Indirect stability of the wave equation with a dynamic boundary control

Denis Mercier, Serge Nicaise, Mohamad Ali Sammoury, Ali Wehbe

Received: 17 January 2017 | Revised: 1 June 2017 | Accepted: 1 July 2017

DOI: 10.1002/mana.201700021

ORIGINAL PAPER

MATHEMATISCHE NACHRICHTEN

Indirect stability of the wave equation with a dynamic boundary control

Denis Mercier[1] | Serge Nicaise[1] | Mohamad Ali Sammoury[2] | Ali Wehbe[2]

[1]Université de Valenciennes et du Hainaut Cambrésis, LAMAV, FR CNRS 2956, 59313 Valenciennes Cedex 9, France

[2]Université Libanaise, EDST et Faculté des Sciences, Hadath, Beyrouth, Liban

Correspondence
Serge Nicaise, Université de Valenciennes et du Hainaut Cambrésis, LAMAV, FR CNRS 2956, 59313 Valenciennes Cedex 9, France.
Email: snicaise@univ-valenciennes.fr

Abstract

In this paper, we consider a damped wave equation with a dynamic boundary control. First, combining a general criteria of Arendt and Batty with Holmgren's theorem we show the strong stability of our system. Next, we show that our system is not uniformly stable in general, since it is the case for the unit disk. Hence, we look for a polynomial decay rate for smooth initial data for our system by applying a frequency domain approach. In a first step, by giving some sufficient conditions on the boundary of our domain and by using the exponential decay of the wave equation with a standard damping, we prove a polynomial decay in $\frac{1}{t^{\frac{1}{4}}}$ of the energy. In a second step, under appropriated conditions on the boundary, called the multiplier control conditions, we establish a polynomial decay in $\frac{1}{t}$ of the energy. Later, we show in a particular case that such a polynomial decay is available even if the previous conditions are not satisfied. For this aim, we consider our system on the unit square of the plane. Using a method based on a Fourier analysis and a specific analysis of the obtained 1-d problems combining Ingham's inequality and an interpolation method, we establish a polynomial decay in $\frac{1}{t}$ of the energy for sufficiently smooth initial data. Finally, in the case of the unit disk, using the real part of the asymptotic expansion of eigenvalues of the damped system, we prove that the obtained decay is optimal in the domain of the operator.

KEYWORDS
Dynamic control, indirect stability

MSC (2010)
35B35, 35L35

1 | INTRODUCTION

Let Ω be a bounded domain of $\mathbb{R}^d$, $d \geq 2$, with a Lipschitz boundary $\Gamma = \bar{\Gamma}_0 \cup \bar{\Gamma}_1$, with Γ_0 and Γ_1 open subsets of Γ such that $\Gamma_0 \cap \Gamma_1 = \emptyset$ and Γ_1 is non empty. In [12,13,17], N. Fourrier, I. Lasiecka and P. Graber studied the following problem (under the assumption that $\overline{\Gamma_0} \cap \overline{\Gamma_1} = \emptyset$):

$$\begin{cases} u_{tt} - \Delta u - k_\Omega \Delta u_t + c_\Omega u_t = 0, & \text{in } \ \Omega \times \mathbb{R}_+^*, \\ u = 0, & \text{on } \ \Gamma_0 \times \mathbb{R}_+^*, \\ u - w = 0, & \text{on } \ \Gamma_1 \times \mathbb{R}_+^*, \\ w_{tt} - k_\Gamma \Delta_T(\alpha w_t + w) + \partial_\nu(u + k_\Omega u_t) + c_\Gamma w_t = 0, & \text{in } \ \Gamma_1 \times \mathbb{R}_+^*, \\ w = 0, & \text{on } \ \partial\Gamma_1 \times \mathbb{R}_+^*, \\ u(\cdot,\cdot,0) = u_0, \quad u_t(\cdot,\cdot,0) = u_1, & \text{in } \ \Omega, \\ w(\cdot,0) = w_0, \quad w_t(\cdot,0) = w_1, & \text{in } \ \Gamma_1, \end{cases} \tag{1.1}$$

where ∂_ν means the normal derivative on Γ_1, ν is the unit outward normal vector along the boundary and Δ_T denotes the Laplace–Beltrami operator on Γ. In system (1.1), two types of dissipation appear: internal (if $c_\Omega > 0$) and boundary (if $k_\Gamma > 0$) frictional ones and internal (if $k_\Omega > 0$) and boundary (if $k_\Gamma \alpha > 0$) viscoelastic ones. A physical description of this model is first described in [28]. In [12,13], it is shown that system (1.1) is exponentially stable if one of the following three conditions is satisfied: $k_\Omega > 0$ (interior viscoelastic damping), or $c_\Omega > 0$ and $c_\Gamma > 0$ (internal and boundary frictional damping) or $c_\Omega > 0$ and $k_\Gamma \alpha > 0$ (internal frictional damping and boundary viscoelastic damping). The first case corresponds to a direct damping, while the other cases correspond to a phenomenon of overdamping. This phenomenon was the motivation of these authors to study the balance between the competiting dampings. On the contrary, in this paper, we are interested in the important case where only a boundary frictional damping occurs, i.e. $k_\Omega = c_\Omega = \alpha = 0$ and $k_\Gamma = c_\Gamma = 1$. More precisely, we consider the following problem

$$\begin{cases} u_{tt} - \Delta u = 0, & \text{in } \Omega \times \mathbb{R}_+^*, \\ u = 0, & \text{on } \Gamma_0 \times \mathbb{R}_+^*, \\ u - w = 0, & \text{on } \Gamma_1 \times \mathbb{R}_+^*, \\ w_{tt} - \Delta_T w + \partial_\nu u + w_t = 0, & \text{in } \Gamma_1 \times \mathbb{R}_+^*, \\ w = 0, & \text{on } \partial\Gamma_1 \times \mathbb{R}_+^*, \\ u(\cdot,0) = u_0, \quad u_t(\cdot,0) = u_1, & \text{in } \Omega, \\ w(\cdot,0) = w_0, \quad w_t(\cdot,0) = w_1, & \text{in } \Gamma_1. \end{cases} \tag{1.2}$$

In this case, the damping term is the term w_t in the fourth equation of (1.2) and therefore the system in Ω is only damped indirectly. The notion of indirect damping mechanisms has been introduced by Russell in [36,37] and since that time it retains the attention of many authors, because several models from acoustic theory enter in this framework. The most popular model is the wave equation with acoustic boundary conditions that takes the following form

$$\begin{cases} u_{tt} - \Delta u = 0, & \text{in } \Omega \times \mathbb{R}_+^*, \\ u = 0, & \text{on } \Gamma_0 \times \mathbb{R}_+^*, \\ \partial_\nu u = w_t, & \text{on } \Gamma_1 \times \mathbb{R}_+^*, \\ m w_{tt} + d w_t + k w + \rho u_t = 0, & \text{on } \Gamma_1 \times \mathbb{R}_+^*, \\ u(\cdot,0) = u_0, \quad u_t(\cdot,0) = u_1, & \text{in } \Omega, \\ w(\cdot,0) = w_0, & \text{in } \Gamma_1. \end{cases} \tag{1.3}$$

In [8], Beale showed that this problem is governed by a C_0-semigroup of contractions, while in [35], the authors obtained, under some geometrical conditions, a polynomial stability of the system.

In [27], S. Micu and E. Zuazua considered the following simple model arising in the control of noise consisting of two coupled hyperbolic equations of dimensions two and one respectively:

$$\begin{cases} u_{tt} - \Delta u = 0, & \text{in } \Omega \times \mathbb{R}_+^*, \\ \partial_\nu u = 0, & \text{on } \Gamma_0 \times \mathbb{R}_+^*, \\ \dfrac{\partial u}{\partial y} = -w_t, & \text{on } \Gamma_1 \times \mathbb{R}_+^*, \\ w_{tt} - w_{xx} + w_t + u_t = 0, & \text{on } \Gamma_1 \times \mathbb{R}_+^*, \\ w_x(0,t) = w_x(1,t) = 0, & \text{for } t > 0, \\ u(0) = u^0, \quad u_t(0) = u_1, & \text{in } \Omega, \\ w(0) = w_0, \quad w_t(0) = w_1, & \text{on } \Gamma_1, \end{cases} \tag{1.4}$$

where $\Gamma_1 = \{(x,0);\ x \in (0,1)\}$ and $\Gamma_0 = \Gamma \backslash \Gamma_0$. This system is nothing else than system (1.3) where the Dirichlet boundary conditions on Γ_0 have been replaced by the Neumann ones. Using a separation of variables method, they studied the asymptotic behavior of the eigenvalues and eigenfunctions of system (1.4). Since there exists a sequence of eigenvalues which approach the imaginary axis, the authors deduced that the decay rate of the energy of (1.4) is not exponential in the energy space. Later, they proved that system (1.4) can be exponentially stable in the subspace of the energy space generated by the eigenfunctions corresponding to all eigenvalues with uniformly bounded negative real parts. For a generalization of system (1.3) and polynomial decay rates, we refer to [1], while an abstract framework is extensively studied in [29]. For other related problems we refer to [2,10,14,16,25].

In this paper, in a first step, using Arendt and Batty's theorem (see [4]) and with the help of Holmgren's theorem we first show the strong stability of system (1.2), but for the simple example like the case when Ω is the unit disc of $\mathbb{R}^2$ and $\Gamma_0 = \emptyset$, we show that our system is not uniformly stable, since the corresponding spatial operator has a sequence of eigenvalues that approach the imaginary axis. Hence, we are interested in proving a weaker decay of the energy, for that purpose, we will apply a frequency domain approach (see [9]) based on the growth of the resolvent on the imaginary axis. More precisely, we will give sufficient conditions that guarantee the polynomial decay of the energy of our system (for sufficiently smooth initial data). We actually obtain two different decay rates. In the first case, we will use the exponential decay of the wave equation with the standard damping

$$\frac{\partial y}{\partial \nu} = -y_t, \quad \text{on} \quad \Gamma_1 \times \mathbb{R}_+^*,$$

and establish a polynomial energy decay rate of type $\frac{1}{t^{\frac{1}{4}}}$. In the second case, under a stronger geometrical conditions on Γ_0 and Γ_1, we establish a polynomial energy decay rate of type $\frac{1}{t}$.

In a second step, we want to show that in some cases such a polynomial decay may be available even if the previous conditions are not satisfied. Therefore, we consider the case of the unit square of the plane where Γ_1 is only one edge of the boundary. In that case, our method is based on a Fourier analysis (compare with [30] where a similar method was used for the wave equation with Ventcel's boundary conditions) and a specific analysis of the corresonding 1-d problems combining Ingham's inequality and an interpolation method from [3]. This leads to a polynomial energy decay rate of type $\frac{1}{t}$ for smooth initial data. Finally, by using the spectral analysis made for the unit disc of $\mathbb{R}^2$ and $\Gamma_0 = \emptyset$, we prove that, in this situation, the obtained energy decay rate is optimal.

The paper is organized as follows. The second section deals with the well-posedness of the problem obtained by using semi-group theory. We further characterize the domain of the associated operator in some particular cases and obtain the strong stability. In Section 3, we show that our system is not uniformly stable in the unit disc. Section 4 is devoted to the proof of the polynomial decay in the general setting by using the frequency domain approach. In Section 5, we obtain the polynomial stability result for a 1-D model. This result is then used in Section 6 to show for the unit square a polynomial decay in $1/t$ of the energy for sufficiently smooth initial data. In Section 7, we show for the unit disc that the polynomial decay in $1/t$ is optimal (for initial data in the domain of the operator).

Let us finish this section with some notations used in the remainder of the paper. For a bounded domain D, the usual norm and semi-norm of $H^s(D)$ $(s \geq 0)$ are denoted by $\|\cdot\|_{s,D}$ and $|\cdot|_{s,D}$, respectively. For $s = 0$, we will drop the index s. Furthermore, the notation $A \lesssim B$ (resp. $A \gtrsim B$) means the existence of a positive constant C_1 (resp. C_2), which is independent of A and B such that $A \leq C_1 B$ (resp. $A \geq C_2 B$). The notation $A \sim B$ means that $A \lesssim B$ and $A \gtrsim B$ hold simultaneously.

2 | WELL-POSEDNESS AND STRONG STABILITY OF THE PROBLEM

If Γ_0 is non empty, we introduce the space $H^1_{\Gamma_0}(\Omega)$ as follows:

$$H^1_{\Gamma_0}(\Omega) = \left\{u \in H^1(\Omega);\, u = 0 \ \text{ on } \ \Gamma_0\right\}, \tag{2.1}$$

which is a Hilbert space with the norm

$$\|u\|_{1,\Omega} = \|\nabla u\|_{\Omega}. \tag{2.2}$$

Next, we introduce the Hilbert space

$$\mathcal{H} = \left\{(u, v, w, z) \in H^1_{\Gamma_0}(\Omega) \times L^2(\Omega) \times H^1_0(\Gamma_1) \times L^2(\Gamma_1) \,:\, \gamma u = w \text{ on } \Gamma_1\right\}, \tag{2.3}$$

endowed with the product

$$\left(\left(u^1, v^1, w^1, z^1\right), \left(u^2, v^2, w^2, z^2\right)\right)_{\mathcal{H}} = \left(\nabla u^1, \nabla u^2\right)_{\Omega} + \left(v^1, v^2\right)_{\Omega} + \left(\nabla_T w^1, \nabla_T w^2\right)_{\Gamma_1} + \left(z^1, z^2\right)_{\Gamma_1}, \tag{2.4}$$

$$\text{for all } \left(u^1, v^1, w^1, z^1\right), \left(u^2, v^2, w^2, z^2\right) \in H^1_{\Gamma_0}(\Omega) \times L^2(\Omega) \times H^1_0(\Gamma_1) \times L^2(\Gamma_1),$$

and associated norm $\|\cdot\|_{\mathcal{H}} = (\cdot,\cdot)^{\frac{1}{2}}_{\mathcal{H}}$, γ being the usual trace operator from $H^1(\Omega)$ into $H^{\frac{1}{2}}(\Gamma)$. In this paper, for simplicity, we will denote γu by u.

If Γ_0 is empty, we define $\mathcal{H}$ is the same manner, but in this case we equip it with its natural norm:

$$\|(u, v, w, z)\|^2 := \|(u, v, w, z)\|^2_{\mathcal{H}} + \|u\|^2_{\Omega} + \|w\|^2_{\Gamma}.$$

The energy of the solution of (1.2) is defined by:

$$E(t) = \frac{1}{2}\|(u, u_t, w, w_t)\|^2_{\mathcal{H}}. \tag{2.5}$$

For smooth solution, a direct computation gives

$$\frac{d}{dt}E(t) = -\|w_t\|^2_{\Gamma_1}. \tag{2.6}$$

Then, system (1.2) is dissipative in the sense that its energy is a nonincreasing function of the time variable t. We can now introduce the unbounded operator $\mathcal{A}$ on $\mathcal{H}$ with domain

$$D(\mathcal{A}) = \left\{\begin{array}{l} U = (u,\ v,\ w,\ z) \in \mathcal{H}; \\ \Delta_T w - \partial_\nu u \in L^2(\Gamma_1), \\ v \in H^1_{\Gamma_0}(\Omega), \quad \Delta u \in L^2(\Omega), \\ z \in H^1_0(\Gamma_1), \quad v = z \text{ on } \Gamma_1, \end{array}\right\}, \tag{2.7}$$

and defined by

$$\mathcal{A}U = \begin{pmatrix} v \\ \Delta u \\ z \\ \Delta_T w - \partial_\nu u - z \end{pmatrix}, \quad \text{for all} \quad U = \begin{pmatrix} u \\ v \\ w \\ z \end{pmatrix} \in D(\mathcal{A}). \tag{2.8}$$

Then, denoting $U = (u, u_t, w, w_t)$ the state of system (1.2), we can rewrite system (1.2) into a first-order evolution equation

$$\begin{cases} U_t(t) = \mathcal{A}U(t), \quad t > 0, \\ U(0) = U_0, \end{cases} \tag{2.9}$$

where $U_0 = (u_0, v_0, w_0, z_0) \in \mathcal{H}$. It is easy to show that $\mathcal{A}$ is a maximal dissipative operator, therefore owing to Lumer–Phillips' theorem (see [33]), it generates a C_0-semigroup $\left(e^{t\mathcal{A}}\right)_{t\geq 0}$ of contractions on the energy space $\mathcal{H}$. Hence, semi-group theory allows to show the next existence and uniqueness results:

Theorem 2.1. *For any initial data $U_0 \in \mathcal{H}$, the problem (2.9) has a unique weak solution $U(t) = e^{t\mathcal{A}}U_0$ such that $U \in C^0([0, +\infty[, \mathcal{H})$. Moreover, if $U_0 \in D(\mathcal{A})$, then the problem (2.9) has a strong solution $U(t) = e^{t\mathcal{A}}U_0$ such that $U \in C^1([0, +\infty[, \mathcal{H}) \cap C^0([0, +\infty[, D(\mathcal{A}))$.*

Now, we characterize the domain $D(\mathcal{A})$ of $\mathcal{A}$ in two different cases: either Γ is smooth enough and $\overline{\Gamma_0} \cap \overline{\Gamma_1} = \emptyset$ or Ω is the unit square. We start with the first situation:

Proposition 2.2. *If the boundary of Ω is $C^{1,1}$ and if $\overline{\Gamma_0} \cap \overline{\Gamma_1} = \emptyset$, then*

$$D(\mathcal{A}) = \left(H^2(\Omega) \cap H^1_{\Gamma_0}(\Omega)\right) \times H^1_{\Gamma_0}(\Omega) \times \left(H^2(\Gamma_1) \cap H^1_0(\Gamma_1)\right) \times H^1_0(\Gamma_1),$$

with

$$\|(u, v, w, z)\|_{D(\mathcal{A})} \sim \|u\|_{2,\Omega} + \|v\|_{1,\Omega} + \|w\|_{2,\Gamma_1} + \|z\|_{1,\Gamma_1}, \quad \textit{for all} \quad (u, v, w, z) \in D(\mathcal{A}).$$

In particular, the resolvent $(I - \mathcal{A})^{-1}$ of $\mathcal{A}$ is compact on the energy space $\mathcal{H}$.

