In stratified problems, we have two main cases: the general case when the stratification may depend on time for which space and time play a similar role and the case when the stratification does not depend on time where the particular structure allows to weaken the assumptions on the time dependence.

Indeed, in this second case, we can write the stratification as

$$\mathbf{M}^{k+1} = \tilde{\mathbf{M}}^k \times \mathbb{R} ,$$

where $\tilde{\mathbb{M}} = (\tilde{\mathbf{M}}^k)_k$ is a stratification of $\mathbb{R}^N$ and $\mathbb{M} = (\mathbf{M}^k)_k$ is the resulting one in $\mathbb{R}^N \times [0,T]$ (which is here presented as the trace on $\mathbb{R}^N \times [0,T]$ of a stratification on $\mathbb{R}^N \times \mathbb{R}$).

As far as Section 3.4 is concerned, the t-variable is always a tangent variable [this is the main difference with the general case] and we can use, as it is classical in all the comparison proofs in viscosity solutions' theory, a "double parameters sup-convolution, i.e. if $u : \mathbb{R}^N \times [0,T] \to \mathbb{R}$ is a sub-solution and if $\tilde{\mathbf{M}}^k$ is identified with $\mathbb{R}^k$ and $x = (y,z)$ with $y \in \mathbb{R}^k$ and $z \in \mathbb{R}^{N-k}$, we set

$$u^{\varepsilon,\beta}(x,t) := \max_{y' \in \mathbb{R}^k, s \in [0,T]} \Big\{ u((y',z),s) - \frac{(|y-y'|^2 + \varepsilon^4)^{\alpha/2}}{\varepsilon^\alpha} - \frac{(|t-s|^2 + \beta^4)^{\alpha/2}}{\beta^\alpha} \Big\},$$

where the parameter β governing the regularization in time satisfies $0 < \beta \ll \varepsilon$.

We drop all the details here but we are sure that they will cause no problem to the reader.

17.2 Unbounded Control Problems

In unbounded control problems, they are two problems: the localization that we treat (in a non-optimal way) in Section 3.2 [cf. the "convex case"] and the sup-convolution regularization.

In order to treat this difficulty, we refer the reader to Section 3.2.4 and in particular to Theorem 3.2.8 and Assumption $(\mathbf{H}_{BA-HJ-U})$. Indeed, in the sup-convolution procedure, if we examine the proof of Theorem 3.4.2, we have to manage the error made by replacing y by y' and this is done by using the dependence in u of the Hamiltonian. This is exactly what Assumption $(\mathbf{H}_{BA-HJ-U})$ means: performing the Kruzkov's change of variable $u \to -\exp(-u)$, one compensates the (too large) terms in "$D_x H$" by (very large) terms in "$D_u H$".

The same ideas can be used in the stratified framework: we drop the details here since a lot of very different situations can occur and it would be impossible (and useless) to describe all of them.

17.3 Lower semicontinuous solutions for lower semicontinuous initial data (à la Barron-Jensen)

The extension of the Barron-Jensen approach to the stratified case requires a change of definition since it is based on the fact that, when considering equations with a convex Hamiltonian, one can just look at minimum points when testing both the sub and supersolutions properties. Of course, the same is true for stratified problems and leads to the following definition.

Definition 17.3.1 (Stratified Barron-Jensen sub and supersolutions)
(i) A locally bounded, lsc function $v : \mathbb{R}^N \times [0,T[\to \mathbb{R}$ is a stratified supersolution of Equation (4.4) iff it is an Ishii's supersolution of this equation.
(ii) A locally bounded, lsc function $u : \mathbb{R}^N \times [0,T[\to \mathbb{R}$ is a stratified subsolution of Equation (4.4) iff
(a) it is an Barron-Jensen subsolution of this equation, i.e. for any smooth function φ, at any minimum point (x,t) of $u - \varphi$, we have

$$\mathbb{F}_*(x,t,u(x,t),(D_t\varphi(x,t),D_x\varphi(x,t))) \leq 0 ,$$

and (b) for any $k = 0, ..., (N+1)$, for any smooth function φ, at any minimum point (x,t) of $u - \varphi$ on $\mathbf{M}^k$, we have

$$\mathbb{F}^k(x,t,u(x,t),(D_t\varphi(x,t),D_x\varphi(x,t))) \leq 0 ,$$

and (c) with analogous definitions

$$\mathbb{F}_{init}(x,u,D_x u) \leq 0 \quad \text{in } \mathbb{R}^N, \text{ for } t = 0.$$

$$\mathbb{F}^k_{init}(x,u,D_x u) \leq 0 \quad \text{on } \mathbf{M}^k_0, \text{ for } t = 0.$$

In addition, we will say that u is a strict stratified subsolution if the ≤ 0-inequalities are replaced by a $\leq -\eta < 0$-inequality where $\eta > 0$ is independent of x and t.

Before providing a comparison result (or trying to do it), we want to examine a key example. If we examine the Eikonal Equation

$$u_t + |D_x u| = 0 \quad \text{in } \mathbb{R}^N \times (0,T)\ ,$$
$$u(x,0) = g(x) = \begin{cases} 1 & \text{if } x \neq 0 \\ 0 & \text{otherwise,} \end{cases}$$

then $w(x,t) = 1$ in $\mathbb{R}^N \times [0,T)$ is an Ishii's viscosity subsolution because $w^*(x,0) \leq g^*(x) = 1$ in $\mathbb{R}^N$. In a similar way compared to the stratification difficulty, the subsolution inequality has to be reinforced but in this framework we cannot impose the "stratification-like" inequality $u(0,0) \leq 0$ because this inequality for an usc subsolution u is clearly too strong.

This is why, if we wish to take into account lsc initial data, we have to super-impose subsolution inequalities at $t = 0$ in order to be sure that this initial data will be seen: indeed in the Barron-Jensen's framework, subsolutions are lsc and the function

$$u(x,t) = \begin{cases} 1 & \text{if } t > 0 \\ g(x) & \text{if } t = 0, \end{cases}$$

is a lsc subsolution of the problem.

In this stratified Barron-Jensen case, we are able to present very general results : we just provide a uniqueness result in a framework which slightly generalizes the one of Ghilli, Rao and Zidani [70] and which uses the following assumptions: first we have a stratification which does not depend on time

$$\mathbf{M}^{k+1} = \tilde{\mathbf{M}}^k \times \mathbb{R}\ ,$$

for any $0 \leq k \leq N$ and we have a classical lsc and bounded initial data g, i.e. our result is valid for lsc sub and supersolutions u and v which satisfy

$$u(x,0) \leq g(x) \leq v(x,0) \quad \text{in } \mathbb{R}^N\ .$$

We also assume that $\mathbb{F}$ is a classical Hamiltonian

$$\mathbb{F}(x,t,r,(p_t,p_x) = p_t + \tilde{\mathbb{F}}(x,t,r,p_x)\ ,$$

with $\tilde{\mathbb{F}}$ being coercive, i.e there exists $\nu > 0$ such that

$$\tilde{\mathbb{F}}(x,t,r,p_x) \geq \nu|p_x| - M|r| - M\ ,$$

for any $x \in \mathbb{R}^N$, $t \in [0,T]$, $r \in \mathbb{R}$ and $p_x \in \mathbb{R}^N$, M being the constant appearing in the assumptions for **BCL**.

In this framework, the result is the

Theorem 17.3.2 *In the above framework and if we are in a "good framework for HJ Equations with discontinuities", for any lsc bounded stratified, Barron-Jensen, sub and supersolution u and v for equation $\mathbb{F}=0$ such that*

$$u(x,0) = \liminf\{u(y,t),\ (y,t) \to (x,0) \text{ with } t>0\}\ , \tag{17.1}$$

then we have

$$u(x,t) \leq v(x,t) \quad \text{in } \mathbb{R}^N \times [0,T)\ .$$

Proof — The main difficulties of this proof are concentrated at $t=0$: indeed, for $t>0$, we can argue in a similar way to the standard case with the help of the two following results which are easy adaptations of Proposition 3.4.1 and 3.4.2.

Proposition 17.3.3 *Under the assumptions of Theorem 17.3.2, for any $(x,t) \in \tilde{\mathbf{M}}^k \times (0,T)$ for $0 \leq k < N$*

$$u(x,t) = \liminf\{u(y,s)\ ;(y,s)) \to (x,t),\ y \in \tilde{\mathbf{M}}^{k+1} \cup \cdots \cup \tilde{\mathbf{M}}^N\}\ . \tag{17.2}$$

Moreover, if $k=N-1$, then locally $\mathbb{R}^N \setminus \tilde{\mathbf{M}}^{N-1}$ has two connected components $(\tilde{\mathbf{M}}^{N-1})_+$, $(\tilde{\mathbf{M}}^{N-1})_-$ and the above result is valid imposing to y to be either in $(\tilde{\mathbf{M}}^{N-1})_+$ or in $(\tilde{\mathbf{M}}^{N-1})_-$.

Proposition 17.3.4 *Under the assumptions of Theorem 17.3.2, if u is a bounded lsc, stratified Barron-Jensen subsolution of $\mathbb{F}=0$, for any $(x,t) \in \tilde{\mathbf{M}}^k \times (0,T)$, there exists a sequence of Lipschitz continuous functions $(u^{\varepsilon,\alpha})_{\varepsilon,\alpha}$ defined in a neighborhood $\mathcal{V}$ of (x,t) such that*

(i) the $u^{\varepsilon,\alpha}$ are stratified Barron-Jensen subsolutions of $\mathbb{F}=0$ in $\mathcal{V}$,

(ii) the $u^{\varepsilon,\alpha}$ are semi-convex and C^1 on $\tilde{\mathbf{M}}^k \times (0,T)$,

(iii) $\sup u^{\varepsilon,\alpha} = \lim_{\varepsilon,\alpha\to 0} u^{\varepsilon,\alpha} = u$ in $\mathcal{V}$.

Proposition 17.3.4 is proved exactly as Proposition 3.4.2 except that we use an inf-convolution instead of a sup-convolution and we treat differently the tangent space variable (with the parameter ε) and the t-variable (with parameter α).

With these two results, we can start the proof of Theorem 17.3.2. We are interested in $M := \sup_{\mathbb{R}^N\times[0,T)}(u-v)$ and we argue by contradiction assuming that $M>0$. With a (by now standard) localization argument, we can assume that $(u-v)(x,t) \to -\infty$ when $|x| \to +\infty$ or $t \to T$ and therefore minimizing sequence $(x_k,t_k)_k$ are bounded, t_k remaining away from T.

Since u is lsc, a priori this supremum is not achieved and therefore the approach of Section 3.2 has to be slightly modified but we skip these modifications here, trusting the reader to do them.

But there are maximizing sequences and, in fact, it is easy to see that the only problem is when all these maximizing sequence are such that $t_k \to 0$; otherwise, repeating the same proof as in the standard case with the help of Proposition 17.3.3 and 17.3.4 yields a contradiction.

We have therefore to argue for t small and show that such t does not cause any problems. But by the coercivity assumption, u is a Barron-Jensen subsolution of the (continuous) equation

$$u_t + \nu|D_x u| - M(||u||_\infty + 1) = 0 \quad \text{in } \mathbb{R}^N \times (0,T) ,$$

and therefore, by the uniqueness for this problem and Oleinik-Lax (or control) formula

$$u(x,t) \leq \inf_{|y-x|\leq \nu t} (u(y,0)) + M(||u||_\infty + 1)t .$$

On the other hand we have for v either by the same arguments or using the Dynamic Programming Principle

$$v(x,t) \geq \inf_{|y-x|\leq Mt} (u(y,0)) - M(||v||_\infty + 1)t .$$

Therefore if $\delta > 0$ is a small constant, we have

$$u(x,\delta+t) \leq v(x,t) + M(||u||_\infty + ||v||_\infty + 2)(t+\delta) ,$$

for any $0 \leq t \leq \dfrac{\nu\delta}{M-\nu}$.

This solves the small time issue and we can compare the subsolution $u(x,\delta+t)$ with the supersolution $v(x,t)$ this gives

$$u(x,\delta+t) \leq v(x,t) + M(||u||_\infty + ||v||_\infty + 2)\delta \quad \text{in } \mathbb{R}^N \times (0,T) ,$$

for any $\delta > 0$ and it remains to let δ tend to 0 to obtain the result.

Q.E.D.

Remark 17.3.5 *Of course, we think that this result holds in a more general framework, under far more general assumptions. But it is worth pointing out that the role of the inf-convolution in the classical Barron-Jensen argument (typically an inf-convolution in x on the solution to treat the lower-semi-continuity and the inf-convolution to take care of the stratifications are not completely compatible; this is what is generating these strong and unnatural assumptions.*

Part IV

State-Constraint Problems

Chapter 18

Introduction to State-Constraints problems

In this part, we are going to extend the results of the first part, and in particular those for stratified problems, to the case of problems set in a bounded or unbounded domain of $\mathbb{R}^N$ with state-constraint boundary conditions.

One of the main reasons to do it, and to choose such boundary conditions, is because the stratified formulation for state-constraint boundary conditions allows to treat within the same global framework several different types of boundary conditions (almost all the usual boundary conditions) for smooth or non-smooth domains, and in rather singular settings.

Other approaches for treating state-constraint problems in stratified situations appear in Hermosilla and Zidani [82], Hermosilla, Wolenski and Zidani [81], Hermosilla, Vinter and Zidani [80].

18.1 A Tanker problem mixing boundary conditions

To give a more concrete idea of what we have in mind, we describe a deterministic control problem proposed by P.L. Lions [101] in his course at the Collège de France in 2016, which was one of our main motivation to look at such formulation.

A controller has to manage a tanker : his aim is to decide when and where it will unload its cargo depending (typically) on the market price for the goods in the cargo. In the simplest modelling, the sea is identified with a smooth domain $\Omega \subset \mathbb{R}^2$ and the harbours are isoled points $P_1, P_2, \cdots, P_L$ on the boundary $\partial\Omega$. He has to control a tanker in such a way that it stays far from the coast and keeps its cargo if prices are low or, on the contrary, come to one of the harbours, unload and sell its cargo if they are higher at this harbour; the choice of the harbour is clearly part of the problem and there is no reason why the harbours should be equivalent. Of course, we have a state constraint boundary conditions on $\partial\Omega$ outside $P_1, P_2, \cdots, P_L$ since the tanker cannot accost to the coast if there is no harbour!

In terms of boundary conditions, we have a non-standard and rather singular problem since we have a state-constraint boundary condition on $\partial\Omega\setminus\{P_1, P_2, \cdots, P_L\}$ and P.L. Lions was suggesting Neumann boundary conditions for the harbours; therefore we have a problem like

$$\begin{aligned} u_t + H(x,t,Du) &= 0 \qquad \text{in } \Omega\times(0,T)\ , \\ u_t + H(x,t,Du) &\geq 0 \qquad \text{on } \partial\Omega\setminus\{P_1, P_2, \cdots, P_L\}\times(0,T)\ , \\ \frac{\partial u}{\partial n} &= g_i(t) \qquad \text{at } P_i\ \text{, for } i=1,\cdots,L. \end{aligned} \tag{18.1}$$

To the best of our knowledge, there is no work on such type of boundary conditions: here the mixing of state-constraint and Neumann boundary conditions (which is already not so standard) is even more complicated since the Neumann boundary conditions take place only at isolated points. And in fact, one quickly realizes that even if one can give a sense to such problems using viscosity solutions' theory, these problems are ill-posed in the sense that no uniqueness result holds in general (see Section 18.2 for a counter-example). The point is that the Neumann boundary conditions, imposed only at isolated points, are not "sufficiently seen" to give sufficient constraints on the solutions to provide the uniqueness.

To overcome this difficulty, we use below a reformulation of such problems in terms of stratified problems since this theory allows discontinuities in the Hamiltonians and it will allow here discontinuities also in the boundary conditions. The point is also that the definition of viscosity solutions for stratified problem consists in "super-imposing" some (subsolutions) inequalities on the discontinuity sets of the Hamiltonians (which can be not only of codimension 1 but also of higher codimension) and this is exactly what is lacking for obtaining uniqueness, as described in the previous paragraph.

Roughly speaking, we are going to show in this part that, in the case of control problems, classical Dirichlet, Neumann or even mixed boundary conditions for the associated Hamilton-Jacobi Bellman Equation can be reformulated as stratified problems with state-constraints boundary conditions, and this can be done also in the case of non-smooth domains since (globally) a stratification is not smooth (since it may contains crosses, for example). Then, treating such state-constraints boundary conditions for stratified problems presents only few additional difficulties, compared to Part III. And even some of the results of this part can be directly used to treat this a priori more general case.

The only additional difficulty (but which already appears when studying Dirichlet boundary conditions, even in the most standard continuous cases) concerns the boundary values of the solutions since some of these boundary values can take "artificial values", reflecting also the fact that the boundary condition in the viscosity sense is not strong enough to impose "natural values". To avoid that, we prove these results for "regular solutions", i.e. for sub and supersolutions which boundary values are (essentially) limits of their values inside Ω. We recall that such properties are also required in $\mathbb{R}^N$ on the different elements of the stratification (meaning on the discontinuity sets of the Hamiltonians) and here this phenomena is doubled since we have at the same time a boundary and a stratified problem which act together. Of course a key question is to identify some (stable) viscosity inequalities implying that subsolutions are "regular" : we refer the reader to Section 19.4 for a discussion.

This study allows us to revisit Dirichlet and Neumann boundary conditions in deterministic control problems and extend some results to far more general frameworks: discontinuous Hamiltonians (of course), non-smooth boundary conditions, mixing of boundary conditions and treatment of rather singular cases (including the above example).

18.2 A counter-example for the Tanker problem

We examine the problem (18.1) in the case when $\Omega = \{x_N > 0\} \subset \mathbb{R}^N$, there is only one harbour $P_1 = 0 \in \partial\Omega$ and the equation is

$$u_t + |Du| = 1 \quad \text{in } \Omega \times (0, +\infty) ,$$

and the boundary condition is

$$\frac{\partial u}{\partial n} = g \quad \text{at } 0, \text{ for all } t \in (0, +\infty) .$$

For the initial data, we choose $u(x, 0) = 0$ on $\overline{\Omega}$.

To compute a solution, we argue formally : the associated control problem is a problem with a reflection at 0 and the controlled trajectory is given by

$$\dot{X}(s) = \alpha(s)ds - 1_{\{X(s)=0\}} n(X(s)) d|k|_s , \; X(0) = x \in \overline{\Omega} ,$$

where $\alpha(\cdot)$ is the control taking values in $B(0,1)$ and the term $-1_{\{X(s)=0\}}n(X(s))d|k|_s$ is the reflection at 0, $(|k|_s)_s$ being the intensity of the reflection. The value function is

$$U(x,t) = \inf_{\alpha(\cdot)} \left\{ \int_0^t 1ds + \int_0^t g 1_{\{X(s)=0\}} d|k|_s \right\} .$$

In this case, the term $1_{\{X(s)=0\}}d|k|_s$ is nothing but $1_{\{X(s)=0\}}\alpha(s)\cdot n(0)ds$.

If $g < 0$, a favorable case to unload the cargo, the clear strategy to minimize the cost is to maximize $\int_0^t g1_{\{X(s)=0\}}d|k|_s$ and therefore to reach 0 as soon as possible and then to have $\alpha(s)\cdot n(0) = 1$, i.e. $\alpha(s) = n(0)$. This gives

$$U(x,t) = t + g(t-|x|)^+ ,$$

since $|x|$ is the time which is necessary to reach 0 from x and then one integrates g till time t.

Now take $g < g' < 0$ and consider $V(x,t) = t + g'(t-|x|)^+$. We claim that V is still a subsolution of (18.1): indeed, since we just change g in g', we have just to check it at $x = 0$, for $t > 0$. But, if $(y,s) \sim (0,t)$, then $(s-|y|)^+ > 0$ and $V(y,s) = s + g'(s-|y|)$. But $g' < 0$ and therefore the super-differential of V is empty at $(0,t)$ and therefore we have no subsolution's inequality to check.

Therefore V is a subsolution of the problem but clearly $V > U$ for $t > |x|$ and this shows that no comparison result can hold.

The interpretation of this counter-example is that the Neumann boundary condition at only one point (or at isolated points) is not seen enough by the notion of viscosity solution, at least not sufficiently to imply comparison/uniqueness. This defect is corrected by the stratified formulation which superimposes an inequality at 0 for all t.

Chapter 19

Stratified Solutions for State-Constraints Problems

19.1 Control problems, stratifications and State-Constraints conditions

In this section, we consider finite horizon, deterministic control problems with state-constraints in $\overline{\Omega} \times [0, T]$ where Ω is a domain in $\mathbb{R}^N$ which a priori is neither bounded nor regular. To formulate it, we are going to assume that the dynamics, discounts and costs are defined in $\mathbb{R}^N \times [0, T]$ (this is not a loss of generality) and may be discontinuous on subsets $\mathbf{M}^k \subset \overline{\Omega}$ for $k < N$ where $\mathbf{M}^k$ is a collection of k-dimensional submanifolds of $\mathbb{R}^N$. More precise assumptions will be given later on.

Following Section 4.2, we first define a general control problem associated to a differential inclusion. As we mention it above, at this stage, we do not need any particular assumption concerning the structure of the stratification, nor on the control sets. We also use the same notations and assumptions as in Section 4.2.

The control problem — as we said, we embed the accumulated cost in the trajectory by solving a differential inclusion in $\mathbb{R}^N \times \mathbb{R}$, namely (4.1) and we introduce the value function which is defined only on $\overline{\Omega} \times [0, T]$ by

$$U(x, t) = \inf_{\mathcal{T}(x,t)} \left\{ \int_0^{+\infty} l\big(X(s), T(s)\big) \exp(-D(s)) ds \right\},$$

where $\mathcal{T}(x, t)$ stands for all the Lipschitz trajectories (X, T, D, L) of the differential inclusion which start at $(x, t) \in \overline{\Omega} \times [0, T]$ and such that $(X(s), T(s)) \in \overline{\Omega} \times [0, T]$ for all $s > 0$.

