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Control Synthesis for Stochastic Switched Systems using the Tamed Euler Method

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Abstract: In this paper, we explain how, under the one-sided Lipschitz (OSL) hypothesis, one can find an error bound for a variant of the Euler-Maruyama approximation method for stochastic switched systems. We then explain how this bound can be used to control stochastic switched switch system in order to stabilize them in a given region. The method is illustrated on several examples of the literature.

Keywords: Stochastic systems, numerical simulation, control system synthesis, switched control systems, nonlinear control systems.

1. INTRODUCTION

Symbolic methods for the verification and control synthesis of hybrid systems (and, particularly, “switched systems”) have received significant attention in the past few years.

One distinguishes two main classes of symbolic methods for hybrid systems: indirect methods and direct methods Asarin et al. (2000).

Indirect methods proceed by constructing a finite abstraction of the original system by discretization of the dense state space \( \mathbb{R}^n \) (where \( n \) is the dimension of the state space). Among the indirect methods, one of the most successful proceeds by approximate bisimulation Girard et al. (2010). This method originally designed for deterministic switched systems has been recently extended for stochastic switched systems Zamani et al. (2015, 2014, 2017). This approach relies on the hypothesis of incremental stability of the stochastic switched system (or existence of a common/multiple Lyapunov function).

A direct method proceeds by working directly at the level of the dense state system \( \mathbb{R}^n \); it computes “trajectory tubes”, which are over-approximations of the set of all the controlled trajectories starting at a given subregion of \( \mathbb{R}^n \). In previous work, We have followed such a direct approach (eg, Fribourg et al. (2014)). The idea is to start with two given hyperrectangles \( R \) and \( S \) of \( \mathbb{R}^n \), (with \( R \subseteq S \)); one covers \( R \) with a finite number of subregions (of the form of balls or sub-rectangles), and finds by exhaustive search, for each subregion, a “control pattern” (i.e., a finite control sequence) such that the trajectories starting from the subregion and controlled by the pattern goes back to \( R \) while never leaving \( S \). Such a direct method ensures the so-called property of “(R,S)-stability”. We have recently applied such a direct method in the deterministic framework, using the Euler approximation scheme, by a variant of the stochastic Euler-Maruyama scheme Le Coënt et al. (2017). We show here how to extend this direct method in order to treat stochastic switched systems. The method is a simple extension of the deterministic method, but replaces the classical Euler approximation scheme, by a variant of the stochastic Euler-Maruyama scheme Hutzenthaler et al. (2012). The correctness of these Euler-based methods does not rely on the hypothesis of incremental stability as in Zamani et al. (2015, 2017), but on the hypothesis of ‘one-sided Lipschitz (OSL)’ condition with constant \( \lambda \in \mathbb{R}^d \) (also called ‘monotonicity’/’dissipativity’, see von Renesse and Scheutzow (2010)). It can be seen that if a stochastic switched system satisfies an OSL condition with \( \lambda < 0 \), then the function \( V(x, x') = \|x - x'\|^2 \) is a common incremental Lyapunov function in the sense of Zamani et al. (2014), from which it follows that the switched system is incrementally stable, and can be treated by approximate bisimulation. However, Euler-based methods also apply when the system is not incrementally stable, in which case the constant \( \lambda \) is necessarily positive. We thus consider a class of systems different from that of Zamani et al. (2014).

The plan of the paper is as follows: In Section 2, we give an explicit upper bound on the mean square error of the tamed Euler method for SDEs under OSL condition. We apply the result in order to ensure properties of stochastic
switched systems, such as \(\text{“(R, S)-stability” (Section 3).}\) We conclude in Section 4.

2. BOUNDING THE ERROR OF THE TAMED EULER METHOD

2.1 Assumptions

The symbol \( | \cdot | \) denotes the Euclidean norm on \( \mathbb{R}^d \). The symbol \( \langle \cdot, \cdot \rangle \) denotes the scalar product of two vectors of \( \mathbb{R}^d \). Given a point \( x \in \mathbb{R}^d \) and a positive real \( r > 0 \), the ball \( B(x, r) \) of centre \( x \) and radius \( r \) is the set \( \{ y \in \mathbb{R}^d \mid ||x - y|| \leq r \} \).