Proof. The proof is based on a bootstrap argument. Let us fix $U = (u, v, w, z) \in D(\mathcal{A})$, and set $h = \Delta_T w - \partial_\nu u - z$, that belongs to $L^2(\Gamma_1)$. Then by definition, $u \in H^1(\Omega)$ with $\Delta u \in L^2(\Omega)$. Hence by a result of Lions and Magenes (see the end of Subsection 1.5 of [18]), we will have $\partial_\nu u \in H^{-\frac{1}{2}}(\Gamma_1)$ (as $\overline{\Gamma_0}$ and $\overline{\Gamma_1}$ are disjoint) with

$$\|\partial_\nu u\|_{-\frac{1}{2},\Gamma_1} \lesssim \|u\|_{1,\Omega} + \|\Delta u\|_{\Omega}. \tag{2.10}$$

Therefore $w \in H^1(\Gamma_1)$ satisfies

$$\Delta_T w = h + \partial_\nu u + z \in H^{-\frac{1}{2}}(\Gamma_1). \tag{2.11}$$

Hence by a standard shift theorem, we deduce that $w \in H^{\frac{3}{2}}(\Gamma_1)$ with

$$\|w\|_{\frac{3}{2},\Gamma_1} \lesssim \|w\|_{1,\Gamma_1} + \|h + \partial_\nu u + z\|_{-\frac{1}{2},\Gamma_1} \lesssim \|w\|_{1,\Gamma_1} + \|h\|_{\Gamma_1} + \|\partial_\nu u\|_{-\frac{1}{2},\Gamma_1} + \|z\|_{\Gamma_1}.$$

Hence by (2.10), we get

$$\|w\|_{\frac{3}{2},\Gamma_1} \lesssim \|w\|_{1,\Gamma_1} + \|h\|_{\Gamma_1} + \|u\|_{1,\Omega} + \|\Delta u\|_\Omega + \|z\|_{\Gamma_1}. \tag{2.12}$$

Now, this improved regularity on w allows to look at $u \in H^1(\Omega)$ as the solution of the next boundary value problem:

$$\begin{cases} \Delta u & \in L^2(\Omega), \\ u = 0, & \text{on} \quad \Gamma_0, \\ u = w \in H^{\frac{3}{2}}(\Gamma_1), & \text{on} \quad \Gamma_1. \end{cases} \tag{2.13}$$

Hence again a standard shift theorem yields $u \in H^2(\Omega)$ with

$$\|u\|_{2,\Omega} \lesssim \|\Delta u\|_\Omega + \|w\|_{\frac{3}{2},\Gamma_1},$$

and hence by (2.12), we get

$$\|u\|_{2,\Omega} \lesssim \|w\|_{1,\Gamma_1} + \|h\|_{\Gamma_1} + \|u\|_{1,\Omega} + \|\Delta u\|_\Omega + \|z\|_{\Gamma_1}. \tag{2.14}$$

By a trace theorem, we deduce that $\partial_\nu u \in H^{\frac{1}{2}}(\Gamma_1)$ and coming back to (2.11), we deduce that

$$\Delta_T w = h + \partial_\nu u + z \in L^2(\Gamma_1).$$

Again a shift theorem yields $w \in H^2(\Gamma_1)$ with

$$\|w\|_{2,\Gamma_1} \lesssim \|w\|_{1,\Gamma_1} + \|\Delta u\|_\Omega + \|h + \partial_\nu u + z\|_{\Gamma_1}.$$

And by (2.14), we deduce that

$$\|w\|_{2,\Gamma_1} \lesssim \|w\|_{1,\Gamma_1} + \|h\|_{\Gamma_1} + \|u\|_{1,\Omega} + \|\Delta u\|_\Omega + \|z\|_{\Gamma_1}. \tag{2.15}$$

We have shown that

$$D(\mathcal{A}) \subset \left(H^2(\Omega) \cap H^1_{\Gamma_0}(\Omega)\right) \times H^1_{\Gamma_0}(\Omega) \times \left(H^2(\Gamma_1) \cap H^1_0(\Gamma_1)\right) \times H^1_0(\Gamma_1).$$

On the other hand the estimates (2.14)–(2.15) yield

$$\|u\|_{2,\Omega} + \|v\|_{1,\Omega} + \|w\|_{2,\Gamma_1} + \|z\|_{1,\Gamma_1} \lesssim \|(u, v, w, z)\|_{D(\mathcal{A})}, \quad \text{for all} \quad (u, v, w, z) \in D(\mathcal{A}),$$

reminding that $\|U\|_{D(\mathcal{A})} = \|U\|_{\mathcal{H}} + \|\mathcal{A}U\|_{\mathcal{H}}$.

The converse inclusion and estimate being trivial, the proof is complete. □

Corollary 2.3. *If the boundary of* Ω *is* $C^{2,1}$ *and if* $\overline{\Gamma_0} \cap \overline{\Gamma_1} = \emptyset$, *then*

$$D\left(\mathcal{A}^2\right) = \left(H^3(\Omega) \cap H^1_{\Gamma_0}(\Omega)\right) \times \left(H^2(\Omega) \cap H^1_{\Gamma_0}(\Omega)\right) \times \left(H^3(\Gamma_1) \cap H^1_0(\Gamma_1)\right) \times \left(H^2(\Gamma_1) \cap H^1_0(\Gamma_1)\right),$$

with

$$\|(u, v, w, z)\|_{D(\mathcal{A}^2)} \sim \|u\|_{3,\Omega} + \|v\|_{2,\Omega} + \|w\|_{3,\Gamma_1} + \|z\|_{2,\Gamma_1}, \quad \textit{for all} \quad (u, v, w, z) \in D\left(\mathcal{A}^2\right).$$

Proof. $U = (u, v, w, z)$ belongs to $D(\mathcal{A}^2)$ if and only if $U \in D(\mathcal{A})$ and $\mathcal{A}U \in D(\mathcal{A})$. Hence by the previous result we will have

$$\Delta u \in H^1(\Omega),$$

and $h = \Delta_T - \partial_\nu u - z \in H_0^1(\Gamma_1)$. As the previous characterization yields $u \in H^2(\Omega)$, we know that $\partial_\nu u$ belongs to $H^{\frac{1}{2}}(\Gamma_1)$ and coming back to (2.11), we deduce that

$$\Delta_T w = h + \partial_\nu u + z \in H^{\frac{1}{2}}(\Gamma_1).$$

A shift theorem will lead to $w \in H^{\frac{5}{2}}(\Gamma_1)$. Then coming back to (2.13), the improved regularity on Δu and w, combined with a shift theorem give $u \in H^3(\Omega)$. Again coming back to (2.11), we deduce that $\Delta_T w = h + \partial_\nu u + z \in H^1(\Gamma_1)$, and therefore $w \in H^3(\Gamma_1)$.

This proves the result (for brevity we have skipped the estimates). □

Proposition 2.4. *If Ω is the unit square with $\Gamma_1 = \{(0, y),\ y \in (0, 1)\}$, and $\Gamma_0 = \Gamma \backslash \overline{\Gamma_1}$, then the statements of Proposition 2.2 and Corollary 2.3 are valid.*

Proof. The difficulty stays on the fact that Ω has a nonmooth boundary and that $\overline{\Gamma_0} \cap \overline{\Gamma_1}$ is not empty. But we take advantage of the particular geometry.

Let us start with the characterization of $D(\mathcal{A})$. Let $U = (u, v, w, z)$ be in $D(\mathcal{A})$. Then by a localization argument and Proposition 2.2, we directly see that u (resp. w) belongs to $H^2(\Omega \setminus W)$ (resp. $H^2(\Gamma_1 \setminus W)$), where W is any neighborhood of the corners. Hence it remains to improve the regularity of u and w near the corners. But in a small neigborhood V of the corner $(1, 0)$ (or $(1, 1)$), as u is solution of a homogeneous Dirichlet problem with $\Delta u \in L^2$, it is well known (see Theorem 3.2.1.2 of [18] for instance) that $u \in H^2(V)$. Hence the main difficulty is to show the regularity of u and w in a neighborhood V of the corner $(0, 0)$ (or $(0, 1)$). By symmetry, it suffices to look at the case of the corner $(0, 0)$. Now fix a cut-off function $\eta \in \mathcal{D}(\mathbb{R}^2)$ such that $\eta = 1$ in the disc of center $(0, 0)$ and radius $1/4$ and equal to 0 outside the disc of center $(0, 0)$ and radius $1/2$. Then we easily check that ηU belongs to $D(\mathcal{A}_0)$, the operator $\mathcal{A}_0$ being our operator $\mathcal{A}$ but defined in the quarter plane $Q = \{(x, y) \in \mathbb{R}^2; x, y > 0\}$, with $\Gamma_1 = \{(0, y) \in \mathbb{R}^2; y > 0\}$ and $\Gamma_0 = \{(x, 0) \in \mathbb{R}^2; x > 0\}$.

Now the first statement holds if we show that

$$D(\mathcal{A}_0) \subset H^2(Q) \times H^1(Q) \times H^2(\Gamma_1) \times H_0^1(\Gamma_1). \tag{2.16}$$

For that purpose, we use a reflexion technique. Let us fix $(u, v, w, z) \in D(\mathcal{A}_0)$ and introduce the function

$$\tilde{u}(x, y) = \begin{cases} u(x, y) & \text{if } y > 0, \\ -u(x, -y) & \text{if } y < 0, \end{cases}$$

defined in the half-plane $\mathbb{R}^2_+ := \{(x, y) \in \mathbb{R}^2 : x > 0\}$, and similarly

$$\tilde{w}(y) = \begin{cases} w(y) & \text{if } y > 0, \\ -w(-y) & \text{if } y < 0, \end{cases}$$

defined in the line $\{(0, y) \in \mathbb{R}^2; y \in \mathbb{R}\}$.

Now we denote by $\tilde{\mathcal{A}}_0$ our operator $\mathcal{A}$ but defined in the half-plane $\mathbb{R}^2_+$, with $\Gamma_0 = \emptyset$. Then denoting by $\tilde{\Gamma}_1 = \{(0, y) \in \mathbb{R}^2; y \in \mathbb{R}\}$, by Proposition 2.2, it is clear that

$$D(\tilde{\mathcal{A}}_0) = H^2(\mathbb{R}^2_+) \times H^1(\mathbb{R}^2_+) \times H^2(\tilde{\Gamma}_1) \times H^1(\tilde{\Gamma}_1).$$

Hence (2.16) holds if we can show that $(\tilde{u}, \tilde{v}, \tilde{w}, \tilde{z})$ belongs to $D(\tilde{\mathcal{A}}_0)$. The only non trivial properties are to check that $\Delta\tilde{u}$ belongs to $L^2(\mathbb{R}^2_+)$ and that $\tilde{w}_{yy} - \partial_\nu \tilde{u}$ belongs to $L^2(\Gamma_1)$.

For the first assertion, we show that

$$\Delta\tilde{u}(x, y) = \begin{cases} \Delta u(x, y) & \text{if } y > 0, \\ -\Delta u(x, -y) & \text{if } y < 0. \end{cases} \tag{2.17}$$

Indeed, we take $\varphi \in \mathcal{D}\big(\mathbb{R}^2_+\big)$, we clearly have

$$\langle \Delta\tilde{u}, \varphi\rangle = \int_Q u\Delta d_\varphi,$$

where $d_\varphi \in H^1_{\Gamma_0}(Q)$ is defined by

$$d_\varphi(x, y) = \varphi(x, y) - \varphi(x, -y), \quad \text{for all} \quad (x, y) \in Q.$$

Since d_φ is zero in a neighborhood of $(0, 0)$, we can apply Theorem 1.5.3.6 of [18] and deduce that

$$\langle \Delta\tilde{u}, \varphi\rangle = \int_Q \Delta u d_\varphi,$$

and (2.17) follows.

Similarly, we show that

$$\tilde{w}_{yy}(y) = \begin{cases} w_{yy}(y) & \text{if } y > 0, \\ -w_{yy}(-y) & \text{if } y < 0. \end{cases} \tag{2.18}$$

Finally for any

$$v \in H^1\big(\mathbb{R}^2_+\big),$$

we have

$$\langle \partial_\nu \tilde{u}, v\rangle = \int_{\mathbb{R}^2_+} (\Delta\tilde{u}v + \nabla\tilde{u} \cdot \nabla v).$$

Hence by the previous argument, we have

$$\langle \partial_\nu \tilde{u}, v\rangle = \int_Q (\Delta u d_v + \nabla\, u \cdot \nabla d_v),$$

where $d_v \in H^1_{\Gamma_0}(Q)$. Hence by the definition of $\partial_\nu u$, we deduce that

$$\langle \partial_\nu \tilde{u}, v\rangle = \langle \partial_\nu u, d_v\rangle,$$

which means that

$$\partial_\nu \tilde{u}(0, y) = \begin{cases} \partial_\nu u(0, y) & \text{if } y > 0, \\ -\partial_\nu u(0, -y) & \text{if } y < 0. \end{cases} \tag{2.19}$$

For the second assertion, if we denote by $h = w_{yy} - \partial_\nu u$ that by assumption belongs to $L^2(\Gamma_1)$, then (2.18) and (2.19) imply that

$$(\tilde{w}_{yy} - \partial_\nu \tilde{u})(y) = \begin{cases} h(y) & \text{if } y > 0, \\ -h(-y) & \text{if } y < 0, \end{cases} \tag{2.20}$$

and consequently it belongs to $L^2\big(\tilde{\Gamma}_1\big)$ as well.

For the characterization of $D\big(\mathcal{A}^2\big)$, it suffices to notice that for $(u, v, w, z) \in D\big(\mathcal{A}^2_0\big)$, then

$$\Delta u \in H^1_{\Gamma_0}(Q).$$

In a neighborhood of the corner $(0, 0)$, we first notice that $\Delta\tilde{u}$ given by (2.17) belongs to $H^1\big(\mathbb{R}^2_+\big)$. Similarly $h = w_{yy} - \partial_\nu u$ belongs to $H^1_0(\Gamma_1)$, and hence $\tilde{w}_{yy} - \partial_\nu \tilde{u}$ given by (2.20) belongs to $H^1\big(\tilde{\Gamma}_1\big)$. This means that $(\tilde{u}, \tilde{v}, \tilde{w}, \tilde{z})$ belongs to $D\big(\tilde{\mathcal{A}}^2_0\big)$ and we conclude by Corollary 2.3.

In a neighborhood of the corners $(1, 0)$ or $(1, 1)$, we simply use the same reflexion technique as before (see Lemma 2.4 of [20]) to get the H^3 regularity of u. □

Now we investigate the strong stability of system (2.9). But before going on, if Γ_0 is empty, we need to introduce the closed subspace

$$\mathcal{H}_0 = \left\{ (u, v, w, z) \in \mathcal{H} \, : \, \int_\Omega v \, dx + \int_{\Gamma_1} z \, d\Gamma + \int_{\Gamma_1} w \, d\Gamma = 0 \right\}$$

of $\mathcal{H}$ and the restriction $\mathcal{B}$ of $\mathcal{A}$ to $\mathcal{H}_0$, defined by $D(\mathcal{B}) = D(\mathcal{A}) \cap \mathcal{H}_0$, and

$$\mathcal{B}U = \mathcal{A}U, \quad \text{for all} \quad U \in D(\mathcal{B}).$$

Note that this definition is meaningful because for all $U \in D(\mathcal{A})$, by some integrations by parts, it is easily checked that $\mathcal{A}U$ belongs to $\mathcal{H}_0$. Hence $\mathcal{B}$ also generates a C_0-semigroup of contractions that is simply the restriction of $\left(e^{t\mathcal{A}}\right)_{t\geq 0}$ to $\mathcal{H}_0$.