Contrarily to Section 4.2, we point out that assumptions are needed in order to have $\mathcal{T}(x,t) \neq \emptyset$ for all $(x,t) \in \mathbb{R}^N \times (0,T]$: indeed, if there is no problem with the boundary $\{t=0\}$, there is a priori no reason why there exists trajectories X satisfying the constraint to remain in $\overline{\Omega}$ for any $x \in \overline{\Omega}$ and $t \in [0,T]$. Therefore, the fact that $\mathcal{T}(x,t)$ is non-empty will be an assumption in all this part: we will say that $(\mathbf{H_U})$ is satisfied if the value-function U is locally bounded on $\overline{\Omega} \times [0,T]$ which is almost equivalent.

A first standard result gathers Theorem 4.2.4 and 4.2.5

Theorem 19.1.1 (Dynamic Programming Principle and Supersolution's Property) *Under Assumptions* $(\mathbf{H_{BCL}})$ *and* $(\mathbf{H_U})$*, then* U *satisfies*

$$U(x,t) = \inf_{\mathcal{T}(x,t)} \Big\{ \int_0^\theta l\big(X(s),T(s)\big) \exp(-D(s))ds + U\big(X(\theta),T(\theta)\big) \exp(-D(\theta)) \Big\} ,$$

for any $(x,t) \in \mathbb{R}^N \times (0,T]$*,* $\theta > 0$*. Moreover, if* $\mathbb{F}$ *is defined by* (4.3)*, then the value function* U *is a viscosity supersolution of*

$$\mathbb{F}(x,t,U,DU) = 0 \quad \text{on } \overline{\Omega} \times [0,T] \, , \tag{19.1}$$

where we recall that $DU = (D_x U, D_t U)$*.*

We point out that, in the same way as Theorem 4.2.4 and 4.2.5, Theorem 19.1.1 holds in a complete general setting, independently of the stratification we may have in mind.

We conclude this first part by the analogue of Lemma 4.2.7 showing that supersolutions always satisfy a super-dynamic programming principle, even in this constrainted setting: again we remark that this result is independent of the possible discontinuities for the dynamic, discount and cost.

Lemma 19.1.2 *Under Assumptions* $(\mathbf{H_{BCL}})$*,* $(\mathbf{H_U})$ *and* $(\mathbf{H_{Sub}})$*, if* v *is a bounded lsc supersolution of* $\mathbb{F}(x,t,v,Dv) = 0$ *on* $\overline{\Omega} \times (0,T]$*, then, for any* $(\bar{x},\bar{t}) \in \overline{\Omega} \times (0,T]$ *and any* $\sigma > 0$*,*

$$v(\bar{x},\bar{t}) \geq \inf_{\mathcal{T}(\bar{x},\bar{t})} \Big\{ \int_0^\sigma l\big(X(s),T(s)\big) \exp(-D(s))\, \mathrm{d}s + v\big(X(\sigma),T(\sigma)\big) \exp(-D(\sigma)) \Big\} \tag{19.2}$$

Proof — The idea is to use Lemma 4.2.7 with a penalization type argument.

To do so, as in the proof of Lemma 4.2.7, we are going to prove Inequality (19.2) for fixed $(\bar{x},\bar{t})$ and σ, and to argue in the domain $B(\bar{x},M\sigma)\times[0,\bar{t}]$ where M is given by $(\mathbf{H_{BCL}})$, thus in a bounded domain. Next, for $\delta>0$ small, we set

$$v_\delta(x,t) := \begin{cases} v(x,t) & \text{if } x\in\overline{\Omega} \\ \delta^{-1} & \text{otherwise} \end{cases}$$

Since we argue in $B(\bar{x},M\sigma)\times[0,\bar{t}]$, v_δ is lsc in $B(\bar{x},M\sigma)\times[0,\bar{t}]$.

Next we change $\mathbf{BCL}$ into $\mathbf{BCL}_\delta$ in the following way: if $x\in\Omega$, $\mathbf{BCL}_\delta(x,t)=\mathbf{BCL}(x,t)$, while if $x\notin\Omega$, then $(b_\delta,c_\delta,l_\delta)\in\mathbf{BCL}_\delta(x,t)$ iff, either $(b_\delta,c_\delta,l_\delta)=(b,c,l+\delta^{-1}d(x,\overline{\Omega}))$ where $(b,c,l)\in BCL(x,t)$ and $d(\cdot,\overline{\Omega})$ denotes the distance to $\overline{\Omega}$, or $(b_\delta,c_\delta,l_\delta)=(0,1,\delta^{-1})$.

If we set for $(x,t)\in B(\bar{x},M\sigma)\times[0,\bar{t}]$

$$\mathbb{F}_\delta(x,t,r,p) := \sup_{(b_\delta,c_\delta,l_\delta)\in\mathbf{BCL}_\delta(x,t)} \big\{-b_\delta\cdot p + c_\delta r - l_\delta\big\}\,,$$

then v_δ is a lsc supersolution of $\mathbb{F}_\delta(x,t,v_\delta,Dv_\delta)=0$ in $B(\bar{x},M\sigma)\times(0,\bar{t})$. Indeed, we have, at the same time $\mathbb{F}_\delta\geq\mathbb{F}$ if $x\in\overline{\Omega}$ and $\mathbb{F}_\delta(x,t,r,p)\geq r-\delta^{-1}$ if $x\notin\Omega$.

Therefore Lemma 4.2.7 implies

$$v_\delta(\bar{x},\bar{t}) \geq \inf\Big\{\int_0^\sigma l_\delta\big(X_\delta(s),t-s\big)\exp(-D_\delta(s))\,\mathrm{d}s + v_\delta\big(X_\delta(\sigma),T_\delta(\sigma)\big)\exp(-D_\delta(\sigma))\Big\}\,,$$

the infimum being taken on all the solutions $(X_\delta,D_\delta,L_\delta)$ of the $\mathbf{BCL}_\delta$ differential inclusion.

To conclude the proof, we have to let δ tend to 0 in the above inequality where we can notice that $v_\delta(\bar{x},\bar{t})=v(\bar{x},\bar{t})$. To do so, we pick an optimal or δ-optimal trajectory $(X_\delta,D_\delta,L_\delta)$.

By the uniform bounds on $\dot{X}_\delta,\dot{D}_\delta,\dot{L}_\delta$, Ascoli-Arzela' Theorem implies that up to the extraction of a subsequence, we may assume that $X_\delta D_\delta,L_\delta$ converges uniformly on $[0,\sigma]$ to (X,D,L). And we may also assume that they derivatives converge in L^∞ weak-* (in particular $\dot{L}^\delta=l^\delta$).

We use the above property for the δ-optimal trajectory, namely

$$\int_0^\sigma l_\delta\big(X_\delta(s),t-s\big)\exp(-D_\delta(s))\,\mathrm{d}s + v_\delta\big(X_\delta(\sigma),T_\delta(\sigma)\big)\exp(-D_\delta(\sigma)) - \delta \leq v(\bar{x},\bar{t})\,,$$

in two ways: first by multiplying by δ, using that $l_\delta\geq -M+\delta^{-1}d(x,\overline{\Omega}))$ and the definition of v_δ outside $\overline{\Omega}$, we get

$$\int_0^\sigma d(X_\delta(s),\overline{\Omega}))\exp(-Ms)ds + \mathbb{1}_{X_\delta(\sigma)\notin\overline{\Omega}}\exp(-M\sigma) = O(\delta)\,.$$

The uniform convergence of X_δ implies that both terms in the left-hand side tend to 0, meaning that $X(s) \in \overline{\Omega}$ for any $s \in [0,\sigma]$. And the proof is complete.

Q.E.D.

19.2 Admissible stratifications for State-Constraints problems

In this section, we extend the notions of *admissible stratification* for a Bellman Equations set on $\overline{\Omega} \times (0,T)$.

Definition 19.2.1 (Admissible Stratification)
We say that a family of subsets $\mathbf{M}^0, \mathbf{M}^1, \cdots, \mathbf{M}^{N+1}$ *of* $\overline{\Omega} \times (0,T)$ *is an* Admissible Stratification of $\overline{\Omega} \times (0,T)$ *if*

$$\overline{\Omega} \times (0,T) = \mathbf{M}^0 \cup \mathbf{M}^1 \cup \cdots \cup \mathbf{M}^{N+1} \;,$$

$$\partial\Omega \times (0,T) \subset \mathbf{M}^0 \cup \mathbf{M}^1 \cup \cdots \cup \mathbf{M}^{N} \;,$$

and the family $\tilde{\mathbb{M}} = (\tilde{\mathbf{M}}^k)_k$ *defined by* $\tilde{\mathbf{M}}^0 = \mathbf{M}^0$, $\tilde{\mathbf{M}}^1 = \mathbf{M}^1, \cdots, \tilde{\mathbf{M}}^N = \mathbf{M}^N$ *and* $\tilde{\mathbf{M}}^{N+1} = (\mathbf{M}^{N+1} \cup [\overline{\Omega}^c \times (0,T)])$ *is an admissible stratification of* $\mathbb{R}^N \times (0,T)$.

In this definition, the only difference comes from the boundary $\partial\Omega \times (0,T)$ and the following lemma explains the structure of the $\mathbf{M}^k$'s in the boundary.

Lemma 19.2.2 *If* $(x,t) \in \mathbf{M}^k \cap \partial\Omega \times (0,T)$ *for some* $0 \le k \le N$, *then for* r *small enough,* $\mathbf{M}^k \cap B((x,t),r) \subset \partial\Omega \times (0,T)$.

As a consequence of this lemma, any connected component of $\mathbf{M}^k$ $(0 \le k \le N)$ is either included in $\partial\Omega \times (0,T)$ or included in $\Omega \times (0,T)$. In other words, the situation where one part of such connected component is in $\partial\Omega \times (0,T)$ and an other part is in $\Omega \times (0,T)$ is excluded by the definition of admissible stratification of $\overline{\Omega} \times (0,T)$.

As we will see it later on, this will have a key importance in the definition of stratified subsolutions since either we will consider interior points and, of course, this will be analogous to the $\mathbb{R}^N \times (0,T)$ case, or we will consider $\mathbb{F}^k$-inequalities at points of the boundary and we will not see any influence from $\Omega \times (0,T)$ since, $\mathbf{M}^k$ being included in $\partial\Omega$ in a neighborhood of these points, these inequalities are just “tangent” inequalities.

Proof — The result being local, we can assume without loss of generality that we are in the (AFS) case, i.e. there exists $r > 0$ such that $\mathbf{M}^k \cap B((x,t),r) = [(x,t) + V_k] \cap B((x,t),r)$ where V_k is a k dimensional vector space.

Lemma 19.2.2 is a consequence of Lemma 3.3.2 using the $\tilde{\mathbb{M}}$-stratification. If, for some $v \in V_k$, $(x,t) + v \in [\Omega \times (0,T)] \cap B((x,t),r)$, then there exists $0 < \delta \ll r$ such that $B((x,t)+v,\delta) \subset [\Omega \times (0,T)] \cap B((x,t),r)$. On the other hand, $B((x,t),\delta) \cap [\Omega^c \times (0,T)] \neq \emptyset$ and if $(x_\delta,t_\delta) \in B((x,t),\delta) \cap [\Omega^c \times (0,T)] \subset B((x,t),r) \cap \tilde{\mathbf{M}}^{N+1}$, we have $(x_\delta,t_\delta) \in \tilde{\mathbf{M}}^{N+1}$. By Lemma 3.3.2, $(x_\delta,t_\delta) + V_k \subset \tilde{\mathbf{M}}^{N+1}$ but $(x_\delta,t_\delta) \in \tilde{\mathbf{M}}^{N+1} \cap [\Omega^c \times (0,T)]$ and $(x_\delta,t_\delta) + v \in \tilde{\mathbf{M}}^{N+1} \cap [\Omega \times (0,T)]$ since $(x_\delta,t_\delta) + v \in B((x,t)+v,\delta) \subset [\Omega \times (0,T)]$. Therefore $(x_\delta,t_\delta) + V_k$ has a point in $\partial\Omega \times (0,T)$ which is a contradiction since there is no point of $\tilde{\mathbf{M}}^{N+1}$ on $\partial\Omega \times (0,T)$.

Q.E.D.

19.3 Stratified solutions and comparison result

Now we turn to the notion of stratified solution.

Definition 19.3.1 (Stratified sub and supersolutions for state-constraint problems)
(i) A locally bounded, lsc function $v : \overline{\Omega} \times [0,T[\to \mathbb{R}$ is a stratified supersolution of

$$\mathbb{F}(x,t,w,Dw) = 0 \quad \text{on } \overline{\Omega} \times [0,T[\ , \tag{19.3}$$

iff it is an Ishii's supersolution of this equation on $\overline{\Omega} \times [0,T[$.
(ii) A locally bounded, usc function $u : \overline{\Omega} \times [0,T[$ is a stratified subsolution of Equation (19.3) iff (a) it is an Ishii's subsolution of this equation in $\overline{\Omega} \times (0,T)$ and (b) for any $k = 0,...,(N+1)$ it is a subsolution of

$$\mathbb{F}^k(x,t,u,(D_t u, D_x u)) \leq 0 \quad \text{on } \mathbf{M}^k,\ \text{for } t > 0,$$

and, for $t = 0$ and $k = 0,...,(N+1)$ it is a subsolution of

$$(\mathbb{F}_{init})_*(x,u,D_x u) \leq 0 \quad \text{in } \Omega\ ,$$

$$\mathbb{F}^k_{init}(x,u,D_x u) \leq 0 \quad \text{on } \mathbf{M}^k_0.$$

In addition, we will say that u is a strict stratified subsolution if the ≤ 0-inequalities are replaced by a $\leq -\eta < 0$-inequality where $\eta > 0$ is independent of x and t.

Several remarks on the definition: for the supersolution, we have the classical Ishii's inequality, and up to the boundary $\partial\Omega \times (0,T)$ as it is classical for state-constraint problems. Of course at time $t=0$, the analogue of Proposition 5.1.1 implies that $\mathbb{F}$ can be replaced by $\mathbb{F}_{init}$. For the subsolution case, the main feature of stratified subsolutions are preserved, i.e. we have to super-impose $\mathbb{F}^k$-inequalities on all $\mathbf{M}^k$ (including at time $t=0$). What is more suprising and unusual in state-constraint framework is the fact that we have subsolutions inequalities on $\partial\Omega \times (0,T)$ but these inequalities are on $\mathbf{M}^k \cap [\partial\Omega \times (0,T)]$ for $k=0,...,N$ and therefore they take into account only the dynamics which stay on $\mathbf{M}^k$, i.e. on $\partial\Omega \times (0,T)$.

We conclude this part by the comparison result.

Theorem 19.3.2 *In the framework of "good framework for HJ Equations with discontinuities" which is described above, we have a comparison result between bounded stratified sub and supersolution for Equation (19.3) provided that the subsolution* $u : \overline{\Omega} \times [0,T[\to \mathbb{R}$ *satisfies, for any* $(x,t) \in [\partial\Omega \times (0,T)] \cap \mathbf{M}^k$*, we have*

$$u(x,t) = \limsup \left\{ u(y,s);\ (y,s) \to (x,t),\ (y,s) \in \mathbf{M}^{k+1} \cup \mathbf{M}^{k+2} \cup \cdots \cup \mathbf{M}^{N+1} \right\}\ , \tag{19.4}$$

and, if $x \in \partial\Omega \cap \mathbf{M}^k_0$*,*

$$u(x,0) = \limsup \left\{ u(y,0);\ (y,0) \to (x,0),\ (y,0) \in \mathbf{M}^{k+1}_0 \cup \mathbf{M}^{k+2}_0 \cup \cdots \cup \mathbf{M}^{N+1}_0 \right\}\ . \tag{19.5}$$

As the reader may imagine it, the proof is (almost) exactly the same as the proof of Theorem 13.2.1 and this is easy to explain why: the fact that some parts of the stratification are on the boundary does not cause any problem and the key ingredients were already used in the $\mathbb{R}^N$-case. The difference comes from a "little detail" that we are going to comment now.

The proof of Theorem 13.2.1 is based (i) regularizing the subsolution u in a neighborhood of each $\mathbf{M}^k$ and then (ii) use Lemma 5.4.1 to conclude. In order to have a proper regularization of u, we need to know that the values of u on $\mathbf{M}^k$ are not "artificial" and more precisely that they are obtained as the limits of those on $\mathbf{M}^{k+1} \cup \mathbf{M}^{k+2} \cup \cdots \cup \mathbf{M}^{N+1}$. In $\mathbb{R}^N \times (0,T)$, this is ensured by the $\mathbb{F}_*$-inequality and Proposition 3.4.1 but, on the boundary, an assumption is required to have the same property and to eliminate "artificial values" of u: this is the role of (19.4)-(19.5).

As we will see it below, this condition is analogous to the "cone's condition" which is used in state-constraint or Dirichlet problems for standard continuous equations. We will see in Section 19.4 how an analogue of the $\mathbb{F}_*$-inequality and Proposition 3.4.1 for boundary points can be used to obtained (19.4) and/or (19.5); this point may be important for stability reasons.

19.4 On the Regularity of Subsolutions

In Part III, the "regularity" of subsolutions was playing a central role in order to obtain continuous subsolution after "tangential regularization" by sup-convolution. This was a keystone of the comparison result in the stratified case. In $\mathbb{R}^N$, this property is a consequence of the (natural since standard) subsolution inequality $u_t+H_* \leq 0$ provided that the normal controllabilty assumption is satisfied (cf. Proposition 3.4.1). Of course, in the present context, the same is true in Ω.

The situation is a little bit different on $\partial\Omega$ since, on one hand, the subsolution inequality $u_t + H_* \leq 0$ cannot hold on the boundary because of the normal controllability assumption and, on the other hand, the analogue of this inequality for boundary point would consist in using inner dynamic but this can be rather complicated to formulate in non-smooth domains and anyway this has the defect of enjoying rather poor stability properties.

We propose two ways to circumvent this difficulty: the first one consists in actually considering cases where we can have ad hoc inequalities on the boundary which plays the role of the "$u_t + H_* \leq 0$" one. The second one, inspired by the "continuous case", is completely different since it consists in redefining the subsolution on the different $\mathbf{M}^k$ of the boundary in order to have (19.4). Of course this second way is far more restrictive since it requires that we have no real discontinuity (in terms of **BCL**) on the boundary but it may be useful since the stratified approach allows non-smooth boundaries.

Before providing our results, we want to point out that such type of difficulty is classical in state-constraint or Dirichlet problem, even if it is, in general, formulated in a slightly different way: from the very first articles of Soner [114, 115] on state-constraint problems, the "cone's condition" (or related properties) is known to play a role in comparison results for such problems since one needs to have some kind of continuity property of the subsolution on the boundary, at least to avoid artificial values.

For Dirichlet problems, Perthame and the first author [16, 17, 18] have worked on this difficulty by either showing such continuity property (even in a weaker sense) or by redefining the subsolution on the boundary in order to have it, two possibilities that we investigate below. Ishii and Koike [90] have formulated the state-constraint boundary condition in a different way, with an unusual subsolution condition on the boundary, by looking only at dynamics which are pointing inside the domain on the boundary: Lemma 19.4.1 below shows that their boundary condition "$u_t + H_{in} \leq 0$" avoids non-regular subsolution if there is an inner dynamic.

Finally we point out that some results for first but also second-order equations are obtained by Katsoulakis [92] or Rouy and the first author [19]: in [19], a blow-up argument allows to show that the cone's condition holds une der suitable assumptions for first-order equations and that we have a related property for the second-order case.

Our first result is the

Lemma 19.4.1 *Assume that Ω is a stratified domain and that $x_0 \in \mathbf{M}^k \cap \partial\Omega$. Assume also that there exist $r, h, M, \bar{t}, k > 0$ and a continuous function $b(\cdot)$ defined on $\partial\Omega \cap B(x_0, r)$ such that, for $0 < \tau < \bar{\tau}t$ and $y \in \partial\Omega \cap B(x_0, r)$, one has*

$$B(y + \tau b(x_0), k\tau) \subset \Omega \, . \tag{19.6}$$

If u is a subsolution of

$$u_t - b(x) \cdot Du \leq M \quad \text{on } [\partial\Omega \cap B(x_0, r)] \times (t_0 - h, t_0 + h) \, , \tag{19.7}$$

for some $t_0 \in (0, T)$, then (19.4) holds at (x_0, t_0), more precisely

$$u(x_0, t_0) = \limsup\{u(y, s), \ (y, s) \to (x_0, t_0), \ y \in \mathbf{M}^{k+1} \cup \cdots \mathbf{M}^N\} \, .$$

Before proving this lemma, we want to point out the following comment: in fact, as the proof is going to show it, this lemma is a very basic result ; a more interesting point is to give general (and if possible, natural) conditions under which a subsolution of the stratified problem is a viscosity subsolution of an equation like (19.7).

Proof — Throughout the proof, we assume that we are in the flat case, namely $\mathbf{M}^k \cap B(x_0, r) = (x_0 + V_k) \cap B(x_0, r)$, where V_k is a k-dimensional vector space in $\mathbb{R}^N$, taking perhaps k smaller in (19.6). We first claim that $b(x_0)$ cannot be in $T_{x_0}\mathbf{M}^k$.

This property is an easy consequence of Lemma 19.2.2: indeed, it is easy to show that $b(x_0) \notin T_{x_0}\mathbf{M}^k$. Indeed, otherwise we would have that the distance from $x_0 + \tau b(x_0)$ to $\mathbf{M}^k$ would be a $o(\tau)$ which would contradict the assumption on b which says that the distance to $\partial\Omega$ is at least $k\tau$ ($k > 0$).