Let \( \tau \in (0, \infty) \) be a fixed real number, let \( (\Omega, \mathcal{F}, \mathbb{P}) \) be a probability space with normal filtration \( (\mathcal{F}_t)_{t \in [0, \tau]} \), let \( d, m \in \mathbb{N} := \{1, 2, \ldots \} \) let \( W = (W^{(1)}, \ldots, W^{(m)}) : [0, \tau] \times \Omega \to \mathbb{R}^m \) be an \( m \)-dimensional standard \( (\mathcal{F}_t)_{t \in [0, \tau]} \) Brownian motion and let \( x_0 : \Omega \to \mathbb{R}^d \) be an \( \mathcal{F}_0 \)-\( \mathcal{B}(\mathbb{R}^d) \)-measurable mapping with \( \mathbb{E}[||x_0||^2] < \infty \) for all \( p \in [1, \infty) \). We also assume that the SDE (1) satisfies the following one-sided Lipschitz (OSL) condition with constant \( \lambda \in \mathbb{R} \): for all \( x, y \in \mathbb{R}^d \) such that, for all \( x, y \in \mathbb{R}^d \)

\[
\langle f(x) - f(y), x - y \rangle \leq \lambda \cdot ||x - y||^2 \tag{H1}
\]

\[
||f(x) - f(y)|| \leq L_g ||x - y|| \tag{H2}
\]

We also assume that the SDE (1) satisfies the following one-sided Lipschitz (OSL) condition with constant \( \lambda \in \mathbb{R} \):

\[
\exists \lambda \in \mathbb{R} \forall x, y \in \mathbb{R}^d : \langle f(y) - f(x), y - x \rangle \leq \lambda \cdot ||y - x||^2 \tag{H3}
\]

Remark 1. Constants \( \lambda, L_g \) and \( D \) can be computed using (constrained) optimization algorithms (see Le Coënt et al. (2017)).

2.2 Tamed Euler approximation

The standard way to extend the classical Euler method for ordinary differential equations to the SDE (1) is the Euler-Maruyama scheme (Maruyama (1955)). More precisely, given \( z : \Omega \to \mathbb{R}^d \) an \( \mathcal{F}_0 \)-\( \mathcal{B}(\mathbb{R}^d) \)-measurable mapping with \( \mathbb{E}[||z||^p] < \infty \) for all \( p \in [1, \infty) \), the explicit Euler-Maruyama (EM) method for the SDE (1) is given by the mappings \( Y_{n+1}^N : \Omega \to \mathbb{R}^d, n \in \{0, 1, \ldots, N\} \), which satisfy \( Y_{0,1}^N = z \) and

\[
Y_{n+1}^N = Y_n^N + \frac{\tau}{N} \cdot f(Y_n^N) + g(Y_n^N)(W_{(n+1)}\tau - W_{n, \tau}/N) \tag{2}
\]

for all \( n \in \{0, 1, \ldots, N - 1\} \) and all \( N \in \mathbb{N} \). See Maruyama (1955). Unfortunately, the convergence results for the EM scheme does not hold when the drift function \( f \) of the SDE (1) behaves polynomially (and not linearly). For the sake of generality, we will now adopt a refined scheme, which has been proposed recently in order to overcome this difficulty Hutzenthaler et al. (2012). Let \( \Delta_{n+1}^N : \Omega \to \mathbb{R}^d \),

\[
X_{n+1}^N = X_n^N + \frac{\tau}{N} \cdot f(X_n^N) + \frac{\tau}{N} \cdot \|f(X_n^N)\| \tag{2}
\]

\[
+ g(X_n^N)(W_{(n+1)\tau} - W_{n, \tau})
\]

for all \( n \in \{0, 1, \ldots, N - 1\} \) and all \( N \in \mathbb{N} \). We refer to the numerical method (2) as a tamed Euler scheme Hutzenthaler et al. (2012). In this method the drift term \( \frac{\tau}{N} \cdot \|f(X_n^N)\| \) is “tamed” by the factor \( 1/(1 + \frac{\tau}{N} \cdot \|f(X_n^N)\|) \) for \( n \in \{0, 1, \ldots, N - 1\} \) and \( N \in \mathbb{N} \).

A time continuous interpolation of the time discrete numerical approximations (2) is also introduced in Hutzenthaler et al. (2012) as follows. Let \( \tilde{X}_t^N : [0, \tau] \times \Omega \to \mathbb{R}^d, N \in \mathbb{N} \), be a sequence of stochastic processes given by

\[
\tilde{X}_{t_N}^N = \frac{t - n \tau}{(n+1)\tau} \cdot f(X_{n,\tau}^N) + \frac{\tau}{(n+1)\tau} \cdot g(X_{n,\tau}^N)(W_{(n+1)\tau} - W_{n, \tau}) \tag{2}
\]

for all \( t \in [\frac{n \tau}{N}, \frac{(n+1)\tau}{N}] \), \( n \in \{0, 1, \ldots, N - 1\} \) and all \( N \in \mathbb{N} \). Note that \( \tilde{X}_\tau^N : [0, \tau] \times \Omega \to \mathbb{R}^d \) is an adapted stochastic process with continuous sample paths for every \( N \in \mathbb{N} \).

