Global subelliptic estimates for Kramers-Fokker-Planck operators with some class of polynomial

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Abstract

In this article we study some Kramers-Fokker-Planck operators with polynomial potential $V(q)$ with degree greater than two having quadratic limiting behavior. This work provide accurate global Subelliptic estimates for KFP operators under some conditions imposed on the potential.

Contents

1 Introduction and main results

The Kramers-Fokker-Planck operator reads

$$K_V = p.\partial_q - \partial_q V(q).\partial_p + \frac{1}{2}(-\Delta_p + p^2), \qquad (q,p) \in \mathbb{R}^{2d}, \tag{1.1}$$

where q denotes the space variable, p denotes the velocity variable, $x.y = \sum_{j=1}^{d} x_j y_j$, $x^2 = \sum_{j=1}^{d} x_j^2$ and the potential $V(q) = \sum_{|\alpha|\leq r} V_\alpha q^\alpha$ is a real-valued polynomial function on $\mathbb{R}^d$ with $d^\circ V = r$.

There have been several works concerned with the operator K_V with diversified approaches. In this article we impose some kind of assumptions on the polynomial potential $V(q)$, so that the Kramers-Fokker-Planck operator K_V admits a global subelliptic estimate and has a compact resolvent. This problem is closely related to the return to the equilibrium for the Kramers-Fokker-Planck operator (see [HeNi], [HerNi]). As mentioned in [HerNi] and [HeNi], the analysis of K_V is also strongly linked to the one of the Witten Laplacian $\Delta_V^{(0)} = -\Delta_q + |\nabla V(q)|^2 - \Delta V(q)$. This relation yielded to the following conjecture established by Helffer-Nier:

$$(1 + K_V)^{-1} \text{ compact} \Leftrightarrow (1 + \Delta_V^{(0)})^{-1} \text{ compact} .$$

This conjecture has been partially resolved in simple cases (see for example [HeNi],[HerNi] and [Li]), whereas for the operator $\Delta_V^{(0)}$ very general criteria of compactness work for polynomial potiential $V(q)$ of arbitrary degree. These last criteria require an analysis of the degeneracies at infinity of the potential and rely on extremely sophisticated tools of hypoellipticity developed by Helffer and Nourrigat in the 1980's (see [HeNo]).

In the case of the Kramers-Fokker-Planck operator, as far as general potential is concerned, different kind of conditions on $V(q)$ had been examined by Hérau-Nier [HerNi], Helffer-Nier [HeNi] and Villani [Vil]. Lately a significant improvement of the results of Helffer-Nier has been done by Wei-Xi Li [Li2] based on some multipliers method.

Denoting

$$O_p = \frac{1}{2}(D_p^2 + p^2) ,$$

and

$$X_V = p.\partial_q - \partial_q V(q).\partial_p ,$$

we can rewrite the Kramers-Fokker-Planck operator K_V defined in (1.1) as $K_V = X_V + O_p$.

Notations: For an arbitrary polynomial $V(q)$ of degree r, we denote for all $q \in \mathbb{R}^d$

$$\mathrm{Tr}_{+,V}(q) = \sum_{\substack{\nu\in\mathrm{Spec}(\mathrm{Hess}\, V)\\ \nu>0}} \nu(q) ,$$

$$\mathrm{Tr}_{-,V}(q) = - \sum_{\substack{\nu\in\mathrm{Spec}(\mathrm{Hess}\, V)\\ \nu\leq 0}} \nu(q) .$$

Futhermore, for a polynomial $P \in E_r := \{P \in \mathbb{R}[X_1, ..., X_d],\ d^\circ P \leq r\}$ and all natural number $n \in \{1, ..., r\}$, we define the functions $R_P^{\geq n} : \mathbb{R}^d \to \mathbb{R}$ and $R_P^{=n} : \mathbb{R}^d \to \mathbb{R}$ by

$$R_P^{\geq n}(q) = \sum_{n\leq|\alpha|\leq r} \left|\partial_q^\alpha P(q)\right|^{\frac{1}{|\alpha|}} , \tag{1.2}$$

$$R_P^{=n}(q) = \sum_{|\alpha|=n} |\partial_q^\alpha P(q)|^{\frac{1}{|\alpha|}} \ . \tag{1.3}$$

For arbitrary real functions A and B, we make also use of the following notation

$$A \asymp B \iff \exists\, c \geq 1 : c^{-1}\,|B| \leq |A| \leq c\,|B| \ .$$

This work is essentially based on the recent publication by Ben Said, Nier, and Viola [BNV], which deals with the analysis of Kramers-Fokker-Planck operators with polynomial of degree less than 3. In this case we define the constants A_V and B_V by

$$\begin{aligned} A_V &= \max\{(1+\mathrm{Tr}_{+,V})^{2/3}, 1+\mathrm{Tr}_{-,V}\} \ , \\ B_V &= \max\{\min_{q\in\mathbb{R}^d} |\nabla\, V(q)|^{4/3} \,, \frac{1+\mathrm{Tr}_{-,V}}{\log(2+\mathrm{Tr}_{-,V})^2}\} \ . \end{aligned}$$