The above property on $b(x_0)$ implies that there exists a vector e, orthogonal to $T_{x_0}\mathbf{M}^k$, such that $b(x_0) \cdot e > 0$. Then, in $\overline{\Omega}$, we consider the function

$$(x, t) \mapsto u(x, t) - \frac{|x - x_0|^2}{\varepsilon^2} + \frac{2}{\varepsilon} e \cdot (x - x_0) - \frac{|t - t_0|^2}{\varepsilon} \, .$$

Using the properties satisfied by a stratification, if (19.4) does not hold then, for $0 < \varepsilon \ll 1$, this function necessarely achieves its maximum on $\mathbf{M}^k$ at $(x_\varepsilon, t_\varepsilon)$ and, as a consequence of the maximum point property we have

$$u(x_0, t_0) \leq u(x_\varepsilon, t_\varepsilon) - \frac{|x_\varepsilon - x_0|^2}{\varepsilon^2} - \frac{|t_\varepsilon - t_0|^2}{\varepsilon} + \frac{1}{\varepsilon} e \cdot (x_\varepsilon - x_0) = u(x_\varepsilon, t_\varepsilon) - \frac{|x_\varepsilon - x_0|^2}{\varepsilon^2} - \frac{|t_\varepsilon - t_0|^2}{\varepsilon} \, ,$$

the equality coming from the definition of e and the fact that we have a flat stratification.

By classical arguments, this implies that the penalisation terms $\dfrac{|x_\varepsilon - x_0|^2}{\varepsilon^2}, \dfrac{|t_\varepsilon - t_0|^2}{\varepsilon}$ tend to 0 when $\varepsilon \to 0$ and in particular, we have $(x_\varepsilon, t_\varepsilon) \to (x_0, t_0)$. Writing the viscosity subsolution inequality yields

$$\frac{2(t_\varepsilon - t_0)}{\varepsilon} - b(x_\varepsilon) \cdot (\frac{2(x_\varepsilon - x_0)}{\varepsilon^2} - \frac{2}{\varepsilon} e) \leq M \ ,$$

which gives, thanks to the previous properties

$$\frac{o(1)}{\varepsilon} + \frac{2}{\varepsilon} b(x_\varepsilon) \cdot e \leq M \ .$$

But, by the continuity of b, $b(x_\varepsilon) \cdot e \to b(x_0) \cdot e > 0$, and we have a contradiction for ε small enough.

Q.E.D.

Lemma 19.4.2 *Assume that Ω is a stratified domain, that* **(NC)**, **(TC)** *hold and that there exist $r, \bar\tau, k > 0$ and a continuous function $b(\cdot)$ defined on $\partial\Omega \cap B(x_0, r)$ such that (19.6) is satisfied. Assume also that, for any $y \in \Omega \cap B(x_0, r)$, for any $x \in \partial\Omega \cap B(x_0, r)$, $dist(b(x), \mathbf{B}(y)) \leq C|x - y|$ for some constant $C > 0$. If u is a subsolution of the stratified state constraint problem in $\overline{\Omega} \cap B(x_0, r)$ and if (19.4) holds for any $(x, t) \in [\partial\Omega \cap B(x_0, r)] \times (t_0 - h, t_0 + h)$ then u is a subsolution of (19.7).*

This corollary means that, in some sense, the property (19.4) is equivalent to a natural "control" inequality (as it is the case in Ω) and that such inequality should be automatically satisfied as a consequence of the normal controllability.

Proof — Using the regularization procedure of Section 3.4, we can assume without loss of generality that u is Lipschitz continuous on $\overline{\Omega} \cap B(x_0, r)$. We point out that (19.4) plays a key role in this property to avoid any discontinuity on the boundary. If ϕ is a test-function and if $(\bar x, \bar t) \in [\partial\Omega \cap B(x_0, r)] \times (t_0 - h, t_0 + h)$ is a strict local maximum point of $u - \phi$ in $[\overline{\Omega} \cap B(x_0, r)] \times (t_0 - h, t_0 + h)$, we consider the function

$$\Psi(x, t, y, s) = u(x, t) - \phi(y, s) - \frac{|x - y - \varepsilon b(x_0)|^2}{\varepsilon^2} - \frac{|t - s|^2}{\varepsilon} \ .$$

which achieves its maximum at $(x_\varepsilon, t_\varepsilon, y_\varepsilon, s_\varepsilon)$. Since u is (Lipschitz) continuous, we have $u(\bar x + \varepsilon b(x_0), \bar t) = u(\bar x, \bar t) + o_\varepsilon(1)$ and therefore

$$u(\bar x, \bar t) - \phi(\bar x, \bar t) + o_\varepsilon(1) \leq \Psi(\bar x + \varepsilon b(x_0), \bar t, \bar x, \bar t) \leq \Psi(x_\varepsilon, t_\varepsilon, y_\varepsilon, s_\varepsilon) \leq u(\bar x, \bar t) - \phi(\bar x, \bar t) + + o_\varepsilon(1) \ ,$$

the last inequality coming from the fact that $x_\varepsilon - y_\varepsilon, t_\varepsilon - s_\varepsilon$ are $O(\varepsilon)$ and therefore $(x_\varepsilon, t_\varepsilon) = (y_\varepsilon, s_\varepsilon) + O(\varepsilon)$.

By classical arguments we deduce that, not only $(x_\varepsilon, t_\varepsilon), (y_\varepsilon, s_\varepsilon) \to (\bar{x}, \bar{t})$ but we also have

$$\frac{|x_\varepsilon - y_\varepsilon - \varepsilon b(x_0)|^2}{\varepsilon^2} + \frac{|t_\varepsilon - s_\varepsilon|^2}{\varepsilon} \to 0 \ ,$$

as $\varepsilon \to 0$. In particular $x_\varepsilon \in B(y_\varepsilon + \varepsilon b(x_0), k\varepsilon) \subset \Omega$ since $|x_\varepsilon - y_\varepsilon - \varepsilon b(x_0)| = o(\varepsilon)$ if ε is small enough.

We can write down the viscosity subsolution inequality for u

$$\alpha_\varepsilon + H_*(x_\varepsilon, t_\varepsilon, u(x_\varepsilon, t_\varepsilon), p_\varepsilon) \leq 0 \ ,$$

where $\alpha_\varepsilon = \dfrac{2(t_\varepsilon - s_\varepsilon)}{\varepsilon} = \phi_t(y_\varepsilon, s_\varepsilon)$ by the maximum point property in s, while $p_\varepsilon = \dfrac{2(x_\varepsilon - y_\varepsilon - \varepsilon b(x_0))}{\varepsilon^2} = D\phi(y_\varepsilon, s_\varepsilon)$ if $y_\varepsilon \in \Omega$ but not necessarely if $y_\varepsilon \in \partial\Omega$.

In order to estimate H_*, we recall that

$$H(x, t, u, p) \geq -b \cdot p + cr - l \ ,$$

where $(b, c, l) \in BCL(x, t)$. In particular $H(x, t, u, p) \geq -b(\tilde{x}) \cdot p + O(|x - \tilde{x}|) - M$ for some constant M since c, u, l are bounded and we have denoted by $\tilde{x}$ (one of) the projection of x on $\partial\Omega$.

Using this estimate, we conclude easily if we know that $y_\varepsilon \in \partial\Omega$ at least for a subsequence of ε tending to 0.

If $y_\varepsilon \in \partial\Omega$, we write for $0 < \tau \ll 1$

$$\Psi(x_\varepsilon, t_\varepsilon, y_\varepsilon, y_\varepsilon + \tau b(y_\varepsilon), t_\varepsilon) \leq \Psi(x_\varepsilon, t_\varepsilon, y_\varepsilon, s_\varepsilon) \ ,$$

and this variation gives $-b(y_\varepsilon) \cdot D\phi(y_\varepsilon, s_\varepsilon) \leq -b(y_\varepsilon) \cdot p_\varepsilon$. The conclusion follows since

$$H_*(x_\varepsilon, t_\varepsilon, u(x_\varepsilon, t_\varepsilon), p_\varepsilon) \geq -b(y_\varepsilon) \cdot p_\varepsilon + O(|x_\varepsilon - y_\varepsilon|)|p_\varepsilon| - M \ ,$$

the term $O(|x_\varepsilon - y_\varepsilon|)|p_\varepsilon|$ being a $o_\varepsilon(1)$. And the proof is complete.

Q.E.D.

Now we turn to the second possibility which is a little bit more restrictive.

Lemma 19.4.3 *Assume that Ω is a stratified domain, that* **(NC)**, **(TC)** *hold. Let $x_0 \in \mathbf{M}^k$ be a point such that there exists $r > 0$ such that $\mathbf{M}^k \cap B(x_0, r) = \partial\mathbf{M}^{k+1} \cap B(x_0, r)$. If, for any $(x, t) \in [\mathbf{M}^k \cap B(x_0, r)] \times (t_0 - h, t_0 + h)$ and any $(y, s) \in [\mathbf{M}^{k+1} \cap$*

$B(x_0,r)] \times (t_0-h,t_0+h)$, we have $dist(\mathbf{BCL}(x,t),\mathbf{BCL}(y,s)) \leq C|x-y|+m(|t-s|)$ for some modulus of continuity m, and if u is a subsolution of the H^{k+1}- equation in $\mathbf{M}^{k+1} \cap B(x_0,r)$ then the function $\tilde{u}$ defined on $[\mathbf{M}^k \cap B(x_0,r)] \times (t_0-h,t_0+h)$ by

$$\tilde{u}(x,t) = \limsup\{u(y,s),\ (y,s) \to (x,t)\ y \in \mathbf{M}^{k+1}\} .$$

satisfies the H^k-inequality in $[\mathbf{M}^k \cap B(x_0,r)] \times (t_0-h,t_0+h)$

Proof — We consider a smooth test-function ϕ and $(x,t) \in [\mathbf{M}^k \cap B(x_0,r)] \times (t_0-h,t_0+h)$, a strict local maximum point of $\tilde{u}-\phi$ on $[\mathbf{M}^k \cap B(x_0,r)] \times (t_0-h,t_0+h)$. By definition of $\tilde{u}$, we can approximate this maximum by maximum points on $[\mathbf{M}^{k+1} \cap B(x_0,r)] \times (t_0-h,t_0+h)$

$$(y,s) \mapsto u(y,s) - \phi(y,s) - \frac{\alpha}{d(y,\mathbf{M}^k)} - \frac{d(y,\mathbf{M}^k)}{\varepsilon} ,$$

by choosing in a proper way the parameters ε,α which are devoted to tend to 0.

If the maximum is achieved at $(\bar{x},\bar{t})$ depending on α and ε, the H^{k+1}-inequality has to be written as a supremum for all $(b,c,l) \in \mathbf{BCL}(\bar{x},\bar{t})$ with $b \in T_{\bar{x}}\mathbf{M}^{k+1}$ but we are going to argue in a different way: if $(\bar{b},\bar{c},\bar{l}) \in \mathbf{BCL}(\bar{y},\bar{t})$ where $\bar{y}$ is the projection of $\bar{x}$ on $\mathbf{M}^k$ then, on one hand, there exists $(b,c,l) \in \mathbf{BCL}(\bar{x},\bar{t})$ such that $|b-\bar{b}| \leq C|\bar{x}-\bar{y}| = Cd(\bar{x},\mathbf{M}^k)$ and $|\bar{c}-c|, |\bar{l}-l|$ are $o(1)$ in α,ε. On the other hand

$$\phi_t(\bar{x},\bar{t}) - b \cdot (D\phi(\bar{x},\bar{t}) + P_{\alpha,\varepsilon}) + cu(\bar{x},\bar{t}) - l \leq 0 ,$$

where $P_{\alpha,\varepsilon}$ denotes the derivatives of the two last terms, for which we have $|P_{\alpha,\varepsilon}|d(y,\mathbf{M}^k) \to 0$ as $\alpha,\varepsilon \to 0$ (again with the proper choice we have to make). Taking into account the fact that $u(\bar{x},\bar{t}) \to u(x,t)$, we are lead to

$$\phi_t(\bar{x},\bar{t}) - \bar{b} \cdot D\phi(\bar{x},\bar{t}) + \bar{c}u(x,t) - \bar{l} \leq o(1) ,$$

and since this is true for any $(\bar{b},\bar{c},\bar{l}) \in \mathbf{BCL}(\bar{y},\bar{t})$ and therefore for any $(\bar{b},\bar{c},\bar{l}) \in \mathbf{BCL}(x,t)$ by tangential continuity, the result follows.

Q.E.D.

The above lemma suggests the following procedure if $\mathbf{M}^k \subset \partial\mathbf{M}^{k+1}$ for any k: we first redefine u on $\mathbf{M}^{N-1}$ and, after this step, (19.4) holds on $\mathbf{M}^{N-1}$. Then we repeat the same operation on $\mathbf{M}^{N-2}$ and inductively until $\mathbf{M}^0$.

Chapter 20

Classical Boundary Conditions and Stratified Formulation

In this chapter, we are going to investigate the connections between stratified problems with state-constraints and classical (or almost classical) problems with boundary conditions (Dirichlet, Neumann or mixed boundary conditions). Of course, the interest of the stratified formulation is to allow to treat cases where the boundary is not smooth or the boundary conditions may present discontinuities, and also both at the same time.

Clearly our aim cannot be to give extremely general results: this would be unreadable and of a poor interest. Instead, we address the following two complementary questions, mainly in very simple frameworks, whose answers may emphasize the role and the interest of the stratified formulation:

(i) in which cases classical Ishii's viscosity solutions and stratified solutions are the same? Of course, in such cases, the theory which is developed in the previous chapter provides complete comparison results;

(ii) on the contrary, in which cases is the stratified formulation needed because the Ishii formulation is not precise enough to identify the "good" solution?

In order to do so, we are just going to consider standard problems where the difficulty only comes from the boundary and boundary data, and for which the equation inside the domain is continuous. Clearly, our results would need to be extended to treat problems where we have also discontinuities inside: some of these extensions are easy using some ideas of this chapter but some other ones are more delicate.

On the other hand, and this is obvious from the definition of stratified solutions, we are going to concentrate on subsolution properties since a stratified supersolution is nothing but a classical viscosity supersolution in the sense of Ishii. Of course, we are going to place ourselves in the "good framework for HJ Equations with discontinuities", which implies that most of the time we will be able to assume without loss of generality that the subsolutions are Lipschitz continuous.

20.1 On the Dirichlet Problem

We are interested in this section in the Dirichlet problem for Hamilton-Jacobi-Bellman Equations, namely

$$\begin{cases} u_t + H(x,t,D_x u) = 0 & \text{in } \Omega \times (0,T) \\ u(x,0) = u_0(x) & \text{in } \Omega \\ u(x,t) = \varphi(x,t) & \text{on } \partial\Omega \times (0,T) \end{cases}, \tag{20.1}$$

where Ω is a domain in $\mathbb{R}^N$ and H is a continuous function.

The classical case is when u_0, φ are continuous functions which satisfy the compatibility condition

$$u_0(x) = \varphi(x,0) \quad \text{on } \partial\Omega\ .$$

In this classical case, there are two kinds of results for such problems in the case when (20.1) is associated to a deterministic exit time problem, i.e. when H has the form (4.3): these two types of results are described in the book [23] and are originated from the works of Perthame and the first author.

- The first one is the "discontinuous approach" where one tries to determine the minimal and maximal solution of (20.1) in full generality, i.e. without any particular assumption on the dynamic and cost, and without assuming the boundary of Ω to be smooth. The result is that there exist a minimal solution U^- and a maximal solution U^+ which are value-functions of exit time problems, U^- being associated to the best stopping time on the boundary, while U^+ is associated to the worst stopping time on the boundary.

- The second one is the "continuous approach" in which one looks for conditions under which the value function is continuous and the unique solution of (20.1): in [18], the result is obtained under classical assumptions on the dynamics and cost, plus an hypothesis of normal controllability on the boundary which looks like very much **(NC)**. This second type of results required some regularity of the boundary ($W^{2,\infty}$ in general).

In this section, our aim is to reformulate the Dirichlet problem in the "stratified" framework, in order to investigate the cases when it is equivalent to the classical viscosity solutions formulation and then to examine the type of results that we can get in that way.

For the reformulation, the idea is very clear and classical: if $(x,t) \in \partial\Omega \times [0,T]$, the Dirichlet condition can be associated to an element of $\mathbf{BCL}(x,t)$ which is $(0,1,\varphi(x,t))$. Indeed, at the level of the Hamiltonian, this produces the expected term

$$-b \cdot p + cu - l = u - \varphi(x,t) \ ,$$

and, for the control point of view, this provides a 0-dynamic allowing to stop at the point (x,t) and pay a cost which is $\varphi(x,t)$, the discount factor being 1. With this point of view, we have just a state-constraint problem since the trajectory exists for all times and stays on $\overline{\Omega}$, the Dirichlet condition just allowing a choice $b=0$ on the boundary.

In the case when H is continuous on $\overline{\Omega}$, the stratified approach consists in considering the stratification $\mathbf{M}^{N+1} = \Omega \times (0,T)$ and $\mathbf{M}^N = \partial\Omega \times (0,T)$. In order to apply the above results, we have to impose

(*i*) some regularity of $\partial\Omega$: here $W^{2,\infty}$ (exactly as in [18]) is natural in general since we have to flatten $\mathbf{M}^N$, by keeping the needed property on H (in particular **(TC)**). But this can be reduced to C^1 if H is coercive, to the cost of sophisticating a little bit our arguments, treating differently the variables t and x.

(*ii*) some normal controlability assumptions which turn out to be also the same as in [18].

We come back later on the advantages of this new approach but let us examine first the boundary condition from the stratified point of view. To do so, we have to compute $\mathbb{F}^N(x,t,p)$ if $(x,t) \in \mathbf{M}^N$ where we recall that $p = (p_x, p_t)$.

At (x,t), the **BCL** set is obtained by considering the convex enveloppe of elements of the form $(b,c,l) = (b,0,l) \in \mathbf{BCL}(x,t)$ [associated to the Hamiltonian H] and of $(0,1,\varphi(x,t))$ associated to the Dirichlet boundary condition; therefore we have to consider all the $(\mu b, (1-\mu), \mu l + (1-\mu)\varphi(x,t))$ for $0 \le \mu \le 1$ but with

$$\mu b = \mu(b^x, -1) \in T_{(x,t)}\mathbf{M}^{N-1} \ ,$$

in other words $b^x \in T_x\partial\Omega$.

In order to compute $\mathbb{F}^N(x,t,p)$, we have to look at the supremum in μ and $(b,0,l) \in \mathbf{BCL}(x,t)$ with $b^x \in T_x\partial\Omega$, of

$$-\mu b\cdot p + (1-\mu)u - (\mu l + (1-\mu)\varphi(x,t)) = \mu(b\cdot p - l) + (1-\mu)(u-\varphi(x,t))\,,$$

and clearly this supremum is achieved either for $\mu = 0$, or $\mu = 1$. Hence the Dirichlet boundary condition

$$\max(u_t + H^N(x,t,D_x u), u - \varphi(x,t)) \leq 0 \quad \text{on } \mathbf{M}^N\,, \tag{20.2}$$

where

$$H^N(x,t,p_x) = \sup_{b^x\in T_x\partial\Omega}\{-b^x\cdot p_x - l\}\,.$$

This is a rather unusual inequality which, to the best of our knowledge, never appears in the study of Dirichlet boundary conditions for HJ-Equations. But, on the other hand, it is rather natural from the control point of view; indeed the inequality $u_t + H^N(x,t,D_x u) \leq 0$ just says that the tangent dynamics are sub-optimal, while the inequality $u - \varphi(x,t) \leq 0$ is an easy consequence of the normal controllability which is natural in this framework. We also point out that the non-tangential dynamics are taken into account in the Ishii viscosity subsolution inequality

$$\min(u_t + H(x,t,D_x u), u - \varphi(x,t)) \leq 0 \quad \text{on } \partial\Omega\times(0,T)\,.$$

Now we turn to the first key question: does a classical viscosity subsolution always satisfy such inequality in the stratified framework? And does an analogous one hold on $\mathbf{M}^k$ for $1 \leq k \leq N$?

We begin with the

Proposition 20.1.1 *Assume that $\overline{\Omega}\times(0,T)$ is a stratified domain, φ is continuous on $\mathbf{M}^N \cap (\partial\Omega\times(0,T))$ and that we are in the good framework for stratified problem. Then,*

(i) if u is any viscosity subsolution of the Dirichlet problem, $u \leq \varphi$ on $\mathbf{M}^N$.

(ii) If $\tilde u$ is given by $\tilde u = u$ in $\Omega\times(0,T)$ and

$$\tilde u(x,t) = \limsup_{\substack{(y,s)\to(x,t)\\ y\in\Omega}} u(y,s)\,,$$

then $\tilde u$ is still a viscosity subsolution of the Dirichlet problem and (20.2) holds for $\tilde u$ on $\mathbf{M}^N$.

The introduction of the function $\tilde{u}$ in order to redefine u on the boundary is classical: in fact, it is needed because the viscosity subsolution inequality is not strong enough to avoid artificial values of u on the boundary. Indeed since the viscosity subsolution property is ensured by the fact that $u \leq \varphi$ on $\mathbf{M}^N$, u could be changed into any u.s.c. function which lies below φ on $\mathbf{M}^N$, with no link whatsoever with the values inside $\Omega \times (0,T)$. The introduction of $\tilde{u}$ consists in imposing the "natural" values of the subsolution on $\mathbf{M}^N$ since they are consistent with those in $\Omega \times (0,T)$. Once this "cleaning" of the boundary values is done, then we have the desired result, namely that viscosity subsolutions are stratified subsolutions.