Let us define \( \tilde{X}_{t_N}^N \) by

\[
\tilde{X}_{t_N}^N = \frac{n \tau}{N} \cdot \tilde{X}_{t_N}^N \tag{2}
\]

for \( t \in [\frac{n \tau}{N}, \frac{(n+1)\tau}{N}) \).

Note that \( \tilde{X}_\tau^N = \tilde{X}_{t_N}^N = X_{n,\tau}^N \) at time \( t = \frac{n \tau}{N} \) for \( n \in \{0, 1, \ldots, N\} \).

The following theorem is proven in Hutzenthaler et al. (2012):

\textbf{Theorem 1.} (Hutzenthaler et al. (2012)). Let us suppose (H1) (H2) and (H3). Let the setting in this section be fulfilled, and \( z : \Omega \to \mathbb{R}^d \) be an \( \mathcal{F}_0 \)-\( \mathcal{B}(\mathbb{R}^d) \)-measurable mapping with \( \mathbb{E}[||z||^p] < \infty \) for all \( p \in [1, \infty) \). Then, for \( p \in [1, \infty) \)

\[
\sup_{N \in \mathbb{N}} \sup_{n \in \{0, 1, \ldots, N\}} \mathbb{E}[||X_{n,\tau}^N - \tilde{X}_{t_N}^N||^p] < \infty
\]

for the sake of simplicity, the number \( N \) of subsampling steps is now left implicit. From Theorem 1, it follows (cf. Lemma 4.3, Higham et al. (2002)):

\textbf{Lemma 1.} Let us suppose (H1) and (H2) and (H3). Let the setting in this section be fulfilled, and \( z : \Omega \to \mathbb{R}^d \) be an \( \mathcal{F}_0 \)-\( \mathcal{B}(\mathbb{R}^d) \)-measurable mapping with \( \mathbb{E}[||z||^p] < \infty \) for all \( p \in [1, \infty) \). Then, for any even integer \( r \geq 2 \), there exist two constants \( E_{r,N} \) and \( F_{r,N} \) such that

\[
\sup_{0 \leq t \leq \tau} \mathbb{E}[|X_{n,\tau}^N - \tilde{X}_{t_N}^N|^r] \leq (\Delta)^{\frac{r}{2}} (E_{r,N} (\Delta)^{\frac{r}{2}} + F_{r,N} (\Delta)^{\frac{r}{2}})
\]
Proof. Given the SDE system (1) satisfying (H1)-(H3), we use the tamed Euler method of Hutzenthaler et al. (2021). Let \( \Delta t_k \) be the time steps at each time \( t = 0, \Delta t, 2\Delta t, \ldots, N\Delta t \). These expected values are computed using a Monte Carlo method (by averaging here the value of 10^4 samplings).

### 2.3 Mean square error bounding

The following Theorem holds for SDEs (1). This corresponds to a stochastic version of Theorem 1 of Le Coent et al. (2017), showing that a similar result holds on average, using the tamed Euler method of Hutzenthaler et al. (2012). It is an adaptation of Theorem 4.4 in Higham et al. (2002).

Theorem 2. Given the SDE system (1) satisfying (H1)-(H2)-(H3). Let \( \delta_0 \in \mathbb{R}_{>0} \). Suppose that \( z \) is a random variable on \( \mathbb{R}^d \) such that

\[
\mathbb{E}[(x_0 - z)^2] \leq \delta_0^2.
\]

Then, we have, for all \( \tau \geq 0 \):

\[
\mathbb{E} \sup_{0 \leq t \leq \tau} \|X_{t,x_0} - \bar{X}_{t,z}\|^2 \leq \delta_0^2 + \frac{\gamma}{\beta(\tau)} e^{\gamma \tau},
\]

where:

\[
\gamma = 2(\sqrt{\Delta t} + 2\lambda + L_g^2 + 128L_d^4) + 4\lambda \Delta t D(F_{1,z} dz + E_{2,z} \Delta t) + 128L_d^4 \Delta t,
\]

\[
\beta(\tau) = 2\delta_0^2 + 2\tau \Delta t L_g^2 (1 + 128L_d^4) + 4\tau \Delta t D(F_{1,z} dz + E_{2,z} \Delta t) + 128L_d^4 \Delta t,
\]

with \( \Delta t = \tau/N \).

proof. The proof closely follows the proof of Theorem 4.4 in Higham et al. (2002). Let \( \beta_t = X_{t,x_0} - \bar{X}_{t,z} \). We have, for all \( 0 \leq t \leq \tau \):

\[
d\beta_t = (f(X_{t,x_0}) - f(z)) dt + (g(X_{t,x_0}) - g(z)) dW_t.
\]

