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Control Synthesis for Stochastic Switched Systems using the Tamed Euler Method

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Abstract: In this paper, we explain how, under the one-sided Lipschitz (OSL) hypothesis, one can find an error bound for a variant of the Euler-Maruyama approximation method for stochastic switched systems. We then explain how this bound can be used to control stochastic switched systems in order to stabilize them in a given region. The method is illustrated on several examples of the literature.

Keywords: Stochastic systems, numerical simulation, control system synthesis, switched control systems, nonlinear control systems.

1. INTRODUCTION

Symbolic methods for the verification and control synthesis of hybrid systems (and, particularly, “switched systems”) have received significant attention in the past few years.

One distinguishes two main classes of symbolic methods for hybrid systems: indirect methods and direct methods Asarin et al. (2000).

Indirect methods proceed by constructing a finite abstraction of the original system by discretization of the dense state space \( \mathbb{R}^n \) (where \( n \) is the dimension of the state space). Among the indirect methods, one of the most successful proceeds by approximate bisimulation Girard et al. (2010). This method originally designed for deterministic switched systems has been recently extended for stochastic switched systems Zamani et al. (2015, 2014, 2017). This approach relies on the hypothesis of incremental stability of the stochastic switched system (or existence of a common/multiple Lyapunov function).

A direct method proceeds by working directly at the level of the dense state system \( \mathbb{R}^n \); it computes “trajectory tubes”, which are over-approximations of the set of all the controlled trajectories starting at a given subregion of \( \mathbb{R}^n \). In previous work, We have followed such a direct approach (eg, Fribourg et al. (2014)). The idea is to start with two given hyperrectangles \( R \) and \( S \) of \( \mathbb{R}^n \), (with \( R \subseteq S \)) one covers \( R \) with a finite number of subregions (of the form of balls or sub-rectangles), and finds by exhaustive search, for each subregion, a “control pattern” (i.e., a finite control sequence) such that the trajectories starting from the subregion and controlled by the pattern goes back to \( R \) while never leaving \( S \). Such a direct method ensures the so-called property of \((R,S)\)-stability. We have recently applied such a direct method in the deterministic framework, using the Euler approximation scheme for calculating over-approximations of tubes of trajectories Le Coënt et al. (2017). We show here how to extend this direct method in order to treat stochastic switched systems. The method is a simple extension of the deterministic method, but replaces the classical Euler approximation scheme, by a variant of the stochastic Euler-Maruyama scheme Hutzenthaler et al. (2012). The correctness of these Euler-based methods does not rely on the hypothesis of incremental stability as in Zamani et al. (2015, 2017), but on the hypothesis of ‘one-sided Lipschitz (OSL)’ condition with constant \( \lambda \in \mathbb{R}^d \) (also called ‘monotonicity’/‘dissipativity’, see von Renesse and Scheutzow (2010)). It can be seen that if a stochastic switched system satisfies an OSL condition with \( \lambda < 0 \), then the function \( V(x,x') = |x-x'|^2 \) is a common incremental Lyapunov function in the sense of Zamani et al. (2014), from which it follows that the switched system is incrementally stable, and can be treated by approximate bisimulation. However, Euler-based methods also apply when the system is not incrementally stable, in which case the constant \( \lambda \) is necessarily positive. We thus consider a class of systems different from that of Zamani et al. (2014).

The plan of the paper is as follows: In Section 2, we give an explicit upper bound on the mean square error of the tamed Euler method for SDEs under OSL condition. We apply the result in order to ensure properties of stochastic...
switched systems, such as “(R, S)-stability” (Section 3). We conclude in Section 4.

2. BOUNDING THE ERROR OF THE TAMED EULER METHOD

2.1 Assumptions

The symbol \( | \cdot | \) denotes the Euclidean norm on \( \mathbb{R}^d \). The symbol \( \langle \cdot, \cdot \rangle \) denotes the scalar product of two vectors of \( \mathbb{R}^d \). Given a point \( x \in \mathbb{R}^d \) and a positive real \( r > 0 \), the ball \( B(x, r) \) of centre \( x \) and radius \( r \) is the set \( \{ y \in \mathbb{R}^d \mid |x - y| < r \} \).