As proved in [BNV], there is a constant $c>0$ such that the following global subelliptic estimate with remainder

$$\begin{aligned} \|K_V u\|^2_{L^2(\mathbb{R}^{2d})} + A_V \|u\|^2_{L^2(\mathbb{R}^{2d})} \geq c\Big(&\|O_p u\|^2_{L^2(\mathbb{R}^{2d})} + \|X_V u\|^2_{L^2(\mathbb{R}^{2d})} \\ &+ \|\langle \partial_q V(q)\rangle^{2/3} u\|^2_{L^2(\mathbb{R}^{2d})} + \|\langle D_q\rangle^{2/3} u\|_{L^2(\mathbb{R}^{2d})}\Big) \end{aligned} \tag{1.4}$$

holds for all $u \in \mathcal{C}_0^\infty(\mathbb{R}^{2d})$. Moreover, if V does not have any local minimum, that is if $\mathrm{Tr}_{-,V} + \min_{q\in\mathbb{R}^d} |\nabla\, V(q)| \neq 0$, there exists a constant $c>0$ such that

$$\|K_V u\|^2_{L^2(\mathbb{R}^{2d})} \geq c\, B_V \|u\|^2_{L^2(\mathbb{R}^{2d})} \ , \tag{1.5}$$

holds for all $u \in \mathcal{C}_0^\infty(\mathbb{R}^{2d})$. Hence combining (1.5) and (1.4), there is a constant $c>0$ so that

$$\begin{aligned} \|K_V u\|^2_{L^2(\mathbb{R}^{2d})} \geq \frac{c}{1+\frac{A_V}{B_V}}\Big(&\|O_p u\|^2_{L^2(\mathbb{R}^{2d})} + \|X_V u\|^2_{L^2(\mathbb{R}^{2d})} \\ &+ \|\langle \partial_q V(q)\rangle^{2/3} u\|^2_{L^2(\mathbb{R}^{2d})} + \|\langle D_q\rangle^{2/3} u\|_{L^2(\mathbb{R}^{2d})}\Big) \end{aligned} \tag{1.6}$$

is valid for all $u \in \mathcal{C}_0^\infty(\mathbb{R}^{2d})$. The constants appearing in (1.4), (1.5) and (1.6) are independent of the potential V. The estimates (1.5) and (1.6) can be seen in [BNV] by combining each result of Theorem 1.1 and Theorem 1.2 along with the inequality (2.1). We recall here that for a smooth potential $V \in \mathcal{C}^\infty(\mathbb{R}^d)$, our operator K_V is essential maximal accretive when endowed with the domain $\mathcal{C}_0^\infty(\mathbb{R}^{2d})$ [HeNi]. As a result the domain of its closure is given by

$$D(K_V) = \big\{u \in L^2(\mathbb{R}^{2d}),\ K_V u \in L^2(\mathbb{R}^{2d})\big\} \ .$$

Consequently by density of $\mathcal{C}_0^\infty(\mathbb{R}^{2d})$ in $D(K_V)$ all estimates stated in this paper, which are checked with $C_0^\infty(\mathbb{R}^{2d})$ functions, can be extended to the domain of K_V.

Given a polynomial $V(q)$ with degree r greater than two, our result will require the following assumption after setting for $\kappa > 0$

$$\Sigma(\kappa) = \Big\{q \in \mathbb{R}^d,\ |\nabla V(q)|^{\frac{4}{3}} \geq \kappa\Big(\,|\mathrm{Hess}\ V(q)| + R_V^{\geq 3}(q)^4 + 1\Big)\Big\} \ .$$

Assumption 1. *There exist large constants* $\kappa_0, C_1 > 1$ *such that for all* $\kappa \geq \kappa_0$ *the polynomial* $V(q)$ *satisfies the following properties*

$$\mathrm{Tr}_{-,V}(q) \geq \frac{1}{C_1}\mathrm{Tr}_{+,V}(q), \quad \textit{for all } q \in \mathbb{R}^d \setminus \Sigma(\kappa) \textit{ with } |q| \geq C_1 \ , \tag{1.7}$$

moreover if $\mathbb{R}^d \setminus \Sigma(\kappa)$ *is not bounded*

$$\lim_{\substack{q\to\infty \\ q\in\mathbb{R}^d\setminus\Sigma(\kappa)}} \frac{R_V^{\geq 3}(q)^4}{|\mathrm{Hess}\, V(q)|} = 0 \ . \tag{1.8}$$

Our main result is the following.