Proof — We start proving that, if u is an u.s.c. viscosity subsolution of the Dirichlet problem, then $u \leq \varphi$ on $\mathbf{M}^N$. We can argue locally and therefore assume that the boundary of Ω is flat since we have a stratified domain: if d denotes the distance to the boundary and $(x,t) \in \mathbf{M}^N$, we consider the function

$$(y,s) \mapsto u(y,s) - \frac{(s-t)^2}{\varepsilon^2} - \frac{|y-x|^2}{\varepsilon^2} - C_\varepsilon d(y) \ ,$$

where $C_\varepsilon > 0$ is a large constant to be chosen later. This function has a maximum point $(y_\varepsilon, t_\varepsilon)$ near (x,t) and, by classical arguments, we have $(y_\varepsilon, t_\varepsilon) \to (x,t)$ and $u(y_\varepsilon, t_\varepsilon) \to u(x,t)$. If the H-inequality holds at $(y_\varepsilon, t_\varepsilon)$, we would have

$$\frac{2(s_\varepsilon - t)}{\varepsilon^2} + H(y_\varepsilon, s_\varepsilon, \frac{2(y_\varepsilon - x)}{\varepsilon^2} + C_\varepsilon D_x d(y_\varepsilon)) \leq 0 \ .$$

But, by the normal controllability assumption, recalling that $|D_x d(y_\varepsilon)| = 1$, this inequality cannot hold if we choose C_ε large enough. As a consequence $(y_\varepsilon, t_\varepsilon) \in \partial\Omega \times (0,T)$ and $u(y_\varepsilon, t_\varepsilon) \leq \varphi(y_\varepsilon, t_\varepsilon)$.

Letting $\varepsilon \to 0$, we obtain the desired result since φ is continuous on $\mathbf{M}^N$.

As we already mentioned it above, the viscosity subsolution inequality being reduced to $u \leq \varphi$ on $\mathbf{M}^N$, since $\tilde{u} \leq u$, it follows that $\tilde{u}$ is also a viscosity subsolution of the Dirichlet problem.

Next we have to show that the $\mathbb{F}^N$-inequality holds for $\tilde{u}$. We may assume without loss of generality that $\tilde{u}$ is Lipschitz continuous because we can perform the regularization in the tangent variables (including t), and then use the normal controllability property. In the same way we can assume that the boundary is flat and use the definition of H^N not only for $x \in \partial\Omega$ but also for $x \in \Omega$ and notice that $H^N \leq H$ since the supremum is taken on a smaller set than $\mathbf{BCL}$.

As in Proposition 16.4.1, if e is a unit vector, normal to the hyperplane $\partial\Omega$ and pointing inside Ω (we recall that we may assume that the boundary is flat since $\overline{\Omega}$ is a stratified domain), it is clear that $\tilde{u}^\varepsilon(x,t) := \tilde{u}(x+\varepsilon e,t)$ is a subsolution of

$$\tilde{u}^\varepsilon_t + H^N(x+\varepsilon e,t,D_x\tilde{u}^\varepsilon) \leq 0 \quad \text{on } \mathbf{M}^N ,$$

and passing to the limit by a standard stability result (since $\tilde{u}^\varepsilon$ converges to $\tilde{u}$ uniformly), we obtain (20.2).

Q.E.D.

Remark 20.1.2 *Two remarks can be made on the above proof.*

(i) The first one concerns the inequality $u(x,t) \leq \varphi(x,t)$. In fact, even if φ is discontinuous, the inequality $u(x,t) \leq \varphi^*(x,t)$ (with the u.s.c. enveloppe of φ) can be proved not only for points in $\mathbf{M}^N$ but for any point where the exterior sphere condition holds, i.e. there exists $\bar{x} \in \mathbb{R}^N, \bar{r} > 0$ such that

$$\overline{B(\bar{x},\bar{r})} \cap \overline{\Omega} = \{x\} .$$

Indeed, it is enough to reproduce the above proof replacing the function $d(y)$ by $y \mapsto |y-\bar{x}| - \bar{r}$. This is therefore a general fact, but unfortunately not convenient for the stratification formulation which requires the more restrictive inequality $u(x,t) \leq \varphi_*(x,t)$.

(ii) On the other hand, in order to obtain the H^N-inequality, we use very few properties, namely the characterization of the stratification (the fact that we have a parallel hyperplane to $\mathbf{M}^N$ inside $\mathbf{M}^{N+1}$) and the inequality $H^N \leq H$, both arguments being true for any $\mathbf{M}^k$.

In order to go further we introduce the

Definition 20.1.3 *Assume that $\overline{\Omega} \times (0,T)$ is a stratified domain and $\varphi : \partial\Omega \times (0,T) \to \mathbb{R}$ is a lower-semicontinuous function. We say that φ is adapted to the stratification if for any $1 \leq k \leq N$, $\varphi|_{\mathbf{M}^k}$ is continuous. Moreover, φ is said to be W-adapted ("well-adapted") to the stratification if, in addition, for any $x \in \mathbf{M}^k$ and for any k*

$$\varphi(x,t) = \liminf_{\substack{(y,s)\to(x,t)\\ (y,s)\in\mathbf{M}^N}} \varphi(y,s) .$$

The result for W-adapted boundary conditions is the following

Proposition 20.1.4 *Assume that $\overline{\Omega} \times (0,T)$ is a stratified domain and that we are in the good framework for stratified problems. If $\varphi : \partial\Omega \times [0,T) \to \mathbb{R}$ is a lower-semicontinuous function which is W-adapted to the stratification and if u is viscosity* *subsolution of the Dirichlet problem, then* $\tilde{u} : \overline{\Omega} \times [0,T) \to \mathbb{R}$ *defined by* $\tilde{u}(x,t) = u(x,t)$ *if* $x \in \Omega$ *and*

$$\tilde{u}(x,t) = \limsup_{\substack{(y,s)\to(x,t)\\ y\in\Omega}} u(y,s) \quad \text{if } x \in \partial\Omega\ ,$$

is a stratified subsolution of the problem.
If, in addition, we assume that

(*i*) $u_0 \in C(\overline{\Omega})$ *and* $u_0(x) \le \varphi_*(x,0)$ *on* $\partial\Omega$,

(*ii*) *For any* $x \in \partial\Omega$, *there exists a* C^1*-function* ϕ *defined in a neighborhood* V *of* x *such that* $\phi(y) = 0$ *if* $y \in \partial\Omega \cap V$, $\phi(y) > 0$ *if* $y \in \Omega \cap V$, *and*

$$\sup_{\mathbf{BCL}(y,s)} \{b^x \cdot D_x\phi(y)\} \ge 0\ ,$$

for any $y \in \Omega \cap V$ *and for small* s,

then, for any viscosity supersolution of the Dirichlet problem, we have

$$\tilde{u} \le v \quad \text{on } \overline{\Omega} \times [0,T)\ .$$

In particular, there exists a unique continuous viscosity solution of the Dirichlet problem (up to a modification of its values on the boundary).

The first part of this proposition says that, under suitable "standard" assumptions and modification of the subsolution on the boundary, then Ishii's viscosity subsolutions and stratified subsolution are the same. For a complete application of this first result, one needs to treat the initial data and, as it will be clear in the proof, Conditions (i) and (ii) imply that

$$\tilde{u}(x,0) \le u_0(x) \le v(x,0) \quad \text{on } \overline{\Omega}\,.$$

Notice that this double inequality prevents maximum points of $u - v$ to be achieved on $\partial\Omega \times \{0\}$ if this maximum is assumed to be strictly positive.

Example 20.1.5 *A standard example where Proposition 20.1.4 can be applied is the square* $[0,1] \times [0,1]$ *in* $\mathbb{R}^2$, *with*

$$\varphi(x) = \varphi_i(x,t) \quad \text{on } S_i,$$

where $S_1 =]0,1[\times\{0\}$, $S_2 = \{1\}\times]0,1[$, $S_3 =]0,1[\times\{1\}$ $S_4 = \{0\}\times]0,1[$, *each* φ_i *being continuous on* S_i*. Of course, in order to have a function* φ *which is W-adapted to the stratification, the values at the four corners are imposed by the values on each* S_i *and obtained by computing their lower semi-continuous extensions. For example, at* $(0,0)$ *we have* $\min(\varphi_1(0,t),\varphi_4(0,t))$*. We point out that* φ *is still adapted if the values at the four corners are below these values.*

If H *satisfies all the controllability conditions, then the first part Proposition 20.1.4 applies.*

For the second one, the compatibility condition on $\partial\Omega\times\{0\}$ *should hold and for* ϕ*, we can choose the distance to the boundary if* x *is not located on one of the corners. In case of a corner, say* $(0,0)$*, we may choose, noting* $x = (x_1,x_2)$*, the function* $\phi(x) = x_1x_2$*, while for* $(0,1)$*, we may choose* $\phi(x) = x_1(1-x_2)$*, i.e. in each case the product of the distances to the adjacent sides. The controllability condition ensures that the requirement on* $D_x\phi$ *is satisfied.*

Proof of Proposition 20.1.4 — The first part of the result is easy: by Proposition 20.1.2, $\tilde{u}$ is a stratified solution on $\mathbf{M}^N$ and it remains to show that it is true on any $\mathbf{M}^k$.

If $(x,t)\in\mathbf{M}^k$, using a tangential regularization in a neighborhood of (x,t), we obtain a Lipschitz continuous function which is below φ on each connected component of $\mathbf{M}^N$ and therefore $\tilde{u}(x,t)\leq\varphi(x,t)$, since (as in the above example) the lower semicontinuous enveloppe of φ can just be computed using points of $\mathbf{M}^N$.

The $\mathbb{F}^k$-inequality of $\mathbf{M}^k$ can be obtained exactly as for the $\mathbb{F}^N$-one in the proof of Proposition 20.1.2 (cf. the second part of Remark 20.1.2): we can also here assume without loss of generality that $\mathbf{M}^k$ is flat and consider inequalities on $\mathbf{M}^k+\varepsilon e\subset\mathbf{M}^{N+1}$, where e is a suitable vector, normal to $\mathbf{M}^k$.

For the comparison, the only additional difficulty is $t=0$ and more precisely the points of $\partial\Omega\times\{0\}$ where we have to show that $\tilde{u}\leq u_0$ and $v\geq u_0$.

The proof for v is easy since the viscosity supersolution inequality reads $v\geq\max(u_0,\varphi_*)\geq u_0$ on $\partial\Omega\times\{0\}$.

But for the subsolution case, we only have $\tilde{u}\leq\max(u_0,\varphi^*)$ on $\partial\Omega\times\{0\}$ and this is not sufficient. To turn around this difficulty and to show that the right inequality holds at $(x,0)$, $x\in\partial\Omega$, we introduce the function

$$(y,s)\mapsto\tilde{u}(y,s)-\frac{s}{\varepsilon}-\frac{|y-x|^2}{\varepsilon}-\frac{\alpha}{\phi(y)}\ ,$$

where $0<\alpha\ll\varepsilon\ll 1$ are parameters devoted to tend to 0 and ϕ is the function associated to x in Assumption (ii).

By classical arguments, this function has a local maximum point $(y_\varepsilon, s_\varepsilon)$ in a neighbordhood of $(x,0)$ and $(y_\varepsilon, s_\varepsilon) \to (x,0)$ with $\tilde{u}(y_\varepsilon, s_\varepsilon) \to \tilde{u}(x,0)$ at least if $\alpha, \varepsilon \to 0$ with $\alpha \ll \varepsilon$ [(1)].

Because of the ϕ-term, $y_\varepsilon \in \Omega$. If $s_\varepsilon > 0$, the H-inequality holds and we have

$$\frac{1}{\varepsilon} + H\left(y_\varepsilon, s_\varepsilon, p_\varepsilon - \frac{\alpha D_x\phi(y_\varepsilon)}{[\phi(y_\varepsilon)]^2}\right) \leq 0 \; ,$$

where $p_\varepsilon = \dfrac{2(y_\varepsilon - x)}{\varepsilon} = \dfrac{o(1)}{\varepsilon}$. Examining the H-term, it can be estimated by

$$\frac{1}{\varepsilon} - M(\frac{o(1)}{\varepsilon} + 1) + \sup_{\mathbf{BCL}(y_\varepsilon, s_\varepsilon)} \left\{ b^x \cdot \frac{\alpha D_x\phi(y_\varepsilon)}{[\phi(y_\varepsilon)]^2} \right\} \leq 0 \; ,$$

where M takes into account the Lipschitz constant of $H(x,t,p)$ in p (coming from boundedness of b) and the boundedness of l.

By the assumption on ϕ the supremum is positive and therefore this inequality cannot hold for ε small enough. Therefore we necessarily have $s_\varepsilon = 0$ and $\tilde{u}(y_\varepsilon, s_\varepsilon) \leq u_0(y_\varepsilon)$. And letting $\alpha, \varepsilon \to 0$ with $\alpha \ll \varepsilon$, we obtain $\tilde{u}(x,0) \leq u_0(x)$.

These inequalities at time $t = 0$ being proved, we have just to apply the comparison result for the stratified problem and the proof is complete.

Q.E.D.

Remark 20.1.6 *We are not going to push further away the question of the existence of the functions ϕ which play the role of a distance function and which are used in a key way to obtain the desired property at time* 0*. But we think that the existence of such functions is not a problem in general, even if it might be difficult to provide a very general result regarding such existence.*

A convincing example is the case when $\overline{\Omega}$ is a convex set given by

$$\overline{\Omega} := \bigcap_i \{x : \; p_i \cdot x \geq q_i\} \; ,$$

where the p_i are in $\mathbb{R}^N$ and the q_i in $\mathbb{R}$. The example of the square above can be generalized in the following way: if $x \in \partial\Omega$ and if $I(x)$ is the set of indices i for which $p_i \cdot x = q_i$, then one can choose

$$\phi(y) := \prod_{i \in I(x)} (p_i \cdot x - q_i) \; .$$

[(1)] By the definition of $\tilde{u}$, the values of $\tilde{u}$ on the boundary are the limits of the values of $\tilde{u}$ in $\Omega \times (0,T)$ and for α small enough, we keep track of the boundary values of $\tilde{u}$

It is easy to check that the condition on $D_x\phi$ is satisfied as an easy consequence of the normal controllability since the p_i's are clearly orthogonal to the space of $\mathbf{M}^k$ at x is in $\mathbf{M}^k$.

This first part where we describe a general framework for which the stratified formulation and the classical viscosity solutions' one are (in some sense) equivalent, also suggests the cases when the stratified formulation is unavoidable: if φ is a lsc function which is adapted but not W-adapted to the stratification, i.e. if, on some connected component of some $\mathbf{M}^k$, we have

$$\varphi(x,t) < \liminf_{\substack{(y,s)\to(x,t)\\(y,s)\in\mathbf{M}^N}} \varphi(y,s) \ ,$$

for some $(x,t)\in\mathbf{M}^k$, there is no way that a subsolution (even after "cleaning" it) should satisfy $u\le\varphi$ on $\mathbf{M}^k$. This property has to be superimposed through the stratification formulation since the Ishii one (using φ^*) will simply erase the small values of φ.

In this case we have the

Proposition 20.1.7 *Assume that $\overline{\Omega}\times(0,T)$ is a stratified domain and that we are in the good framework for stratified problem. Let $\varphi:\partial\Omega\times[0,T)\to\mathbb{R}$ be a lower-semicontinuous function which is adapted to the stratification.*

If u is an u.s.c. viscosity subsolution of the Dirichlet problem such that

$$u(x,t)\le\varphi(x,t)\quad \text{for any } (x,t)\in\mathbf{M}^{N-1}\cup\cdots\cup\mathbf{M}^1 \ , \tag{20.3}$$

then $\tilde u:\overline{\Omega}\times[0,T)\to\mathbb{R}$ defined by $\tilde u(x,t)=u(x,t)$ if $x\in\Omega$ and

$$\tilde u(x,t)=\limsup_{\substack{(y,s)\to(x,t)\\ y\in\Omega}} u(y,s)\quad \text{if } x\in\partial\Omega \ ,$$

is a stratified subsolution of the problem.
If, in addition, we have

(*i*) *$u_0\in C(\overline{\Omega})$ and $u_0(x)\le\varphi_*(x,0)$ on $\partial\Omega$,*

(*ii*) *For any $x\in\partial\Omega$, there exists a C^1-function ϕ defined in a neighborhood V of x such that $\phi(y)=0$ if $y\in\partial\Omega\cap V$, $\phi(y)>0$ if $y\in\Omega\cap V$, and*

$$\sup_{\mathbf{BCL}(y,s)}\{b^x\cdot D_x\phi(y)\}\ge 0 \ ,$$

for any $y\in\Omega\cap V$ and for small s, *then, for any viscosity supersolution of the Dirichlet problem, we have*

$$\tilde{u} \le v \quad on\ \overline{\Omega} \times [0,T)\ .$$

In particular, there exists a unique continuous viscosity solution of the Dirichlet problem which satisfies (20.3).

As we already explain it above, the key difference between Propositions 20.1.4 and 20.1.7 is that the first one applies to all Ishii viscosity solutions while, in the second case, Condition 20.3 has to be imposed.

The *Proof of Proposition 20.1.7* follows the ideas of the proof of Proposition 20.1.4, namely

(i) For any k, the condition on $\mathbf{M}^k$, i.e.

$$\max(u_t + H^k(x,t,D_x u), u - \varphi(x,t)) \le 0 \quad \text{on } \mathbf{M}^k\ ,$$

where

$$H^k(x,t,p_x) = \sup_{b \in T_x\mathbf{M}^k} \{-b^x \cdot p_x - l\}\ ,$$

is obtained by combining (20.3) with an approximation "from inside", following Remark 20.1.2.

(ii) The comparison result follows from the stratified formulation, while the existence is provided by the value-function of the associated control problem.

20.2 On the Neumann Problem

20.2.1 The case of continuous data

As for the Dirichlet problem, we begin with the most standard framework: an oblique derivative problem in a smooth enough domain, namely

$$\begin{cases} u_t + H(x,t,D_x u) = 0 & \text{in } \Omega \times (0,T) \\ u(x,0) = u_0(x) & \text{in } \Omega \\ \dfrac{\partial u}{\partial \gamma} = g(x,t) & \text{on } \partial\Omega \times (0,T)\ , \end{cases} \tag{20.4}$$

where u_0, g but also H and γ are, at least, continuous functions and γ satisfies

$$\gamma(x,t) \cdot n(x) \ge \nu > 0\ ,$$

for some $\nu > 0$ and for any $(x,t) \in \partial\Omega \times (0,T)$, where $n(x)$ is the unit outward normal to $\partial\Omega$ at x.

Therefore $\mathbf{M}^{N+1} = \Omega \times (0,T)$ and $\mathbf{M}^N = \partial\Omega \times (0,T)$. The first key difference with the Dirichlet problem is that a viscosity subsolution u does not need to be redefined on the boundary if H satisfies the usual normal controllability assumption. More precisely, for any $(x,t) \in \partial\Omega \times (0,T)$, any viscosity subsolution necessarily satisfies

$$u(x,t) = \limsup_{\substack{(y,s)\to(x,t)\\ (y,s)\in\mathbf{M}^{N+1}}} u(y,s) \; .$$

Indeed, if we assume by contradiction that $u(x,t)$ is strictly bigger than the above right-hand side, we can first consider, for $0 < \varepsilon \ll 1$, the function defined on $\mathbf{M}^N$ by

$$(y,s) \mapsto u(y,s) - \frac{(s-t)^2}{\varepsilon^2} - \frac{|y-x|^2}{\varepsilon^2} \; .$$

This function has a local maximum point at $(y_\varepsilon, s_\varepsilon)$ near (x,t) and $u(y_\varepsilon, s_\varepsilon) \to u(x,t)$ as $\varepsilon \to 0$. But the jump of u on the boundary implies that $(y_\varepsilon, s_\varepsilon)$ is also a local maximum point of the function defined on $\overline{\Omega} \times (0,T)$

$$(y,s) \mapsto u(y,s) - \frac{(s-t)^2}{\varepsilon^2} - \frac{|y-x|^2}{\varepsilon^2} - \lambda d(y) \; ,$$

FOR ANY $\lambda \in \mathbb{R}$, where d denotes the distance function to the boundary $\partial\Omega$. Of course, for λ large enough, this would lead to an easy contradiction because of the normal controllability assumption on H.

In this simple case, it remains to identify the $\mathbb{F}^N$-inequality on $\mathbf{M}^N$ and to show the equivalence between Ishii's viscosity (sub)solutions and stratified (sub)solutions. As we did for the Dirichlet case, we enlarge the set **BCL** on the boundary to take into account the boundary condition. Here, the enlargement consists in adding triplets of the form $((-\gamma(x,t),0),0,g(x,t))$, assigning the cost $g(x,t)$ to a reflection-type boundary dynamic $-\gamma(x,t)$ on $\partial\Omega$.

The result is the following

Proposition 20.2.1 *Assume that $\partial\Omega$ is a C^1-domain and that H, γ and g are continuous functions. Then if u is a viscosity subsolution of the oblique derivative problem, it is a stratified subsolution of the problem with*

$$\mathbb{F}^N(x,t,(p_x,p_t)) = \sup\left\{\theta p_t - (\theta b^x - (1-\theta)\gamma)\cdot p_x - (\theta l + (1-\theta)g)\right\} \;\; on \; \mathbf{M}^N \; ,$$

where the supremum is taken on all $(b,0,l) \in \mathbf{BCL}(x,t)$ such that there exists $\theta \in (0,1)$ such that $(\theta b^x - (1-\theta)\gamma)\cdot n(x) = 0$, where $n(x)$ is the unit outward normal to $\partial\Omega$ at x.