Let \( \tau \in (0, \infty) \) be a fixed real number, let \( (\Omega, \mathcal{F}, \mathbb{P}) \) be a probability space with normal filtration \( (\mathcal{F}_t)_{t \in [0, \tau]} \), let \( d, m \in \mathbb{N} := \{1, 2, \ldots \} \) let \( W = (W^{(1)}, \ldots, W^{(m)}) : [0, \tau] \times \Omega \to \mathbb{R}^m \) be an \( m \)-dimensional standard \( (\mathcal{F}_t)_{t \in [0, \tau]} \)-Brownian motion and let \( x_0 : \Omega \to \mathbb{R}^d \) be an \( \mathcal{B}(\mathbb{R}^d) \)-measurable mapping with \( \mathbb{E}(|x_0|^p) < \infty \) for all \( p \in [1, \infty) \). Moreover, let \( f : \mathbb{R}^d \to \mathbb{R}^d \) be a continuously differentiable and globally one-sided Lipschitz continuous function whose derivative grows at most polynomially and let \( g = (g_{ij})_{i \in \{1, \ldots, d\}, j \in \{1, \ldots, m\}} : \mathbb{R}^d \to \mathbb{R}^{d \times m} \) be a globally Lipschitz continuous function.

Then consider the Stochastic Differential Equations (SDE): \( dX_t = f(X_t)dt + g(X_t)dW_t, \quad X_0 = x_0 \) for \( t \in [0, \tau] \). The drift coefficient \( f \) is the infinitesimal mean of the process \( X \) and the diffusion coefficient \( g \) is the infinitesimal standard deviation of the process \( X \). Under the above assumptions, the SDE (1) is known to have a unique strong solution. More formally, there exists an adapted stochastic process \( X : [0, \tau] \times \Omega \to \mathbb{R}^d \) with continuous sample paths fulfilling

\[
X_{t,x_0} = x_0 + \int_0^t f(X_s)ds + \int_0^t g(X_s)dw_s
\]

for all \( t \in [0, \tau] \) \( \mathbb{P} \)-a.s. (see, e.g., Øksendal (2002)).

We denote by \( X_{t,x_0} \) the solution of Equation (1) at time \( t \) from initial condition \( X_{0,x_0} = x_0 \) \( \mathbb{P} \)-a.s., in which \( x_0 \) is a random variable that is measurable in \( \mathcal{F}_0 \).

We suppose that \( f \) behaves polynomially and \( g \) is Lipschitz, i.e.: there exist constants \( D \in \mathbb{R}_{\geq 0}, \, q \in \mathbb{N} \) and \( L_g \in \mathbb{R}_{\geq 0} \) such that, for all \( x, y \in \mathbb{R}^d \)

\[
|f(x) - f(y)|^2 \leq D|x - y|^2(1 + |x|^q + |y|^q) \quad (H1)
\]

\[
|g(x) - g(y)| \leq L_g|x - y| \quad (H2)
\]

We also assume that the SDE (1) satisfies the following one-sided Lipschitz (OSL) condition with constant \( \lambda \in \mathbb{R} \):

\[
\exists \lambda \in \mathbb{R} \forall x, y \in \mathbb{R}^d : \langle f(y) - f(x), y - x \rangle \leq \lambda |y - x|^2 \quad (H3)
\]

Remark 1. Constants \( \lambda, L_g \) and \( D \) can be computed using (constrained) optimization algorithms (see Le Coënt et al. (2017)).

2.2 Tamed Euler approximation

The standard way to extend the classical Euler method for ordinary differential equations to the SDE (1) is the Euler-Maruyama scheme Maruyama (1955). More precisely, given \( z : \Omega \to \mathbb{R}^d \) an \( \mathcal{F}_0/B(\mathbb{R}^d) \)-measurable mapping with \( \mathbb{E}(|z|^p) < \infty \) for all \( p \in [1, \infty) \), the explicit Euler-Maruyama (EM) method for the SDE (1) is given by the mappings \( Y_{n+1}^z : \Omega \to \mathbb{R}^d, \, n \in \{0, 1, \ldots, N\} \), which satisfy

\[
Y_{0,z}^N = z \quad \text{and} \quad Y_{n+1,z}^N = Y_{n,z}^N + \frac{\tau}{N} f(Y_{n,z}^N) + g(Y_{n,z}^N)(W_{(n+1)\tau/N} - W_{n\tau/N})
\]