Then, by using Equation (4) and the integral version of Itô formula applied to function \( x \mapsto \|x\|^2 \) we obtain

\[
\|\beta_t\|^2 = \|\beta_0\|^2 + \int_0^t \left( 2\epsilon_s, f(X_{s,x_0}) - f(\bar{X}_{s,z}) \right) ds + M(t),
\]

where \( \epsilon_0 = x_0 - z \), and

\[
M(t) = \int_0^t \left( \left( g(X_{s,x_0}) - g(\bar{X}_{s,z}) \right) ds + (\bar{X}_{s,z} - \bar{X}_{t,z}) dW_s.
\]

So we have using (H2):

\[
\|e_t\|^2 \leq \|e_0\|^2 + \int_0^t \left( 2\epsilon_s, f(X_{s,x_0}) - f(\bar{X}_{s,z}) \right) ds + M(t).
\]

Thus, we have

\[
\|e_t\|^2 \leq \|e_0\|^2 + \int_0^t \left( 2\epsilon_s, f(X_{s,x_0}) - f(\bar{X}_{s,z}) \right) ds + M(t).
\]

So we have using (H3) and Young’s inequality:

\[
\|e_t\|^2 \leq \|e_0\|^2 + \int_0^t \left( 2\epsilon_s, f(X_{s,x_0}) - f(\bar{X}_{s,z}) \right) ds + M(t).
\]

So we have using (H1), for all \( 0 \leq t \leq \tau \):

\[
\|e_t\|^2 \leq \|e_0\|^2 + \int_0^t \left( 2\epsilon_s, f(X_{s,x_0}) - f(\bar{X}_{s,z}) \right) ds + M(t).
\]

It follows using Lemma 1 for \( r = 2 \), and Cauchy-Schwarz inequality:

\[
\mathbb{E} \sup_{0 \leq s \leq t} \|e_s\|^2 \leq \mathbb{E} \sup_{0 \leq s \leq t} \|e_0\|^2 + \int_0^t \mathbb{E} |e_s|^2 ds + M(t).
\]

Therefore, using Lemma 1 for \( r = 4 \), and inequality \( (a + b)^r \leq 2^r (a^r + b^r) \):

\[
\mathbb{E} \sup_{0 \leq s \leq t} \|e_s\|^2 \leq \mathbb{E} \sup_{0 \leq s \leq t} \|e_0\|^2 + \int_0^t \mathbb{E} |e_s|^2 ds + M(t).
\]
On the other hand, from the Burkholder-Davis-Gundy inequality, we get:
\[ m(t) \leq 16\mathbb{E}\left[ \int_0^t \|e_s\|^2 \|g(X_{s,x_0}) - g(X_{s,x})\|^2 ds \right]^\frac{1}{2} \]
Hence, using (H2):
\[ m(t) \leq 16L_g^2 \mathbb{E}\left[ \sup_{0 \leq s \leq t} \|e_s\|^2 \int_0^t \|X_{s,x_0} - X_{s,x}\|^2 ds \right]^\frac{1}{2} \]
Then, using Young’s inequality (for any \( \alpha > 0 \)):
\[ m(t) \leq 8L_g^2 (\alpha \mathbb{E}\left[ \sup_{0 \leq s \leq t} \|e_s\|^2 \right] + \frac{1}{\alpha} \mathbb{E}\left[ \int_0^t \|X_{s,x_0} - X_{s,x}\|^2 ds \right] ) \]
Hence, by using Lemma 1 for \( r = 2 \):
\[ m(t) \leq 8\alpha L_g^2 \mathbb{E}\left[ \sup_{0 \leq s \leq t} \|e_s\|^2 \right] + \frac{8L_g^2}{\alpha} \int_0^t \mathbb{E}\left[ \sup_{0 \leq r \leq s} \|e_r\|^2 \right] ds \]
\[ + \frac{8L_g^2}{\alpha} \tau \Delta t (E_{t_2} - \Delta t + F_{t_2} d) \]
Hence, letting \( \alpha = \frac{1}{16L_g^2} \), we have by replacing in (10):
\[ \frac{1}{2} \mathbb{E}\left[ \sup_{0 \leq s \leq t} \|e_s\|^2 \right] \leq \delta_0 + (\sqrt{\Delta t} + 2\lambda + L_g^2 + 128L_d^4) \int_0^t \mathbb{E}\left[ \sup_{0 \leq r \leq s} \|e_r\|^2 \right] ds \]
\[ + \tau (L_g^2 + 128L_d^4) \Delta t (E_{t_2} - \Delta t + F_{t_2} d) \]
\[ + \frac{1}{2} \tau \Delta t (E_{t_2} - \Delta t + F_{t_2} d) \]
\[ (1 + 4\mathbb{E}\left[ \sup_{0 \leq s \leq t} \|X_{s}\|^2 \right] + 4\mathbb{E}\left[ \sup_{0 \leq s \leq t} \|X_{s}\|^2 \right] ) \]
It results from Gronwall’s inequality:
\[ \mathbb{E}\left[ \sup_{0 \leq t \leq \tau} \|e_t\|^2 \right] = \beta(\tau)e^{\gamma \tau}, \]
with
\[ \gamma = 2(\sqrt{\Delta t} + 2\lambda + L_g^2 + 128L_d^4), \] and
\[ \beta(\tau) = 2\alpha \]
\[ + 2\tau (\Delta t L_g^2 + 128L_d^4) (F_{t_2} d + E_{t_2} \Delta t) \]
\[ + \tau \Delta t (E_{t_2} - \Delta t + F_{t_2} d) \]
\[ (1 + 4\mathbb{E}\left[ \sup_{0 \leq s \leq \tau} \|X_{s}\|^2 \right] + 4\mathbb{E}\left[ \sup_{0 \leq s \leq \tau} \|X_{s}\|^2 \right]) \]
\[ = \frac{1}{\tau} \int_0^\tau \mathbb{E}\left[ \sup_{0 \leq s \leq \tau} \|e_s\|^2 \right] ds \]
(13)