Theorem 1.1. *Let* $V(q)$ *be a polynomial of degree* r *greater than two verifying Assumption 1. Then there exists a strictly positive constant* $C_V > 1$ *(depending on* V*) such that*

$$\begin{aligned}\|K_V u\|_{L^2}^2 + C_V\|u\|_{L^2}^2 \geq \frac{1}{C_V}\Big(&\|L(1+O_p)u\|_{L^2}^2 + \|L(\langle 1+|\nabla V(q)|\rangle^{\frac{2}{3}})u\|_{L^2}^2 \\ &+ \|L(\langle 1+|\mathrm{Hess}\, V(q)|\rangle^{\frac{1}{2}})u\|_{L^2}^2 + \|L(\langle 1+|D_q|\rangle^{\frac{2}{3}})u\|_{L^2}^2\Big) \ ,\end{aligned} \tag{1.9}$$

holds for all $u \in D(K_V)$ *where* $L(s) = \frac{s}{\log(s)}$ *for any* $s \geq 2$.

Corollary 1.2. *If* $V(q)$ *is polynomial of degree greater than two that satisfies Assumption 1, then the Kramers-Fokker-Planck operator* K_V *has a compact resolvent.*

Proof. Proof of Corollary 1.2
Assume $0 < \delta < 1$. Define the functions $f_\delta : \mathbb{R}^d \to \mathbb{R}$ by

$$f_\delta(q) = |\nabla V(q)|^{\frac{4}{3}(1-\delta)} + |\mathrm{Hess}\, V(q)|^{1-\delta} \ .$$

From (1.9) in Theorem 1.1 there is a constant $C_V > 1$ such that

$$\|K_V u\|^2 + C_V\|u\|^2 \geq \frac{1}{C_V}\langle u, f_\delta u\rangle + \|L(1+O_p)u\|_{L^2}^2 + \|L(\langle 1+|D_q|\rangle^{\frac{2}{3}})u\|_{L^2}^2 \ ,$$

holds for all $u \in \mathcal{C}_0^\infty(\mathbb{R}^{2d})$ and all $\delta \in (0,1)$. In order to prove that the operator K_V has a compact resolvent it is sufficient to show that $\lim_{q\to+\infty} f_\delta(q) = +\infty$.

To do so, assume $A > 0$ and denote $\kappa = A^{\frac{1}{1-\delta}}$. If $q \in \Sigma(\kappa)$, one has

$$|\nabla V(q)|^{\frac{4}{3}(1-\delta)} \geq \kappa^{1-\delta} = A \ .$$

Else if $q \in \mathbb{R}^d \setminus \Sigma(\kappa)$ by (1.8) in Assumption 1, $\lim_{\substack{q\to\infty \\ q\in\mathbb{R}^d\setminus\Sigma(\kappa)}} |\mathrm{Hess}\, V(q)| = +\infty$. Hence there exists a constant $\eta > 0$ such that $|\mathrm{Hess}\, V(q)|^{1-\delta} \geq A$ for all $q \in \mathbb{R}^d \setminus \Sigma(\kappa)$ with $|q| \geq \eta$.

□

2 Preliminary results

This work is essentially based on two main strategies. The first one consists in the use of a partition of unity which is the most important tool that allows one to pass from local to global estimates.

In this paper, given a polynomial $V(q)$ we make use of a locally finite partition of unity with respect to the position variable $q \in \mathbb{R}^d$

$$\sum_{j\in\mathbb{N}} \chi_j^2(q) = \sum_{j\in\mathbb{N}} \widetilde{\chi}_j^2\Big(R_V^{\geq 3}(q_j)^{-1}(q-q_j)\Big) = 1 \tag{2.1}$$

where

$$\text{supp } \widetilde{\chi}_j \subset B(q_j, a) \text{ and } \widetilde{\chi}_j \equiv 1 \text{ in } \quad B(q_j, b)$$

for some $q_j \in \mathbb{R}^d$ with $0 < b < a$ independent of $j \in \mathbb{N}$. Such a partition is described more precisely in Lemma A.6 after taking $n = 3$. In our study introducing this partition yields to errors to be well controlled.

The second approach lies in the decomposition of the operator K_V onto two parts so that the first one be a Kramers-Fokker-Planck operator with polynomial potential of degree less than three. On this way, based on [BNV], we derive the result of Theorem 1.1.

In order to prove Theorem 1.1 we need the following basic lemmas.