Proof — We have to show that, if ϕ is a smooth function and if $(x,t) \in \mathbf{M}^N$ is a strict local maximum point of $u - \phi$ then

$$\theta\phi_t(x,t) - (\theta b^x - (1-\theta)\gamma) \cdot D_x\phi(x,t) - (\theta l + (1-\theta)g) \leq 0 \,,$$

for any b, l, θ satisfying the conditions of Proposition 20.2.1.

To do so, we introduce $\lambda \in \mathbb{R}$ which is the unique solution of the equation

$$\gamma(x,t) \cdot (D_x\phi(x,t) - \lambda n(x)) = g(x,t)\,, \tag{20.5}$$

which is well-defined since $\gamma(x,t) \cdot n(x) \neq 0$. Then, we consider the function

$$(y,s) \mapsto u(y,s) - \phi(y,s) - (\lambda - \delta)d(y) - \frac{[d(y)]^2}{\varepsilon^2}\,,$$

for $0 < \varepsilon \ll 1$ and for some small $\delta > 0$. We recall that, as above, d denotes the distance function to the boundary $\partial\Omega$ and we point out that $D_x d(x) = -n(x)$ on $\partial\Omega$; we will use the notation $n(x)$ for $-D_x d(x)$ even if x is not on the boundary.

We first fix δ. If ε is small enough, this function has a local maximum point at $(x_\varepsilon, t_\varepsilon)$ and $(x_\varepsilon, t_\varepsilon) \to (x,t)$ as $\varepsilon \to 0$ by the maximum point property of (x,t).

If $(x_\varepsilon, t_\varepsilon) \in \partial\Omega \times (0,T)$, we claim that the γ-inequality cannot hold. Indeed otherwise

$$\gamma(x_\varepsilon, t_\varepsilon) \cdot (D_x\phi(x_\varepsilon, t_\varepsilon) - (\lambda - \delta)n(x_\varepsilon)) \leq g(x_\varepsilon, t_\varepsilon)\,,$$

but this inequality cannot be valid for ε small enough because of the definition of λ and the facts that $\delta > 0$ and $\gamma(x_\varepsilon, t_\varepsilon) \cdot n(x_\varepsilon) \geq \nu > 0$.

Hence the H-inequality holds. The continuity of H implies that there exists $(b_\varepsilon, 0, l_\varepsilon) \in \mathbf{BCL}(x_\varepsilon, t_\varepsilon)$ such that $(b_\varepsilon, 0, l_\varepsilon) \to (b, 0, l)$ and we have, in particular

$$\phi_t(x_\varepsilon, t_\varepsilon) - b^x_\varepsilon \cdot [D_x\phi(x_\varepsilon, t_\varepsilon) - (\lambda - \delta)n(x_\varepsilon) - \frac{2d(x_\varepsilon)}{\varepsilon^2} n(x_\varepsilon)] - l_\varepsilon \leq 0\,.$$

But, from the property $(\theta b^x - (1-\theta)\gamma) \cdot n(x) = 0$, we first deduce that $\theta > 0$ since $\gamma(x,t) \cdot n(x) > 0$ and then that $b^x \cdot n(x) > 0$. Therefore, if ε is small enough, $b^x_\varepsilon \cdot n(x_\varepsilon) > 0$ and we can drop the term $\dfrac{2d(x_\varepsilon)}{\varepsilon^2} b^x_\varepsilon \cdot n(x_\varepsilon)$ in the above inequality. Letting $\varepsilon \to 0$, this yields

$$\phi_t(x,t) - b^x \cdot [D_x\phi(x,t) - (\lambda - \delta)n(x)] - l \leq 0\,.$$

Next we let $\delta \to 0$ in this inequality and the conclusion follows by using the θ-convex combination of this inequality with (20.5).

Q.E.D.

Several remarks after this result.

(i) It is clear enough from the proof that the case of *sliding boundary conditions*, i.e.

$$u_t + \frac{\partial u}{\partial \gamma} = g(x,t) \quad \text{on } \partial\Omega \times (0,T)\ ,$$

can be treated exactly in the same way.

(ii) Less obviously (but this is still easy), the case where there is a control on the reflection

$$\sup_\beta \{\gamma_\beta \cdot D_x u - g_\beta\} = 0 \quad \text{on } \partial\Omega \times (0,T)\ ,$$

where the set of (γ_β, g_β) is convex and continuous in (x,t), can also be treated[(2)]. It is easy to check that one has just to repeat the above arguments for (b,l) and (γ_β, g_β) such that $(\theta b^x - (1-\theta)\gamma_\beta)\cdot n(x) = 0$.

(iii) But, as it may be expected, the stratified formulation does not bring new results as long as all data are continuous...

Remark 20.2.2 *For a Neumann boundary condition of the form*

$$\frac{\partial u}{\partial n} = g(x,t) \quad \textit{on } \partial\Omega \times (0,T)\ ,$$

we have

$$\mathbb{F}^N(x,t,(p_x,p_t)) = \sup\{\theta p_t - (\theta b^x - (1-\theta)n(x))\cdot p_x - (\theta l + (1-\theta)g)\} \ \textit{on } \mathbf{M}^N\ ,$$

where the supremum is taken on all $(b,0,l) \in \mathbf{BCL}(x,t)$ *such that there exists* $\theta \in (0,1)$ *such that* $(\theta b^x - (1-\theta)n(x))\cdot n(x) = 0$.

This means that we have

$$\theta b^x \cdot n(x) - (1-\theta) = 0\ .$$

and therefore $b^x \cdot n(x) \geq 0$ *and* $\theta = (1 + b^x\cdot n(x))^{-1}$. *If* $b^x = b^{x,\perp} + b^{x,\top}$ *where* $b^{x,\perp}$ *is projection of* b^x *on the normal direction and* $b^{x,\top}$ *on the tangent space of* $\partial\Omega$ *at* x, *we have to look at the supremum of* $\theta(p_t - b^{x,\top}\cdot p_x - (l + b\cdot ng(x,t)))$ *for* $b^x \cdot n \geq 0$ *since* $(1-\theta) = \theta b^x \cdot n(x)$. *But* θ *cannot vanish and the condition reduces to*

$$u_t + \sup_{b^x\cdot n(x)\geq 0} \left(b^{x,\top}\cdot D_x u - (l + b\cdot ng(x,t))\right) \leq 0\ .$$

[(2)] In general this set is not convex but we can take a convex enveloppe and this does not change the “sup” in the boundary condition.

When looking at the reflected trajectory for a control problem, one has to solve an ode like

$$\dot{X}(s) = b^x(s) - 1_{\{X(s)\in\partial\Omega\}} n(X(s)).d|k|_s \; ,$$

where $|k|_s$ is the process with bounded variation which keeps the trajectory inside $\overline{\Omega}$ and the associated cost is

$$\int_0^t l(s)ds + \int_0^t g(X(s),s)d|k|_s \; .$$

It is easy to see that $d|k|_s = 1_{\{X(s)\in\partial\Omega\}} b(s).n(X(s))ds$ if $b(s)^x \cdot n(X(s)) \geq 0$ and the cost becomes

$$\int_0^t (l(s) + 1_{\{X(s)\in\partial\Omega\}} b^x(s).n(X(s)))ds \; ,$$

which is exactly what the stratified formulation is seeing on the boundary.

20.2.2 Oblique derivative problems in stratified domains

In [52], Dupuis and Ishii prove comparison results for oblique derivative problems in non-smooth domains: roughly speaking, they treat the case of a smooth enough direction of reflection in domains which satisfy only an interior-exterior cone condition. Our aim is to provide here a stratified version of their result.

In the following result, we assume that $\overline{\Omega} \times (0,T)$ is a stratified domain with a t-independent stratification such that

$$\mathbf{M}^{N+1} = \Omega \times (0,T) \quad \text{and} \quad \mathbf{M}^k \subset \partial\Omega \times (0,T) \text{ for } 1, 2, \cdots, N.$$

We assume that $\gamma = \gamma(x) \in C^1(\mathbb{R}^N)$ and that, for any $x \in \partial\Omega$, there exists $\bar{\delta}, \eta > 0$ such that, for any $0 < \delta \leq \bar{\delta}$

$$B(x + \delta\gamma(x), \eta\delta) \subset \Omega^c \quad \text{and} \quad B(x - \delta\gamma(x), \eta\delta) \subset \Omega \; .$$

It is worth pointing out that this condition on γ provides a regularity property on Ω since it implies an exterior-interior cone condition.

Proposition 20.2.3 *Under the above assumptions and if g is a continuous function, a classical viscosity subsolution of* (20.4) *is also a stratified subsolution of* (20.4). *Therefore we have a comparison result for classical viscosity sub and supersolutions of* (20.4).

Proof — Let u be an Ishii subsolution of (20.4). We have to prove that if $(x,t) \in \mathbf{M}^k$, for any smooth function, if $u-\phi$ has a strict local maximum point at $(\bar x,\bar t)$ on $\mathbf{M}^k$ then
$$\theta\phi_t(\bar x,\bar t) - (\theta b^x - (1-\theta)\gamma(\bar x))\cdot D_x\phi(\bar x,\bar t) \leq (\theta l + (1-\theta)g(\bar x,\bar t))\ ,$$
for any $((b^x,-1),0,l) \in \mathbf{BCL}(\bar x,\bar t)$ and $0 \leq \theta \leq 1$ such that $(\theta b^x - (1-\theta)\gamma(\bar x)) \in T_{(\bar x,\bar t)}\mathbf{M}^k$. In fact, we have to perform the proof only for elements $((b^x,-1),0,l)$ which are in the interior of $\mathbf{BCL}(\bar x,\bar t)$, and therefore, by the continuity of H, such elements are in $\mathbf{BCL}(x,t)$ for (x,t) close enough to $(\bar x,\bar t)$.

Of course, because of the assumptions on the stratification, we can assume that $\mathbf{M}^k$ is an affine space but since the coordinates on $\mathbf{M}^k$ are not playing a key role, we can just examine the case when, $k=1$, $\bar x = 0$, $\gamma(0,\bar t) = -e_N = -e_2$. Then, the inequality we want ot obtain reduces to
$$\theta\phi_t(\bar t) \leq (\theta l + (1-\theta)g(0,\bar t))\ ,$$
for a smooth function ϕ of t such that $u(0,t)-\phi(t)$ has a strict local maximum point at $\bar t$, and for any $((b^x,-1),0,l) \in \mathbf{BCL}(0,\bar t)$, $0 \leq \theta \leq 1$ such that $(\theta b^x - (1-\theta)\gamma(\bar x)) = 0$.

We borrow the arguments Dupuis and Ishii [52] for proving this inequality. By the conditions on γ, we have
$$\{(x',x_N):\ x_N > 0,\ |x'| \leq \eta x_N\} \subset \Omega \quad \text{and}\{(x',x_N):\ x_N < 0,\ |x'| \leq -\eta x_N\} \subset \Omega^c\ .$$

We consider the function
$$u(x,t) - \phi(t) - \frac{1}{\varepsilon}\psi_\varepsilon(x',x_N) + (g(0,\bar t)-\delta)x_N\ ,$$
for $0 < \varepsilon \ll 1$, where $x = (x',x_N)$ and
$$\psi_\varepsilon(x',x_N) = \sqrt{[(\eta^2 x_N^2 - |x'|^2)^+]^2 + \varepsilon^4} + |x'|^2\ .$$
This function is an ad hoc regularized version of $\max(|x'|,\eta x_N)$. Since $\psi_\varepsilon(x',x_N) \geq \nu|x|^2$ for ν small enough, for all x, the above function has a maximum point at $(x_\varepsilon,t_\varepsilon)$ and $(x_\varepsilon,t_\varepsilon) \to (0,\bar t)$ as $\varepsilon \to 0$.

Now we examine the different possibilities for the subsolution inequality: if $x_\varepsilon \in \partial\Omega$ then, the condition on γ implies that $|(x_\varepsilon)'| > \eta|(x_\varepsilon)_N|$ and therefore the space-derivative of the test-function is nothing but $2(x_\varepsilon)'/\varepsilon - (g(0,\bar t)-\delta)e_N$. So, if the oblique derivative inequality were satisfied, we would get
$$(\frac{2(x_\varepsilon)'}{\varepsilon} - (g(0,\bar t)-\delta)e_N)\cdot\gamma(x_\varepsilon) \leq g(x_\varepsilon,t_\varepsilon)\ .$$

But, using the Lipschitz continuity of γ and the continuity of g we have

$$\begin{aligned}\frac{2(x_\varepsilon)'}{\varepsilon}\cdot\gamma(x_\varepsilon) &= \frac{2(x_\varepsilon)'}{\varepsilon}\cdot(\gamma(x_\varepsilon)-\gamma(0)) + \frac{2(x_\varepsilon)'}{\varepsilon}\cdot\gamma(0)\\ &= \frac{2(x_\varepsilon)'}{\varepsilon}\cdot(\gamma(x_\varepsilon)-\gamma(0))\\ &= O\left(\frac{|x_\varepsilon|^2}{\varepsilon}\right) = o(1)\ ,\end{aligned}$$

while

$$-(g(0,\bar t)-\delta)e_N\cdot\gamma(x_\varepsilon,t_\varepsilon) = g(x_\varepsilon,t_\varepsilon)+\delta+o(1)\ .$$

Therefore such γ-inequality cannot hold and necessarily the H-one does. In particular, we see that

$$\phi'(t_\varepsilon) - \frac{1}{\varepsilon}D_x\psi_\varepsilon\cdot b + (g(0,\bar t)-\delta)b\cdot e_N \leq l\ .$$

Assume for the moment that

$$-\frac{1}{\varepsilon}D_x\psi_\varepsilon\cdot b \geq o(1)\ ,$$

letting $\varepsilon\to 0$, we obtain

$$\phi'(\bar t) + (g(0,\bar t)-\delta)b\cdot e_N \leq l\ ,$$

which is the desired inequality (up to letting $\delta\to 0$) since $\theta b^x = -(1-\theta)e_N$: in order to conclude, it is enough to multiply the above inequality by θ.

It remains to prove the claim. For the $|x'|^2$-term in ψ_e, the proof is already done above while checking that the oblique derivative inequality cannot hold: we get a $o(1)$ as $\varepsilon\to 0$.

It remains to treat the other term (with the root), when $\eta^2x_N^2 > |x'|^2$. Notice that the sign of $-D_x\psi_\varepsilon\cdot b$ is the same as the one of $-(2\eta^2x_Ne_N - 2x')\cdot b$ and therefore the same as for $(2\eta^2x_Ne_N-2x')\cdot e_N$. Now, on this last formula, it appears clearly that this sign is positive, which concludes the proof.

Q.E.D.

20.2.3 Discontinuities in the direction of reflection and domains with corners: the $\mathbb{R}^2$-case.

We start by a standard evolution problem in 2-d described by Fig. 20.1

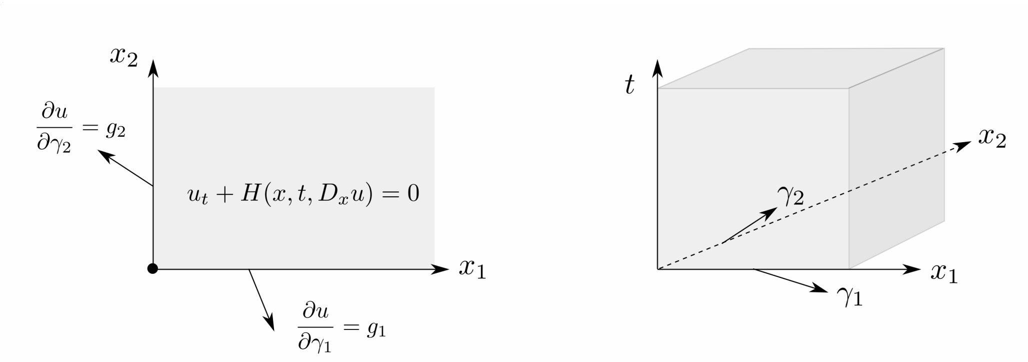


Figure 20.1: Standard Neuman problem with corner

Here we have

$$\begin{aligned}
\mathbf{M}^3 &= \{(x_1,x_2);\ x_1>0,\ x_2>0\}\times(0,T)\ ,\\
\mathbf{M}^2 &= \{(x_1,x_2);\ x_1=0, x_2>0 \text{ or } x_1>0, x_2=0\}\times(0,T)\ ,\\
\mathbf{M}^1 &= \{(0,0)\}\times(0,T)\ .
\end{aligned}$$

Of course the analysis of the previous section gives the stratified formulation on all the boundary except on $\mathbf{M}^1$, i.e at the points $((0,0),t)$ for $t\in(0,T)$, which require a specific treatment.

For $\mathbf{M}^1$, the answer is given by the following result in which we denote by **BCL** the set of dynamic-discount factor and cost related to H. We also point out that, in order to simplify, we argue as if $\gamma_1,\gamma_2,g_1,g_2$ were constants but the reader can check that all the arguments work if they are continuous functions of x and t.

Proposition 20.2.4 *We assume that*

(i) either $\gamma_1\cdot e_1=\gamma_2\cdot e_2=0$

(ii) or $\gamma_1\cdot e_1$, $\gamma_2\cdot e_2$ *have the same strict sign and* $\det(\gamma_1,\gamma_2)<0$.

If u is a viscosity subsolution of the above oblique derivative problem, it is a stratified subsolution of the problem with

$$\mathbb{F}^1((p_x,p_t)) = \sup\{\theta_3 p_t - (\theta_3 l - \theta_1 g_1 - \theta_2 g_2)\}\ \text{ on } \mathbf{M}^1\ ,$$

where the supremum is taken on all $(b,0,l)\in\mathbf{BCL}(0,t)$ *such that there exists* $\theta_1,\theta_2,\theta_3\in(0,1)$ *such that* $\theta_1+\theta_2+\theta_3=1$ *and* $\theta_1\gamma_1+\theta_2\gamma_2-\theta_3 b^x=0$.

Proof — Let ϕ be a C^1-function on $\mathbb{R}$ and t be a strict local maximum point of the function $s \mapsto u(0,s) - \phi(s)$. We have to show that, if $\theta_1, \theta_2, \theta_3, \gamma_1, \gamma_2, b$ satisfy the property which is required in Proposition 20.2.4, then

$$\theta_3 \phi'(t) - (\theta_3 l - \theta_1 g_1 - \theta_2 g_2) \leq 0 \ .$$

It is worth pointing out that we can do that only if $(b, 0, l)$ is in the interior of $\mathbf{BCL}(0,t)$, a point that we will use in the proof.

To do so, we first construct p_δ such that

$$p_\delta \cdot \gamma_1 = g_1 + \delta \quad , \quad p_\delta \cdot \gamma_2 = g_2 + \delta \ , \tag{20.6}$$

notice that such a p_δ exists because of the assumptions on γ_1, γ_2.

Next, we introduce the function

$$(y,s) \mapsto u(y,s) - \phi(s) - p_\delta \cdot y - \frac{Ay \cdot y}{\varepsilon^2} \ .$$

where A is a symmetric, positive definite matrix A. Additional properties on A will be needed and described all along the proof. At the end, we will show that such a matrix exists.

This function has a maximum point at $(x_\varepsilon, t_\varepsilon)$ and $(x_\varepsilon, t_\varepsilon) \to (0,t)$ as $\varepsilon \to 0$ by the maximum point property of $(0,t)$. Now we examine the different possibilities: if $x_\varepsilon = ((x_\varepsilon)_1, 0) = (x_\varepsilon)_1 e_1$ with $(x_\varepsilon)_1 \geq 0$ and $e_1 = (1,0)$, we have

$$\begin{aligned} \big[p_\delta + \frac{2Ax_\varepsilon}{\varepsilon^2}\big] \cdot \gamma_1 &= g_1 + \delta + \frac{2x_\varepsilon}{\varepsilon^2} \cdot A\gamma_1 \\ &= g_1 + \delta + \frac{2(x_\varepsilon)_1 e_1}{\varepsilon^2} \cdot A\gamma_1 \ . \end{aligned}$$

Hence if $A\gamma_1 \cdot e_1 \geq 0$, the inequality "$\dfrac{\partial u}{\partial \gamma_1} \leq g_1$" cannot hold.

In the same way, if $x_\varepsilon = (0, (x_\varepsilon)_2) = (x_\varepsilon)_2 e_2$, $(x_\varepsilon)_2 \geq 0$ and $e_2 = (0,1)$, the inequality "$\dfrac{\partial u}{\partial \gamma_2} \leq g_2$" cannot hold provided $A\gamma_2 \cdot e_2 \geq 0$.

Therefore, whereever x_ε is, the H-inequality holds and, since $(b,0,l)$ is in the interior of $\mathbf{BCL}(0,t)$, for ε small enough, $(b,0,l) \in \mathbf{BCL}(x_\varepsilon, t_\varepsilon)$ and we have

$$\phi'(t_\varepsilon) - b^x \cdot [p_\delta + \frac{2Ax_\varepsilon}{\varepsilon^2}] \leq l \ .$$

Now we examine the b^x-term, remarking that θ_3 cannot be 0

$$-b^x \cdot \frac{2Ax_\varepsilon}{\varepsilon^2} = -\frac{1}{\theta_3}(\theta_1\gamma_1 + \theta_2\gamma_2) \cdot \frac{2Ax_\varepsilon}{\varepsilon^2}$$
$$= -\frac{1}{\theta_3}(\theta_1 A\gamma_1 + \theta_2 A\gamma_2) \cdot \frac{2x_\varepsilon}{\varepsilon^2}\ ,$$

(recall that, being a symmetric matrix, the transpose of A is A itself). Since we want this term to be positive for any $x_\varepsilon = ((x_\varepsilon)_1, (x_\varepsilon)_2)$ with $(x_\varepsilon)_1, (x_\varepsilon)_2 \geq 0$, we have to require that all the coordinates of the vector $\theta_1 A\gamma_1 + \theta_2 A\gamma_2$ be negative.