It follows from Theorem 2 and Jensen’s inequality:

**Proposition 1.** Consider two points \( x_0 \) and \( z \) of \( \mathbb{R}^d \), and a positive real number \( \delta_0 \). Suppose that \( x_0 \in B(z, \delta_0) \) (i.e. \( \|x_0 - z\| \leq \delta_0 \)). Then \( \mathbb{E}X_{t_0} \in B(\hat{X}_{t_0}, \delta_{t, \delta_0}) \) for all \( t \in [0, \tau] \).

It also follows from Theorem 2:

**Proposition 2.** In the setting of Theorem 2, the expression \( \delta_{t, \delta_0} \) tends to
\[ \delta_0 \sqrt{2e^{2\lambda \tau + L_g^2 + 128L_d^4}} \]
when \( \Delta t \) tends to 0 (i.e., when \( N \) tends to \( \infty \)).

### 2.4 Implementation

This method has been implemented in the interpreted language Octave, and the experiments performed on a

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1 Note that, in the figures, all the end points (at \( t = \tau \)) of the sampling trajectories lie in the final ball, but this is not true in general; we only know by Proposition 1 that, for any starting point \( x_0 \) in the initial ball, the expected value of the end point lies in the final ball.
We now consider a finite number of SDEs. Each SDE is an element of "(control) pattern of length \( n \)" (i.e., a sequence of lengths \( n \) of \( U^k \)). Each pattern \( j \) corresponds to the selection of mode \( j \) for time \( t \in [0, \tau] \), then mode \( j_2 \) for \( t \in [\tau, 2\tau] \), and so on, until \( t = k\tau \). We assume that the solution of the system is continuous at sampling instants \( t = \tau, 2\tau, \ldots \) (which means that there is no "reset" of the system at sampling instants).

Given a stochastic switched system, a pattern \( \pi \) of length \( k \) and an initial random variable \( z \), one constructs the "approximate solution controlled by \( \pi \)" by composing together the approximations obtained by successive application of the modes of \( \pi \). Formally, the "continuous" approximate solution \( \tilde{X}^{\pi}_{t,z} \) is defined at time \( t \in [0, k\tau] \) as follows:

- \( \tilde{X}^{\pi}_{t,z} = \tilde{X}^j_{t,z} \) if \( \pi = j \in U \), \( k = 1 \) and \( t \in [0, \tau] \), and
- \( \tilde{X}^{\pi}_{(k-1)\tau+t,z} = \tilde{X}^j_{t,z} \) with \( j' = \pi^{\prime} \) if \( k \geq 2 \), \( t' \in [0, \tau] \), \( \pi = \pi^{\prime} * j \) for some \( j \in U \) and \( \pi^{\prime} \in U^{k-1} \).

The "staircase" approximate solution \( X^{\pi}_{t,z} \) is defined analogously. Likewise, given an initial error radius \( \delta_0 > 0 \) and a pattern \( \pi \) of length \( k \geq 1 \), one defines the error radius \( \delta_{t,\pi,\delta_0} \) as follows:

- \( \delta_{t,\pi,\delta_0} = \delta_{1,\delta_0} \) if \( \pi = j \in U \), \( k = 1 \) and \( t \in [0, \tau] \), and
- \( \delta_{(k-1)\tau+t,\pi,\delta_0} = \delta_{t,\pi,\delta_0} \) with \( \delta' = \delta_{(k-1)\tau,\pi',\delta_0} \), if \( k \geq 2 \), \( t' \in [0, \tau] \), \( \pi = \pi^{'} * j \) for some \( j \in U \) and \( \pi^{'} \in U^{k-1} \).