Lemma 2.1. *Assume $V \in E_r$ with degree $r \in \mathbb{N}$. Consider the Kramers-Fokker-Planck operator K_V defined as in (1.1). For a locally finite partition of unity namely $\sum_{j\in\mathbb{N}} \chi_j^2(q) = 1$ one has*

$$\|K_V u\|^2_{L^2(\mathbb{R}^{2d})} = \sum_{j\in\mathbb{N}} \|K_V(\chi_j u)\|^2_{L^2(\mathbb{R}^{2d})} - \|(p\partial_q \chi_j)u\|^2_{L^2(\mathbb{R}^{2d})} \ , \tag{2.2}$$

for all $u \in \mathcal{C}_0^\infty(\mathbb{R}^{2d})$.

In particular when the degree of V is larger than two and the cutoff functions χ_j have the form (2.1), there exists a constant $c_d > 0$ (depending on the dimension d) so that

$$\|K_V u\|^2_{L^2(\mathbb{R}^{2d})} \geq \sum_{j\in\mathbb{N}} \|K_V(\chi_j u)\|^2_{L^2(\mathbb{R}^{2d})} - c_d R_V^{\geq 3}(q_j)^2 \|p\chi_j u\|^2_{L^2(\mathbb{R}^{2d})} \ , \tag{2.3}$$

holds for all $u \in \mathcal{C}_0^\infty(\mathbb{R}^{2d})$.

Proof. First let $V \in E_r$ with $r \in \mathbb{N}$ is the degree of V. Assume that $u \in \mathcal{C}_0^\infty(\mathbb{R}^{2d})$. On the one hand,

$$\|K_V u\|^2_{L^2} = \sum_{j\in\mathbb{N}} \langle K_V u, \chi_j^2 K_V u\rangle = \sum_{j\in\mathbb{N}} \langle u, K_V^* \chi_j^2 K_V u\rangle \ .$$

On the other hand,

$$\sum_{j\in\mathbb{N}} \|K_V(\chi_j u)\|^2_{L^2} = \sum_{j\in\mathbb{N}} \langle u, \chi_j K_V^* K_V \chi_j u\rangle \ .$$

Putting the above equalities together

$$\|K_V u\|_{L^2}^2 - \sum_{j\in\mathbb{N}} \|K_V(\chi_j u)\|_{L^2}^2 = \sum_{j\in\mathbb{N}} \langle u, (K_V^*\chi_j^2 K_V - \chi_j K_V^* K_V \chi_j)u\rangle \ .$$

Using commutators, we compute

$$\begin{aligned} K_V^*\chi_j^2 K_V &= K_V^*\chi_j[\,\chi_j, K_V\,] + K_V^*\chi_j K_V\chi_j \\ &= K_V^*\chi_j[\,\chi_j, K_V\,] + [\,K_V^*, \chi_j\,]K_V\chi_j + \chi_j K_V^* K_V\chi_j \\ &= K_V^*\chi_j[\,\chi_j, K_V\,] + [\,K_V^*, \chi_j\,]\Big([\,K_V, \chi_j\,] + \chi_j K_V\Big) + \chi_j K_V^* K_V \chi_j \ . \end{aligned}$$

Thus

$$K_V^*\chi_j^2 K_V - \chi_j K_V^* K_V\chi_j = K_V^*\chi_j[\,\chi_j, K_V\,] + [\,K_V^*, \chi_j\,]\chi_j K_V + [\,K_V^*, \chi_j\,]\circ[\,K_V, \chi_j\,] \ .$$

Now it is easy to check the following commutation relations

$$\left\{ \begin{array}{l} [\,\chi_j, K_V\,] = -[\,K_V, \chi_j\,] = -[\,p\partial_q, \chi_j(q)\,] = -p\partial_q\chi_j \\ [\,K_V^*, \chi_j\,] = [\,-p\partial_q, \chi_j(q)\,] = -p\partial_q\chi_j \\ [\,K_V^*, \chi_j\,]\circ[\,K_V, \chi_j\,] = -(p\partial_q\chi_j)^2 \ . \end{array} \right.$$

Collecting the terms, we obtain

$$\begin{aligned} \sum_{j\in\mathbb{N}} (K_V^*\chi_j^2 K_V - \chi_j K_V^* K_V\chi_j) &= \sum_{j\in\mathbb{N}} K_V^*\chi_j(-p\partial_q\chi_j) + (-p\partial_q\chi_j)\chi_j K_V - (p\partial_q\chi_j)^2 \\ &= \sum_{j\in\mathbb{N}} K_V^*\Big(\partial_q(\frac{\chi_j^2}{2})\Big) - p\partial_q(\frac{\chi_j^2}{2})K_V - (p\partial_q\chi_j)^2 \\ &= -(p\partial_q\chi_j)^2 \ , \end{aligned}$$

where in the last line we make use simply $\sum\limits_{j\in\mathbb{N}} \chi_j^2(q) = 1$.