for all \( n \in \{0, 1, \ldots, N-1\} \) and all \( N \in \mathbb{N} \). See Maruyama (1955). Unfortunately, the convergence results for the EM scheme does not hold when the drift function \( f \) of the SDE (1) behaves polynomially (and not linearly). For the sake of generality, we will now adopt a refined scheme, which has been proposed recently in order to overcome this difficulty Hutzenthaler et al. (2012). Let \( \Delta_n^z : \Omega \to \mathbb{R}^d \),

\[
\Delta_{n+1}^z = \Delta_n^z + \frac{\tau}{N} f(\Delta_n^z) + \frac{1}{N} \cdot g(\Delta_n^z)(W_{(n+1)\tau/N} - W_{n\tau/N})
\]

for all \( n \in \{0, 1, \ldots, N-1\} \) and all \( N \in \mathbb{N} \). We refer to the numerical method (2) as a tamed Euler scheme Hutzenthaler et al. (2012). In this method the drift term \( \frac{\tau}{N} \cdot f(\Delta_n^z) \) is “tamed” by the factor \( 1/(1 + \frac{\tau}{N} \cdot |f(\Delta_n^z)|) \) for \( n \in \{0, 1, \ldots, N-1\} \) and \( N \in \mathbb{N} \). A time continuous interpolation of the time discrete numerical approximations (2) is also introduced in Hutzenthaler et al. (2012) as follows. Let \( \hat{X}^N : [0, \tau] \times \Omega \to \mathbb{R}^d \), \( N \in \mathbb{N} \), be a sequence of stochastic processes given by

\[
\hat{X}^N_{t,z} = \hat{X}^N_{n,z} + \frac{(t - n\tau/N)}{1 + \frac{\tau}{N} \cdot |f(\Delta_n^z)|} \cdot g(\Delta_n^z)(W_t - W_{n\tau/N})
\]

for all \( t \in \left[ \frac{n\tau}{N}, \frac{(n+1)\tau}{N} \right] \), \( n \in \{0, 1, \ldots, N-1\} \) and all \( N \in \mathbb{N} \). Note that \( \hat{X}^N_{t,z} : [0, \tau] \times \Omega \to \mathbb{R}^d \) is an adapted stochastic process with continuous sample paths for every \( N \in \mathbb{N} \).

Let us define \( \tilde{X}^N_{t,z} \) by

\[
\tilde{X}^N_{t,z} := \hat{X}^N_{t,z} \quad \text{for } t \in \left[ \frac{n\tau}{N}, \frac{(n+1)\tau}{N} \right].
\]

Note that \( \tilde{X}^N_{t,z} = X_{t,z} = \hat{X}^N_{t,z} \) at time \( t = \frac{n\tau}{N} \) for \( n \in \{0, 1, \ldots, N\} \).

The following theorem is proven in Hutzenthaler et al. (2012):

Theorem 1. (Hutzenthaler et al. (2012)). Let us suppose (H1) (H2) and (H3). Let the setting in this section be fulfilled, and \( z : \Omega \to \mathbb{R}^d \) be an \( \mathcal{F}_0/B(\mathbb{R}^d) \)-measurable mapping with \( \mathbb{E}(|z|^p) < \infty \) for all \( p \in [1, \infty) \). Then, for all \( p \in [1, \infty) \)

\[
\limsup_{N \to \infty} \sup_{0 \leq t \leq \tau} \mathbb{E}(|\tilde{X}^N_{t,z} - \tilde{X}^N_{t,z}|^p) \leq (\Delta_t)^{\frac{p}{2}} (E_r^N(\Delta_t)^{\frac{p}{2}} + F_r^N dt).
\]
with $\Delta_t = \tau/N$ and:

$$E_{r,z} = 2^r|f(0)|^2 + 2D_{2^r}^2 \left(1 + \mathbb{E}\sup_{0 \leq t \leq \tau} |X_{t,z}|^{2^r}\right),$$

$$F_{r,z} = 2^r|g(0)|^2 + L_g^2 \mathbb{E}\sup_{0 \leq t \leq \tau} |X_{t,z}|^{2^r}.$$ 

**Proof.** see Appendix.

**Remark 2.** Constants $E_{r,z}$ and $F_{r,z}$ are computed using the constants $\lambda$ and $L_g$ (see Remark 1), and the expected values of $X_{t,z}$ at each time $t = 0, \Delta t, 2\Delta t, \ldots, N\Delta t$. These expected values are computed using a Monte Carlo method (by averaging here the value of $10^4$ samplings).