3.3 Controlled \((R,S)\)-stability

Given a rectangle \( R \subset \mathbb{R}^d \) and a rectangle \( S \subset \mathbb{R}^d \) such that \( R \subseteq S \), we now extend the problem of "controlled \((R,S)\)-stability", as defined in Le Coënt et al. (2017) for deterministic switched systems, to SDEs, as follows:

For any starting point \( x_0 \in R \), find a pattern \( \pi \) of length \( k \) such that

- \( \mathbb{E}X_{t,x_0}^{\pi} \in R \) for \( t = k\tau \)
- \( \mathbb{E}X_{t,x_0}^{\pi} \in S \) for all \( t = \tau, 2\tau, 3\tau, \ldots \).

It is easy to see that, in order to solve this problem, it suffices to exhibit a finite set of points \( z_1, \ldots, z_p \) of \( S \), and a positive real \( \delta_0 > 0 \) such that:

1. all the balls \( B(z_i, \delta_0) \), \( i = 1, \ldots, p \), cover \( R \), and are included into \( S \) (i.e. \( R \subseteq \bigcup_{i=1}^p B(z_i, \delta_0) \subseteq S \));
2. for each \( i = 1, \ldots, p \), there is a pattern \( \pi \) of length \( k \) such that:
   - \( B_{i,\tau,\pi} \subseteq S \) for \( t = \tau, 2\tau, \ldots, (k-1)\tau \), and
   - \( B_{i,k\tau,\pi} \subseteq R \) for \( t = k\tau \).

where \( B_{i,\tau,\pi} := B(\mathbb{E}\tilde{X}^{\pi}_{t,z_i,\delta_{t,\pi,\delta_0}}) \).

The program mentioned in Section 2.4, has been extended in order to find, by exhaustive enumeration, patterns that make the balls covering \( R \) return to \( R \), and such that the intermediate balls (at \( t = \tau, 2\tau, \ldots \)) belong to \( S \). Please refer to Appendix B for illustrations and to Le Coënt et al. (2016) for more information on the algorithm.

We now give an application of this program.

Example 2. Consider the system (see Zamani et al. (2014, 2015)):

\[
\begin{align*}
\mathrm{d}x_1 &= (-0.25x_1 + u x_2 + (-1)^u 0.25) \mathrm{d}t + 0.01 x_1 \mathrm{d}W_t^1 \\
\mathrm{d}x_2 &= ((u - 3)x_1 - 0.25x_2 + (-1)^u (3 - u)) \mathrm{d}t + 0.01 x_2 \mathrm{d}W_t^2
\end{align*}
\]

where \( u = 1, 2 \).

For \( \tau = 0.5 \), \( \Delta_1 = 10^{-4} \), one finds (for all modes \( u = 1, 2 \)):

\footnote{For the sake of notation conciseness, we suppose that the number of steps of subsampling \( N \) is the same whatever the mode \( j \) of the pattern \( \pi \) is, hence the stepsize of the subsampling is always equal to \( \Delta_t = \tau/N \); in full generality, we should write \( N_j \) instead of \( N \) to express the dependence.

\footnote{The enumeration can be accelerated using different branch-and-bound heuristics (see Le Coënt et al. (2017)).}
Example 3. (the slit problem) sketched out this point in the following example.

3.4 Other applications

Our Euler-based method can also be used to control systems in order to achieve reachability properties. We sketched out this point in the following example.

Example 3. (the slit problem)

\[ q = 0, D = 1.36, L_q = 0.01, \lambda = 0.25; \text{ for } z = (-4, -3.8): \]
\[ E_{2,z} = 893.31, E_{4,z} = 2.14 \cdot 10^5, F_{2,z} = 0.002, F_{4,z} = 4.9 \cdot 10^{-6}; \text{ and for } z = (0, 3): E_{2,z} = 543.22, E_{4,z} = 7.94 \cdot 10^4, F_{2,z} = 0.0442, F_{4,z} = 0.00178. \]

Our program shows \((R, S)\)-stability of the system for \( R = [-5.5] \times [-4.4] \) and \( S = [-8.8] \times [-7.7] \); given a covering of \( R \) with balls of radius \( \delta_0 = 0.1 \), the program finds, by exhaustive search, patterns of length \( \leq 5 \) that make the balls return to \( R \). It takes 6 hours of CPU time. Figures 3, 4, 5 and 6 depict in black the initial balls (at \( t = 0 \)) centered at the corners of \( R \); and for each initial ball, the pattern that sends the ball back to \( R \) (at time \( t = k\tau \)); the intermediate balls (at \( t = \tau, 2\tau, \ldots, (k-1)\tau \)) are depicted in red, and 200 sampling trajectories drawn in blue.