If these properties hold true, we end up with

$$\phi'(t_\varepsilon) - b^x \cdot p_\delta \leq l\ .$$

Letting ε tend to 0, and using a convex combination with (20.6) provides the answer, after letting δ tend to 0.

It remains to show that such matrix A exists under the conditions of Proposition 20.2.4. We point out that this matrix may depend on the convex combination, hence on $\theta_1, \theta_2, \theta_3$ since the above proof is done for any fixed such convex combination.

We recall that the three conditions are

$$A\gamma_1 \cdot e_1 \geq 0 \quad , \quad A\gamma_2 \cdot e_2 \geq 0\ ,$$

$$\theta_1 A\gamma_1 + \theta_2 A\gamma_2 \leq 0\ ,$$

this last condition meaning that all the components of the vector are negative.

Writing the first two ones as $\gamma_1 \cdot A^{-1}e_1 \geq 0$ and $\gamma_2 \cdot A^{-1}e_2 \geq 0$ and keeping in mind that γ_1, γ_2 are directions of reflection on the axes, the natural choice is to choose

$$A^{-1}e_1 = -\lambda_1\gamma_2 \quad \text{and} \quad A^{-1}e_2 = -\lambda_2\gamma_1\ ,$$

where λ_1, λ_2 are non-negative constants which have to be chosen properly. Hence

$$A^{-1} = \begin{pmatrix} -\lambda_1\gamma_{2,1} & -\lambda_2\gamma_{1,1} \\ -\lambda_1\gamma_{2,2} & -\lambda_2\gamma_{1,2} \end{pmatrix}\ .$$

In order that A satisfies the required conditions, we need that A^{-1} satisfies them. And this leads to the following conditions

(i) A^{-1} is symmetric if either $\gamma_{1,1} = \gamma_{2,2} = 0$ or $\gamma_{1,1}, \gamma_{2,2}$ have the same strict sign. Then we can choose $\lambda_1 = |\gamma_{2,2}|^{-1}$, $\lambda_2 = |\gamma_{1,1}|^{-1}$.

(*ii*) The trace of A is non-negative since $\gamma_{2,1}, \gamma_{1,2} < 0$ by the conditions on the directions of reflection.

(*iii*) $\det(A^{-1}) = \lambda_1\lambda_2 \det(\gamma_2, \gamma_1) > 0$ by assumption.

Hence we can conclude if one of the two conditions holds
1. $\gamma_{1,1} = \gamma_{2,2} = 0$ with $A = Id$.
2. $\gamma_{1,1}, \gamma_{2,2}$ have the same strict sign and $\det(\gamma_1, \gamma_2) < 0$.

In order to investigate the other cases and to show that A does not exist in these cases, we assume (without loss of generality) that $\gamma_1 = (\gamma_{1,1}, -1)$, $\gamma_2 = (-1, \gamma_{2,2})$ and we write A as

$$A = \begin{pmatrix} \alpha & \beta \\ \beta & \gamma \end{pmatrix} ,$$

where β can be chosen as $0, 1$ or -1 since A can be replaced by λA for $\lambda > 0$.

The constraint can be written as

$$\alpha\gamma_{1,1} - \beta \geq 0 ,$$

$$-\beta + \gamma\gamma_{2,2} \geq 0 ,$$

$$\theta_1(\alpha\gamma_{1,1} - \beta) + \theta_2(-\alpha + \beta\gamma_{2,2}) \leq 0 ,$$

$$\theta_1(\beta\gamma_{1,1} - \gamma) + \theta_2(-\beta + \gamma\gamma_{2,2}) \leq 0 .$$

We begin with the case when $\gamma_{1,1} \geq 0$, $\gamma_{2,2} \leq 0$. In this case, the (necessary) choice $\beta = -1$ yields

$$\alpha\gamma_{1,1} + 1 \geq 0 ,$$

$$1 + \gamma\gamma_{2,2} \geq 0 ,$$

$$\theta_1(\alpha\gamma_{1,1} + 1) + \theta_2(-\alpha - \gamma_{2,2}) \leq 0 ,$$

$$\theta_1(-\gamma_{1,1} - \gamma) + \theta_2(1 + \gamma\gamma_{2,2}) \leq 0 .$$

The first constraint gives no limitation on α, while the second one imposes (a priori) γ to be small enough. For the two next ones we recall that $\theta_3 b = \theta_1\gamma_1 + \theta_2\gamma_2$ and therefore

$$\theta_3 b_1 = \theta_1\gamma_{1,1} - \theta_2 \quad , \quad \theta_3 b_2 = -\theta_1 + \theta_2\gamma_{2,2} < 0 .$$

Hence the two last constraints can be written as

$$\alpha b_1 - b_2 \leq 0 ,$$

$$-b_1 + \gamma b_2 \leq 0 .$$

Clearly one can conclude only if $b_1 < 0$ by choosing α large and with

$$\frac{b_1}{b_2} \leq \gamma \leq -\frac{1}{\gamma_{2,2}} \, .$$

We have indeed the existence of such γ since

$$\frac{b_1}{b_2} \leq -\frac{1}{\gamma_{2,2}}$$

is equivalent to $\det(b, \gamma_2) \leq 0$ and $\det(b, \gamma_2) = \dfrac{\theta_1}{\theta_3} \det(\gamma_1, \gamma_2) \leq 0$.

But, given γ_1, γ_2, in order to have $b_1 < 0$ for any choice of the convex combination, the only possibility is $\gamma_{1,1} = 0$. And the same conclusion holds in the case $\gamma_{1,1} \leq 0$, $\gamma_{2,2} \geq 0$.

The proof is then complete.

Q.E.D.

Example 20.2.5 *We consider the following problem where $Q = (0,1) \times (0,1)$*

$$\begin{cases} u_t + a(x,t)|D_x u| = f(x) & \text{in } Q \times (0,T) \\ u(x,0) = u_0(x) & \text{in } \Omega \\ \dfrac{\partial u}{\partial n_i} = g_i(x,t) & \text{on } \partial Q_i \times (0,T) \quad , \end{cases} \tag{20.7}$$

where $\partial Q_1 = (0,1) \times \{0\}$, $\partial Q_2 = \{1\} \times (0,1)$, , $\partial Q_3 = (0,1) \times \{1\}$, ,$\partial Q_4 = \{1\} \times (0,1)$ and n_i is the exterior unit normal vector to ∂Q_i.

If a is a Lipschitz continuous function (in particular in x) satisfying $a(x,t) > 0$ on $\overline{Q} \times [0,T]$, and $u_0, f, g_1, \cdots g_4$ are continuous, there exists a unique viscosity solution of this problem which coincides with the stratified solution. This result is a straightforward consequence of the preceeding results which shows that the notions of viscosity solutions and stratified solutions are the same. It is worth remarking on this example that, despite we did not insist above on that point, the Hamiltoniant $\mathbb{F}^1, \mathbb{F}^2$ satisfy the right conditions: indeed these Hamiltonians fullfill the required continuity assumptions because in the above convex combinations like $\theta_1 + \theta_2 + \theta_3 = 1$ and $\theta_1\gamma_1 + \theta_2\gamma_2 - \theta_3 b^x = 0$, θ_3 is bounded away from 0.

In the present exemple, on ∂Q_i

$$\mathbb{F}^2(x,t,(p_x,p_t)) = \max(\theta(p_t - a(x,t)v \cdot p_x - f(x)) + (1-\theta)(n_i \cdot p_x - g_i)) \, ,$$

the maximum being taken on all $|v| \leq 1$ *and* $\theta \in [0,1]$ *such that* $[\theta a(x,t)v - (1-\theta)n_i] \cdot n_i = 0$

Writing $v = v^\perp + v^\top$, *where* $v^\perp$ *is the normal part of* v *(i.e. the part which is colinear to* n_i*) and* $v^\top$ *the tangent part, we have* $\theta a(x,t)v^\perp \cdot n_i = (1-\theta)$ *and we take divide by* θ *to have*

$$\mathbb{F}^2(x,t,(p_x,p_t)) = \max_{|v^\top|^2+|v^\perp|^2=1} (p_t - a(x,t)v^\top \cdot p_x - f(x) + a(x,t)v^\perp \cdot n_i g_i) \; .$$

On an other hand, at $x=0$, *a simple computation gives*

$$\mathbb{F}^1(0,t,p_t) = \max(p_t - f(x) - g_1; p_t - f(x) - g_4) \; .$$

Remark 20.2.6 *We do not know if the conditions given in Proposition 20.2.4 are optimal or not. Clearly they are stronger than those given in Dupuis and Ishii [52, 51] inspired by those of Harrison and Reiman [79] and Varadhan and Williams [120]. Maybe a different choice of test-function, namely the term* $\dfrac{Ay \cdot y}{\varepsilon^2}$, *could lead to more general cases but we have no idea how to build such a function which necessarely will be* C^1 *but not* C^2 *at* 0.

Now we turn to the case of a "flat" discontinuity in $\mathbb{R}^2$, as depicted in Fig. 20.2 below

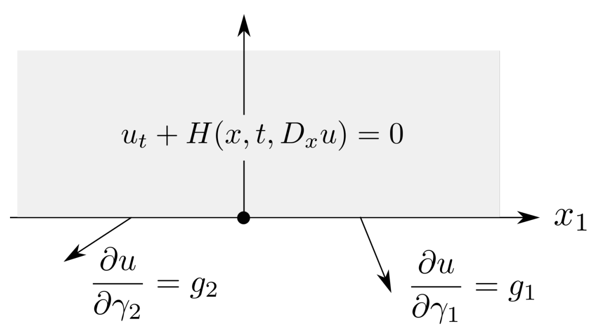


Figure 20.2: Flat discontinuous Neuman problem

Proposition 20.2.7 *Assume that* γ_1, γ_2 *satisfy* $\det(\gamma_1,\gamma_2) \leq 0$. *Then the same result as in Proposition 20.2.4 holds.*

Proof — The proof follows exactly along the same lines, only the constraints on A are different. Indeed, we should have

$$A\gamma_1 \cdot e_1 \geq 0 \; , \; A\gamma_2 \cdot e_1 \leq 0 \; ,$$

and $Ab = -\lambda e_2$ for some $\lambda > 0$. In this last property, we can take $\lambda = 1$ without loss of generality and therefore the symmetric matrix A^{-1} has the form

$$\begin{pmatrix} \alpha & -b_1 \\ -b_1 & -b_2 \end{pmatrix} ,$$

for some suitable parameter $\alpha > 0$. Moreover,

$$A = [\det(A^{-1})]^{-1} \begin{pmatrix} -b_2 & b_1 \\ b_1 & \alpha \end{pmatrix} .$$

Hence $A\gamma_1 \cdot e_1 = \gamma_1 \cdot Ae_1$ has the same sign as $\det(b, \gamma_1)$ and $A\gamma_2 \cdot e_1 = \gamma_2 \cdot Ae_1$ has the same sign as $\det(b, \gamma_2)$.

Therefore A exists (taking $\alpha > 0$ large enough) if $\det(b, \gamma_1) \geq 0$ and $\det(b, \gamma_2) \leq 0$. But $\theta_3 b = \theta_1\gamma_1 + \theta_2\gamma_2$ and therefore these conditions reduce to $\det(\gamma_1, \gamma_2) \leq 0$.

Q.E.D.

Remark 20.2.8 *If we assume that* $\gamma_1{\cdot}e_2 = \gamma_2{\cdot}e_2 = -1$*, then the condition* $\det(\gamma_1, \gamma_2) \leq 0$ *reduces to the tangential components inequality* $\gamma_{2,1} \leq \gamma_{1,1}$*. Fig. 20.3 below shows different types of situation where* γ_1, γ_2 *and their oppposite (in dashed lines) are shown, those opposites being involved in the reflexion process that occurs on the boundary.*

From the first two examples it could be guessed that the trajectories in the good case are more of a "regular" type. However, the other examples show that we can also allow some cases where the reflexions go in the same direction, provided some "squeezing" effect holds.

20.2.4 Discontinuities in the direction of reflection and domains with corners: the $\mathbb{R}^N$-case.

Of course, in $\mathbb{R}^N$, there exists a lot of possibilities and we are going to investigate the following three situations

Case 1: a simple 1-dimensional corner

$$\begin{aligned} \mathbf{M}^{N+1} &= \{(x_1, \cdots, x_N);\ x_1 > 0, x_2 > 0\} \times (0,T), \\ \mathbf{M}^{N} &= \{(x_1, \cdots, x_N);\ x_1 = 0, x_2 > 0 \text{ or } x_1 > 0, x_2 = 0\} \times (0,T) , \\ \mathbf{M}^{N-1} &= \{(x_1, \cdots, x_N);\ x_1 = 0, x_2 = 0\} \times (0,T) . \end{aligned}$$

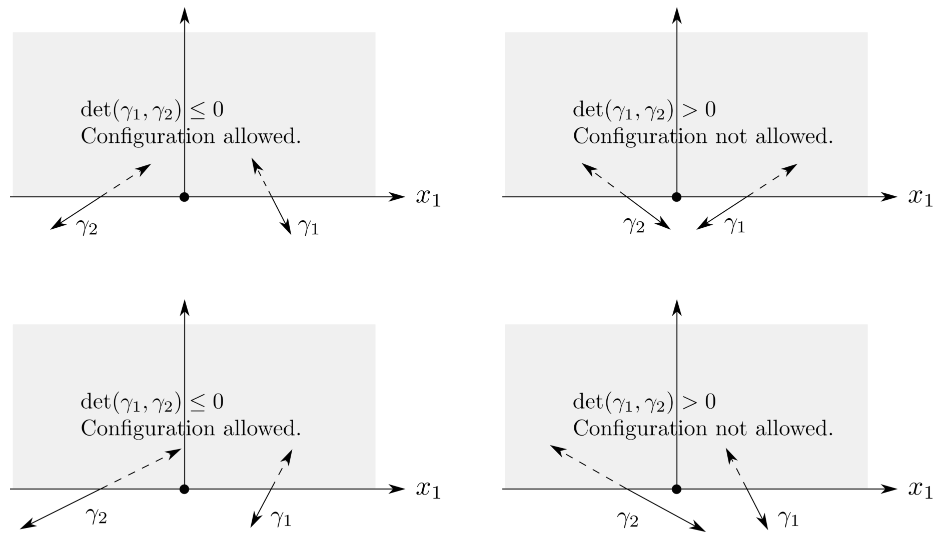


Figure 20.3: Configurations for discontinuous Neuman problem

Case 2: a simple discontinuity in the oblique derivative boundary condition

$$\begin{aligned}
\mathbf{M}^{N+1} &= \{(x_1,\cdots,x_N);\ x_2>0\}\times(0,T),\\
\mathbf{M}^{N} &= \{(x_1,\cdots,x_N);\ x_1\neq 0, x_N=0\}\times(0,T)\ ,\\
\mathbf{M}^{N-1} &= \{(x_1,\cdots,x_N);\ x_1=0, x_2=0\}\times(0,T)\ .
\end{aligned}$$

Case 3: a multi-dimensional corner

$$\begin{aligned}
\mathbf{M}^{N+1} &= \{(x_1,\cdots,x_N);\ x_1>0\cdots x_N>0\}\times(0,T),\\
\mathbf{M}^{N} &= \bigcup_i \{(x_1,\cdots,x_N);\ x_1\geq 0\cdots x_N\geq 0, x_i=0\}\times(0,T)\ ,\\
\mathbf{M}^{N-1} &= \{(x_1,\cdots,x_N);\ x_{N-1}=0, x_N=0\}\times(0,T)\ .
\end{aligned}$$

In each case, the question is: when is the classical notion of subsolution equivalent to the stratification formulation?

The answer is simple in the two first cases. Let us write

$$\gamma_1 = (\gamma_1^{(1)},\gamma_2^{(1)},\cdots,\gamma_N^{(1)}) \quad \text{and} \quad \gamma_2 = (\gamma_1^{(2)},\gamma_2^{(2)},\cdots,\gamma_N^{(2)})\ ,$$

and introduce the two vectors of $\mathbb{R}^2$

$$\tilde\gamma_1 = (\gamma_1^{(1)},\gamma_2^{(1)}) \quad \text{and} \quad \tilde\gamma_2 = (\gamma_1^{(2)},\gamma_2^{(2)})\ .$$

The result is

Proposition 20.2.9 *In Case 1 and 2, the classical viscosity formulation and the stratified formulation are equivalent if $\tilde\gamma_1, \tilde\gamma_2$ satisfy the condition of Proposition 20.2.4 in Case 1 and Proposition 20.2.7 in Case 2.*

Proof — In Case 1, we have to show that a viscosity subsolution u is also a stratified subsolution on $\mathbf{M}^{N-1}$. To do so, we denote any $x \in \mathbb{R}^N$ by (x_1, x_2, x') where $x' = (x_3, \cdots, x_N)$.

If $(\bar x, \bar t) \in \mathbf{M}^{N-1}$ is a maximum point of $x' \mapsto u(0,0,x',t) - \phi(x',t)$ where ϕ is a smooth function, we have to show that, if we have a convex combination $(-\theta_1\gamma_1 - \theta_2\gamma_2 + \theta_3 b^x, -\theta_3) \in T_{(\bar x,\bar t)}\mathbf{M}^{N-1}$ with $(b,0,l) \in \mathbf{BCL}(\bar x, \bar t)$ and $b = (b^x, -1)$. Then

$$\theta_3\phi_t + (\theta_1\gamma_1 + \theta_2\gamma_2 - \theta_3 b^x)\cdot D_x\phi(\bar x', \bar t) \leq -\theta_1 g_1 - \theta_2 g_2 + \theta_3 l \ .$$

As in the proof of Proposition 20.2.4, we introduce p_δ such that

$$p_\delta\cdot\gamma_1 = g_1 + \delta \quad , \quad p_\delta\cdot\gamma_2 = g_2 + \delta \ ,$$

and the function

$$(y,s) \mapsto u(y,s) - \phi(y',s) - p_\delta\cdot y - \frac{A\tilde y\cdot\tilde y}{\varepsilon^2} \ ,$$

where $\tilde y = (y_1, y_2)$ and A is a 2×2 symmetric, positive definite matrix.

It is clear on this formulation that, only the $\tilde y$ terms plays a real role and we are in the same situation as in $\mathbb{R}^2$. This explains the statement of the result and shows that the proof for Case 2 follows from the same arguments.

Q.E.D.

Example 20.2.10 *A standard example for Case 2 is the case when we look at an oblique derivative problem in a smooth domain $\Omega \subset \mathbb{R}^N$ whose boundary is splitted into three components*

$$\partial\Omega = \partial\Omega_1 \cup \partial\Omega_2 \cup \Gamma \ ,$$

where $\partial\Omega_1, \partial\Omega_2$ are smooth $(N-1)$-dimensional manifolds and Γ a smooth $(N-2)$-dimensional manifold. The idea is to have the oblique derivative boundary condition

$$\frac{\partial u}{\partial\gamma_i} = g_i \quad \text{on } \partial\Omega_i \times (0,T) \ ,$$

for $i = 1, 2$.

The question is when the classical viscosity solution coincides with the stratified one and therefore is unique?

To answer to this question is not completely obvious since we have to apply the above result for Case 2 in the right way on Γ*. To do so, we consider* $x \in \Gamma$ *and we introduce two unit vectors:* n *the unit outward normal to* $\partial\Omega$ *at* x *and* $r \in T_x\partial\Omega$ *a unit vector which is normal to* $T_x\Gamma$ *and which is pointing toward* Ω_1*.*

With these notations, the answer to the above question is yes if the determinant

$$\begin{vmatrix} \gamma_1 \cdot r & \gamma_2 \cdot r \\ -\gamma_1 \cdot n & -\gamma_2 \cdot n \end{vmatrix} \leq 0 \,.$$

For Case 3, we introduce the $N\times N$-matrix Γ whose columns are given by $\gamma_1, \gamma_2, \cdots \gamma_n$ and we formulate the

Proposition 20.2.11 *In Case 3, the classical viscosity formulation and the stratified formulation are equivalent if there exists a* $N \times N$*-diagonal matrix* D *with strictly positive diagonal terms such that* $\Gamma.D^{-1}$ *is a symmetric, negative definite matrix.*

Proof — The proof follows along the arguments of the proof of Proposition 20.2.4: the key (and only) point is to find a symmetric, positive definite matrix A such that

$$A\gamma_i = -d_i e_i \quad \text{with } d_i > 0, \quad \text{for any } 1 \leq i \leq N \,.$$

This property can be written as $A.\Gamma = -D$ and therefore $A = -D\Gamma^-1 = -(\Gamma.D^{-1})^{-1}$. The assumption ensures the existence of A.

Q.E.D.

This result can, of course, be extended to the case of more general convex domains like

$$\Omega := \bigcap_i \{x : \; n_i \cdot x < q_i\} \,,$$

with a direction of reflection γ_i on $\{x : \; n_i \cdot x = q_i\}$ by the

Proposition 20.2.12 *The classical viscosity formulation and the stratified formulation are equivalent if there exists a* $N \times N$ *symmetric, positive definite matrix* A *such that*

$$A\gamma_i = d_i n_i \quad \textit{with } d_i > 0, \quad \textit{for any } 1 \leq i \leq N \,.$$

.

Remark 20.2.13 *Clearly, as in the case of the* 2*-dimensional corner we have no idea if these conditions are optimal or not but, at least, they are obviously satisfied if* $\gamma_i = n_i$ *with* $A = Id$ *and all* $d_i = 1$.