### 2.3 Mean square error bounding

The following Theorem holds for SDE (1). This corresponds to a stochastic version of Theorem 1 of Le Coënt et al. (2017), showing that a similar result holds on average, using the tamed Euler method of Hutzenthaler et al. (2012). It is an adaptation of Theorem 4.4 in Higham et al. (2002).

**Theorem 2.** Given the SDE system (1) satisfying (H1)-(H2)-(H3). Let $\Delta_0 \in \mathbb{R}_{>0}$. Suppose that $z$ is a random variable on $\mathbb{R}^d$ such that

$$\mathbb{E}[|x_0 - z|^2] \leq \delta_0^2,$$

Then, we have, for all $\tau \geq 0$:

$$\mathbb{E}\sup_{0 \leq t \leq \tau} |X_{t,x_0} - \tilde{X}_{t,z}|^2 \leq \delta_0^2,$$

with $\delta_{r,\delta_0} := \beta(\gamma)e^{r\tau}$, where:

$$\gamma = 4(\sqrt{\gamma} + 2\lambda + L_g^2 + 128L_0^4),$$

$$\beta(\tau) = 2\delta_0^2 + 2\tau\Delta_0 L_g^2(1 + 128L_0^2)(F_{2,z}d + E_{2,z}\Delta_t) + 4\tau\sqrt{\Delta_0}D(F_{1,z}d + E_{1,z}\Delta_t)^{\frac{1}{2}} + (1 + 4\mathbb{E}\sup_{0 \leq t \leq \tau} |X_{t,z}|^{2^r} + 4\mathbb{E}\sup_{0 \leq t \leq \tau} |\tilde{X}_{t,z}|^{2^r})^{\frac{1}{2}}.$$ 

with $\Delta_t = \tau/N$.

**Proof.** The proof closely follows the proof of Theorem 4.4 in Higham et al. (2002). Let $e_t = X_{t,x_0} - \tilde{X}_{t,z}$. We have, for all $0 \leq t \leq \tau$:

$$de_t = (f(X_{t,x_0}) - \tilde{f}(z))dt + (g(X_{t,x_0}) - g(z))dW_t.$$ 

Then, by using Equation (4) and the integral version of Itô formula applied to function $x \mapsto |x|^2$ we obtain

$$\|e_t\|^2 = \|e_0\|^2 + \int_0^t 2\langle e_s, f(X_{s,x_0}) - f(\tilde{X}_{s,z})\rangle ds + \int_0^t \|g(X_{s,x_0}) - g(\tilde{X}_{s,z})\|^2 ds + M(t),$$

where $e_0 = x_0 - z$, and

$$M(t) = \int_0^t 2\langle e_s, g(X_{s,x_0}) - g(\tilde{X}_{s,z})\rangle dW_s.$$

So we have using (H2):

$$\|e_t\|^2 \leq \|e_0\|^2 + \int_0^t 2\langle e_s, f(X_{s,x_0}) - f(\tilde{X}_{s,z})\rangle ds + \int_0^t \|g(X_{s,x_0}) - g(\tilde{X}_{s,z})\|^2 ds + M(t).$$ 

So we have using (H3) and Young's inequality:

$$\|e_t\|^2 \leq \|e_0\|^2 + \int_0^t (2\|e_s\|^2 + L_g^2\|e_s\|^2) ds + \int_0^t \|g(X_{s,x_0}) - g(\tilde{X}_{s,z})\|^2 ds + M(t).$$ 