Fig. 3. initial ball \( B(z, \delta_0) \) with \( z = (-5, 4) \) and \( \delta_0 = 0.1; \)
\( \text{pattern} = (1 \cdot 1 \cdot 1); \tau = 0.5 \)

Fig. 4. initial ball \( B(z, \delta_0) \) with \( z = (5, 4) \) and \( \delta_0 = 0.1; \)
\( \text{pattern} = (2 \cdot 2 \cdot 2); \tau = 0.5 \)

Fig. 5. initial ball \( B(z, \delta_0) \) with \( z = (5, -4) \) and \( \delta_0 = 0.1; \)
\( \text{pattern} = (2 \cdot 2); \tau = 0.5 \)

Fig. 6. initial ball \( B(z, \delta_0) \) with \( z = (-5, -4) \) and \( \delta_0 = 0.1; \)
\( \text{pattern} = (1 \cdot 1 \cdot 1 \cdot 1 \cdot 1); \tau = 0.5 \)

The problem is adapted from Morzfeld (2015). The controlled dynamics is:

\[ dX = ud\tau + dW, \quad X_0 = 1 \]

with mode \( u \in \{-6, -5, -4, -3, -2, -1, 0, 1, 2, 3, 4, 5, 6\} \). We have (at \( t = 0.5 \)) a slit at \( x \in [-1, -4] \). The objective is thus to control the system so that \( x(t) \in S = [-1, -4] \) at \( t = 0.5 \).

One has, for all modes: \( q = 0, D = 0, L_q = 0, \lambda = 0. \)
For \( \delta_0 = 0.5 \), an initial point \( z = 1 \) and a sampling time \( \tau = 0.5 \) with subsampling \( \Delta_s = 10^{-3} \), one has for mode \( u = -6: \)
\[ E_{2,z} = 144, E_{4,z} = 20736, F_{2,z} = 4, F_{4,z} = 16; \]
and for mode \( u = 0: E_{2,z} = 0, E_{4,z} = 0, F_{2,z} = 4, F_{4,z} = 16. \]
Suppose that all the trajectories start at \( x_0 \) with \( x_0 \in B(z, \delta_0) \) (i.e., \( |x_0 - z| \leq 0.5 \)), with \( z = 1 \) and \( \delta_0 = 0.5 \). When there is no control \( (u = 0) \), at time \( t = 0.5 \), the expected value of \( X_{t,x_0} \) is in \( B(z_1, \delta_{1,x_0}) \) with \( z_1 = 1 \) and \( \delta_{1,x_0} = 2 \). From Markov’s inequality, it follows that the trajectories pass by \( S = [-1, -4] \) at \( t = 0.5 \) with low probability; see Figure 7. On the other hand, with control \( u = -6 \), at time \( t = \tau = 0.5 \), the expected value of \( X_{t,x_0} \) is in \( B(z_1, \delta_{1,x_0}) \) with \( z_1 = -2 \) and \( \delta_{x_0} = 2 \). This


**Appendix A. APPENDIX: PROOF OF LEMMA 1**

**Proof.** Let \( t \in [k \Delta_t, (k+1) \Delta_t) \). Then (using the inequality \((a+b)^r \leq 2^r (a^r + b^r)\)):

\[
|X_t - \hat{X}_t|^r = |(t-t_k) f(X_k) + g(X_k)(W_t - W_{t_k})|^r \\
\leq 2^r((\Delta_t)^r(|f(X_k)|^r + |g(X_k)(W_t - W_{t_k})|^r) \\
+ |g(X_k)||W_t - W_{t_k}|^r) ||X_t - \hat{X}_t|^r \\
\leq 2^r((\Delta_t)^r(|f(X_k)|^r + |g(X_k)|^r + |g(X_k)(W_t - W_{t_k})|^r) \\
+ |g(X_k)||W_t - W_{t_k}|^r) ||X_t - \hat{X}_t|^r \\
\leq 2^r((\Delta_t)^r(D(1 + \|X_k\|)^2 ||X_k||^2 + |f(0)|^r) \\
+ (L''_f ||X_k||^r + |g(0)|^r)|W_t - W_{t_k}|^r) ||X_t - \hat{X}_t|^r \\
\leq 2^r((\Delta_t)^r(D(1 + \|X_k\|)^2 ||X_k||^r + |f(0)|^r) \\
+ (L''_f ||X_k||^r + |g(0)|^r)|W_t - W_{t_k}|^r) ||X_t - \hat{X}_t|^r. \\
(A.1)
\]