We conclude this section by an open question in the case of a non-convex domain, the model case being in 2-d

$$\Omega = \{(x_1, x_2) : \ x_1 > 0 \text{ or } x_2 > 0\} ,$$

with normal reflection on the two parts of the boundary, $\{(x_1, 0) : \ x_1 > 0\}$ and $\{(0, x_2) : \ x_2 > 0\}$, or different oblique derivative boundary conditions.

The strategy we follow above clearly fails due to the non-convexity of the domain and,maybe surprisingly, we were unable to obtain any general result in this case (some particular cases can, of course, be treated). We do not know if this is just a technical problem or if really they are counterexample where Ishii's solutions are not unique since, otherwise, they coincide with the unique stratified solution.

20.3 Mixing the Dirichlet and Neumann Problems

20.3.1 The most standard case

The most standard case is the case when the boundary $\partial\Omega$ is smooth and can be written as

$$\partial\Omega = \partial\Omega_1 \cup \partial\Omega_2 \cup \mathcal{H} ,$$

where $\partial\Omega_1, \partial\Omega_2$ are open sets of $\partial\Omega$ and $\mathcal{H}$ is a $(N-2)$ submanifold of $\partial\Omega$, and where the boundary condition is of the type

$$\begin{cases} u = \varphi \quad \text{on } \partial\Omega_1 \times (0, T) , & (20.8) \\ \dfrac{\partial u}{\partial \gamma} = g \quad \text{on } \partial\Omega_2 \times (0, T) , & (20.9) \end{cases}$$

where φ, γ and g are continuous functions on the boundary, γ satisfying the usual condition, i.e.

$$\gamma(x, t) \cdot n(x) \geq \nu > 0 ,$$

for some $\nu > 0$ and for any $(x, t) \in \partial\Omega \times (0, T)$, where $n(x)$ is the unit outward normal to $\partial\Omega$ at x.

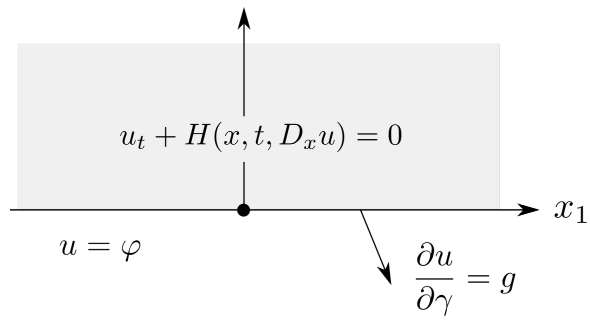


Figure 20.4: Flat Dirichlet-Neuman problem

Locally, after flattening the boundary in the neighborhood of a point in $\mathcal{H}$, we have a picture like Fig. 20.4 below.

But we may also be in a slightly less standard case if $\partial\Omega$ is not smooth and if, in terms of stratification, we have

$$\mathbf{M}^N = \partial\Omega_1 \cup \partial\Omega_2 \times (0,T) \quad , \quad \mathbf{M}^{N-1} = \mathcal{H} \times (0,T) \; ,$$

typically a picture like Fig. 20.5

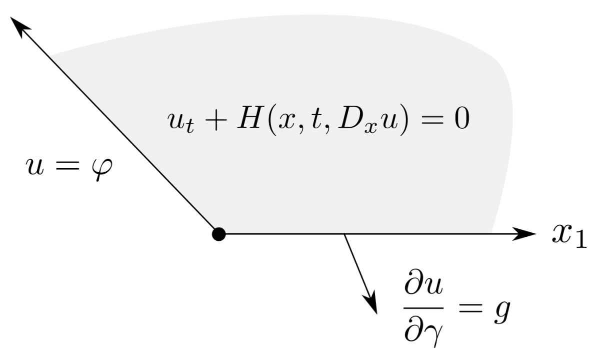


Figure 20.5: Angular Dirichlet-Neuman problem

Maybe surprisingly both cases can be treated in the same way and the main property needed is the

Lemma 20.3.1 *Assume that $\overline{\Omega}\times(0,T)$ is a stratified domain with the above described stratification and that we are in the good framework for stratified problems. Then*

$$u \le \varphi \quad \textit{on } \mathbf{M}^{N-1} = \mathcal{H} \times (0,T) \; .$$

Proof — Let $(\bar x, \bar t) \in \mathbf{M}^{N-1}$; we want to prove the inequality $u(\bar x,\bar t) \le \varphi(\bar x,\bar t)$.

We first remark that $u(x,t) \leq \varphi(x,t)$ if $x \in \partial\Omega_1$, $t > 0$ as a consequence of the results for the Dirichlet problem. Hence, if we assume by contradiction that $u(\bar{x},\bar{t}) > \varphi(\bar{x},\bar{t})$ and if we redefine u on $\partial\Omega_1$ by introducing

$$\tilde{u}(x,t) = \limsup_{\substack{(y,s)\to(x,t)\\ y\in\Omega}} u(y,s) \quad \text{if } x \in \partial\Omega_1 \ ,$$

$\tilde{u}$ being equal to u otherwise, then $\tilde{u}$ is still an usc subsolution of the problem.

Next, following the arguments of the Neumann part, we have

$$\tilde{u}(x,t) = \limsup_{\substack{(y,s)\to(x,t)\\ y\in\Omega}} \tilde{u}(y,s) \quad \text{if } x \in \partial\Omega_2 \ .$$

With all these properties, the regularization procedure of Section 3.4.1 together with normal controllability properties of H implies that the partial sup-convolution procedure in the $\mathbf{M}^{N-1}$-direction provides a function, still denoted by $\tilde{u}$ which is continuous except perhaps on $\mathbf{M}^{N-1}$ and such that

$$\limsup_{\substack{(y,s)\to(\bar{x},\bar{t})\\ y\in\Omega\cup\partial\Omega_1\cup\partial\Omega_2}} \tilde{u}(y,s) \leq \varphi(\bar{x},\bar{t}) < \tilde{u}(\bar{x},\bar{t}) \ .$$

Now we consider the function $(x,t) \mapsto \tilde{u}(x,t) - \dfrac{|x-\bar{x}|}{\varepsilon^2} - \dfrac{|t-\bar{t}|}{\varepsilon^2}$: for $\varepsilon > 0$ small enough, this function has a maximum point at $(x_\varepsilon, t_\varepsilon)$ near $(\bar{x},\bar{t})$ and the above properties implies:

(i) $(x_\varepsilon, t_\varepsilon) \in \mathbf{M}^{N-1}$,

(ii) $\tilde{u}(x_\varepsilon, t_\varepsilon) \to \tilde{u}(\bar{x},\bar{t})$ and $\dfrac{|x_\varepsilon-\bar{x}|}{\varepsilon^2} + \dfrac{|t_\varepsilon-\bar{t}|}{\varepsilon^2} \to 0$ as $\varepsilon \to 0$

(iii) for any vector p which is normal to $\mathbf{M}^{N-1}$, $(x_\varepsilon, t_\varepsilon)$ is still a maximum point of

$$(x,t) \mapsto \tilde{u}(x,t) - \frac{|x-\bar{x}|}{\varepsilon^2} - \frac{|t-\bar{t}|}{\varepsilon^2} - \frac{p\cdot(x-\bar{x})}{\varepsilon} \ .$$

Choosing $p = n_2(\bar{x})$ the outward unit normal vector to $\partial\Omega_2$ at $\bar{x}$ and using the normal controlability assumption, it is clear that the viscosity subsolution inequality at $(\bar{x},\bar{t})$ cannot hold for ε small enough, giving the desired contradiction.

Q.E.D.

The above lemma gives us the

Proposition 20.3.2 *Assume that $\overline{\Omega}\times(0,T)$ is a stratified domain with the above described stratification and that we are in the good framework for stratified problem. Assume that φ,γ,g are continuous functions on $\partial\Omega\times[0,T)$ such that* *(i)* $\varphi(x,0) = u_0(x)$ *on* $\partial\Omega \times \{0\}$,

(ii) γ *is Lipschitz continuous on* $\partial\Omega \times [0,T)$.

If u *is an u.s.c. viscosity subsolution of the mixed problem, then the function* $\tilde{u} : \overline{\Omega} \times [0,T) \to \mathbb{R}$ *defined by* $\tilde{u}(x,t) = u(x,t)$ *if* $x \in \Omega \cup \partial\Omega_2 \cup \mathcal{H}$ *and*

$$\tilde{u}(x,t) = \limsup_{\substack{(y,s)\to(x,t)\\ y\in\Omega}} u(y,s) \quad \text{if } x \in \partial\Omega_1 \,,$$

is a stratified subsolution of the associated stratified problem with the hamiltonian defined on $\mathbf{M}^{N-1}$

$$\mathbb{F}^{N-1}(x,t,Du) := \\ \max\Big(u - \varphi(x,t)\,;\, \sup\left\{\theta p_t - (\theta b^x - (1-\theta)\gamma)\cdot p_x - (\theta l + (1-\theta)g)\right\}\Big)\,,$$

where the supremum is taken on all $(b,0,l) \in \mathbf{BCL}(x,t)$ *such that there exists* $\theta \in (0,1)$ *such that* $(\theta b^x - (1-\theta)\gamma) \in T_{(x,t)}\mathbf{M}^{N-1}$.

As a consequence, up to a modification of the value of the subsolution on $\partial\Omega_1$*, one has a comparison result for the mixed problem and therefore there exists a unique continuous viscosity solution of the mixed problem up to this modification.*

The proof of this result is simple since we use both the ingredients for the Dirichlet and Neumann problems in $\partial\Omega_1$ and $\partial\Omega_2$, the only difficulty being $\mathcal{H}\times(0,T) = \mathbf{M}^{N-1}$.

The lemma provides the inequality $\tilde{u} \leq \varphi$ on $\mathbf{M}^{N-1}$, while the other part is obtained by a stability result from "inside" $\mathbf{M}^N$ in the spirit of Remark 20.1.2 or Proposition 16.4.1.

We leave these details to the reader but we provide the following result result which treats the difficulty connected to the initial data.

Lemma 20.3.3 *Under the assumptions of Proposition 20.3.2, if* u *and* v *are respectively an u.s.c. viscosity subsolution and a l.s.c. supersolution of the mixed problem, we have*

$$u(x,0) \leq u_0(x) \leq v(x,0)\,.$$

Proof — Of course, the difficulty comes from the points of $\partial\Omega \times \{0\}$ which are on $\mathcal{H}$. We only prove the result for the subsolution, the proof for the supersolution being analogous.

If $\bar{x} \in \mathcal{H}$, we want to prove that $u(\bar{x},0) \leq u_0(\bar{x})$. Since we are in a stratified framework, we can assume that $\partial\Omega_2 \subset \{x: \ (x-\bar{x})\cdot n_2 = 0\}$ where $n_2 \cdot \gamma > 0$ on $\partial\Omega_2$.

For $0 < \varepsilon \ll 1$, we consider the function

$$(x,t) \mapsto u(x,t) - \frac{|x-\bar{x}|^2}{\varepsilon^2} - C_\varepsilon t - \varepsilon\psi\left(\frac{(x-\bar{x})\cdot n_2}{\varepsilon^2}\right) .$$

where $C_\varepsilon > \varepsilon^{-1}$ is a large constant to be chosen later on and $\psi : \mathbb{R} \to \mathbb{R}$ is a C^1, increasing function such that $\psi(t) = -1$ if $t \leq -1$, $\psi(t) = 1$ if $t \geq 1$ and, $\psi(0) = 0$, $\psi'(0) = 1$.

This function has a maximum point at $(x_\varepsilon, t_\varepsilon)$ near $(\bar{x},0)$ and we have $(x_\varepsilon, t_\varepsilon) \to (\bar{x},0)$, $u(x_\varepsilon, t_\varepsilon) \to u(\bar{x},0)$ and $\dfrac{|x_\varepsilon - \bar{x}|}{\varepsilon^2} + C_\varepsilon t_\varepsilon \to 0$ as $\varepsilon \to 0$.

If $x_\varepsilon \in \partial\Omega_2$, then the oblique derivative inequality cannot holds since in the viscous sense,

$$Du\cdot\gamma = \frac{2(x-\bar{x})}{\varepsilon^2} + \frac{1}{\varepsilon}\psi'(0) = \frac{o(1)}{\varepsilon} + \frac{1}{\varepsilon} > 0 ,$$

for ε small enough. On the other hand, by choosing C_ε large enough, the H-inequality cannot hold wherever is $(x_\varepsilon, t_\varepsilon)$. Hence one of the inequalities $u(x_\varepsilon, t_\varepsilon) \leq \varphi(x_\varepsilon, t_\varepsilon)$ or $u(x_\varepsilon, t_\varepsilon) \leq u_0(x_\varepsilon)$ holds and the conclusion follows by letting ε tend to 0.

Q.E.D.

20.3.2 The Tanker Problem

As an example where the classical Ishii's viscosity solutions formulation cannot be sufficient for treating singular discontinuities, we come back to the example given by P.L. Lions in his course at the Collège de France, namely the problem (18.1).

At P_1, P_2, P_L, one would like to impose Neumann boundary conditions

$$\frac{\partial u}{\partial n} = g_i(t) \quad \text{at } P_i \ \ ,$$

(see Fig. 20.6) but such a boundary condition is far from being classical. However, we can handle it through the stratified formulation by setting $\mathbf{M}^{N+1} = \Omega \times (0,T)$, $\mathbf{M}^1 = \{P_1, P_2, \cdots, P_L\} \times (0,T)$ and $\mathbf{M}^N = (\partial\Omega \setminus \{P_1, P_2, \cdots, P_L\}) \times (0,T)$.

The only point is compute $\mathbb{F}^1$ and the computation is done as in the previous section, except that we are in $\mathbf{M}^1$ and we look for dynamics consisting in staying at P_i for some i.

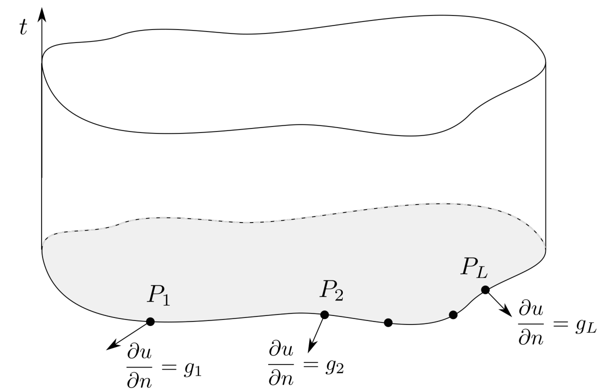


Figure 20.6: The tanker problem

At any P_i, we have to consider the convex combinations of $(b, c, l) = ((b^x, -1), 0, l) \in \mathbf{BCL}(x, t)$ and $((-n(P_i), 0), 0, g_i(t))$, i.e $(\mu b - (1-\mu)(n(P_i), 0), 0, \mu l + (1-\mu)g(x, t))$ for $0 \leq \mu \leq 1$ with (since we are on $\mathbf{M}^1$) the constraint $\mu b^x - (1-\mu)n(P_i) = 0$ which leads to

$$\mu b^x = (1-\mu)n(P_i) \ .$$

To compute $\mathbb{F}^1(x, t, p_t)$, we have to look at the supremum of $\mu p_t - (\mu l + (1-\mu)g_i(t))$ but taking into account the fact that $1 - \mu = \mu b.n(P_i)$ and since μ cannot vanish and the condition reduces to

$$u_t + \sup_{b^x = \lambda n,\ \lambda \geq 0} \Big\{ -(l + b^x \cdot n(P_i) g_i(t)) \Big\} \leq 0 \ .$$

Adequate controllability assumptions yield uniqueness of the stratified solution for such problem and of course, we can weaken the regularity assumptions on Ω which can be a square in $\mathbb{R}^2$ if the corners are harbours.

Part V

Appendices

Appendix A

Notations and terminology

F, G, H	Generic Hamiltonians
$\mathcal{A}, \mathcal{E}$	Generic subsets of $\mathbb{R}^N$
$\mathcal{O}, \mathcal{F}$	Generic open and closed subsets of $\mathbb{R}^N$
$\mathcal{K}$	Generic compact subset of $\mathbb{R}^N$
$B(y,r)$	Open ball of center $y \in \mathbb{R}^k$ and of radius $r > 0$ for the Euclidian norm.
A	a compact, convex subset of $\mathbb{R}^p$
$\mathbb{A}$	the space of controls, $\mathbb{A} = L^\infty(0,T;A)$
$\mathbf{BCL}(\cdot,\cdot)$	set-valued map combining all the dynamics, costs, discount factors, p.67
(X,T,D,L)	a generic trajectory of the differential inclusion, p.68
$\mathcal{T}(x,t)$	space of controled trajectories such that $(X,T,D,L)(0) = (x,t,0,0)$, p.70
$\mathcal{T}^{\mathrm{reg}}(x,t)$	space of regular controlled trajectories such that $(X,T,D,L)(0) = (x,t,0,0)$, p.113
z_*, z^*	semi-continuous enveloppes, p.27
V_k	k-dimensional vectorial subspace $V_k = \mathbb{R}^k \times \{0\}^{N-k}$, p.50
$Q^{x,t}_{r,h}$	the open cylinder $B(x,r) \times (t-h,t)$.
$Q^{x,t}_{r,h}[\mathcal{F}]$	the open cylinder $\big(B(x,r) \cap \mathcal{F}\big) \times (t-h,t)$, p.39.
$u.s.c., l.s.c.$	upper/lower semi-continuous function, p.27
$\mathrm{USCS}(\mathcal{F})$	set of u.s.c. subsolutions on $\mathcal{F}$, p.34
$\mathrm{LSCS}(\mathcal{F})$	set of l.s.c. supersolutions on $\mathcal{F}$, p.34
$\mathrm{PC}^1(\mathbb{R}^N \times [0,T])$	piecewise C^1-smooth test functions, p.140

$\mathbb{M}$	a general regular stratification of $\mathbb{R}^N$, p.53
(AFS)	Admissible Flat Stratification, p.50
(RS)	Regular Stratification, p.53
(HJB-SD)	Hamilton-Jacobi-Bellman in Stratified Domains
(AHG)	Assumptions on the Hamiltonian in the General case
(LAHF)	Local Assumptions on the Hamiltonians in the Flat case
(TC)	Tangential Continuity, p.55
(NC)	Normal Controllability, p.56
(Mon)	Monotonicity Assymption, p.56
(**SCR**)	Strong Comparison Result, p.31,34
(**LCR**)	Local Comparison Result, p.35
(**GCR**)	Global Comparison Result, p.35

Notions of solutions (see also appendix B for quick reference)

(CVS)	Ishii solutions / Classical viscosity solutions, p.100 (see also Section 3.1.1)
(FLS)	Flux-Limited Solution, p.142
(JVS)	Junction Viscosity solution, p.144
(JVS)-(KC)	Viscosity solutions for the Kirchhoff condition, p.145
	Stratified solutions, p.226

NB: The "good framework for HJ Equations with discontinuities" is defined p.95.

Appendix B

Assumption, hypotheses, notions of solutions

The page number refers to the page where the assumption is stated for the first time in the book.

Basic (or Fundamental) Assumptions

- $(\mathbf{H}_{\mathbf{BA-CP}}^{\text{class.}})$ *Basic Assumptions on the Control Problem – Classical case*: p. 20

  (i) The function $u_0 : \mathbb{R}^N \to \mathbb{R}$ is a bounded, uniformly continuous function.

  (ii) The functions b, c, l are bounded, uniformly continuous on $\mathbb{R}^N \times [0, T] \times A$.

  (iii) There exists a constant $C_1 > 0$ such that, for any $x, y \in \mathbb{R}^N$, $t \in [0, T]$, $\alpha \in A$, we have

$$|b(x, t, \alpha) - b(y, t, \alpha)| \leq C_1 |x - y| \ .$$

- $(\mathbf{H}_{\mathbf{BA-CP}})$ *Basic Assumptions on the Control Problem*: p. 22

  (i) The function $u_0 : \mathbb{R}^N \to \mathbb{R}$ is a bounded, continuous function.

  (ii) The functions b, c, l are bounded, continuous functions on $\mathbb{R}^N \times [0, T] \times A$ and the sets $(b, c, l)(x, t, A)$ are convex compact subsets of $\mathbb{R}^{N+2}$ for any $x \in \mathbb{R}^N$, $t \in [0, T]$ [(1)].

[(1)] The last part of this assumption which is not a loss of generality will be used for the connections with the approach by differential inclusions.

(*iii*) For any ball $B \subset \mathbb{R}^N$, there exists a constant $C_1(B) > 0$ such that, for any $x, y \in \mathbb{R}^N$, $t \in [0,T]$, $\alpha \in A$, we have

$$|b(x,t,\alpha) - b(y,s,\alpha)| \leq C_1(B)\left(|x-y| + |t-s|\right) \ .$$

- $(\mathbf{H_{BA-HJ}})$ *Basic Assumptions on the Hamilton-Jacobi equation*: p. 23

  There exists a constant $C_2 > 0$ and, for any ball $B \subset \mathbb{R}^N \times [0,T]$, for any $R > 0$, there exists constants $C_1(B,R) > 0, \gamma(R) \in \mathbb{R}$ and a modulus of continuity $m(B,R) : [0,+\infty) \to [0,+\infty)$ such that, for any $x, y \in B$, $t, s \in [0,T]$, $-R \leq r_1 \leq r_2 \leq R$ and $p, q \in \mathbb{R}^N$

$$|H(x,t,r_1,p) - H(y,s,r_1,p)| \leq C_1(B,R)[|x-y|+|t-s|]|p| + m(B,R)(|x-y|+|t-s|) \ ,$$

$$|H(x,t,r_1,p) - H(x,t,r_1,q)| \leq C_2|p-q| \ ,$$

$$H(x,t,r_2,p) - H(x,t,r_1,p) \geq \gamma(R)(r_2 - r_1) \ .$$

- $(\mathbf{H_{BA-}}p_t)$ *Basic Assumption on the p_t-dependence,* p. 41

  For any $(x,t,r,p_x,p_t) \in \mathcal{F} \times (0,T] \times \mathbb{R} \times \mathbb{R}^N \times \mathbb{R}$, the function $p_t \mapsto G\big(x,t,r,(p_x,p_t)\big)$ is increasing and $G\big(x,t,r,(p_x,p_t)\big) \to +\infty$ as $p_t \to +\infty$, uniformly for bounded x, t, r, p_x.