From this follows immediately the identity

$$\|K_V u\|_{L^2}^2 = \sum_{j\in\mathbb{N}} \Big(\|K_V(\chi_j u)\|_{L^2}^2 - \|(p\partial_q\chi_j)u\|_{L^2}^2\Big)$$

for all $u \in \mathcal{C}_0^\infty(\mathbb{R}^{2d})$.

Next, suppose that the degree of V is greater than two and $\chi_j(q) = \widetilde{\chi}_j\Big(R_V^{\geq 3}(q_j)^{-1}(q-q_j)\Big)$ for all index j and any $q \in \mathbb{R}^d$ with

$$\operatorname{supp}\,\widetilde{\chi}_j \subset B(q_j, a) \text{ and } \ \widetilde{\chi}_j \equiv 1 \text{ in } \ B(q_j, b) \ .$$

Then we can write

$$\begin{aligned} \sum_{j\in\mathbb{N}} \|(p\partial_q\chi_j)u\|^2 &= \sum_{j\in\mathbb{N}}\sum_{j'\in\mathbb{N}} \|(p\partial_q\chi_j)\chi_{j'}u\|^2 \\ &\leq c_d \sum_{j\in\mathbb{N}} R_V^{\geq 3}(q_j)^2 \|p\chi_j u\|^2 \end{aligned}$$

where c_d is a constant that depends only on the dimension d. Here the last inequality is due to the fact that for each index j there are finitely many j' such that $(\partial_q\chi_j)\chi_{j'}$ is nonzero. $\square$

Before stating the following lemma, we fix and remind some notations.

Notations 2.2. *Let V be a polynomial of degree r larger than two. Consider a locally finite partition of unity $\sum_{j\in\mathbb{N}} \chi_j^2(q) = 1$ described as in (2.1).*

Set for all $\kappa > 0$

$$J(\kappa) = \Big\{ j \in \mathbb{N},\ \textit{such that}\ \operatorname{supp} \chi_j \subset \Sigma(\kappa) \Big\} ,$$

where we recall that

$$\Sigma(\kappa) = \Big\{ q \in \mathbb{R}^d,\ |\nabla V(q)|^{\frac{4}{3}} \geq \kappa\Big(|\mathrm{Hess}\, V(q)| + R_V^{\geq 3}(q)^4 + 1 \Big) \Big\} .$$

For a given $\kappa > 0$ and all index $j \in \mathbb{N}$, let V_j^2 be the polynomial of degree less than three given by

$$V_j^2(q) = \sum_{0\leq|\alpha|\leq 2} \frac{\partial_q^\alpha V(q_j')}{\alpha!}(q - q_j')^\alpha , \tag{2.4}$$

where

$$\begin{cases} q_j' = q_j & \textit{if } j \in J(\kappa) \\ q_j' \in (\operatorname{supp} \chi_j) \cap \Big(\mathbb{R}^d \setminus \Sigma(\kappa) \Big) & \textit{else.} \end{cases}$$

Lemma 2.3. *Assume V a polynomial of degree r larger than two. Consider a locally finite partion of unity described as in (2.1). For a multi-index $\alpha \in \mathbb{N}^d$ of length $|\alpha| \in \{1, 2\}$ and all $j \in \mathbb{N}$, one has*

$$\left| \partial_q^\alpha V(q) - \partial_q^\alpha V_j^2(q) \right| \leq c_{\alpha,d,r} \Big(R_V^{\geq 3}(q_j') \Big)^{|\alpha|} \tag{2.5}$$

for any $q \in \operatorname{supp} \chi_j = B(q_j, aR_V^{\geq 3}(q_j)^{-1})$, where $c_{\alpha,d,r} = \sum_{3\leq|\beta|\leq r} \beta!\ a^{-|\beta|+|\alpha|}$.

As a consequence, if V satisfies Assumption 1, there exists a large constant $\kappa_1 \geq \kappa_0$ so that for all $\kappa \geq \kappa_1$

- *if $j \in J(\kappa)$*

$$2^{-1} \left| \partial_q V_j^2(q) \right| \leq |\partial_q V(q)| \leq 2 \left| \partial_q V_j^2(q) \right| \qquad \textit{for every } q \in \operatorname{supp} \chi_j , \tag{2.6}$$

- *if $j \notin J(\kappa)$*

$$2^{-1} \left| \mathrm{Hess} V_1^2(q) \right| \leq |\mathrm{Hess}\, V| \leq 2 \left| \mathrm{Hess}\, V_j^2(q) \right| , \tag{2.7}$$

for any $q \in \operatorname{supp} \chi_j$ with $|q| \geq C_2(\kappa)$ where $C_2(\kappa) > 0$ is a large constant that depends on κ.