It follows using Lemma 1 for $r = 2$, and Cauchy-Schwarz inequality:

$$\mathbb{E}\sup_{0 \leq s \leq t} |e_s|^2 \leq \mathbb{E}\sup_{0 \leq s \leq t} |e_0|^2 + \int_0^t \mathbb{E}|e_s|^2 ds + \int_0^t \|g(X_{s,x_0}) - g(\tilde{X}_{s,z})\|^2 ds + M(t),$$

where

$$M(t) = \mathbb{E}\sup_{0 \leq s \leq t} |M(s)|.$$ 

Hence, using using Lemma 1 for $r = 4$, and inequality $(a + b)r \leq 2^r(a^r + b^r)$:

$$\mathbb{E}\sup_{0 \leq s \leq t} |e_s|^2 \leq \mathbb{E}|e_0|^2 + \int_0^t \mathbb{E}|e_s|^2 ds + \int_0^t \|g(X_{s,x_0}) - g(\tilde{X}_{s,z})\|^2 ds + \int_0^t \|g(X_{s,x_0}) - g(\tilde{X}_{s,z})\|^2 ds + M(t).$$
On the other hand, from the Burkholder-Davis-Gundy inequality, we get:

\[ m(t) \leq 16L_g^2 \mathbb{E} \left[ \sup_{0 \leq s \leq t} \| e_s \|^2 \right] \int_0^t \| g(X_{s,x_0}) - g(X_{s,z}) \|^2 ds \frac{1}{2} \]

Hence, using (H2):

\[ m(t) \leq 16L_g^2 \mathbb{E} \left[ \sup_{0 \leq s \leq t} \| e_s \|^2 \right] \int_0^t \| X_{s,x_0} - X_{s,z} \|^2 ds \frac{1}{2} \]

Then, using Young’s inequality (for any \( \alpha > 0 \)):

\[ m(t) \leq 8L_g^2 (\alpha \mathbb{E} \left[ \sup_{0 \leq s \leq t} \| e_s \|^2 \right] + \frac{1}{\alpha} \mathbb{E} \left[ \int_0^t \| X_{s,x_0} - X_{s,z} \|^2 ds \right]). \]

Hence, by using Lemma 1 for \( r = 2 \):

\[ m(t) \leq 8L_g^2 \mathbb{E} \left[ \sup_{0 \leq s \leq t} \| e_s \|^2 \right] + \frac{8L_g^2}{\alpha} \int_0^t \mathbb{E} \left[ \sup_{0 \leq s \leq t} \| e_s \|^2 \right] ds \]

\[ + \frac{8L_g^2}{\alpha} \tau \Delta t + F_{2z} \]

(11)

Hence, letting \( \alpha = \frac{1}{2m_L^2} \), we have by replacing in (10):

\[ \frac{1}{2} \mathbb{E} \left[ \sup_{0 \leq s \leq t} \| e_t \|^2 \right] \leq \]

\[ \delta_0^2 + (\sqrt{\Delta t} + 2\lambda + L_g^2 + 128L_g^4) \int_0^t \mathbb{E} \left[ \sup_{0 \leq s \leq t} \| e_s \|^2 \right] ds \]

\[ + \tau (L_g^2 + 128L_g^4) \Delta t + F_{2z} \]

\[ + \tau 2D \sqrt{\Delta t} (E_{2z} \Delta t + F_{2z} d) \]

\[ (1 + 4E \sup_{0 \leq s \leq t} \| X_{s,t} \|^2 + 4E \sup_{0 \leq s \leq t} \| \tilde{X}_{s,t} \|^2 \frac{1}{2}). \]

It results from Gronwall’s inequality:

\[ \mathbb{E} \left[ \sup_{0 \leq s \leq t} \| e_t \|^2 \right] = \beta(\tau)e^{\gamma \tau}, \]

with

\[ \gamma = 2(\sqrt{\Delta t} + 2\lambda + L_g^2 + 128L_g^4), \]

\[ \beta(\tau) = 2\delta_0^2 \]

\[ + 2\tau(D_0^2 + 128L_g^4) (F_{2z} d + E_{2z} \Delta t) \]

\[ + \tau 2D \sqrt{\Delta t} (E_{2z} \Delta t + F_{2z} d) \]

\[ (1 + 4E \sup_{0 \leq s \leq t} \| X_{s,t} \|^2 + 4E \sup_{0 \leq s \leq t} \| \tilde{X}_{s,t} \|^2 \frac{1}{2}). \]

(13)

It follows from Theorem 2 and Jensen’s inequality:

**Proposition 1.** Consider two points \( x_0 \) and \( z \) of \( \mathbb{R}^d \), and a positive real number \( \delta_0 \). Suppose that \( x_0 \in B(z, \delta_0) \) (i.e. \( \| x_0 - z \| \leq \delta_0 \)). Then \( \mathbb{E} X_{t,x_0} \in B(\tilde{X}_{t,z}, \delta_{t,\delta_0}) \) for all \( t \in [0, \tau] \).