- $(\mathbf{H_{BA-Conv}})$ *Basic Assumption in the convex case*, p.46

  $H(x,t,r,p)$ is a locally Lipschitz function which is convex in (r,p). Moreover, for any ball $B \subset \mathbb{R}^N \times [0,T]$, for any $R > 0$, there exists constants $L(B,R), K(B,R) > 0$ and a function $G(B,R) : \mathbb{R}^N \to [1,+\infty[$ such that, for any $x, y \in B$, $t, s \in [0,T]$, $-R \leq u \leq v \leq R$ and $p \in \mathbb{R}^N$

$$D_pH(x,t,r,p) \cdot p - H(x,t,u,p) \geq G(B,R)(p) - L(B,R) \ ,$$

$$|D_xH(x,t,r,p)|, |D_tH(x,t,r,p)| \leq K(B,R)G(B,R)(p)(1+|p|) \ ,$$

$$D_rH(x,t,r,p) \geq 0 \ .$$

Stratification assumptions

- $(\mathbf{H_{ST}})_{flat}$ *Structure of an admissible flat stratification*, p.50

  The family $\mathbb{M} = (\mathbf{M}^k)_{k=0..N}$ of disjoint submanifolds of $\mathbb{R}^N$ is said to be an *Admissible Flat Stratification* of $\mathbb{R}^N$ if $\mathbb{R}^N = \mathbf{M}^0 \cup \mathbf{M}^1 \cup \cdots \cup \mathbf{M}^N$ and

(i) For any $x \in \mathbf{M}^k$, there exists $r > 0$ and V_k a k-dimensional linear subspace of $\mathbb{R}^N$ such that

$$B(x,r) \cap \mathbf{M}^k = B(x,r) \cap (x + V_k) \ .$$

Moreover $B(x,r) \cap \mathbf{M}^l = \emptyset$ if $l < k$.

(ii) If $\mathbf{M}^k \cap \overline{\mathbf{M}^l} \neq \emptyset$ for some $l > k$ then $\mathbf{M}^k \subset \overline{\mathbf{M}^l}$.

(iii) We have $\overline{\mathbf{M}^k} \subset \mathbf{M}^0 \cup \mathbf{M}^1 \cup \cdots \cup \mathbf{M}^k$.

- $(\mathbf{H_{ST}})_{reg}$ *Structure of a general regular stratification*, p.53

  The family $\mathbb{M} = (\mathbf{M}^k)_{k=0..N}$ of disjoint submanifolds of $\mathbb{R}^N$ is said to be a general regular stratification (RS) of $\mathbb{R}^N$ if

  (i) $\mathbb{R}^N = \mathbf{M}^0 \cup \mathbf{M}^1 \cup \cdots \cup \mathbf{M}^N$,

  (ii) for any $x \in \mathbb{R}^N$, there exists $r = r(x) > 0$ and a $C^{1,1}$-change of coordinates $\Psi^x : B(x,r) \to \mathbb{R}^N$ such that the $\Psi^x(\mathbf{M}^k \cap B(x,r))$ form an $(\mathbf{H_{ST}})_{flat}$ in $\Psi^x(B(x,r))$.

- **N.B.** *Convergence in the sense of stratification is defined* pp. 239 and 241.

Assumptions for the differential inclusion and the value-function

- $(\mathbf{H_{BCL}})_{fund}$ *Fundamental assumptions on the set-valued map* **BCL**, p.67

  The set-valued map $\mathbf{BCL} : \mathbb{R}^N \times [0,T] \to \mathcal{P}(\mathbb{R}^{N+3})$ satisfies

  (i) The map $(x,t) \mapsto \mathbf{BCL}(x,t)$ has compact, convex images and is upper semi-continuous;

  (ii) There exists $M > 0$, such that for any $x \in \mathbb{R}^N$ and $t > 0$,

  $$\mathbf{BCL}(x,t) \subset \left\{(b,c,l) \in \mathbb{R}^{N+1} \times \mathbb{R} \times \mathbb{R} : |b| \leq M; |c| \leq M; |l| \leq M\right\},$$

- $(\mathbf{H_{BCL}})_{struct}$ *Structure assumptions on the set-valued map* **BCL**, p.68

  There exists $\underline{c}, K > 0$ such that

  (i) For all $x \in \mathbb{R}^N$, $t \in [0,T]$ and $b = (b^x, b^t) \in \mathbf{B}(x,t)$, $-1 \leq b^t \leq 0$. Moreover, there exists $b = (b^x, b^t) \in \mathbf{B}(x,t)$ such that $b^t = -1$.

  (ii) For all $x \in \mathbb{R}^N$, $t \in [0,T]$, if $((b^x, b^t), c, l) \in \mathbf{BCL}(x,t)$, then $-Kb^t + c \geq 0$.

(iii) For any $x \in \mathbb{R}^N$, there exists an element in $\mathbf{BCL}(x,0)$ of the form $((0,0),c,l)$ with $c \geq \underline{c}$.

(iv) For all $x \in \mathbb{R}^N$, $t \in [0,T]$, if $(b,c,l) \in \mathbf{BCL}(x,t)$ then $\max(-b^t,c,l) \geq \underline{c}$.

- $(\mathbf{H_{BCL}})$ is just the conjunction of $(\mathbf{H_{BCL}})_{fund}$ and $(\mathbf{H_{BCL}})_{struct}$.
- $(\mathbf{H_U})$: the value-function U is locally bounded on $\overline{\Omega} \times [0,T]$, p.282.

Normal controllability, tangential continuity, Monotonicity

- **(TC)** *Tangential Continuity, HJ version*, p.55

  for any $x_1 = (y_1, z), x_2 = (y_2, z) \in B_\infty(\bar{x}, r)$, $|u| \leq R$, $p \in \mathbb{R}^N$, then

$$|G(x_1,u,p) - G(x_2,u,p)| \leq C_1^R |y_1 - y_2|.|p| + m^R\big(|y_1 - y_2|\big)\,.$$

- **(TC-s)** *Strong Tangential Continuity*, p. 62

  For any $x_1 = (y_1, z), x_2 = (y_2, z) \in B_\infty(\bar{x}, r)$, $|u| \leq R$, $p = (p_y, p_z) \in \mathbb{R}^N$, then

$$|G(x_1,u,p) - G(x_2,u,p)| \leq C_1^R |y_1 - y_2|.|p_y| + m^R\big(|y_1 - y_2|\big)\,.$$

- **(TC-BCL)** *Tangential Continuity, controle version*, p.227

  For any $j \geq k$, if $(y_1,t_1),(y_2,t_2) \in \mathbf{M}^j \cap B(x,r)$ with $(y_1,t_1) - (y_2,t_2) \in V_k$,

$$\begin{cases} \mathrm{dist_H}\left(\mathbf{B}(y_1,t_1), \mathbf{B}(y_2,t_1)\right) \leq C_1(|y_1 - y_2| + |t_1 - t_2|)\,, \\ \mathrm{dist_H}\left(\mathbf{BCL}(y_1,t_1), \mathbf{BCL}(y_2,t_2)\right) \leq m\big(|y_1 - y_2| + |t_1 - t_2|\big)\,, \end{cases}$$

  where $\mathrm{dist_H}$ denotes the Hausdorff distance.

- **(NC)** *Normal Controllability, HJ version*, p. 56

  for any $x = (y,z) \in B_\infty(\bar{x}, r)$, $|u| \leq R$, $p = (p_y, p_z) \in \mathbb{R}^N$, then

$$G(x,u,p) \geq C_2^R |p_z| - C_3^R |p_y| - C_4^R\,.$$

- $(\mathbf{NC}_{\mathcal{H}})$ *Normal Controllability, codimension* 1 *case*, p. 102

  For any $(x,t) \in \mathcal{H} \times [0,T]$, there exists $\delta = \delta(x,t)$ and a neighborhood $\mathcal{V} = \mathcal{V}(x,t)$ such that, for any $(y,s) \in \mathcal{V}$

$$[-\delta, \delta] \subset \{b_1(y,s,\alpha_1) \cdot e_N,\ \alpha_1 \in A_1\} \quad \text{if } (y,s) \in \overline{\Omega_1}\ ,$$

$$[-\delta, \delta] \subset \{b_2(y,s,\alpha_2) \cdot e_N,\ \alpha_2 \in A_2\} \quad \text{if } (y,s) \in \overline{\Omega_2}\ ,$$

  where $e_N = (0, 0 \cdots, 0, 1) \in \mathbb{R}^N$.

- **(NC-BCL)** *Normal Controllability, multi-D case controle version*, p.227

  There exists $\delta = \delta(x,t) > 0$, such that, for any $(y,s) \in B((x,t),r) \setminus \mathbf{M}^k$, one has

$$B(0,\delta) \cap V_k^{\perp} \subset P^{\perp}\left(\mathbf{B}(y,t)\right)\ .$$

- **(Mon)** *Monotonicity property*, p. 56

  For any $R > 0$, there exists $\lambda_R, \mu_R \in \mathbb{R}$, such that we have either $\Lambda_R > 0$ and for any $x \in B_\infty(\bar{x}, r)$, $p = (p_y, p_z) \in \mathbb{R}^N$,

$$G(x,u_2,p) - G(x,u_1,p) \geq \lambda^R (u_2 - u_1) \tag{B.1}$$

  for any $-R \leq u_1 \leq u_2 \leq R$, or this inequality holds with $\lambda_R = 0$, we have $\mu_R > 0$ and

$$G(x,u_1,q) - G(x,u_1,p) \geq \mu^R (q_{y_1} - p_{y_1})\ , \tag{B.2}$$

  for any $q = (q_y, p_z)$ with $p_{y_1} < q_{y_1}$ and $p_{y_i} = q_{y_i}$ for $i = 2, ..., p$.

  We say that (**Mon**-u) is satisfied if (B.1) holds and (**Mon**-p) is satisfied if (B.2) holds.

Localisation, convexity, subsolutions

- (**LOC1**) *localization hypothesis 1*, p.35

  If $\mathcal{F}$ is unbounded, for any $u \in USCS(\mathcal{F})$, for any $v \in LSCS(\mathcal{F})$, there exists a sequence $(u_\alpha)_{\alpha>0}$ of usc subsolutions of (3.3) such that $u_\alpha(x) - v(x) \to -\infty$ when $|x| \to +\infty$, $x \in \mathcal{F}$. Moreover, for any $x \in \mathcal{F}$, $u_\alpha(x) \to u(x)$ when $\alpha \to 0$.

- **(LOC2)** *localization hypothesis 2*, p.35

  For any $x \in \mathcal{F}$, if $u \in USCS(\mathcal{F}^{x,r})$, there exists a sequence $(u^\delta)_{\delta>0}$ of functions in $USCS(\mathcal{F}^{x,r})$ such that $u^\delta(x) - u(x) \geq u^\delta(y) - u(y) + \eta(\delta)$ if $y \in \partial\mathcal{F}^{x,r}$, where $\eta(\delta) > 0$ for all δ. Moreover, for any $y \in \mathcal{F}$, $u^\delta(y) \to u(y)$ when $\delta \to 0$.

- **(LOC1)**-evol *localization hypothesis 1, evolution version*, p.39

  If $\mathcal{F}$ is unbounded, for any $u \in \mathrm{USCS}(\mathcal{F} \times [0,T])$, for any $v \in \mathrm{LSCS}(\mathcal{F} \times [0,T])$, there exists a sequence $(u_\alpha)_{\alpha>0}$ of usc subsolutions of (3.3) such that $u_\alpha(x,t) - v(x,t) \to -\infty$ when $|x| \to +\infty$, $x \in \mathcal{F}$. Moreover, for any $x \in \mathcal{F}$, $u_\alpha(x,t) \to u(x,t)$ when $\alpha \to 0$.

- **(LOC2)**-evol *localization hypothesis 2, evolution version*, p.39

  For any $x \in \mathcal{F}$, if $u \in \mathrm{USCS}(Q^{x,t}_{r,h}[\mathcal{F}])$, there exists a sequence $(u^\delta)_{\delta>0}$ of functions in $\mathrm{USCS}(Q^{x,t}_{r,h}[\mathcal{F}])$ such that if $y \in (\partial\mathcal{F}^{x,r}) \times [t, t-h]$, then $u^\delta(y, t-h) \leq u(y, t-h) + \tilde\eta(\delta)$ where $\tilde\eta(\delta) \to 0$ as $\delta \to 0$. Moreover, for any $y \in \mathcal{F}$, $u^\delta(y) \to u(y)$ when $\delta \to 0$.

- **($\mathbf{H_{Sub-HJ}}$)** *Existence of a subsolution*, p.47

  There exists a C^1-function $\psi : \mathbb{R}^N \times [0,T] \to \mathbb{R}$ which is a subsolution of (3.9) and which satisfies $\psi(x,t) \to -\infty$ as $|x| \to +\infty$, uniformly for $t \in [0,T]$ and $\psi(x,0) \leq u_0(x)$ in $\mathbb{R}^N$.

- **($\mathbf{H_{Conv}}$)** *convexity for a general Hamiltonian*, p. 61

  For any $x \in B_\infty(\bar x, r)$, the function $(u,p) \mapsto G(x,u,p)$ is convex.

- **($\mathbf{H_{QC}}$)** *Quasiconvex Hamiltonians*, p. 162

  For $i = 1,2$, $H_i = \max(H_i^+, H_i^-)$ where H_i^+, H_i^- are bounded from below, Lipschitz continuous functions such that, for any x in a neighborhood of $\mathcal{H}$, $t \in [0,T]$ and $p \in \mathbb{R}^N$

  $\lambda \mapsto H_1^+(x,t,p+\lambda e_N)$ is decreasing, $\lambda \mapsto H_1^-(x,t,p+\lambda e_N)$ is increasing and tends to $+\infty$ as $\lambda \to +\infty$, locally uniformly w.r.t. x, t and p, and

  $\lambda \mapsto H_2^+(x,t,p+\lambda e_N)$ is increasing, $\lambda \mapsto H_2^-(x,t,p+\lambda e_N)$ is decreasing and tends to $+\infty$ as $\lambda \to -\infty$, locally uniformly w.r.t. x, t and p.

Comparison results

- $(\mathbf{GCR})^{\mathcal{F}}$ *Global Comparison Result in $\mathcal{F}$*, p.35

For any $u \in \mathrm{USCS}(\mathcal{F})$, for any $v \in \mathrm{LSCS}(\mathcal{F})$, we have $u \leq v$ on $\mathcal{F}$.

- $(\mathbf{LCR})^{\mathcal{F}}$ *Local Comparison Result in* $\mathcal{F}$, p.35

  For any $x \in \mathcal{F}$, there exists $r > 0$ such that, if $u \in \mathrm{USCS}(\mathcal{F}^{x,r})$, $v \in \mathrm{LSCS}(\mathcal{F}^{x,r})$ and $\max_{\overline{\mathcal{F}^{x,r}}}(u-v) > 0$, then

$$\max_{\overline{\mathcal{F}^{x,r}}}(u-v) \leq \max_{\partial \mathcal{F}^{x,r}}(u-v)\,.$$

- **(LCR)**-evol *Local Comparison Result, evolution case*, p.39

  For any $(x,t) \in \mathcal{F} \times (0,T]$, there exists $r > 0$, $0 < h < t$ such that, if $u \in \mathrm{USCS}(Q^{x,t}_{r,h}[\mathcal{F}])$, $v \in \mathrm{LSCS}(Q^{x,t}_{r,h}[\mathcal{F}])$ and $\max_{\overline{Q^{x,t}_{r,h}[\mathcal{F}]}}(u-v) > 0$, then

$$\max_{\overline{Q^{x,t}_{r,h}[\mathcal{F}]}}(u-v) \leq \max_{\partial_p Q^{x,t}_{r,h}[\mathcal{F}]}(u-v)\,.$$

  N.B. here, $\partial_p Q^{x,t}_{r,h}[\mathcal{F}]$ stands for the parabolic boundary: $(\partial B(x,r) \cap \mathcal{F}) \times [t-h,t] \cup (\overline{B(x,r)} \cap \mathcal{F}) \times \{t-h\}$.

- $\mathbf{LCR}^{\psi}(\bar{x},\bar{t})$ *Local Comparison Result around* (x,t) *in the stratified case,* p.230

  There exists $r = r(\bar{x},\bar{t}) > 0$ and $h = h(\bar{x},\bar{t}) \in (0,\bar{t})$ such that, if u and v are respectively a strict stratified subsolution and a stratified supersolution of some ψ–Equation in $Q^{\bar{x},\bar{t}}_{r,h}$ and if $\max_{\overline{Q^{\bar{x},\bar{t}}_{r,h}}}(u-v) > 0$, then

$$\max_{\overline{Q^{\bar{x},\bar{t}}_{r,h}}}(u-v) \leq \max_{\partial_p Q^{\bar{x},\bar{t}}_{r,h}}(u-v)\,.$$

  N.B. here, ψ-equation means an equation with obstacle ψ, a continuous function: $\max(\mathbb{F}(x,t,w,Dw), w-\psi) = 0$.

Notions of solutions

N.B. The following definitions are just gathered here as a quick reminder, the reader will find more details and the precise definition on the page given in reference.

- (CVS) *Ishii Solution for the hyperplane case*, p.100

This is the "classical" notion of viscosity solution (hence the acronym (CVS)) where on the hyperplane the relaxed condition reads (in the viscosity sense)

$$\begin{cases} \max\Big(u_t + H_1(x,t,u,Du), u_t + H_2(x,t,u,Du)\Big) \geq 0\,, \\ \min\Big(u_t + H_1(x,t,u,Du), u_t + H_2(x,t,u,Du)\Big) \leq 0\,. \end{cases}$$

The notion is "classical" in the sense that testing is done with test-functions in $C^1(\mathbb{R}^N \times [0,T])$ contrary to (FLS) and (JVS) below.

- (FLS) *Flux-Limited Solution*, p.142

  We are given a flux-limiter G on $\mathcal{H}$ (codim-1 discontinuity). Here, we use the extended $\mathrm{PC}^1(\mathbb{R}^N \times [0,T])$-test-functions. At a max point of $u - \psi$ located on the discontinuity, a subsolution satifies

$$\max\Big(\psi_t + G(x,t,u,D_{\mathcal{H}}\psi), \psi_t + H_1^+(x,t,u,D\psi_1), \psi_t + H_2^-(x,t,u,D\psi_2))\Big) \leq 0\,.$$

  while for a supersolution, at a min point of $v - \psi$, there holds

$$\max\Big(\psi_t + G(x,t,v,D_{\mathcal{H}}\psi), \psi_t + H_1^+(x,t,v,D\psi_1), \psi_t + H_2^-(x,t,v,D\psi_2))\Big) \geq 0\,.$$

- (JVS) Junction Viscosity solution, p.144

  This notion is somehow "consistent" with the usual viscosity definitions. Given a flux-limiter G we use test-functions in $\mathrm{PC}^1(\mathbb{R}^N \times [0,T])$ for which we test as usual: a subsolution u satisfies that if $u - \psi$ has a max at a point (x,t) on $\mathcal{H} \times [0,T]$, then

$$\min\Big(\psi_t + G(x,t,v,D_{\mathcal{H}}\psi), \psi_t + H_1(x,t,u,D\psi_1), \psi_t + H_2(x,t,u,D\psi_2))\Big) \leq 0\,.$$

  For a supersolution v, at a min point of $v - \psi$, we have

$$\max\Big(\psi_t + G(x,t,v,D_{\mathcal{H}}\psi), \psi_t + H_1(x,t,v,D\psi_1), \psi_t + H_2(x,t,v,D\psi_2))\Big) \geq 0\,.$$

- *Viscosity solutions for the Kirchhoff condition*, p.145

  The notion of solution follows the (JVS) definition just above, except that we replace the flux-limiter G by the Kirchhoff condition on the junction: for a subsolution u we have

$$\min\Big(\frac{\partial\psi_1}{\partial n_1} + \frac{\partial\psi_2}{\partial n_2}, \psi_t + H_1(x,t,u,D\psi_1), \psi_t + H_2(x,t,u,D\psi_2))\Big) \leq 0\,,$$

  and for a subsolution v we have

$$\max\Big(\frac{\partial\psi_1}{\partial n_1} + \frac{\partial\psi_2}{\partial n_2}, \psi_t + H_1(x,t,v,D\psi_1), \psi_t + H_2(x,t,v,D\psi_2))\Big) \geq 0\,.$$

- *Stratified solutions*, p.226

  First, a stratified *supersolution* is just an Ishii supersolution of the equation. But the subsolution definition involves the various tangential hamiltonians on each manifold of the stratification: a stratified *subsolution* is an Ishii subsolution of the equation such that for any $k = 0, ..., (N+1)$ it is also a subsolution of

$$\mathbb{F}^k(x,t,u,(D_t u, D_x u)) \leq 0 \quad \text{on } \mathbf{M}^k, \text{ for } t > 0,$$

  and for the initial condition,

$$(\mathbb{F}_{init})_*(x,u,D_x u) \leq 0 \quad \text{in } \mathbb{R}^N, \text{ for } t = 0.$$

$$\mathbb{F}^k_{init}(x,u,D_x u) \leq 0 \quad \text{on } \mathbf{M}^k_0, \text{ for } t = 0.$$

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Index