\[
E||X_t - \hat{X}_t||^r \\
\leq 2^r((\Delta_t)^r(|f(0)|^r + D2^\frac{r+1}{2}) \\
(1 + E[||X_k||^9])^\frac{1}{2} (E[||X_k||^2]^r)) \\
\leq 2^r((\Delta_t)^r(|f(0)|^r + D2^\frac{r+1}{2}) \\
(1 + E[||X_k||^9])^\frac{1}{2} (E[||X_k||^2]^r) + (L''_f E[||X_k||^r])^\frac{1}{2} \tau \\
\leq 2^r((\Delta_t)^r(|f(0)|^r + D2^\frac{r+1}{2}) \\
(1 + E[||X_k||^9])^\frac{1}{2} (E[||X_k||^2]^r) + \tau (E[||X_k||^2]^r)) \\
\leq 2^r((\Delta_t)^r(|f(0)|^r + D2^\frac{r+1}{2}) \\
(1 + E[||X_k||^9])^\frac{1}{2} (E[||X_k||^2]^r) + \tau (E[||X_k||^2]^r)) \\
\leq 2^r((\Delta_t)^r(|f(0)|^r + D2^\frac{r+1}{2}) \\
(1 + E[||X_k||^9])^\frac{1}{2} (E[||X_k||^2]^r) + \tau (E[||X_k||^2]^r)) \\
\leq 2^r((\Delta_t)^r(|f(0)|^r + D2^\frac{r+1}{2}) \\
(1 + E[||X_k||^9])^\frac{1}{2} (E[||X_k||^2]^r) + \tau (E[||X_k||^2]^r)) \\
\leq 2^r((\Delta_t)^r(|f(0)|^r + D2^\frac{r+1}{2}) \\
(1 + E[||X_k||^9])^\frac{1}{2} (E[||X_k||^2]^r) + \tau (E[||X_k||^2]^r)) \\
\leq 2^r((\Delta_t)^r(|f(0)|^r + D2^\frac{r+1}{2}) \\
(1 + E[||X_k||^9])^\frac{1}{2} (E[||X_k||^2]^r) + \tau (E[||X_k||^2]^r)) \\
\leq 2^r((\Delta_t)^r(|f(0)|^r + D2^\frac{r+1}{2}) \\
(1 + E[||X_k||^9])^\frac{1}{2} (E[||X_k||^2]^r) + \tau (E[||X_k||^2]^r)) \\
\leq 2^r((\Delta_t)^r(|f(0)|^r + D2^\frac{r+1}{2}) \\
(1 + E[||X_k||^9])^\frac{1}{2} (E[||X_k||^2]^r) + \tau (E[||X_k||^2]^r)) \\
\leq 2^r((\Delta_t)^r(|f(0)|^r + D2^\frac{r+1}{2}) \\
(1 + E[||X_k||^9])^\frac{1}{2} (E[||X_k||^2]^r) + \tau (E[||X_k||^2]^r)) \\
\leq 2^r((\Delta_t)^r(|f(0)|^r + D2^\frac{r+1}{2}) \\
(1 + E[||X_k||^9])^\frac{1}{2} (E[||X_k||^2]^r) + \tau (E[||X_k||^2]^r)) \\
\leq 2^r((\Delta_t)^r(|f(0)|^r + D2^\frac{r+1}{2}) \\
(1 + E[||X_k||^9])^\frac{1}{2} (E[||X_k||^2]^r) + \tau (E[||X_k||^2]^r)) \\
\leq 2^r((\Delta_t)^r(|f(0)|^r + D2^\frac{r+1}{2}) \\
(1 + E[||X_k||^9])^\frac{1}{2} (E[||X_k||^2]^r) + \tau (E[||X_k||^2]^r)) \\
\leq 2^r((\Delta_t)^r(|f(0)|^r + D2^\frac{r+1}{2}) \\
(1 + E[||X_k||^9])^\frac{1}{2} (E[||X_k||^2]^r) + \tau (E[||X_k||^2]^r)) \\
\leq 2^r((\Delta_t)^r(|f(0)|^r + D2^\frac{r+1}{2}) \\
(1 + E[||X_k||^9])^\frac{1}{2} (E[||X_k||^2]^r) + \tau (E[||X_k||^2]^r))
\](A.2)

Hence:

\[
\sup_{0 \leq t \leq \tau} E||X_t - \hat{X}_t||^r \leq (\Delta_t)^r (E_{r,z}(|\Delta_t|^r + F_{r,z}d))
\]

with \( E_{r,z} = 2^r(|f(0)|^r + D2^\frac{r+1}{2} + 2^r \sup_{0 \leq t \leq \tau} ||X_k||^9)^\frac{1}{2} \\
(\sup_{0 \leq t \leq \tau} ||X_k||^2)^\frac{1}{2}) \\
F_{r,z} = 2^r(|g(0)|^r + L''_g \sup_{0 \leq t \leq \tau} ||X_k||^r).
\]

**Appendix B. CONTROL SYNTHESIS ALGORITHM**