Let us also notice that the semi-norm $\|\cdot\|_{\mathcal{H}}$ is actually a norm in $\mathcal{H}_0$. By using the compact embedding of $H^1(\Omega)$ (resp. $H^1(\Gamma)$) into $L^2(\Omega)$ (resp. $L^2(\Gamma)$), $\|\cdot\|_{\mathcal{H}}$ is even equivalent to the norm $\|\cdot\|$.

Theorem 2.5. *If Γ_0 is non empty, then the semigroup of contractions $\left(e^{t\mathcal{A}}\right)_{t\geq 0}$ is strongly stable on the energy space $\mathcal{H}$, i.e., for any $U_0 \in \mathcal{H}$, we have*

$$\lim_{t\to+\infty} \left\| e^{t\mathcal{A}} U_0 \right\|_{\mathcal{H}} = 0. \tag{2.21}$$

If $\Gamma_0 = \emptyset$, then the semigroup of contractions $\left(e^{t\mathcal{A}}\right)_{t\geq 0}$ is strongly stable on the space $\mathcal{H}_0$. Further, for any $U_0 = (u_0, v_0, w_0, z_0) \in \mathcal{H}$ if $\alpha = \frac{1}{|\Gamma_1|}\left(\int_\Omega v_0 \, dx + \int_{\Gamma_1} z_0 \, d\Gamma + \int_{\Gamma_1} w_0 \, d\Gamma \right)$ (where $|\Gamma_1|$ means the measure of Γ_1), then

$$\lim_{t\to+\infty} \left\| e^{t\mathcal{A}} U_0 - \alpha(1,0,1,0) \right\| = 0, \tag{2.22}$$

as well as

$$\lim_{t\to+\infty} \left\| e^{t\mathcal{A}} U_0 \right\|_{\mathcal{H}} = 0. \tag{2.23}$$

To prove the theorem above, we apply the strategy used in [31]. First we need to prove the following two lemmas:

Lemma 2.6. *For all $\lambda \in \mathbb{R}^*$, we have*

$$\ker(i\lambda I - \mathcal{A}) = \{0\},$$

while

$$\ker \mathcal{A} = \{0\},$$

if Γ_0 is non empty, and

$$\ker \mathcal{A} = \text{Span}\,\{(1,0,1,0)\},$$

if Γ_0 is empty, but

$$\ker(i\lambda I - \mathcal{B}) = \{0\}, \quad \textit{for all} \quad \lambda \in \mathbb{R}. \tag{2.24}$$

Proof. Let $U = (u, v, w, z) \in D(\mathcal{A})$ and let $\lambda \in \mathbb{R}$ be such that

$$\mathcal{A}U = i\lambda U. \tag{2.25}$$

First, by detailing (2.25) we get

$$\begin{cases} v = i\lambda u, \\ \Delta u = i\lambda v, \\ z = i\lambda w, \\ \Delta_T w - \dfrac{\partial u}{\partial \nu} - z = i\lambda z. \end{cases} \tag{2.26}$$

Next, a straightforward computation gives

$$\Re(\mathcal{A}U, U)_{\mathcal{H}} = -\int_{\Gamma_1} |z|^2 d\Gamma. \tag{2.27}$$

Then, using (2.25) and (2.27) we deduce that

$$z = 0, \quad \text{on} \quad \Gamma_1. \tag{2.28}$$

Now we distinguish two cases:

Case 1 ($\lambda \neq 0$). Using (2.28) and the third equation of system (2.26), we deduce that $u = w = 0$ on Γ_1. Thus, by eliminating v, the system (2.26) implies that

$$\begin{cases} \Delta u + \lambda^2 u = 0, & \text{in} \quad \Omega, \\ u = 0, & \text{on} \quad \Gamma_1, \\ \partial_\nu u = 0, & \text{on} \quad \Gamma_1. \end{cases} \tag{2.29}$$

Therefore, using Holmgren's theorem, we deduce that $u = 0$ and consequently, $U = 0$.

Case 2 ($\lambda = 0$). The system (2.26) becomes

$$\begin{cases} v = 0, & \text{in} \quad \Omega, \\ \Delta u = 0, & \text{in} \quad \Omega, \\ z = 0, & \text{in} \quad \Gamma_1, \\ \Delta_T w - \partial_\nu u = 0, & \text{on} \quad \Gamma_1. \end{cases} \tag{2.30}$$

By integrating by parts and using the boundary conditions $u = 0$ on Γ_0 and $w = 0$ on $\partial\Gamma_1$, we have

$$0 = \int_\Omega \Delta u\bar{u} = -\int_\Omega |\nabla u|^2 + \int_{\Gamma_1} \partial_\nu u\bar{u} = -\int_\Omega |\nabla u|^2 - \int_{\Gamma_1} |\nabla_T u|^2.$$

Hence u is constant in the whole domain Ω. Therefore if Γ_0 is non empty we deduce that $u = w = 0$ and directly conclude that $\ker(i\lambda I - \mathcal{A}) = \{0\}$. On the other hand, if Γ_0 is empty, then $u = w$ constant is allowed and we find that $(1, 0, 1, 0)$ is the sole eigenvector of $\mathcal{A}$ of eigenvalue 0. But since $(1, 0, 1, 0)$ does not belong to $\mathcal{H}_0$, 0 is not an eigenvalue of $\mathcal{B}$ and consequently we deduce that (2.24) holds. □

Lemma 2.7. *If* $\Gamma_0 \neq \emptyset$, *for all* $\lambda \in \mathbb{R}$, *we have*

$$\mathrm{R}(i\lambda I - \mathcal{A}) = \mathcal{H},$$

while if $\Gamma_0 = \emptyset$, *for all* $\lambda \in \mathbb{R}$, *we have*

$$\mathrm{R}(i\lambda I - \mathcal{B}) = \mathcal{H}_0.$$

Proof. We give the proof in the case $\Gamma_0 \neq \emptyset$, the proof of the second statement is fully similar by using (2.24). Let $\lambda \in \mathbb{R}$ and $F = (f, g, h, k) \in \mathcal{H}$, then we look for $U = (u, v, w, z) \in D(\mathcal{A})$ such that

$$i\lambda U - \mathcal{A}U = F, \tag{2.31}$$

or equivalently

$$\begin{cases} i\lambda u - v = f, & \text{in} \quad \Omega, \\ i\lambda v - \Delta u = g, & \text{in} \quad \Omega, \\ i\lambda w - z = h, & \text{on} \quad \Gamma_1, \\ i\lambda z - \Delta_T w + \partial_\nu u + z = k, & \text{on} \quad \Gamma_1. \end{cases} \tag{2.32}$$

From the first and the third identities of (2.32) and the fact that $w = u$ on Γ_1, we get

$$\begin{cases} -\Delta u - \lambda^2 u = g + i\lambda f, & \text{in} \quad \Omega, \\ -\lambda^2 u - \Delta_T u + \partial_\nu u + i\lambda u = k + (i\lambda - 1)h, & \text{on} \quad \Gamma_1. \end{cases} \tag{2.33}$$

Next, define the space V by

$$V = \left\{ u \in H^1_{\Gamma_0}(\Omega) : u \in H^1_0(\Gamma_1) \right\}$$

endowed with the norm

$$\|u\|_V^2 = \|\nabla u\|_\Omega^2 + \|\nabla_T u\|_{\Gamma_1}^2.$$

Multiplying the first equation of (2.33) by $\tilde{u} \in V$, integrating in Ω and using the second equation of the same problem, and formal integration by parts, we get formally the following identity:

$$a_\lambda(u, \tilde{u}) = L_\lambda(\tilde{u}), \tag{2.34}$$

where a_λ is the sesquilinear form from $V \times V$ into $\mathbb{C} \times \mathbb{C}$ given by

$$a_\lambda(u, \tilde{u}) = \int_\Omega (\nabla u \cdot \nabla \tilde{u} - \lambda^2 u\tilde{u})\, dx + \int_{\Gamma_1} \left(\nabla_T u \cdot \nabla_T \tilde{u} + \left(i\lambda - \lambda^2\right) u\tilde{u}\right) d\Gamma, \tag{2.35}$$

and L_λ is the linear form from V into $\mathbb{C}$ defined by

$$L_\lambda(\tilde{u}) = \int_\Omega (g + i\lambda f)\tilde{u}\, dx + \int_{\Gamma_1} (k + (i\lambda - 1)h)\tilde{u}\, d\Gamma. \tag{2.36}$$

Now, we introduce the operator $\mathcal{A}_\lambda : V \to V'$ by

$$\langle \mathcal{A}_\lambda u, \tilde{u} \rangle_{V',V} = a_\lambda(u, \tilde{u}), \quad \text{for all} \quad \tilde{u} \in V.$$

For λ, $\lambda' \in \mathbb{R}$, we have

$$\begin{aligned} \left|\langle (\mathcal{A}_\lambda - \mathcal{A}_{\lambda'})u, \tilde{u} \rangle_{V',V}\right| &= \left|a_\lambda(u, \tilde{u}) - a_{\lambda'}(u, \tilde{u})\right| \\ &= \left| \int_\Omega \left(\lambda'^2 - \lambda^2\right) u\tilde{u}\, dx + \int_{\Gamma_1} \left(i(\lambda - \lambda') + \left(\lambda'^2 - \lambda^2\right) \right) u\tilde{u}\, d\Gamma \right| \\ &\leq C_{\lambda,\lambda',\Omega} \|u\|_V \left(\|\tilde{u}\|_{L^2(\Omega)} + \|\tilde{u}\|_{L^2(\Gamma)} \right) \\ &\leq C_{\lambda,\lambda',\Omega} \|u\|_V \|\tilde{u}\|_{H^{1/2+\epsilon}_{\Gamma_0}(\Omega)}. \end{aligned}$$

This implies that

$$\mathcal{A}_\lambda - \mathcal{A}_{\lambda'} \in \mathcal{L}\left(V; H^{1/2+\epsilon}_{\Gamma_0}(\Omega)' \right)$$

and thus $\mathcal{A}_\lambda - \mathcal{A}_{\lambda'}$ is a compact operator from V into V'. On the other hand, since $\Gamma_0 \neq \emptyset$, then, it is easy to see that, the operator $\mathcal{A}_0$ is an isomorphism and consequently, it is a Fredholm operator of index zero. It follows, from the compactness of $\mathcal{A}_\lambda - \mathcal{A}_{\lambda'}$, that $\mathcal{A}_\lambda$ is also a Fredholm operator of index zero for all λ. Therefore, $\mathcal{A}_\lambda$ is surjective if and only if it is injective. Using Lemma

2.6, we deduce the injectivity of the operator $\mathcal{A}_\lambda$ (compare with Proposition 3.3 in [31]). This means that $\mathcal{A}_\lambda$ is an isomorphism for all $\lambda \in \mathbb{R}$ and therefore problem (2.35) has a unique solution $u \in V$. By choosing appropriated test functions in (2.35), we see that u satisfies (2.33). By defining $w = u$, $z = i\lambda w - h$ on Γ_1 and $v = i\lambda u - f$ in Ω, we deduce that $U = (u, v, w, z)$ belongs to $D(\mathcal{A})$ and is solution of (2.31). This completes the proof. □

Proof of Theorem 2.5. We distinguish two cases:

Case 1 ($\Gamma_0 \neq \emptyset$). Using Lemmas 2.6 and 2.7, we directly deduce that the imaginary axis is included in the resolvent set of $\mathcal{A}$. We then conclude with the help of Arendt-Batty's theorem [4].

Case 2 ($\Gamma_0 = \emptyset$). As before using Lemmas 2.6 and 2.7 and Arendt–Batty's theorem, we conclude that the semigroup generated by $\mathcal{B}$ is stable, in other words

$$\lim_{t\to+\infty} \left\| e^{t\mathcal{B}} \widetilde{U}_0 \right\|_{\mathcal{H}} = 0, \quad \text{for all} \quad \widetilde{U}_0 \in \mathcal{H}_0.$$

But, for $U_0 \in \mathcal{H}$ and α given as in the second statement of Theorem 2.5, we notice that

$$\widetilde{U}_0 := U_0 - \alpha(1, 0, 1, 0)$$

belongs to $\mathcal{H}_0$. The conclusion then follows by noticing that $e^{t\mathcal{A}}(1, 0, 1, 0) = (1, 0, 1, 0)$. The proof is thus completed (compare with Theorem 4.3.2 of [12]). □

3 | A NON-UNIFORM STABILITY RESULT

In this section we show that uniform stability (i.e., exponential stability) does not hold in general, since it is already the case for the unit disc D of $\mathbb{R}^2$ and $\Gamma_0 = \emptyset$ as shown below. This result is due to the fact that a subsequence of eigenvalues of $\mathcal{A}$ is close to the imaginary axis. First, let $\lambda \in \mathbb{C}$ and $U = (u, v, w, z) \in D(\mathcal{A})$ be such that $\mathcal{A}U = \lambda U$. Equivalently we have

$$\begin{cases} v = \lambda u, & \text{in} \quad D, \\ \Delta u = \lambda v, & \text{in} \quad D, \\ z = \lambda w, & \text{on} \quad \partial D, \\ \Delta_T w - \partial_\nu u - z = \lambda z & \text{on} \quad \partial D. \end{cases}$$

As before, by eliminating v and z from the above system and using the fact that $u = w$ on Γ_1 we get the following system:

$$\begin{cases} \Delta u - \lambda^2 u = 0, & \text{in} \quad D, \\ \Delta_T u - \partial_\nu u - \lambda(\lambda + 1)u = 0, & \text{on} \quad \partial D. \end{cases} \tag{3.1}$$

A radial solution $u(r, \theta) = U(r)$ of (3.1) is a solution of

$$\begin{cases} U''(r) + \dfrac{1}{r} U'(r) - \lambda^2 U(r) = 0, & r \in (0, 1), \\ [6pt] U'(1) + (\lambda^2 + \lambda) U(1) = 0. \end{cases} \tag{3.2}$$

If $\lambda \neq 0$, the general solution of the differential equation of (3.2) is given by

$$f(r) = c_J J_0(i\lambda r) + c_Y Y_0(i\lambda r), \quad \text{with} \;\; c_J, c_Y \in \mathbb{C},$$

where J_0 (resp. Y_0) is the Bessel function of the first (resp. second) kind of order 0. Since u is regular in D, necessarily we have $c_Y = 0$ and $c_J \neq 0$. Therefore, using the second equation of (3.2), we find that if $\lambda \in \mathbb{C} \setminus \{0\}$ satisfies

$$-i\lambda J_1(i\lambda) + \left(\lambda^2 + \lambda\right) J_0(i\lambda) = 0, \tag{3.3}$$

then λ is an eigenvalue of $\mathcal{A}$ (recalling that $J_1 = -J_0'$ is the Bessel function of the first kind of order 1). Our goal is to find large eigenvalues which are close to the imaginary axis and to give their asymptotic expansion. For that reason, we fix $c > 0$ large enough and consider the solution of (3.3) which are in the strip

$$S = \{\lambda \in \mathbb{C}; -c \leq \Re\lambda \leq c\}.$$

For convenience, we set $\phi(\lambda) = \frac{1}{\lambda^2}\sqrt{\frac{i\pi\lambda}{2}}\left(-i\lambda J_1(i\lambda) + (\lambda^2+\lambda)J_0(i\lambda)\right)$, and notice that (3.3) is equivalent to

$$\phi(\lambda) = 0. \tag{3.4}$$

In the following proposition we give the asymptotic behavior of some high frequency eigenvalues corresponding to radial solutions of problem (3.1).

Proposition 3.1. *There exist $k_0 \in \mathbb{N}^*$ and a sequence $(\lambda_k)_{k\geq k_0}$ of simple roots of ϕ (that are also simple eigenvalues of $\mathcal{A}$) and satisfying the following asymptotic behavior:*

$$\lambda_k = i\left(k\pi - \frac{\pi}{4} + \frac{9}{8k\pi} + \frac{9}{32k^2\pi^2}\right) - \frac{1}{k^2\pi^2} + o\left(\frac{1}{k^2}\right), \tag{3.5}$$

for k large enough.

Proof. For clarity, the proof is divided into two steps.