It also follows from Theorem 2:

**Proposition 2.** In the setting of Theorem 2, the expression \( \delta_{t,\delta_0} \) tends to

\[ \delta_0 \sqrt{2e^{2\lambda \tau} + L_g^2 + 128L_g^4} \]

when \( \Delta t \) tends to 0 (i.e., when \( N \) tends to \( \infty \)).

### 2.4 Implementation

This method has been implemented in the interpreted language Octave, and the experiments performed on a

2.80 GHz Intel Core i7-4810MQ CPU with 8 GB of memory. The implementation is an adaptation of the program described in Le Coënt et al. (2017) for controlling deterministic switched systems, but makes use of the tamed Euler scheme for SDEs (with the error function \( \delta \) given in Theorem 2) instead of the classical Euler scheme.

**Example 1.** Consider the following system, corresponding to the example in Section 6.2 of Zamani et al. (2015) (cf. Zamani et al. (2014)) for mode \( u = 1 \):

\[
\begin{align*}
\frac{dx_1}{dt} &= (-0.25x_1 + x_2 + 0.25)dt + 0.05x_1 dW_t^1 \\
\frac{dx_2}{dt} &= (-2x_1 - 0.25x_2 - 2)dt + 0.05x_2 dW_t^2
\end{align*}
\]

The program given (for \( \tau = 1 \), \( \Delta_t = \tau/10^4 \)): \( q = 0 \), \( D = 1.36 \), \( L_g = 0.05 \), \( \lambda = 0.25 \), and for \( z = (-4, -3.8) \): \( E_{2z} = 893.3 \), \( E_{4z} = 2.14 \times 10^5 \), \( F_{2z} = 0.002 \), \( F_{4z} = 4.9 \times 10^{-6} \).

Consider now the system corresponding to the example of Zamani et al. (2015) for mode \( u = 2 \):

\[
\begin{align*}
\frac{dx_1}{dt} &= (-0.25x_1 + x_2 - 0.25)dt + 0.05x_1 dW_t^1 \\
\frac{dx_2}{dt} &= (-x_1 - 0.25x_2 + 1)dt + 0.05x_2 dW_t^2
\end{align*}
\]

The program gives (for \( \tau = 1 \), \( \Delta_t = \tau/10^4 \)): \( q = 0 \), \( D = 1.36 \), \( L_g = 0.05 \), \( \lambda = 0.25 \), and for \( z = (0, 3) \): \( E_{2z} = 543.2 \), \( E_{4z} = 7.94 \times 10^4 \), \( F_{2z} = 0.0442 \), \( F_{4z} = 0.00178 \).

Both computations take less than 10 s. of CPU time. Simulations of the two systems are given in Figure 1 for mode \( u = 1 \) and starting point \( z = (-4, 3.8) \), and in Figure 2 for mode \( u = 2 \) and starting point \( z = (0, 3) \). On each figure, the initial ball \( (t = 0) \) is depicted in black, the final ball \( (t = \tau) \) in red, and 200 random sampling trajectories in blue for \( t \in [0, \tau] \).

---

1 Note that, in the figures, all the end points (at \( t = \tau \)) of the sampling trajectories lie in the final ball, but this is not true in general; we only know by Proposition 1 that, for any starting point \( x_0 \) in the initial ball, the expected value of the end point lies in the final ball.
We now consider a finite number of SDEs. Each SDE corresponds to the selection of mode $\pi_j$ as an element of $U$ as follows:

- $\tilde{X}_{t,z}^{\pi} = \tilde{X}_{t,z}^j$ if $\pi = j \in U$, $k = 1$ and $t \in [0, \tau]$, and
- $\tilde{X}_{(k-1)\tau+t',z}^{\pi} = \tilde{X}_{t',z}^j$ with $t' = (k-1)\tau + t$ if $k \geq 2$, $t' \in [0, \tau]$, $\pi = \pi' \ast j$ for some $j \in U$ and $\pi' \in U^{k-1}$.