Proof. Let V be a polynomial of degree r greater than two. In this proof we are going to need the following equivalence

$$R_V^{\geq 3}(q) \asymp R_V^{\geq 3}(q') \,, \tag{2.8}$$

satisfied for all $q, q' \in \operatorname{supp} \chi_j$ and proved in Lemma A.4. That is there is a constant $C > 1$ such that for every $q, q' \in \operatorname{supp} \chi_j$,

$$\Big(\frac{R_V^{\geq 3}(q)}{R_V^{\geq 3}(q')}\Big)^{\pm 1} \leq C \,. \tag{2.9}$$

Assume $\alpha \in \mathbb{N}^d$ of length $|\alpha| \in \{1, 2\}$. For every $j \in \mathbb{N}$, observe that

$$\begin{aligned}\left|\partial_q^\alpha V(q) - \partial_q^\alpha V_j^2(q)\right| &= \Big|\sum_{\substack{3\leq|\beta|\leq r\\ \beta\geq\alpha}} \frac{\beta!}{(\beta-\alpha)!}\partial_q^\beta V(q_j')(q-q_j')^{\beta-\alpha}\Big| \\ &\leq \sum_{\substack{3\leq|\beta|\leq r\\ \beta\geq\alpha}} \frac{\beta!}{(\beta-\alpha)!}\left|\partial_q^\beta V(q_j')\right| \, \left|q-q_j'\right|^{|\beta|-|\alpha|} \,,\end{aligned}$$

for any $q \in \mathbb{R}^d$. Hence regarding the equivalence (2.9), there exists a constant $c_{\alpha,d,r} > 0$ (depending as well on the multi-index α, the dimesion d and the degree r of V) so that

$$\begin{aligned}\left|\partial_q^\alpha V(q) - \partial_q^\alpha V_j^2(q)\right| &\leq \sum_{\substack{3\leq|\beta|\leq r\\ \beta\geq\alpha}} \frac{\beta!}{(\beta-\alpha)!}\Big(R_V^{\geq 3}(q_j')\Big)^{|\beta|}\Big(aR_V^{\geq 3}(q_j)\Big)^{-|\beta|+|\alpha|} \\ &\leq \sum_{\substack{3\leq|\beta|\leq r\\ \beta\geq\alpha}} \frac{\beta!}{(\beta-\alpha)!}a^{-|\beta|+|\alpha|}\Big(R_V^{\geq 3}(q_j')\Big)^{|\alpha|} \\ &\leq c_{\alpha,d,r}\Big(R_V^{\geq 3}(q_j')\Big)^{|\alpha|} \,,\end{aligned} \tag{2.10}$$

holds for all q in the support of χ_j, where $c_{\alpha,d,r} = \sum_{3\leq|\beta|\leq r} \beta! \; a^{-|\beta|+|\alpha|}$.

In the rest of the proof, let the polynomial $V(q)$ satisfies Assumption 1. In vue of (2.10), we get when $|\alpha| = 1$

$$\left|\nabla V(q) - \nabla V_j^2(q)\right| \leq c_{1,d,r} \; R_V^{\geq 3}(q_j') \,, \tag{2.11}$$

for all $j \in \mathbb{N}$ and any $q \in \operatorname{supp} \chi_j$, where $c_{1,d,r} = \sum_{3\leq|\beta|\leq r} \beta! \; a^{-|\beta|+1}$. Given $\kappa \geq \kappa_0$, assume first that $j \in J(\kappa)$. By virtue of the equivalence (2.9), it results from (2.11)

$$\left|\nabla V(q) - \nabla V_j^2(q)\right| \leq c_{1,d,r}C \; R_V^{\geq 3}(q) \,, \tag{2.12}$$

for every $q \in \text{supp}\ \chi_j$. Then we obtain

$$\begin{aligned}\left|\nabla V(q) - \nabla V_j^2(q)\right| &\leq \frac{c_{1,d,r}C}{\kappa^{\frac{1}{4}}}\,|\nabla V(q)|^{\frac{1}{3}} \\ &\leq \frac{c_{1,d,r}C}{\kappa^{\frac{1}{4}}}\,|\nabla V(q)| \end{aligned} \tag{2.13}$$

for all $q \in \text{supp}\ \chi_j$. For the above second inequality we know that $|\nabla V(q)| \geq 1$ for every $q \in \text{supp}\ \chi_j$, indeed since $j \in J(\kappa)$,

$$|\nabla V(q)| \geq \kappa^{\frac{3}{4}} \geq \kappa_0^{\frac{3}{4}} \geq 1\ .$$