Step 1. First, using the asymptotic expansions of Bessel's functions (see formula 10.7.3 of [32] for instance), we have

$$\sqrt{\frac{i\pi\lambda}{2}}J_0(i\lambda) = i\cos\left(\frac{\pi}{4}+i\lambda\right) + \frac{1}{8\lambda}\cos\left(\frac{\pi}{4}-i\lambda\right) + \frac{9i}{128\lambda^2}\cos\left(\frac{\pi}{4}+i\lambda\right) + O\left(\frac{1}{|\lambda|^3}\right), \quad \text{as } |\lambda| \to \infty, \tag{3.6}$$

and

$$\sqrt{\frac{i\pi\lambda}{2}}(-i\lambda J_1(i\lambda)) = \lambda\cos\left(\frac{\pi}{4}-i\lambda\right) - \frac{3}{8}i\cos\left(\frac{\pi}{4}+i\lambda\right) + O\left(\frac{1}{|\lambda|}\right), \quad \text{as } |\lambda| \to \infty. \tag{3.7}$$

From (3.6) and (3.7) it follows that for $\lambda \in S$ with $|\lambda|$ large enough, we have

$$\begin{aligned}\phi(\lambda) = {} & i\cos\left(\frac{\pi}{4}+i\lambda\right) + \left[\left(\frac{9}{8}\cos\left(\frac{\pi}{4}-i\lambda\right) + i\cos\left(\frac{\pi}{4}+i\lambda\right)\right)\right]\frac{1}{\lambda} \\ & + \left[-\frac{39}{128}i\cos\left(\frac{\pi}{4}+i\lambda\right) + \frac{1}{8}\cos\left(\frac{\pi}{4}-i\lambda\right)\right]\frac{1}{\lambda^2} + O\left(\frac{1}{|\lambda|^3}\right).\end{aligned}$$

Since the roots of the analytic function $\lambda \mapsto \cos(\frac{\pi}{4}+i\lambda)$ are $\lambda_k^0 = ik\pi - i\frac{\pi}{4}$, for any $k \in \mathbb{Z}$, using Rouché's theorem, we deduce that ϕ admits an infinity of simple roots in S denoted by λ_k, with $|k| \geq k_0$, for k_0 large enough, such that

$$\lambda_k = \lambda_k^0 + o(1) = ik\pi - i\frac{\pi}{4} + o(1), \quad \text{as } |k| \to \infty.$$

Equivalently we have

$$\lambda_k = ik\pi - i\frac{\pi}{4} + \epsilon_k \quad \text{with} \quad \lim_{|k|\to\infty} \epsilon_k = 0. \tag{3.8}$$

Step 2. **Asymptotic behavior of ϵ_k**: First, using (3.8) we obtain

$$\cos\left(\frac{\pi}{4}+i\lambda_k\right) = -i(-1)^k\epsilon_k + o(\epsilon_k), \tag{3.9}$$

$$\cos\left(\frac{\pi}{4}-i\lambda_k\right) = (-1)^k + O\left(\epsilon_k^2\right), \tag{3.10}$$

and

$$\frac{1}{\lambda_k} = -\frac{i}{k\pi} + o\left(\frac{1}{k}\right). \tag{3.11}$$

Next, by inserting (3.9)–(3.11) in the identity $\phi(\lambda_k) = 0$ and keeping only the terms of order $\frac{1}{k}$, we find after a simplification

$$(-1)^k \epsilon_k + o(\epsilon_k) - \frac{9i(-1)^k}{8k\pi} + o\left(\frac{1}{k}\right) = 0,$$

and thus

$$\epsilon_k = \frac{9i}{8k\pi} + o\left(\frac{1}{k}\right).$$

Later, from the above identity we can write $\lambda_k = ik\pi - i\frac{\pi}{4} + \frac{9i}{8k\pi} + \frac{\tilde{\epsilon}_k}{k}$, with $\lim_{|k|\to\infty} \tilde{\epsilon}_k = 0$. That implies

$$\cos\left(\frac{\pi}{4} + i\lambda_k\right) = \frac{9(-1)^k}{8k\pi} - \frac{i(-1)^k\tilde{\epsilon}_k}{k} + o\left(\frac{1}{k^2}\right), \tag{3.12}$$

$$\cos\left(\frac{\pi}{4} - i\lambda_k\right) = (-1)^k - (-1)^k\frac{81}{128k^2\pi^2} + o\left(\frac{1}{k^2}\right), \tag{3.13}$$

$$\frac{1}{\lambda_k} = -\frac{i}{k\pi} - \frac{i}{4k^2\pi} + o\left(\frac{1}{k^2}\right), \tag{3.14}$$

and

$$\frac{1}{\lambda_k^2} = -\frac{1}{k^2\pi^2} + o\left(\frac{1}{k^2}\right). \tag{3.15}$$

Inserting (3.12)–(3.15) in the equation $\phi(\lambda_k) = 0$ and keeping only the terms of order $\frac{1}{k^2}$, we find after simplifications

$$\frac{(-1)^k\tilde{\epsilon}_k}{k} + \frac{(-1)^k}{k^2\pi^2} - \frac{9i(-1)^k}{32k^2\pi^2} + o\left(\frac{1}{k^2}\right) = 0,$$

thus

$$\tilde{\epsilon}_k = \frac{9i}{32k\pi^2} - \frac{1}{k\pi^2} + o\left(\frac{1}{k}\right).$$

This expression in the identity $\lambda_k = ik\pi - i\frac{\pi}{4} + \frac{9i}{8k\pi} + \frac{\tilde{\epsilon}_k}{k}$ leads to (3.5). □

As (3.5) shows that the eigenvalues λ_k of $\mathcal{A}$ approach the imaginary axis as k goes to infinity, clearly system (2.9) in the unit disc is not uniformly stable.

The asymptotic behavior of λ_k in (3.5) can be numerically validated, namely from (3.5) we have

$$-\lim_{k\to+\infty} k^2\pi^2\Re(\lambda_k) = 1.$$

The table below confirms numerically this behavior.

k	100	150	200	250	300	350	400	450	500
$-\pi^2k^2\Re(\lambda_k)$	1.00495	1.00331	1.00249	1.00199	1.00166	1.00142	1.00125	1.00111	1.001

In addition, Figure 1 represents some of these eigenvalues.

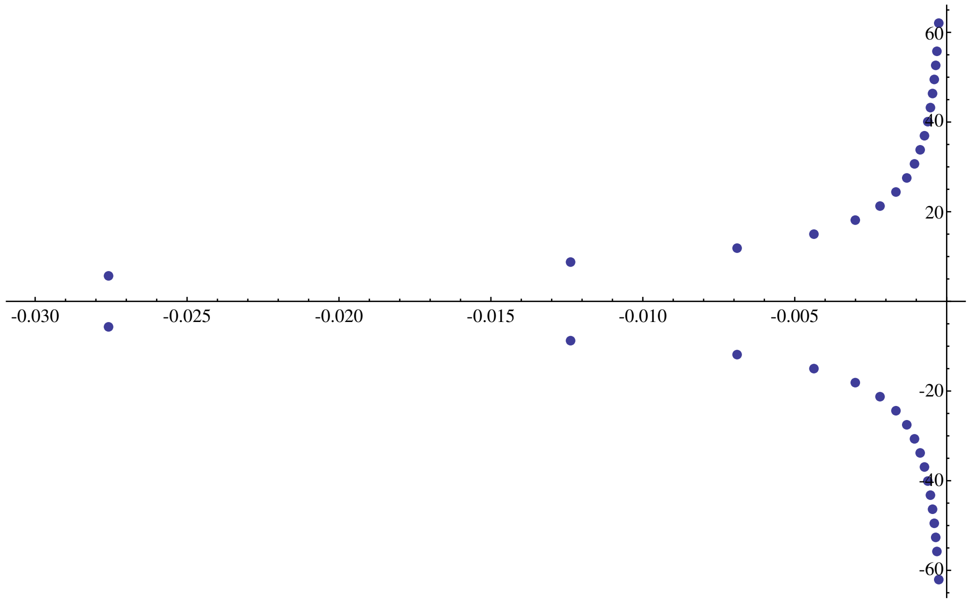

FIGURE 1 Some eigenvalues of $\mathcal{A}$ for the unit disc

4 | POLYNOMIAL ENERGY DECAY RATE

In this section we study the polynomial decay rate of the energy of problem (2.9) under appropriated conditions. For that purpose, we will use a frequency domain approach, namely we will use Theorem 2.4 of [9] (see also [6,7,26]) that we partially recall.

Theorem 4.1. *Let $(T(t))_{t\geq 0}$ be a bounded C_0-semigroup on a Hilbert space H with generator A such that $i\mathbb{R} \subset \rho(A)$. Then for a fixed $\ell > 0$ the following conditions are equivalent*

$$\left\|(is - A)^{-1}\right\| = O\left(|s|^{\ell}\right), \quad s \to \infty, \tag{4.1}$$

$$\left\|T(t)A^{-1}\right\| = O\left(t^{-1/\ell}\right), \quad t \to \infty. \tag{4.2}$$

As the condition $i\mathbb{R} \subset \rho(A)$ was already checked in Theorem 2.5, it remains to prove that condition (4.1) holds. This is made under two different assumptions. For the first one, we consider the following auxiliary problem, namely the wave equation with a standard boundary damping on Γ_1:

$$\begin{cases} \varphi_{tt}(x,t) - \Delta\varphi(x,t) = 0, & x \in \Omega, \quad t > 0, \\ \varphi(x,t) = 0, & x \in \Gamma_0, \ t > 0, \\ \partial_\nu \varphi(x,t) = -\varphi_t(x,t), & x \in \Gamma_1, \ t > 0. \end{cases} \tag{4.3}$$

First, we introduce the following condition:

$$(H): \ \text{the problem (4.3) is uniformly stable in the energy space } H^1_{\Gamma_0}(\Omega) \times L^2(\Omega),$$

or equivalently there exist two positive constants C and ω such that for any $(\varphi_0, \varphi_1) \in H^1_{\Gamma_0}(\Omega) \times L^2(\Omega)$, the solution φ of (4.3) with initial conditions

$$\varphi(\cdot, 0) = \varphi_0, \quad \varphi_t(\cdot, 0) = \varphi_1,$$

satisfies

$$|\varphi(\cdot,t)|^2_{1,\Omega} + \|\varphi_t(\cdot,t)\|^2_\Omega \leq Ce^{-\omega t}\left(|\varphi_0|^2_{1,\Omega} + \|\varphi_1\|^2_\Omega\right), \quad \text{for all} \quad t \geq 0.$$

Alternatively, we recall the multiplier control condition **MCC** in the following definition:

Definition 4.2. We say that the multiplier control condition **MCC** holds if there exist $x_0 \in \mathbb{R}^d$ and a positive constant $m_0 > 0$ such that

$$m \cdot \nu \leq 0 \text{ on } \Gamma_0 \quad \text{and} \quad m \cdot \nu \geq m_0 \text{ on } \Gamma_1,$$

with $m(x) = x - x_0$, for all $x \in \mathbb{R}^d$.

Remark 4.3. In [5], Bardos et al. proved that (H) holds if Γ is smooth (of class C^∞), $\overline{\Gamma_0} \cap \overline{\Gamma_1} = \emptyset$ and under a geometric control condition, called **GCC** (namely, the geometric control condition **GCC** holds if there exists $T > 0$ such that every ray of geometrical optics starting at any point $x \in \Omega$ at time $t = 0$, hits Γ_1 in the finite time T). For less regular domains, namely of class C^2, (H) holds if the vector field assumptions described in [23] (see (i), (ii), (iii) of Theorem 1 in [23]) hold. Moreover, in Theorem 1.2 of [24] the authors prove that (H) holds for smooth domains under weaker geometric conditions than in [23] (without (ii) of Theorem 1). It is easy to see that the multiplier control condition **MCC** implies that the vector field assumptions described in [23] are satisfied and therefore the condition (H) holds if **MCC** holds.

Next, we present the main result of this section.

Theorem 4.4.

1. *Assume that $\overline{\Gamma_0} \cap \overline{\Gamma_1} = \emptyset$ and that the condition (H) holds. Then for all initial data $U_0 \in D(\mathcal{A})$, there exists a constant $c > 0$ independent of U_0, such that the solution of the problem* (2.9) *satisfies the following estimate*

$$E(t) \leq \frac{c}{t^{\frac{1}{4}}} \|U_0\|^2_{D(\mathcal{A})}, \quad \textit{for all} \quad t > 0. \tag{4.4}$$

2. *Assume that $\overline{\Gamma_0} \cap \overline{\Gamma_1} = \emptyset$, that the multiplier control condition* MCC *holds and that the boundary of Ω is $C^{1,1}$. Then for all initial data $U_0 \in D(\mathcal{A})$, there exists a constant $c > 0$ independent of U_0, such that the solution of problem* (2.9) *satisfies the following estimate*

$$E(t) \leq \frac{c}{t} \|U_0\|^2_{D(\mathcal{A})}, \quad \textit{for all} \quad t > 0. \tag{4.5}$$

Proof. As announced, the proof is based on Theorem 4.1. Owing to Theorem 2.5, we are then reduced to show that condition

$$\sup_{|\beta| \geq 1} \frac{1}{|\beta|^l} \left\| (i\beta I - \mathcal{A})^{-1} \right\|_{\mathcal{L}(\mathcal{H})} < +\infty \tag{4.6}$$

hold with $l = 8$ (respectively with $l = 2$). This is checked by using a contradiction argument. Indeed assume that it does not hold, then there exist a sequence $\beta_n \in \mathbb{R}, n \in \mathbb{N}$, such that $\beta_n \to +\infty$ as $n \to +\infty$, and a sequence $U_n = (u_n, v_n, w_n, z_n) \in D(\mathcal{A})$ such that

$$\|U_n\|_{\mathcal{H}} = 1, \tag{4.7}$$

$$\beta_n^l \|(i\beta_n I - \mathcal{A})U_n\|_{\mathcal{H}} \longrightarrow 0, \text{ as } n \to +\infty. \tag{4.8}$$

For simplificity, we replace β_n by β; $U_n = (u_n, v_n, w_n, z_n)$ by $U = (u, v, w, z)$ and $F_n = \beta_n^l (i\beta_n I - \mathcal{A})U_n = (f_{1,n}, f_{2,n}, f_{3,n}, f_{4,n})$ by $F = (f_1, f_2, f_3, f_4)$. Next, by detailing (4.8) we obtain

$$\begin{cases} \beta^l (i\beta u - v) = f_1 \longrightarrow 0 & \text{in } H^1_{\Gamma_0}(\Omega), \\ \beta^l (i\beta v - \Delta u) = f_2 \longrightarrow 0 & \text{in } L^2(\Omega), \\ \beta^l (i\beta w - z) = f_3 \longrightarrow 0 & \text{in } H^1_0(\Gamma_1), \\ \beta^l \left(i\beta z - \Delta_T w + \partial_\nu u + z \right) = f_4 \longrightarrow 0 & \text{in } L^2(\Gamma_1). \end{cases} \tag{4.9}$$

Later, by eliminating v and z from system (4.9) and since $u = w$ on Γ_1 we obtain

$$\begin{cases} \beta^2 u + \Delta u = -\dfrac{f_2 + i\beta f_1}{\beta^l} & \text{in} \quad \Omega, \\ \beta^2 u + \Delta_T u - \partial_\nu u - i\beta u = -\dfrac{f_4 + (1 + i\beta) f_3}{\beta^l} & \text{on} \quad \Gamma_1. \end{cases} \tag{4.10}$$

□

Lemma 4.5. *The solution $(u, v, w, z) \in D(\mathcal{A})$ of system (4.9) satisfies the following estimate*

$$\int_{\Gamma_1} |u|^2 d\Gamma = \frac{o(1)}{\beta^{l+2}}. \tag{4.11}$$

Proof. First, multiplying Equation (4.8) by U in $\mathcal{H}$, we get

$$\int_{\Gamma_1} |z|^2 d\Gamma = \Re(i\beta U - \mathcal{A}U, U)_{\mathcal{H}} = \frac{o(1)}{\beta^l}. \tag{4.12}$$

Next, using the third equation of system (4.9) and using (4.12), we get

$$\int_{\Gamma_1} |w|^2 d\Gamma = \frac{o(1)}{\beta^{l+2}}. \tag{4.13}$$

Finally, since $u = w$ on Γ_1, from (4.13) we deduce directly (4.11). □

Before going on, we give a relation between $\nabla_T u$ and $\partial_\nu u$.