The “staircase” approximate solution $\tilde{X}_{t,z}^{\pi}$ is defined analogously. Likewise, given an initial error radius $\delta_0 > 0$ and a pattern $\pi$ of length $k \geq 1$, one defines the error radius $\delta_{t,\delta_0}$ as follows:

- $\delta_{t,\delta_0} = \delta_{t,\delta_0}^j$ if $\pi = j \in U$, $k = 1$ and $t \in [0, \tau]$, and
- $\delta_{(k-1)\tau+t',\delta_0} = \delta_{t',\delta_0}^j$ with $t' = (k-1)\tau + t$ if $k \geq 2$, $t' \in [0, \tau]$, $\pi = \pi' \ast j$ for some $j \in U$ and $\pi' \in U^{k-1}$.

### 3.3 Controlled $(R,S)$-stability

Given a rectangle $R \subset \mathbb{R}^d$ and a rectangle $S \subset \mathbb{R}^d$ such that $R \subseteq S$, we now extend the problem of “controlled $(R,S)$-stability”, as defined in Le Coënt et al. (2017) for deterministic switched systems, to SDEs, as follows:

For any starting point $x_0 \in R$, find a pattern $\pi$ of length $k$ such that

- $\mathbb{E}X_{0}^{\pi} \in R$ for $t = k\tau$
- $\mathbb{E}X_{0}^{\pi} \in S$ for all $t = \tau, 2\tau, 3\tau, \ldots$

It is easy to see that, in order to solve this problem, it suffices to exhibit a finite set of points $z_1, \ldots, z_p$ of $S$, and a positive real $\delta_0 > 0$ such that:

1. all the balls $B(z_i, \delta_0)$, $i = 1, \ldots, p$, cover $R$, and are included into $S$ (i.e. $R \subseteq \bigcup_{i=1}^{p} B(z_i, \delta_0) \subseteq S$);
2. for each $i = 1, \ldots, p$, there is a pattern $\pi$ of length $k$ such that:
   - $B_{i,\pi,t} \subseteq S$ for $t = \tau, 2\tau, \ldots, (k-1)\tau$, and
   - $B_{i,\pi,t} \subseteq R$ for $t = k\tau$.

where $B_{i,\pi,t} := B(\mathbb{E}\tilde{X}_{t,z}^{\pi}, \delta_{t,\delta_0}^i)$.

The program mentioned in Section 2.4, has been extended in order to find, by exhaustive enumeration, patterns that make the balls covering $R$ return to $R$, and such that the intermediate balls (at $t = \tau, 2\tau, \ldots$) belong to $S$. Please refer to Appendix B for illustrations and to Le Coënt et al. (2016) for more information on the algorithm.

We now give an application of this program.

**Example 2.** Consider the system (see Zamani et al. (2014, 2015)):

$$dx_1 = (0.25x_1 + u_1 + (1)^{u_2})dt + 0.01x_1dW_t^1$$
$$dx_2 = ((u-3)x_1 - 0.25x_2 + (1)^{u}(3-u))dt + 0.01x_2dW_t^2$$

where $u = 1, 2$.

For $\tau = 0.5$, $\Delta_t = 10^{-4}$, one finds (for all modes $u = 1, 2$): 2

2. For the sake of notation conciseness, we suppose that the number of steps of subsampling $N$ is the same whatever the mode $j$ of the pattern $\pi$ is, hence the stepsize of the subsampling is always equal to $\Delta_t = \pi/N$; in full generality, we should write $N_j$ instead of $N$ to express the dependence.

3. The enumeration can be accelerated using different branch-and-bound heuristics (see Le Coënt et al. (2017)).
Our problem is adapted from Morzfeld (2015). The controlled dynamics is:
\[ dX = ud\tau + dW, \quad X_0 = 1 \]
with mode \( u \in \{-6, -5, -4, -3, -2, 1, 0, 1, 2, 3, 4, 5, 6\} \). We have (at \( t = 0.5 \)) a slit at \( x \in [-1, -4] \). The objective is thus to control the system so that \( x(t) \in S = [-1, -4] \) at \( t = 0.5 \).

One has, for all modes: \( q = 0, D = 0, L_q = 0, \lambda = 0 \). For \( \delta_0 = 0.5 \), an initial point \( z = 1 \) and a sampling time \( \tau = 0.5 \) with subsampling \( \Delta = 10^{-3} \), one has for mode \( u = -6 \): \( E_{2,z} = 144, E_{4,z} = 20736, F_{2,z} = 4, F_{4,z} = 16 \); and for mode \( u = 0 \): \( E_{2,z} = 0, E_{4,z} = 0, F_{2,z} = 4, F_{4,z} = 16 \).