Taking the constant $\kappa_1 \geq \kappa_0$ such that $\frac{c_{1,d,r}C}{\kappa_1^{\frac{1}{4}}} \leq \frac{1}{2}$, we get for every $\kappa \geq \kappa_1$

$$\Big|\,|\nabla V(q)| - |\nabla V_j^2(q)|\Big| \leq |\nabla V(q) - \nabla V_j^2(q)| \leq \frac{1}{2}|\nabla V(q)|\ ,$$

for any $q \in \text{supp}\ \chi_j$ when $j \in J(\kappa)$. Therefore

$$\frac{1}{2}|\nabla V_j^2(q)| \leq |\nabla V(q)| \leq \frac{3}{2}|\nabla V_j^2(q)|$$

holds for all $q \in \text{supp}\ \chi_j$ when $j \in J(\kappa)$.

On the other hand when $|\alpha| = 2$, by (2.10) and (2.9) there is a constant $c_{2,d,r} > 0$ so that for all $j \in \mathbb{N}$

$$|\partial_q^\alpha V(q) - \partial_q^\alpha V_j^2(q)| \leq c_{2,d,r} C^2 R_V^{\geq 3}(q)^2\ . \tag{2.14}$$

holds for every $q \in \text{supp}\ \chi_j$, where $c_{2,d,r} = \sum\limits_{3 \leq |\beta| \leq r} \beta!\ a^{-|\beta|+2}$. Given $\kappa \geq \kappa_0$ assume now $j \notin J(\kappa)$. Using the fact that $R_V^{\geq 3}(q) \geq R_V^{=r}(0)$ for every $q \in \mathbb{R}^d$, we derive from (2.14) that

$$|\partial_q^\alpha V(q) - \partial_q^\alpha V_j^2(q)| \leq c_{2,d,r} C^2 \frac{R_V^{\geq 3}(q)^4}{R_V^{=r}(0)^2}\ ,$$

for all $q \in \text{supp}\ \chi_j$.

Assuming $\kappa \geq \kappa_0$ and $j \notin J(\kappa)$, we obtain using the previous inequality and applying Lemma B.6

$$\Big|\sum_{|\alpha|=2} |\partial_q^\alpha V(q)| - \sum_{|\alpha|=2} |\partial_q^\alpha V_j^2(q)|\Big| \leq \sum_{|\alpha|=2} |\partial_q^\alpha V(q) - \partial_q^\alpha V_j^2(q)| \leq \frac{1}{2}|\text{Hess}\ V(q)|\ ,$$

for any $q \in \text{supp}\ \chi_j$ with $|q| \geq C_2(\kappa)$ where $C_2(\kappa)$ is a strictly positive large constant depending on κ. In other words,

$$\frac{1}{2}|\text{Hess}\ V_j^2(q)| \leq |\text{Hess}\ V(q)| \leq \frac{3}{2}|\text{Hess}\ V_j^2(q)|$$

holds for all $q \in \text{supp}\ \chi_j$ with $|q| \geq C_2(\kappa)$ and $j \notin J(\kappa)$. □

Lemma 2.4. *Given two positive operators A and B such that*

$$2\|u\|^2 \leq \langle u, Au\rangle \leq \langle u, Bu\rangle$$

for all $u \in \mathcal{D}$ where $\mathcal{D}$ is dense in $D(A^{1/2})$, one has

$$\langle u, \frac{A^{\alpha_0}}{(\log(A^{\alpha_0/2}))^k} u\rangle \leq \langle u, \frac{B^{\alpha_0}}{(\log(B^{\alpha_0/2}))^k} u\rangle \ , \tag{2.15}$$

for all $u \in \mathcal{D}$, any $\alpha_0 \in [0, 1]$ and every natural number k.

Proof. Assume that A, B are two positive operators so that

$$2\|u\|^2 \leq \langle u, Au\rangle \leq \langle u, Bu\rangle \ , \tag{2.16}$$

holds for all $u \in \mathcal{D}$. Referring to [Sim] (see Proposition 6.7 and Example 6.8), for any positive operator C and every $\alpha \in (0, 1)$ we can write

$$C^\alpha = \frac{2\sin(\pi\alpha)}{\pi} \int_0^{+\infty} w^{\alpha-1}(C + w)^{-1} C dw \ . \tag{2.17}$$

From (2.16) and (2.17)

$$2^\alpha\|u\|^2 \leq \langle u, A^\alpha u\rangle \leq \langle u, B^\alpha u\rangle \ , \tag{2.18}$$

for any $u \in \mathcal{D}$ and every $\alpha \in [0, 1]$.

Furthermore, for any positive operator C with domain $D(C)$ we define its logarithm for all $u \in D(C)$ by

$$\langle u, \log(C)u\rangle = \lim_{\alpha\to 0^+} \langle u, \frac{C^\alpha - 1}{\alpha} u\rangle \ , \tag{2.19}$$

where the operator C^α is given in (2.17).