Lemma 4.6. *Let $u \in H^1(\Omega)$ be such that $\Delta u \in L^2(\Omega)$. Then*

$$u \in H^1(\Gamma) \Longleftrightarrow \partial_\nu u \in L^2(\Gamma) \tag{4.14}$$

and in this case we have

$$\|\Delta u\|_\Omega + \|u\|_{1,\Gamma} \sim \|\partial_\nu u\|_\Gamma + \|\Delta u\|_\Omega. \tag{4.15}$$

Proof. First, we denote by $h = \Delta u$ and we set

$$\tilde{h} = \begin{cases} h & \text{in} \quad \Omega, \\ 0 & \text{in} \quad \mathbb{R}^d \backslash \Omega. \end{cases}$$

Moreover, we consider O a smooth domain such that $\overline{\Omega} \subset O$. Next, let $w \in H_0^1(O)$ be a solution of

$$\Delta w = \tilde{h} \quad \text{in} \quad O.$$

Then $w \in H^2(O)$ and we have

$$\|w\|_{2,O} \lesssim \left\|\tilde{h}\right\|_O \lesssim \|h\|_\Omega. \tag{4.16}$$

Consequently $v = u - w \in H^1(\Gamma)$ and satisfies $\Delta v = 0$ in Ω. On the other hand, using Lemma 1 of [11], we deduce that

$$v \in H^1(\Gamma) \Longleftrightarrow \partial_\nu v \in L^2(\Gamma) \tag{4.17}$$

and

$$\|v\|_{1,\Gamma} \sim \|\partial_\nu v\|_\Gamma. \tag{4.18}$$

As $u = v + w$ and $\partial_\nu u = \partial_\nu v + \partial_\nu w$ and since by (4.16) $w \in H^1(\Gamma)$ and $\partial_\nu w \in L^2(\Gamma)$, using (4.18) we deduce that

$$u \in H^1(\Gamma) \Longleftrightarrow \partial_\nu u \in L^2(\Gamma).$$

Now, to prove the estimates from (4.15), we notice that

$$\|u\|_{1,\Gamma} = \|v + w\|_{1,\Gamma} \leq \|v\|_{1,\Gamma} + \|w\|_{1,\Gamma}.$$

Hence, using (4.18) we get

$$\|u\|_{1,\Gamma} \lesssim \|\partial_\nu v\|_\Gamma + \|w\|_{1,\Gamma} \lesssim \|\partial_\nu u\|_\Gamma + \|\partial_\nu w\|_\Gamma + \|w\|_{1,\Gamma}.$$

Finally, by using a trace theorem and (4.16) we obtain

$$\|u\|_{1,\Gamma} \lesssim \|\partial_\nu u\|_\Gamma + \|w\|_{2,\Omega} \lesssim \|\partial_\nu u\|_\Gamma + \|h\|_\Omega.$$

The converse inequality is proved similarly. □

Lemma 4.7. *Assume that $l \geq 1$. Then the solution $(u, v, w, z) \in D(\mathcal{A})$ of system (4.9) satisfies the following estimate*

$$\int_{\Gamma_1} |\partial_\nu u|^2 d\Gamma = O\left(\beta^2\right). \tag{4.19}$$

Proof. First, since $u \in H_0^1(\Gamma_1)$, we have $u \in H^1(\Gamma)$. Next, as $\Delta u \in L^2(\Omega)$, using (4.15) we obtain

$$\|\partial_\nu u\|_\Gamma \lesssim \|\Delta u\|_\Omega + \|u\|_{1,\Gamma_1}.$$

Therefore if $\Gamma_0 \neq \emptyset$, Poincaré's inequality (in Γ_1) implies that

$$\|\partial_\nu u\|_\Gamma \lesssim \|\Delta u\|_\Omega + \|\nabla_T u\|_{\Gamma_1}. \tag{4.20}$$

On the contrary if $\Gamma_0 = \emptyset$, then we can subtract to u, the mean of u on Γ, namely consider $u - m$, with $m = \frac{1}{|\Gamma|}\int_\Gamma u(x)\, d\Gamma$ and applying (4.15) and Poincaré's inequality to $u - m$, we have

$$\|\partial_\nu (u-m)\|_\Gamma \lesssim \|\Delta(u-m)\|_\Omega + \|u-m\|_{1,\Gamma} \lesssim \|\Delta(u-m)\|_\Omega + \|\nabla_T u\|_\Gamma,$$

leading also to (4.20) since m is a constant.

Next, using the first equation of system (4.10) we have

$$\|\Delta u\|_\Omega \lesssim \beta^2 \|u\|_\Omega + \frac{o(1)}{\beta^{l-1}}. \tag{4.21}$$

Finally, since βu and $\nabla_T u$ are uniformly bounded in $L^2(\Omega)$ and in $L^2(\Gamma_1)$ respectively, then by combining (4.20) and (4.21) with $l \geq 1$, we deduce (4.19). □

Lemma 4.8. *Assume that $\overline{\Gamma_0} \cap \overline{\Gamma_1} = \emptyset$, that Γ is $C^{1,1}$ and that the multiplier control condition* **MCC** *holds. Then, the solution $(u, v, w, z) \in D(\mathcal{A})$ of system (4.9) with $l = 2$ satisfies the following estimate*

$$\int_{\Gamma_1} |\partial_\nu u|^2 d\Gamma = O(1). \tag{4.22}$$

Proof. First, we introduce a cut-off function $\eta \in C^2\left(\overline{\Omega}\right)$ that satisfies $0 \leq \eta \leq 1$ and

$$\eta(x) = \begin{cases} 1 & \text{for all} \quad x \in \Gamma_1, \\ 0 & \text{for all} \quad x \in \Omega \backslash O_\alpha, \end{cases} \tag{4.23}$$

where O_α is a neighborhood of Γ_1 given by

$$O_\alpha = \left\{ x \in \Omega;\ \inf_{y \in \Gamma_1} |x - y| \leq \alpha \right\} \tag{4.24}$$

and where α is a positive constant small enough such that $\overline{\Gamma_0} \cap O_\alpha = \emptyset$. Next, multiplying the first equation of system (4.10) by $2\eta m \cdot \nabla \overline{u}$ we get

$$2\beta^2 \int_\Omega \eta u(m \cdot \nabla \overline{u})\, dx + 2 \int_\Omega \eta \Delta u(m \cdot \nabla \overline{u})\, dx = \frac{o(1)}{\beta}. \tag{4.25}$$

On the other hand, by integrating by parts we obtain

$$2\beta^2 \Re \int_\Omega \eta u(m \cdot \nabla \overline{u})\, dx = -d \int_\Omega \eta |\beta u|^2 dx - \int_\Omega (m \cdot \nabla \eta)|\beta u|^2 dx + \int_{\Gamma_1} (m \cdot \nu)|\beta u|^2 d\Gamma. \tag{4.26}$$

Moreover, since $U \in D(\mathcal{A})$, Proposition 2.2 yields $\eta u \in H^2(\Omega)$. Then, using Green's formula, we can easily check that

$$\begin{aligned} 2\Re \int_\Omega \eta \Delta u(m \cdot \nabla \overline{u})\, dx = (d-2) \int_\Omega \eta |\nabla u|^2 dx - 2\Re \int_\Omega (\nabla u \cdot \nabla \eta)(m \cdot \nabla \overline{u})\, dx + 2\Re \int_{\Gamma_1} \partial_\nu u(m \cdot \nabla \overline{u})\, d\Gamma \\ - \int_{\Gamma_1} (m \cdot \nu)|\nabla u|^2 d\Gamma + \int_\Omega (m \cdot \nabla \eta)|\nabla u|^2 dx. \end{aligned} \tag{4.27}$$

By taking the real part of (4.25), writing $\overrightarrow{\nabla u} = \partial_\nu u \vec{\nu} + \nabla_T u$ on Γ_1 and using (4.26)–(4.27) we obtain

$$\begin{aligned} &\int_{\Gamma_1} (m \cdot \nu)|\partial_\nu u|^2 d\Gamma + \int_{\Gamma_1} (m \cdot \nu)|\beta u|^2 d\Gamma + (d-2) \int_\Omega \eta |\nabla u|^2 dx \\ &= d \int_\Omega \eta |\beta u|^2 dx + \int_\Omega (m \cdot \nabla \eta)|\beta u|^2 dx + 2\Re \int_\Omega (\nabla u \cdot \nabla \eta)(m \cdot \nabla \overline{u})\, dx \\ &\quad -2\Re \int_{\Gamma_1} \partial_\nu u\big(m \cdot \nabla_T \overline{u}\big)\, d\Gamma + \int_{\Gamma_1} (m \cdot \nu)|\nabla_T u|^2 d\Gamma - \int_\Omega (m \cdot \nabla \eta)|\nabla u|^2 dx + \frac{o(1)}{\beta}. \end{aligned}$$

Later, using the multiplier control condition **MCC** we get

$$\begin{aligned} &m_0 \int_{\Gamma_1} |\partial_\nu u|^2 d\Gamma + m_0 \int_{\Gamma_1} |\beta u|^2 d\Gamma + (d-2) \int_\Omega \eta |\nabla u|^2 dx \\ &\le d \int_\Omega \eta |\beta u|^2 dx + \int_\Omega (m \cdot \nabla \eta)|\beta u|^2 dx + 2\Re \int_\Omega (\nabla u \cdot \nabla \eta)(m \cdot \nabla \overline{u})\, dx \\ &\quad -2\Re \int_{\Gamma_1} \partial_\nu u\big(m \cdot \nabla_T \overline{u}\big)\, d\Gamma + \int_{\Gamma_1} (m \cdot \nu)|\nabla_T u|^2 d\Gamma - \int_\Omega (m \cdot \nabla \eta)|\nabla u|^2 dx + \frac{o(1)}{\beta}. \end{aligned}$$

Its follows that

$$\begin{aligned} m_0 \int_{\Gamma_1} |\partial_\nu u|^2 d\Gamma \le d \int_\Omega \eta |\beta u|^2 dx + \int_\Omega (m \cdot \nabla \eta)|\beta u|^2 dx + 2\Re \int_\Omega (\nabla u \cdot \nabla \eta)(m \cdot \nabla \overline{u})\, dx \\ - 2\Re \int_{\Gamma_1} \partial_\nu u\big(m \cdot \nabla_T \overline{u}\big)\, d\Gamma + \int_{\Gamma_1} (m \cdot \nu)|\nabla_T u|^2 d\Gamma - \int_\Omega (m \cdot \nabla \eta)|\nabla u|^2 dx + \frac{o(1)}{\beta}. \end{aligned}$$

Thus, applying Cauchy–Schwarz's and Young's inequalities we obtain

$$(m_0 - \epsilon) \int_{\Gamma_1} |\partial_\nu u|^2 d\Gamma \le \left(\frac{R^2}{\epsilon} + R\right) \int_{\Gamma_1} |\nabla_T u|^2 d\Gamma + C_1 \int_\Omega |\beta u|^2 dx + C_2 \int_\Omega |\nabla u|^2 dx + \frac{o(1)}{\beta} \tag{4.28}$$

where ϵ is an arbitrary positive constant, $R = \|m\|_\infty, C_1 = C\big(R, \|\eta\|_\infty\big)$ and $C_2 = C\big(\|\eta\|_\infty, \|\nabla\eta\|_\infty, R\big)$. Now, since $U \in D(\mathcal{A})$, we have $u = w$ and therefore $\nabla_T u = \nabla_T w$ on Γ_1. Thus, from (4.7) we deduce that $\|\nabla u\|_\Omega$ and $\|\nabla_T u\|_{\Gamma_1}$ are uniformly bounded. Further, using the first equation of system (4.9) we deduce that $\|\beta u\|_\Omega$ is uniformly bounded. Finally, setting $\epsilon = \frac{m_0}{2}$ in (4.28) we directly get (4.22). □

Lemma 4.9. *Assume that $l = 8$. Then, the solution $(u, v, w, z) \in D(\mathcal{A})$ of system* (4.9) *satisfies the following estimate*

$$\int_{\Gamma_1} |\nabla_T u|^2 d\Gamma = \frac{o(1)}{\beta^4}. \tag{4.29}$$

On the other hand, assume that $\overline{\Gamma_0} \cap \overline{\Gamma_1} = \emptyset$, the control boundary condition **MCC** *holds and $l = 2$. Then, the solution $(u, v, w, z) \in D(\mathcal{A})$ of system* (4.9) *satisfies the following estimate*

$$\int_{\Gamma_1} |\nabla_T u|^2 d\Gamma = \frac{o(1)}{\beta^2}. \tag{4.30}$$

Proof. Multiplying the second identity of system (4.10) by $\bar{u}$, integrating by parts and using (4.11), we obtain

$$\int_{\Gamma_1} |\nabla_T u|^2 d\Gamma + \int_{\Gamma_1} \partial_\nu u \bar{u}\, d\Gamma + i\beta \int_{\Gamma_1} |u|^2 d\Gamma - \int_{\Gamma_1} |\beta u|^2 d\Gamma = \frac{o(1)}{\beta^{\frac{3l}{2}}}. \tag{4.31}$$

In the first case, using (4.11) and (4.19) with $l = 8$, we get

$$\int_{\Gamma_1} |\beta u|^2 d\Gamma = \frac{o(1)}{\beta^8}, \quad \text{and} \quad \int_{\Gamma_1} \partial_\nu u \bar{u}\, d\Gamma = \frac{o(1)}{\beta^4}. \tag{4.32}$$

Thus, substituting (4.32) into (4.31) with $l = 8$ we get directly (4.29).

In the second case, using (4.11) and (4.22) with $l = 2$, we obtain

$$\int_{\Gamma_1} |\beta u|^2 d\Gamma = \frac{o(1)}{\beta^2}, \quad \text{and} \quad \int_{\Gamma_1} \partial_\nu u \bar{u}\, d\Gamma = \frac{o(1)}{\beta^2}. \tag{4.33}$$

Finally, substituting (4.33) into (4.31) with $l = 2$ we strictly get (4.30). □

Now, for any $s \in \mathbb{R}$, we consider the following auxiliary problem:

$$\begin{cases} -(s^2 + \Delta)\varphi_u = u, & \text{in} \quad \Omega, \\ \varphi_u = 0, & \text{on} \quad \Gamma_0, \\ \partial_\nu \varphi_u + is\varphi_u = 0, & \text{on} \quad \Gamma_1, \end{cases} \tag{4.34}$$

where u is solution of system (4.10).

Lemma 4.10. *Assume that the condition (H) holds. Then, for any $s \in \mathbb{R}$ such that $|s| > 1$, the solution φ_u of problem* (4.34) *satisfies the following estimate*

$$|s| \|\varphi_u\|_\Omega + \|\varphi_u\|_{1,\Omega} + |s| \|\varphi_u\|_{\Gamma_1} \lesssim \|u\|_\Omega. \tag{4.35}$$

Proof. If Γ_0 is non empty, the result was proved in Proposition 2.2 of [1]. Hence let us concentrate on the case $\Gamma_0 = \emptyset$. In this case, we define the Hilbert space

$$H = \left\{ (u, v) \in H^1(\Omega) \times L^2(\Omega) : \int_\Omega v\, dx + \int_\Gamma u\, d\Gamma = 0 \right\},$$

with norm

$$\|(u, v)\|_H^2 = \|u\|_{1,\Omega}^2 + \|v\|_\Omega^2,$$

and introduce the operator A on H by

$$AU = (v, \Delta u)^\top, \quad U = (u, v)^\top \in D(A), \tag{4.36}$$

with

$$D(A) = \left\{ U \in H : \Delta u \in L^2(\Omega), v \in H^1(\Omega), \partial_\nu u = -v \text{ on } \Gamma \right\}.$$

We let the reader check that AU is indeed in H if $U \in D(A)$.

By Huang-Prüss' theorem (see [15,21,34]) the exponential stability of system (4.3) implies that there exists $M > 0$ such that

$$\left\|(isI - A)^{-1}\right\|_{\mathcal{L}(H)} \leq M < +\infty, \tag{4.37}$$

for all $s \in \mathbb{R}$. Now for $f \in L^2(\Omega)$, take $k_0 = \kappa\theta$ and $k_1 = isk_0$, with $\theta \in \mathcal{D}(\Omega)$ such that $\int_\Omega \theta\, dx = 1$ and $\kappa \in \mathbb{C}$ fixed such that

$$\int_\Omega (f - k_1)\, dx + \int_\Gamma k_0\, d\Gamma = 0,$$

or equivalently such that

$$\kappa = (is)^{-1} \int_\Omega f\, dx. \tag{4.38}$$

With such a choice, the pair $(k_0, f - k_1)$ belongs to H and due to (4.37), there exists a unique $(\varphi, \psi) \in D(A)$ solution of

$$(is - A)(\varphi, \psi) = (k_0, f - k_1),$$

and such that

$$\|(\varphi, \psi)\|_H \leq M \|(k_0, f - k_1)\|_H. \tag{4.39}$$

We then deduce that

$$\begin{cases} is\varphi - \psi = k_0, \\ is\psi - \Delta\varphi = f - k_1, \end{cases}$$

which gives $\psi = is\varphi - k_0$ and

$$\left(s^2 + \Delta\right)\varphi = -f.$$

In view of (4.38), we deduce that

$$k_0 = (is)^{-1}\left(\int_\Omega f\, dx\right)\theta, \quad k_1 = \left(\int_\Omega f\, dx\right)\theta.$$

This implies that

$$\|k_0\|_{1,\Omega} \leq C|s|^{-1}\|f\|_\Omega,$$
$$\|k_1\|_\Omega \leq C\|f\|_\Omega,$$

for some positive constant C independent of s. These two estimates and (4.39) imply that

$$|s|\|\varphi\|_\Omega + \|\varphi\|_{1,\Omega} \leq M_1\|f\|_\Omega, \tag{4.40}$$

for some positive constant M_1 independent of s.