Suppose that all the trajectories start at \( x_0 \) with \( x_0 \in B(z, \delta_0) \) (i.e., \( |x_0 - z| \leq 0.5 \)), with \( z = 1 \) and \( \delta_0 = 0.5 \). When there is no control \( (u = 0) \), at time \( t = 0.5 \), the expected value of \( X_{t,x_0} \) is in \( B(z_1, \delta_{t,\delta_0}) \) with \( z_1 = 1 \) and \( \delta_{t,\delta_0} = 2 \). From Markov’s inequality, it follows that the trajectories pass by \( S = [-1, -4] \) at \( t = 0.5 \) with low probability: see Figure 7. On the other hand, with control \( u = -6 \), at time \( t = \tau = 0.5 \), the expected value of \( X_{t,x_0} \) is now in \( B(z_1, \delta_{t,\delta_0}) \) with \( z_1 = -2 \) and \( \delta_{t,\delta_0} = 2 \). This
4. FINAL REMARKS AND FUTURE WORK

We have explained how to use an Euler-based method in order to control stochastic switched systems. We have focused our work on the property of $(R,S)$-stability, but it can also be used for achieving reachability properties. In the future, we plan to experiment the method with examples where the drift functions behave polynomially. We would like also to find bounds not only for the expected values of the solutions, but for their variance.

REFERENCES


Appendix A. APPENDIX: PROOF OF LEMMA 1

**Proof.** Let \( t \in [k \Delta t, (k + 1) \Delta t] \). Then (using the inequality \((a + b)^r \leq 2^r (a^r + b^r)\)):

\[
|X_t - \bar{X}_t|^r = |(t - t_k) f(X_t) + g(X_t) (W_t - W_{t_k})|^r \\
\leq 2^r ((\Delta_t)^r \|f(X_t)\|^r + D_2^{\frac{r}{r-1}} (1 + \|X_t\|)^\frac{r}{r-1} (E\|X_t\|^{2r})^{\frac{1}{r}} + (E\|W_t - W_{t_k}\|^2)^{\frac{3}{2}}) \\
+ \|g(X_t)\|^r \|W_t - W_{t_k}\|^r) (X_t - \bar{X}_t)^r \\
\leq 2^r ((\Delta_t)^r \|f(0)\|^r + D_2^{\frac{r}{r-1}} (1 + \|X_t\|)^\frac{r}{r-1} (E\|X_t\|^{2r})^{\frac{1}{r}} + (E\|W_t - W_{t_k}\|^2)^{\frac{3}{2}}) \\
+ \|g(0)\|^r \|W_t - W_{t_k}\|^r |W_t - W_{t_k}|^r)
\]

(A.1)

\[
E|X_t - \bar{X}_t|^r \\
\leq 2^r ((\Delta_t)^r \|f(0)\|^r + D_2^{\frac{r}{r-1}} (1 + \|X_t\|)^\frac{r}{r-1} (E\|X_t\|^{2r})^{\frac{1}{r}} + (E\|W_t - W_{t_k}\|^2)^{\frac{3}{2}}) \\
+ \|g(0)\|^r \|W_t - W_{t_k}\|^r |W_t - W_{t_k}|^r)
\]

(A.2)

\[
\text{Hence: } \sup_{0 \leq t \leq T} E|X_t - \bar{X}_t|^r \leq (\Delta_t)^{\frac{r}{2}} (E_{r,z}((\Delta_t)^{\frac{r}{2}} + F_{r,z}d)
\]

with \( E_{r,z} = 2^r \|f(0)\|^r + D_2^{\frac{r}{r-1}} (1 + \sup_{0 \leq t \leq T} \|X_t\|^{2r})^{\frac{1}{r}} (E\sup_{0 \leq t \leq T} \|X_t\|^{2r})^{\frac{1}{r}} \),

\( F_{r,z} = 2^r \|g(0)\|^r + L_g^{\frac{3}{2}} \sup_{0 \leq t \leq T} \|X_t\|^{\frac{3}{2}} \).

Appendix B. CONTROL SYNTHESIS ALGORITHM

Fig. B.1. Scheme of a set covering of \( R \).

Fig. B.2. Control of one of the covering sets.