To obtain the estimate

$$|s|\|\varphi\|_\Gamma \lesssim \|f\|_\Omega, \tag{4.41}$$

we write

$$\int_\Omega (-s^2\varphi - f)\bar{\varphi}\, dx + \int_\Omega |\nabla\varphi|^2 dx = \int_\Omega \Delta\varphi\bar{\varphi}\, dx + \int_\Omega |\nabla\varphi|^2 dx = \int_\Gamma \partial_\nu\varphi\bar{\varphi}\, d\Gamma,$$

hence

$$\int_\Gamma \partial_\nu\varphi\bar{\varphi}\, d\Gamma = \int_\Omega \left(|\nabla\varphi|^2 - s^2|\varphi|^2 - f\bar{\varphi}\right) dx. \tag{4.42}$$

As $\psi = is\varphi$ and $\partial_\nu \varphi = -\psi$ on Γ, then taking the imaginary part of (4.42) we get

$$|s| \int_\Gamma |\varphi|^2 d\Gamma = |\Im \int_\Omega f \bar{\varphi}\, dx| \le \|f\|_\Omega \|\varphi\|_\Omega \lesssim \frac{\|f\|_\Omega^2}{|s|},$$

which proves (4.41). □

Lemma 4.11. *Assume that* $\overline{\Gamma_0} \cap \overline{\Gamma_1} = \emptyset$ *and that the condition* (H) *holds. Then, the solution* φ_u *of system* (4.34) *satisfies the following estimate*

$$\int_{\Gamma_1} |\nabla_T \varphi_u|^2 d\Gamma = O(1). \tag{4.43}$$

Proof. First, let $h = \Delta(\eta\varphi_u) = \eta\Delta\varphi_u + 2\nabla\eta \cdot \nabla\varphi_u + \Delta\eta\varphi_u$ where η is defined in (4.23)–(4.24). Next, it is easy to check that

$$\partial_\nu(\eta\varphi_u) = \begin{cases} \partial_\nu \varphi_u & \text{on } \Gamma_1, \\ 0 & \text{on } \Gamma_0, \end{cases} \tag{4.44}$$

and therefore

$$\partial_\nu(\eta\varphi_u) \in L^2(\Gamma). \tag{4.45}$$

Then, using (4.45) and applying Lemma 4.6 we obtain

$$\int_\Gamma |\nabla_T \varphi_u|^2 d\Gamma \lesssim \int_\Omega |\Delta(\eta\varphi_u)|^2 dx + \int_\Gamma |\partial_\nu(\eta\varphi_u)|^2 d\Gamma = \int_\Omega |h|^2 dx + \int_{\Gamma_1} |\partial_\nu \varphi_u|^2 d\Gamma. \tag{4.46}$$

On the other hand, using (4.35) and the third equation of system (4.3) we get

$$\int_\Omega |h|^2 dx \le \|\eta\|_\infty^2 \int_\Omega |\Delta\varphi_u|^2 dx + \|\nabla\eta\|_\infty^2 \int_\Omega |\nabla\varphi_u|^2 dx + \|\Delta\eta\|_\infty^2 \int_\Omega |\varphi_u|^2 dx \lesssim \beta^2 \int_\Omega |u|^2 dx \tag{4.47}$$

and

$$\int_{\Gamma_1} |\partial_\nu \varphi_u|^2 d\Gamma = \beta^2 \int_{\Gamma_1} |\varphi_u|^2 d\Gamma \lesssim \int_\Omega |u|^2 dx. \tag{4.48}$$

Finally, since $\|\beta u\|_\Omega$ is uniformly bounded, combining (4.46)–(4.48), we deduce (4.43). □

Lemma 4.12. *Assume that* $\overline{\Gamma_0} \cap \overline{\Gamma_1} = \emptyset$*, that* Γ *is* $C^{1,1}$ *and that the multiplier control condition* **MCC** *holds. Then, the solution* φ_u *of system* (4.34) *satisfies the following estimate*

$$\int_{\Gamma_1} |\nabla_T \varphi_u|^2 d\Gamma = \frac{O(1)}{\beta^2}. \tag{4.49}$$

Proof. First, multiplying the first equation of system (4.34) by $2\eta m \cdot \nabla\overline{\varphi_u}$ where η is the cut-off function defined in (4.23)–(4.24), we get

$$-2\beta^2 \int_\Omega \varphi_u \eta\big(m \cdot \nabla\overline{\varphi_u}\big)\, dx - 2\int_\Omega \Delta\varphi_u \eta\big(m \cdot \nabla\overline{\varphi_u}\big)\, dx = 2\int_\Omega u\eta\big(m \cdot \nabla\overline{\varphi_u}\big)\, dx.$$

Then, by taking the real part of the above equation and using (4.26)–(4.27) with φ_u instead of u, we obtain

$$\begin{aligned} &d\int_\Omega \eta|\beta\varphi_u|^2 dx + \int_\Omega (m\cdot\nabla\eta)|\beta\varphi_u|^2 dx - \int_{\Gamma_1} (m\cdot\nu)|\beta\varphi_u|^2 d\Gamma - (d-2)\int_\Omega \eta|\nabla\varphi_u|^2 dx \\ &+2\,\Re \int_\Omega (\nabla\varphi_u \cdot \nabla\eta)\big(m\cdot\nabla\overline{\varphi_u}\big) dx - 2\Re \int_{\Gamma_1} \partial_\nu\varphi_u\big(m\cdot\nabla\overline{\varphi_u}\big) d\Gamma + \int_{\Gamma_1} (m\cdot\nu)|\nabla\varphi_u|^2 d\Gamma \\ &- \int_\Omega (m\cdot\nabla\eta)|\nabla\varphi_u|^2 dx = 2\Re \int_\Omega u\big(m\cdot\nabla\overline{\varphi_u}\big) dx. \end{aligned}$$

Next, using the first equation of system (4.9) we get $\|u\|^2_\Omega = \frac{O(1)}{\beta^2}$. Then, using (4.35) we obtain

$$\int_{\Gamma_1}(m\cdot\nabla\eta)|\beta\varphi_u|^2 d\Gamma \leq R\|\eta\|_\infty \int_{\Gamma_1}|\beta\varphi_u|^2 d\Gamma \lesssim R\|\nabla\eta\|_\infty \int_\Omega |u|^2 d\Gamma = \frac{O(1)}{\beta^2},$$

where $R = \|m\|_\infty$ and therefore

$$\int_{\Gamma_1}(m\cdot\nabla\eta)|\beta\varphi_u|^2 d\Gamma = \frac{O(1)}{\beta^2}. \tag{4.51}$$

Similarly, we get

$$2\Re\int_\Omega(\nabla\varphi_u\cdot\nabla\eta)\left(m\cdot\nabla\overline{\varphi_u}\right)dx = \frac{O(1)}{\beta^2}, \tag{4.52}$$

$$(d-2)\int_\Omega \eta|\nabla\varphi_u|^2 dx = \frac{O(1)}{\beta^2}, \tag{4.53}$$

$$\int_{\Gamma_1}(m\cdot\nu)|\beta\varphi_u|^2 d\Gamma = \frac{O(1)}{\beta^2}, \tag{4.54}$$

$$\int_\Omega(m\cdot\nabla\eta)|\nabla\varphi_u|^2 dx = \frac{O(1)}{\beta^2}, \tag{4.55}$$

and

$$2\Re\int_\Omega u\left(m\cdot\nabla\overline{\varphi_u}\right)dx = \frac{O(1)}{\beta^2}. \tag{4.56}$$

Later, inserting (4.51)–(4.56) into (4.50) we obtain

$$\int_{\Gamma_1}(m\cdot\nu)|\nabla\varphi_u|^2 d\Gamma - 2\Re\int_{\Gamma_1}\partial_\nu\varphi_u\left(m\cdot\nabla\overline{\varphi_u}\right)d\Gamma = \frac{O(1)}{\beta^2}, \tag{4.57}$$

which implies, using the multiplier control condition **MCC**, that

$$\begin{aligned} m_0\int_{\Gamma_1}|\nabla_T\varphi_u|^2 d\Gamma &\leq \int_{\Gamma_1}(m\cdot\nu)|\nabla_T\varphi_u|^2 d\Gamma \\ &\leq \int_{\Gamma_1}(m\cdot\nu)|\nabla\varphi_u|^2 d\Gamma \\ &\leq 2\left|\int_{\Gamma_1}\partial_\nu\varphi_u\left(m\cdot\nabla_T\overline{\varphi_u}\right)d\Gamma\right| + \frac{O(1)}{\beta^2} \\ &\leq 2R\int_{\Gamma_1}|\nabla_T\varphi_u||\beta\varphi_u|\,d\Gamma + \frac{O(1)}{\beta^2}. \end{aligned}$$

Applying Cauchy–Schwarz's and Young's inequalities and using (4.35) we directly deduce (4.49). □

We can now finish the proof of Theorem 4.4.

(1) First, multiplying the first equation of system (4.10) by $\overline{\varphi_u}$ and applying Green's formula we obtain

$$\int_\Omega u(\beta^2+\Delta)\overline{\varphi_u}\,dx + \int_{\Gamma_1}\left(\partial_\nu u\overline{\varphi_u} - u\partial_\nu\overline{\varphi_u}\right)d\Gamma = -\int_\Omega\left(\frac{f_2+i\beta f_1}{\beta^l}\right)\overline{\varphi_u}\,dx. \tag{4.58}$$

Moreover, using the second equation of system (4.10) we have

$$\partial_\nu u = \frac{f_4+(1+i\beta)f_3}{\beta^l} + \beta^2 u + \Delta_T u - i\beta u. \tag{4.59}$$

Then, substituting (4.59) into (4.58), using the first equation of problem (4.34) and integrating by parts yields

$$\int_\Omega |\beta u|^2 dx = \int_\Omega \left(\frac{f_2 + i\beta f_1}{\beta^{l-2}}\right)\overline{\varphi_u}\, dx + \int_{\Gamma_1}\left(\frac{f_4 + (1+i\beta)f_3}{\beta^{l-2}}\right)\overline{\varphi_u}\, d\Gamma + \int_{\Gamma_1}\beta^4 u\overline{\varphi_u}\, d\Gamma$$
$$- \int_{\Gamma_1}(\beta\nabla_T u)(\beta\nabla_T\overline{\varphi_u})\, d\Gamma. \tag{4.60}$$

On the other hand, multiplying the first equation of system (4.10) by $\overline{u}$ and applying Green's formula and using (4.59), we get

$$\int_\Omega |\nabla u|^2 dx = \beta^2\left(\int_\Omega |u|^2 dx - \int_{\Gamma_1}|u|^2 d\Gamma\right) - \int_{\Gamma_1}|\nabla_T u|^2 d\Gamma - i\beta\int_{\Gamma_1}|u|^2 d\Gamma$$
$$+ \int_{\Gamma_1}\left(\widetilde{f}_4 - (1+i\beta)\widetilde{f}_3\right)\overline{u}\, d\Gamma + \int_\Omega\left(\widetilde{f}_2 + i\beta\widetilde{f}_1\right)\overline{u}\, dx. \tag{4.61}$$

Now, using Lemmas 4.5, 4.9, 4.10 and 4.11, then from (4.60) and (4.61) we obtain

$$\int_\Omega |\beta u|^2 dx = \int_\Omega |\nabla u|^2 dx = o(1). \tag{4.62}$$

This implies from the first equation of (4.9) that

$$\int_\Omega |z|^2 dx = o(1) \tag{4.63}$$

and therefore

$$\|U\|_{\mathcal{H}} = o(1), \tag{4.64}$$

which is a contradiction with (4.7).

(2) Using Lemmas 4.5, 4.9 and 4.12, then from (4.60) and (4.61) we still get (4.62), (4.63) and (4.64), which is a contradiction with (4.7).
In both cases, we have shown that (4.6) holds with $l = 8$ and $l = 2$ respectively, hence the decays (4.4) and (4.5) directly follow in the case $\Gamma_0 \neq \emptyset$. On the contrary if $\Gamma_0 = \emptyset$, we get the same decays in $\mathcal{H}_0$. But since for $U_0 \in D(\mathcal{A})$, $U_0 - \alpha(1,0,1,0)$ (with α given in Theorem 2.5) clearly belongs to $D(\mathcal{B})$, we will have

$$\|U(t) - \alpha(1,0,1,0)\|_{\mathcal{H}} \le Ct^{-1/l}(\|U_0 - \alpha(1,0,1,0)\|_{\mathcal{H}} + \|\mathcal{B}(U_0 - \alpha(1,0,1,0))\|_{\mathcal{H}}, \quad \text{for all} \quad t > 0.$$

We then conclude by noticing that $\|U(t) - \alpha(1,0,1,0)\|_{\mathcal{H}} = \|U(t)\|_{\mathcal{H}}$, and $\|\mathcal{B}(U_0 - \alpha(1,0,1,0))\|_{\mathcal{H}} = \|\mathcal{A}U_0\|_{\mathcal{H}}$.

5 | POLYNOMIAL ENERGY DECAY RATE OF 1-d MODEL WITH A PARAMETER

The aim of this section is to established a polynomial energy decay rate of 1-d model with a parameter associated with problem (1.2) on the unit square $\Omega = (0,1)^2$ with $\Gamma_1 = \{(0,y),\ y \in (0,1)\}$ and $\Gamma_0 = \Gamma\backslash\overline{\Gamma_1}$. First, we fix a real parameter $L \ge \pi$ and consider the solution (u^L, w^L) of the following wave equation with damping at 0:

$$\begin{cases} u^L_{tt} - u^L_{xx} + L^2u^L = 0, & \text{in} \quad (0,1), \quad \text{for all } t > 0, \\ u^L(1,t) = 0, & \text{for all } t > 0, \\ u^L(0,t) = w^L, & \text{for all } t > 0, \\ w^L_{tt} + L^2w^L - u_x(0,t) + w^L_t = 0, & \text{for all } t > 0, \\ u^L(\cdot,0) = u^L_0, \quad u^L_t(\cdot,0) = u^L_1, & \text{in} \quad (0,1), \\ \ w^L(0) = w^L_0, \quad w^L_t(0) = w^L_1. & \end{cases} \tag{5.1}$$

Next, we introduce the energy associated with (5.1) by

$$E_L(t) = \frac{1}{2}\int_0^1 \left(\left|u_t^L(x,t)\right|^2 + \left|u_x^L(x,t)\right|^2 + L^2\left|u^L(x,t)\right|^2\right)dx + \left|w_t^L(t)\right|^2 + L^2\left|w^L(t)\right|^2. \tag{5.2}$$

A simple integration by parts gives

$$\frac{d}{dt}E_L(t) = -\left|w_t^L(t)\right|^2. \tag{5.3}$$

Later, we split up the solution $U^L = \left(u^L, w^L\right)$ of system (5.1) as follows:

$$U^L = U_1 + U_2, \tag{5.4}$$

where $U_1 = (u_1, w_1)$ is solution of the same problem than $\left(u^L, w^L\right)$ but without damping and (u_2, w_2) is the remainder (for shortness we do not write the dependence of (u_i, w_i), $i = 1, 2$ with respect to L). This means that they are respective solutions of

$$\begin{cases} u_{1,tt} - u_{1,xx} + L^2 u_1 = 0, & \text{in} \quad (0,1), \quad \text{for all} \quad t > 0,\\ u_1(1,t) = 0, & \text{for all} \quad t > 0,\\ u_1(0,t) = w_1, & \text{for all } t > 0,\\ w_{1,tt} + L^2 w_1 - u_{1,x}(0,t) = 0, & \text{for all } t > 0,\\ u_1(\cdot,0) = u_0^L, \quad u_{1,t}(\cdot,0) = u_1^L, & \\ w_1(0) = w_0^L, \quad w_{1,t}(0) = w_1^L, & \end{cases} \tag{5.5}$$

and

$$\begin{cases} u_{2,tt} - u_{2,xx} + L^2 u_2 = 0, & \text{in} \quad (0,1), \quad \text{for all } t > 0,\\ u_2(1,t) = 0, & \text{for all } t > 0,\\ u_2(0,t) = w_2, & \text{for all } t > 0,\\ w_{2,tt} + L^2 w_2 - u_{2,x}(0,t) + w_t^L = 0, & \text{for all} \quad t > 0,\\ u_2(\cdot,0) = 0, \quad u_{2,t}(\cdot,0) = 0, & \\ w_2(0) = 0, \quad w_{2,t}(0) = 0. & \end{cases} \tag{5.6}$$

The above splitting is quite standard and is based on the following idea: First, for the problem (5.5), we prove an observability inequality for the solution via a spectral analysis and Ingham's inequality. Next, by a perturbation argument based on the dependence of the constants with respect to the time T and the parameter L, we find the requested observability estimate for the starting problem (5.1).

First, the problem (5.5) is related to the positive self-adjoint operator A_L from $H = L^2(0,1) \times \mathbb{C}$ into itself (with a compact inverse) with domain

$$D(A_L) = \left\{U_1 = (u_1, w_1) \in H^2(0,1) \times \mathbb{C};\ u_1(1) = 0 \text{ and } u_1(0) = w_1\right\} \tag{5.7}$$

and defined by

$$A_L U_1 = \left(-u_{1,xx} + L^2 u_1, L^2 w_1 - u_{1,x}(0)\right). \tag{5.8}$$

Therefore, we can formulate problem (5.5) into a second order evolution equation

$$\begin{cases} U_{1,tt}(t) + A_L U_1(t) = 0,\\ U_1(0) = U_0^L,\\ U_{1,t}(0) = U_1^L, \end{cases} \tag{5.9}$$

where $U_0^L = \left(u_0^L, w_0^L\right)$ and $U_1^L = \left(u_1^L, w_1^L\right)$. The spectrum of A_L is characterized as follows:

Theorem 5.1. *The eigenvalues λ^2 of A_L are strictly larger than L^2 and are the roots of the transcendental equation*

$$\tan\sqrt{\lambda^2 - L^2} = \frac{1}{\sqrt{\lambda^2 - L^2}}. \tag{5.10}$$

Writing $\left\{\lambda_k^2\right\}_{k=0}^{\infty}$ the sequence of these roots in increasing order, it forms the set of eigenvalues of A_L which are simple and of associated normalized eigenvectors given by

$$U_{1,k} = \frac{1}{\delta_k}\left(\sin(\theta_k(1-x)), \sin\theta_k\right), \quad \delta_k = \frac{\sqrt{1+\sin^2\theta_k}}{\sqrt{2}} \tag{5.11}$$

with $\theta_k = \sqrt{\lambda_k^2 - L^2}$. Furthermore, the next gap condition

$$\lambda_{k+1} - \lambda_k \geq \frac{\widetilde{\gamma}}{L}, \quad \textit{for all} \quad k \in \mathbb{N}, \tag{5.12}$$

holds for a positive real number $\widetilde{\gamma}$ independent of k.

Proof. First, let λ^2 be an eigenvalue of A_L and $U_1 = (u_1, w_1)$ an associated eigenvector. Then, using Green's formula and the boundary conditions in (5.7), we obtain

$$\langle A_L U_1, U_1\rangle_H = \int_0^1 |u_{1,x}|^2 dx + L^2 \int_0^1 |u_1|^2 dx + L^2|w_1|^2 \geq L^2\|U_1\|_H^2, \tag{5.13}$$

which clearly implies that the eigenvalues of A_L are larger than L^2. Next, for $\lambda^2 \geq L^2$, we look for (u_1, w_1) solution of

$$\begin{cases} -u_{1,xx} + L^2 u_1 = \lambda^2 u_1, & \text{in } (0,1), \\ L^2 w_1 - u_{1,x}(0) = \lambda^2 w_1, \\ u_1(0) = w_1, \\ u_1(1) = 0. \end{cases} \tag{5.14}$$

We easily check that if $\lambda^2 = L^2$, the only solution of problem (5.14) is $u_1 = w_1 = 0$, hence $\lambda^2 = L^2$ cannot be an eigenvalue of A_L. Now for $\lambda^2 > L^2$, there exists $\alpha \in \mathbb{R}$ such that the solution of the first equation of system (5.14) with the boundary condition $u_1(1) = 0$ is given by

$$u_1(x) = \alpha \sin(\theta(1-x)), \tag{5.15}$$

with $\theta = \sqrt{\lambda^2 - L^2}$ and some complex number α different from zero. Then, the second boundary condition of (5.14) becomes

$$\theta \sin\theta = \cos\theta. \tag{5.16}$$

Therefore a nontrivial solution (u_1, w_1) exists if and only if (5.16) holds. The form (5.11) of the eigenvectors also follows from this consideration. As (5.16) is equivalent to (5.10), we deduce that its roots are simple and verify, with the notation $\theta_k = \sqrt{\lambda_k^2 - L^2}$,

$$0 < \theta_0 < \frac{\pi}{2}, \qquad \frac{\pi}{2} + (k-1)\pi < \theta_k < \frac{\pi}{2} + k\pi, \quad \text{for all} \quad k \in \mathbb{N}^*. \tag{5.17}$$

Now, we check the gap between the eigenvalues. Setting $\varphi(t) = \sqrt{t^2 + L^2}$ and using the mean value theorem, we deduce that there exists $\theta_c \in (\theta_k, \theta_{k+1})$ such that

$$\lambda_{k+1} - \lambda_k = \varphi(\theta_{k+1}) - \varphi(\theta_k) = \partial_t\varphi(\theta_c)\left(\theta_{k+1} - \theta_k\right) = \frac{\theta_c}{\sqrt{\theta_c^2 + L^2}}\left(\theta_{k+1} - \theta_k\right). \tag{5.18}$$

Since $\frac{t}{\sqrt{\frac{t^2}{L^2}+1}}$ is an nondecreasing function of the variable t we obtain

$$\frac{\theta_c}{\sqrt{\theta_c^2+L^2}} \geq \frac{\theta_0}{L\sqrt{\frac{\theta_0^2}{L^2}+1}} \geq \frac{C}{L}, \quad \text{with } C = \frac{\theta_0}{\sqrt{\frac{\theta_0^2}{\pi^2}+1}}. \tag{5.19}$$

Finally, setting $\widetilde{\gamma} = C \min_{k\in\mathbb{N}} \left(\theta_{k+1} - \theta_k\right)$, we obtain (5.12) from (5.18)–(5.19). □

Before going on, we recall Lemma 3.3 from [30] which gives a variant of Ingham's inequality [22] (see also [19]), where the dependence of the constants of equivalence are given with respect to the gap condition.

Lemma 5.2. *Let $(\xi_n)_{n\in\mathbb{Z}}$ be a sequence of real numbers and a positive real number γ such that the following gap condition:*

$$\xi_{n+1} - \xi_n \geq \gamma, \quad \textit{for all} \quad k \in \mathbb{Z},$$

holds. Then, there exist two positive constants c, C independent of γ such that for all function f in the form

$$f(t) = \sum_{n\in\mathbb{Z}} a_n e^{i\xi_n t}$$

with $a_n \in \mathbb{C}$, we have

$$\frac{c}{\gamma}\sum_{n\in\mathbb{Z}} |a_n|^2 \leq \int_0^{\frac{4\pi}{\gamma}} |f(t)|^2 dt \leq \frac{C}{\gamma}\sum_{n\in\mathbb{Z}} |a_n|^2.$$

□

Now, we set $V_L = D\left(A_L^{\frac{1}{2}}\right)$. We will bound a weak energy of system (5.5) with respect to an appropriate boundary term in the following theorem:

Theorem 5.3. *Let $\widetilde{E}_{U_1}(t)$ be a weak energy of U_1 solution of (5.9) defined by*

$$\widetilde{E}_{U_1}(t) = \frac{1}{2}\|U_1(x,t)\|_H^2 + \frac{1}{2}\|U_{1,t}(x,t)\|_{V_L'}^2. \tag{5.20}$$

Then, there exist two positive constants C_1, C_2 independent of L such that for all $T \geq C_1 L$ we have

$$C_2 L \widetilde{E}_{U_1}(0) \leq \int_0^T |w_{1,t}(t)|^2 dt. \tag{5.21}$$

Proof. By the spectral theorem, the solution U_1 of (5.9) is given by

$$U_1(\cdot,t) = \sum_{k=0}^{+\infty}\left(u_0^k \cos\left(\lambda_k t\right) + u_1^k \frac{\sin\left(\lambda_k t\right)}{\lambda_k}\right)U_{1,k}, \tag{5.22}$$

where u_0^k (resp. u_1^k) is the Fourier coefficient of U_0^L (resp. U_1^L), *i.e.*,

$$U_0^L = \sum_{k=0}^{+\infty} a_0^k U_{1,k} \quad \text{and} \quad U_1^L = \sum_{k=0}^{+\infty} a_1^k U_{1,k}.$$

Writting $U_{1,k} = \left(u_{1,k}, w_{1,k}\right)$, this implies that

$$w_1(t) = \sum_{k=0}^{+\infty}\left(u_0^k \cos\left(\lambda_k t\right) + u_1^k \frac{\sin\left(\lambda_k t\right)}{\lambda_k}\right)w_{1,k}, \tag{5.23}$$

and therefore

$$w_{1,t}(t) = \sum_{k=0}^{+\infty} \left(-u_0^k \lambda_k \sin\left(\lambda_k t\right) + u_1^k \cos\left(\lambda_k t\right)\right) w_{1,k}. \tag{5.24}$$

Then according to the gap condition (5.12) and using Lemma 5.2, we deduce that there exist two positive constants C_1, C_3 independent of L such that for all $T \geq C_1 L$ we have

$$C_3 L \sum_{k=0}^{+\infty} \left(\lambda_k^2 \left|u_0^k\right|^2 + \left|u_1^k\right|^2 \right) |w_{1,k}|^2 \leq \int_0^T |w_{1,t}(t)|^2 dt. \tag{5.25}$$

Next, using (5.10) and (5.11) we get

$$|w_{1,k}|^2 = \frac{1}{\delta_k^2} \times \sin^2\theta_k = \frac{2}{(1+\sin^2\theta_k)} \times \frac{\cos^2\theta_k}{\theta_k^2} \sim \frac{C}{\theta_k^2} \sim \frac{C}{\lambda_k^2}, \quad k \to \infty, \tag{5.26}$$

since $\theta_k \sim k\pi$ as k goes to infinity. This equivalence in (5.25) yields the existence of a positive constant C_2 independent of L such that for $T \geq C_1 L$ we have:

$$C_2 L \sum_{k=0}^{+\infty} \left(|u_0^k|^2 + \frac{|u_1^k|^2}{\lambda_k^2} \right) \leq \int_0^T |w_{1,t}(t)|^2 dt. \tag{5.27}$$

We conclude by the identity

$$\sum_{k=0}^{+\infty} \left(\left|u_0^k\right|^2 + \frac{|u_1^k|^2}{\lambda_k^2} \right) = \left\|U_0^L\right\|_H^2 + \left\|U_1^L\right\|_{V_L'}^2.$$

□

We go on with an estimate on w_2.

Theorem 5.4. *There exists a positive constant C_4 independent of L such that*

$$\int_0^T |w_{2,t}(t)|^2 dt \leq C_4^2 T^2 \int_0^T \left|w_t^L(t)\right|^2 dt, \quad \textit{for all} \quad T > 0. \tag{5.28}$$

Proof. First, we start by rewriting problem (5.6) as follows:

$$\begin{cases} U_{2,tt}(t) + A_L U_2(t) = K(t) H_0, \\ U_2(0) = 0, \\ U_{2,t}(0) = 0, \end{cases} \tag{5.29}$$

with $H_0 = (0,1) \in H$ and $K(t) = w_t^L(t)$. Remark that H_0 is given by

$$H_0 = \sum_{k=0}^{\infty} w_{1,k} U_{1,k}.$$

Indeed we have

$$\langle H_0, U_{1,k} \rangle_H = \left\langle (0,1), \left(u_{1,k}, w_{1,k}\right) \right\rangle_H = w_{1,k}. \tag{5.30}$$

Next, using the orthonormal basis $\{U_{1,k}\}_{k=0}^{+\infty}$ of H, we can write the solution $U_2 = (u_2, w_2)$ of problem (5.29) as follows:

$$U_2(t) = \sum_{k=0}^{+\infty} u_{2,k}(t) U_{1,k}. \tag{5.31}$$

From (5.29)–(5.31) we deduce that for every fixed $k \in \mathbb{N}$, $u_{2,k}$ is solution of the following boundary value problem

$$\begin{cases} u_{2,k,tt}(t) + \lambda_k^2 u_{2,k}(t) = K(t) w_{1,k}, \\ u_{2,k}(0) = 0, \\ u_{2,k,t}(0) = 0. \end{cases} \tag{5.32}$$

Consequently $u_{2,k}$ is given by

$$u_{2,k}(t) = w_{1,k} \int_0^t \frac{\sin\left(\lambda_k s\right)}{\lambda_k} K(t-s)\,ds. \tag{5.33}$$

Thus, we obtain

$$U_2(t) = \int_0^t u(s) K(t-s)\,ds, \tag{5.34}$$

where $u(s) = \sum_{k=0}^{\infty} \frac{\sin\left(\lambda_k s\right)}{\lambda_k} w_{1,k} U_{1,k}$. It follows that

$$w_2(t) = \int_0^t \psi(s) K(t-s)\,ds, \tag{5.35}$$

where $\psi(s) = \sum_{k=0}^{\infty} \frac{\sin\left(\lambda_k s\right)}{\lambda_k} w_{1,k}^2$, which implies that

$$w_{2,t}(t) = \int_0^t \psi_t(s) K(t-s)\,ds. \tag{5.36}$$

On the other hand, using (5.26) we have

$$|\psi_t(s)| = \left| \sum_{k=0}^{+\infty} \cos\left(\lambda_k s\right) w_{1,k}^2 \right| \le \sum_{k=0}^{+\infty} |w_{1,k}|^2 \le C_4 < \infty. \tag{5.37}$$

Later, using (5.36)–(5.37) and applying Cauchy–Schwarz's inequality we obtain

$$|w_{2,t}(t)|^2 \le C_4^2 t \int_0^t |K(t-s)|^2 ds. \tag{5.38}$$

Finally, by integrating (5.38) between 0 and T for $t \le T$ and by a change of variable we deduce that

$$\int_0^T |w_{2,t}(t)|^2 dt \le C_4^2 T^2 \int_0^T |K(t)|^2 dt = C_4^2 T^2 \int_0^T |w_t^L(t)|^2 dt. \tag{5.39}$$

□

We are ready to prove the main result of this section.

Theorem 5.5. *There exists a positive constant C_5 independent of L such that for all initial data $\left(U_0^L, U_1^L\right) \in D(A_L) \times V_L$ we have*

$$E_L(t) \le \frac{C_5 L^2}{t} E_L^1(0), \tag{5.40}$$

where $E_L^1(t)$ is given by

$$E_L^1(t) = \left\| U^L(t) \right\|_{D(A_L)}^2 + \left\| U_t^L(t) \right\|_{V_L}^2. \tag{5.41}$$

Proof. According to Theorem 5.3, we fix $T = C_1 L$. Now using the splitting (5.4) we obtain

$$|w_{1,t}(t)|^2 \leq 2\left(\left|w_t^L(t)\right|^2 + |w_{2,t}(t)|^2\right). \tag{5.42}$$

Then, integrating (5.42) between 0 and T and using the inequalities (5.21) and (5.28) we get

$$\int_0^T \left|w_t^L(t)\right|^2 dt \geq \frac{C_6 L}{T^2} \widetilde{E}_{U_1}(0), \tag{5.43}$$

for $C_6 = \frac{C_2}{2C_4^2}$. Next, since $\widetilde{E}_{U_1}(0) = \widetilde{E}_{U^L}(0)$ and using (5.3), the estimate (5.43) becomes

$$E_L(0) - E_L(T) \geq \frac{C_6 L}{T^2} \widetilde{E}_{U^L}(0). \tag{5.44}$$

On the other hand, using interpolation theory we can show that

$$\left\|U_0^L\right\|_H^2 \geq \frac{\|U_0^L\|_{V_L}^4}{\|U_0^L\|_{D(A_L)}^2} \quad \text{and} \quad \left\|U_1^L\right\|_{V_L'}^2 \geq \frac{\|U_1^L\|_H^4}{\|U_1^L\|_{V_L}^2}. \tag{5.45}$$

Thus, combining (5.20) and (5.45) we obtain

$$\widetilde{E}_{U^L}(0) \geq \frac{1}{2} \frac{\|U_0^L\|_{V_L}^4 + \|U_1^L\|_H^4}{\|U_0^L\|_{D(A_L)}^2 + \|U_1^L\|_{V_L}^2} = \frac{E_L^2(0)}{2E_L^1(0)}, \tag{5.46}$$

where $E_L^1(t)$ is defined by (5.41). Later, substituting (5.46) into (5.44) and using the fact that $E_L(t)$ is a nonincreasing function of the variable t we get

$$E_L(T) \leq E_L(0) - C_7 \frac{E_L^2(T)}{E_L^1(0)}, \tag{5.47}$$

where $C_7 = \frac{C_6 L}{2T^2}$. Now, we introduce the sequence $\xi_k = \frac{E_L(kT)}{E_L^1(0)}$ for $k \in \mathbb{N}$. Then, since $E_L(t)$ is a nonincreasing function of the variable t, dividing (5.47) by $E_L^1(0)$ we can easily check that the sequence $(\xi_k)_{k\in\mathbb{N}}$ verifies the following inequalities

$$\xi_{k+1} \leq \xi_k - C_7 \xi_{k+1}^2, \quad \text{for all} \quad k \in \mathbb{N}. \tag{5.48}$$

Our goal is to determine a constant M such that $\xi_k \leq \frac{M}{k+1}$. For this aim, we introduce the sequence $(F_k)_{k\in\mathbb{N}}$ as follows:

$$F_k = \frac{M}{k+1}, \quad k \in \mathbb{N}.$$

First, we notice that

$$F_k - F_{k+1} = \frac{M}{(k+1)(k+2)} \leq \frac{2}{M} F_{k+1}^2. \tag{5.49}$$

Now, if we assume that

$$\frac{2}{C_7} \leq M \quad \text{and} \quad \xi_0 \leq F_0 = M, \tag{5.50}$$

then we can prove by induction that

$$\xi_k \leq F_k, \quad \text{for all} \quad k \in \mathbb{N}. \tag{5.51}$$

Hence (5.51) holds as soon as $M = \max\left\{\frac{2}{C_7}, \xi_0\right\}$. Clearly (5.51) is equivalent to

$$E_L(kT) \leq \frac{M}{k+1} E_L^1(0), \tag{5.52}$$

and therefore, for any $t > 0$, as there exists $k \in \mathbb{N}$ such that $kT \leq t \leq (k+1)T$, we deduce that

$$E_L(t) \leq \frac{MC_1L}{t} E_L^1(0) = \begin{cases} \dfrac{C_1 L E_L(0)}{t} & \text{if} \quad M = \xi_0, \\ \text{or} & \\ \dfrac{C_8 L^2 E_L^1(0)}{t} & \text{if} \quad M = \dfrac{2}{C_7}, \end{cases} \tag{5.53}$$

where $C_8 = \frac{4C_1^2}{C_6}$. On the other hand, from (5.10) we know that $\lambda_k^2 \geq L^2$, from which we easily prove that

$$E_L(0) \leq \frac{1}{L^2} E_L^1(0). \tag{5.54}$$

Finally, combining (5.53) and (5.54), we conclude that (5.40) holds. □

6 | POLYNOMIAL ENERGY DECAY RATE ON THE UNIT SQUARE

In this section, we establish the polynomial stability of the system (1.2) in the unit square of $\mathbb{R}^2$ with $\Gamma_1 = \{0\} \times (0,1)$. This case does not satisfy the assumptions of Theorem 4.4 since neither the condition (H) holds nor $\overline{\Gamma_0} \cap \overline{\Gamma_1} = \emptyset$ holds. Nevertheless, combining a Fourier analysis and the results from the previous section we obtain a polynomial decay rate (compare with [30]).

Consequently, we perform the partial Fourier expansion of the solution $U = (u, w)$ of system (1.2)

$$U(x, y, t) = \sum_{p=1}^{+\infty} U^{p\pi}(x, t) \sin(p\pi y), \tag{6.1}$$

where $U^{p\pi}(x, t) = \left(u^{p\pi}(x, t), w^{p\pi}(t)\right)$ is then solution of system (5.1) with $L = p\pi$. Recalling that the energy of system (5.1) is given by

$$E_{p\pi}(t) = \frac{1}{2} \int_0^1 \left(|u_t^{p\pi}(x,t)|^2 + |u_x^{p\pi}(x,t)|^2 + p^2\pi^2 |u^{p\pi}(x,t)|^2\right) dx + |w_t^{p\pi}(t)|^2 + p^2\pi^2 |w^{p\pi}(t)|^2, \tag{6.2}$$

we directly check that

$$E(t) = \sum_{p=1}^{+\infty} E_{p\pi}(t). \tag{6.3}$$

Using a Fourier synthesis and Theorem 5.5, we obtain the following result.

Theorem 6.1. *There exists a positive constant $C > 0$, such that for all initial data $U_0 = (u_0, u_1, w_0, w_1) \in D\left(\mathcal{A}^2\right)$, the energy of the solution of* (1.2) *in the unit square of* $\mathbb{R}^2$ *with* $\Gamma_1 = \{0\} \times (0,1)$ *satisfies*

$$E(t) \leq \frac{C}{t} \|U_0\|^2_{D(\mathcal{A}^2)}. \tag{6.4}$$

Proof. First, combining (5.40) and (6.3) we obtain

$$E(t) = \sum_{p=1}^{+\infty} E_{p\pi}(t) \leq \frac{C_5}{t} \sum_{p=1}^{+\infty} p^2\pi^2 E_{p\pi}^1(0), \tag{6.5}$$

where

$$E^1_{p\pi}(0) = \|U^{p\pi}(0)\|_{D(A_L)} + \|U^{p\pi}_t(0)\|_{V_L}, \quad U^{p\pi}(0) = \left(u^{p\pi}_0, w^{p\pi}_0\right), \quad U^{p\pi}_t(0) = \left(u^{p\pi}_1, w^{p\pi}_1\right). \tag{6.6}$$

By integrating by parts and by using the boundary conditions of system (5.1) we obtain

$$\begin{aligned}&\left\|A_L U^{p\pi}(x,0)\right\|^2_H \\ &= \int_0^1 |u^{p\pi}_{0,xx}(x)|^2 dx + p^4\pi^4 \int_0^1 |u^{p\pi}_0(x)|^2 dx + 2p^2\pi^2 \int_0^1 |u^{p\pi}_{0,x}(x)|^2 dx + p^4\pi^4 |w^{p\pi}_0|^2 + |u^{p\pi}_{0,x}(0)|^2,\end{aligned} \tag{6.7}$$

hence $\|\cdot\|_{D(A_L)} \lesssim \|A_L \cdot\|_H$, while by definition, we have

$$\left\|U^{p\pi}_t(x,0)\right\|^2_{V_L} = \int_0^1 |u^{p\pi}_{1,x}(x)|^2 dx + p^2\pi^2 \int_0^1 |u^{p\pi}_1(x)|^2 dx + p^2\pi^2 |w^{p\pi}_1|^2. \tag{6.8}$$

Now, combining (6.5), (6.6), (6.7) and (6.8) we get

$$E(t) \le \frac{C_5}{t} E_2(0) \tag{6.9}$$

where

$$\begin{aligned}E_2(0) = &\sum_{p=1}^{+\infty} \int_0^1 \left(p^2\pi^2 |u^{p\pi}_{0,xx}(x)|^2 + p^6\pi^6 |u^{p\pi}_0(x)|^2 + 2p^4\pi^4 |u^{p\pi}_{0,x}(x)|^2\right) dx + \sum_{p=1}^{+\infty} p^6\pi^6 |w^{p\pi}_0|^2 \\ &+ \sum_{p=1}^{+\infty} p^2\pi^2 |u^{p\pi}_{0,x}(0)|^2 + \sum_{p=1}^{+\infty} \int_0^1 \left(p^2\pi^2 |u^{p\pi}_{1,x}(x)|^2 + p^4\pi^4 |u^{p\pi}_1(x)|^2\right) dx + \sum_{p=1}^{+\infty} p^4\pi^4 |w^{p\pi}_1|^2.\end{aligned} \tag{6.10}$$

Finally, by Parseval's identity and the result of Proposition 2.4 we deduce that

$$E_2(0) \le \|u_0\|^2_{H^3(\Omega)} + \|w_0\|^2_{H^3(\Gamma_1)} + \|u_1\|^2_{H^2(\Omega)} + \|w_1\|^2_{H^2(\Gamma_1)} = \|U_0\|^2_{D(\mathcal{A}^2)}. \tag{6.11}$$

□

7 | OPTIMAL POLYNOMIAL ENERGY DECAY RATE FOR THE UNIT DISC

Theorem 4.4 yields the energy decay rate of type $\frac{1}{t}$ for the system (1.2) in the unit disc with $\Gamma_0 = \emptyset$ and initial data $U_0 \in D(\mathcal{A})$ (since this case enters into the second case with $m(x) = x$). In this section, we will prove that this decay rate is optimal.

Theorem 7.1. *(Optimal decay rate) The energy decay rate* (4.5) *is optimal in the unit disc with* $\Gamma_0 = \emptyset$*, in the sense that for any* $\epsilon > 0$*, we cannot expect the decay rate* $\frac{1}{t^{1+\epsilon}}$ *for all initial data* $U_0 \in D(\mathcal{A})$*.*

Proof. Let $\epsilon > 0$ and set $\hat{l} = \frac{\epsilon}{1+\epsilon}$. First, let λ_k, with $k \ge k_0$, be the sequence of eigenvalues of the operator $\mathcal{A}$ described in Proposition 3.1 and let $U_k \in D(\mathcal{A})$ be the associated normalized eigenfunction. Moreover, set

$$\beta_k = \Im(\lambda_k), \quad \text{for all} \quad k \ge k_0.$$

Next, using (3.5) we have

$$(i\beta_k I - \mathcal{A})U_{l,k} = \left(-\frac{2}{k^2\pi^2} + o(\frac{1}{k^2})\right)U_k, \quad \text{for all} \quad k \ge k_0,$$

and therefore

$$\beta_k^{2-2\hat{l}} \|(i\beta_k I - \mathcal{A})U_k\|_{\mathcal{H}} \sim \frac{1}{k^{\frac{2\epsilon}{1+\epsilon}}}, \quad \text{for all} \quad k \ge k_0.$$

Thus, we deduce that

$$\lim_{k\to+\infty} \beta_k^{2-2\hat{l}} \|(i\beta_k I - \mathcal{A})U_{l,k}\|_{\mathcal{H}} = 0.$$

Thanks to Theorem 4.1 (see also Theorem 2.4 in [9]), we deduce that for $U_0 \in D(\mathcal{A})$, $\left\|e^{t\mathcal{A}}U_0\right\|_{\mathcal{H}}$ decays slower than $\frac{1}{t^{\frac{1}{2-2\hat{l}}}}$ as the time $t \to +\infty$. The proof is thus complete. □

ORCID

Serge Nicaise http://orcid.org/0000-0003-3673-3495

REFERENCES

[1] Z. Abbas and S. Nicaise, *The multidimensional wave equation with generalized acoustic boundary conditions ii: Polynomial stability*, SIAM J. Control Optim. **53** (2015), no. 4, 2582–2607.

[2] N. Aissa and D. Hamroun, *Stabilization of a wave-wave system*, Port. Math. (N.S.) **61** (2004), no. 2, 147–159.

[3] K. Ammari and M. Tucsnak, *Stabilization of second order evolution equations by a class of unbounded feedbacks*, ESAIM Control Optim. Calc. Var. **6** (2001), 361–386 (electronic).

[4] W. Arendt and C. J. K. Batty, *Tauberian theorems and stability of one-parameter semigroups*, Trans. Amer. Math. Soc. **306** (1988), no. 2, 837–852.

[5] C. Bardos, G. Lebeau, and J. Rauch, *Sharp sufficient conditions for the observation, control, and stabilization of waves from the boundary*, SIAM J. Control Optim. **30** (1992), no. 5, 1024–1065.

[6] A. Bátkai et al., *Polynomial stability of operator semigroups*, Math. Nachr. **279** (2006), no. 13–14, 1425–1440.

[7] C. J. K. Batty and T. Duyckaerts, *Non-uniform stability for bounded semi-groups on Banach spaces*, J. Evol. Equ. **8** (2008), no. 4, 765–780.

[8] J. T. Beale, *Spectral properties of an acoustic boundary condition*, Indiana Univ. Math. J. **25** (1976), no. 9, 895–917.

[9] A. Borichev and Y. Tomilov, *Optimal polynomial decay of functions and operator semigroups*, Math. Ann. **347** (2010), no. 2, 455–478.

[10] M. M. Cavalcanti, I. Lasiecka, and D. Toundykov, *Geometrically constrained stabilization of wave equations with Wentzell boundary conditions*, Appl. Anal. **91** (2012), no. 8, 1427–1452.

[11] M. Costabel, *A remark on the regularity of solutions of Maxwell's equations on Lipschitz domains*, Math. Methods Appl. Sci. **12** (1990), no. 4, 365–368.

[12] N. Fourrier, Analysis of existence, regularity and stability of solutions to wave equations with dynamic boundary conditions, PhD thesis, University of Virginia, 2013.

[13] N. Fourrier and I. Lasiecka, *Regularity and stability of a wave equation with a strong damping and dynamic boundary conditions*, Evol. Equ. Control Theory **2** (2013), no. 4, 631–667.

[14] C. G. Gal, G. R. Goldstein, and J. A. Goldstein, *Oscillatory boundary conditions for acoustic wave equations*, J. Evol. Equ. **3** (2003), no. 4, Dedicated to Philippe Bénilan. 623–635.

[15] L. Gearhart, *Spectral theory for contraction semigroups on Hilbert space*, Trans. Amer. Math. Soc. **236** (1978), 385–394.

[16] S. Gerbi and B. Said-Houari, *Asymptotic stability and blow up for a semilinear damped wave equation with dynamic boundary conditions*, Nonlinear Anal. **74** (2011), no. 18, 7137–7150.

[17] P. J. Graber and I. Lasiecka, *Analyticity and Gevrey class regularity for a strongly damped wave equation with hyperbolic dynamic boundary conditions*, Semigroup Forum **88** (2014), no. 2, 333–365.

[18] P. Grisvard, *Elliptic problems in nonsmooth domains*, Monographs and Studies in Mathematics, vol. 24, Pitman, Advanced Publishing Program, Boston, MA, 1985.

[19] A. Haraux, *Séries lacunaires et contrôle semi-interne des vibrations d'une plaque rectangulaire*, J. Math. Pures Appl. (9) **68** (1990), no. 4, 457–465 1989.

[20] T. Hell, A. Ostermann, and M. Sandbichler, *Modification of dimension-splitting methods—overcoming the order reduction due to corner singularities*, IMA J. Numer. Anal. **35** (2015), no. 3, 1078–1091.

[21] F. L. Huang, *Characteristic conditions for exponential stability of linear dynamical systems in Hilbert spaces*, Ann. Differential Equations **1** (1985), no. 1, 43–56.

[22] A. E. Ingham, *Some trigonometrical inequalities with applications to the theory of series*, Math. Z. **41** (1936), no. 1, 367–379.

[23] J. Lagnese, *Decay of solutions of wave equations in a bounded region with boundary dissipation*, J. Differential Equations **50** (1983), no. 2, 163–182.

[24] I. Lasiecka and R. Triggiani, *Uniform stabilization of the wave equation with Dirichlet or Neumann feedback control without geometrical conditions*, Appl. Math. Optim. **25** (1992), no. 2, 189–224.

[25] W. Littman and L. Markus, *Stabilization of a hybrid system of elasticity by feedback boundary damping*, Ann. Mat. Pura Appl. (4) **152** (1988), 281–330.

[26] Z. Liu and B. Rao, *Characterization of polynomial decay rate for the solution of linear evolution equation*, Z. Angew. Math. Phys. **56** (2005), no. 4, 630–644.

[27] S. Micu and E. Zuazua, *Asymptotics for the spectrum of a fluid/structure hybrid system arising in the control of noise*, SIAM J. Math. Anal. **29** (1998), no. 4, 967–1001 (electronic).

[28] P. Morse and K. Ingard, *Theoretical acoustics*, International Series in Pure and Applied Physics, Princeton University Press, 1968.

[29] D. Mugnolo, *Abstract wave equations with acoustic boundary conditions*, Math. Nachr. **279** (2006), no. 3, 299–318.

[30] S. Nicaise and K. Laoubi, *Polynomial stabilization of the wave equation with Ventcel's boundary conditions*, Math. Nachr. **283** (2010), no. 10, 1428–1438.

[31] S. Nicaise and C. Pignotti, *Stability of the wave equation with localized Kelvin–Voigt damping and boundary delay feedback*, Discrete Contin. Dyn. Syst. Ser. S **9** (2016), no. 3, 791–813.

[32] N. of standards and D. l. o. m. f. h. technology, available at http://dlmf.nist.gov/.

[33] A. Pazy, *Semigroups of linear operators and applications to partial differential equations*, Appl. Math. Sci., vol. 44, Springer-Verlag, New York, 1983.

[34] J. Prüss, *On the spectrum of C_0-semigroups*, Trans. Amer. Math. Soc. **284** (1984), no. 2, 847–857.

[35] J. M. Rivera and Y. Qin, *Polynomial decay for the energy with an acoustic boundary condition*, Appl. Mathe. Lett. **16** (2003), no. 2, 249–256.

[36] D. L. Russell, *Decay rates for weakly damped systems in Hilbert space obtained with control-theoretic methods*, J. Differential Equations **19** (1975), no. 2, 344–370.

[37] D. L. Russell, *A general framework for the study of indirect damping mechanisms in elastic systems*, J. Math. Anal. Appl. **173** (1993), no. 2, 339–358.

How to cite this article: Mercier D, Nicaise S, Sammoury MA, Wehbe A. Indirect stability of the wave equation with a dynamic boundary control. *Mathematische Nachrichten*. 2017;1–33. https://doi.org/10.1002/mana.201700021