Using (2.16) and (2.19)

$$\log(2)\|u\|^2 \leq \langle u, \log(A)u\rangle \leq \langle u, \log(B)u\rangle \ , \tag{2.20}$$

holds for all $u \in \mathcal{D}$. Integrating (2.18) with respect to α over $[0, \alpha_0]$ where $\alpha_0 \in [0, 1]$ we get

$$\langle u, \frac{1}{\log(A)}(A^{\alpha_0} - I)u\rangle \leq \langle u, \frac{1}{\log(B)}(B^{\alpha_0} - I)u\rangle \ . \tag{2.21}$$

Furthermore by (2.20)

$$\langle u, \frac{1}{\log(B)} u\rangle \leq \langle u, \frac{1}{\log(A)} u\rangle \leq \frac{1}{\log(2)}\|u\|^2 \ . \tag{2.22}$$

Therefore from (2.21) and (2.22)

$$\langle u, \frac{A^{\alpha_0}}{\log(A)} u\rangle \leq \langle u, \frac{B^{\alpha_0}}{\log(B)} u\rangle.$$

holds for any $\alpha_0 \in [0,1]$. Then by induction on $k \in \mathbb{N}$, we obtain

$$\langle u, \frac{A^{\alpha_0}}{(\log(A))^k} u\rangle \leq \langle u, \frac{B^{\alpha_0}}{(\log(B))^k} u\rangle,$$

for all $\alpha_0 \in [0,1]$ and every natural number k. Or equivalently

$$\langle u, \frac{A^{\alpha_0}}{(\log(A^{\alpha_0/2}))^k} u\rangle \leq \langle u, \frac{B^{\alpha_0}}{(\log(B^{\alpha_0/2}))^k} u\rangle,$$

for every $\alpha_0 \in [0,1],\ k \in \mathbb{N}$.

□

Lemma 2.5. *Assume $V(q)$ a polynomial of degree greater than two. Let $\sum_{j\in\mathbb{N}} \chi_j^2(q)$ be a locally finite partition of unity defined as in (2.1).*

There is a constant $c > 0$ such that

$$\langle u, (2 + D_q^2 + R_V^{\geq 3}(q)^4)^\alpha u\rangle \leq c \sum_{j\in\mathbb{N}} \langle u, \chi_j (2 + D_q^2 + R_V^{\geq 3}(q_j')^4)^\alpha \chi_j u\rangle \ , \tag{2.23}$$

is valid for all $u \in \mathcal{C}_0^\infty(\mathbb{R}^{2d})$ and any $\alpha \in [0,1]$.

As a consequence, there exists a constant $c > 0$ so that

$$\sum_{j\in\mathbb{N}} \|L\Big((2 + D_q^2 + R_V^{\geq 3}(q_j')^4)^{\frac{1}{3}}\Big)\chi_j u\|^2 \geq \frac{1}{c}\|L\Big((2 + D_q^2 + R_V^{\geq 3}(q)^4)^{\frac{1}{3}}\Big) u\|^2 \ , \tag{2.24}$$

holds for all $u \in \mathcal{C}_0^\infty(\mathbb{R}^{2d})$, where $L(s) = \frac{s}{\log(s)}$ for all $s \geq 2$.

Proof. We first set $E_0 = L^2(\mathbb{R}^{2d})$ and $E_1 = \Big\{ u \in L^2(\mathbb{R}^{2d}),\ \langle u, (2 - \Delta_q + R_V^{\geq 3}(q)^4) u\rangle < +\infty \Big\}$ endowed respectively with the norms $\|\cdot\|_{E_0} = \|\cdot\|_{L^2(\mathbb{R}^{2d})}$ and $\|\cdot\|_{E_1}$ defined as follows for all $u \in L^2(\mathbb{R}^{2d})$

$$\begin{aligned}\|u\|_{E_1}^2 &= 2\|u\|_{L^2(\mathbb{R}^{2d})}^2 + \|D_q u\|_{L^2(\mathbb{R}^{2d})}^2 + \|R_V^{\geq 3}(q)^2 u\|_{L^2(\mathbb{R}^{2d})}^2 \\ &= \|(2 - \Delta_q + R_V^{\geq 3}(q)^4)^{1/2} u\|_{L^2(\mathbb{R}^{2d})}^2 \ .\end{aligned}$$

By Simader theorem (which states that if $W \in \mathcal{C}^\infty(\mathbb{R}^d)$ and $-\Delta + W(x)$ is a symetric non negative operator on $\mathcal{C}_0^\infty(\mathbb{R}^d)$ then $-\Delta + W(x)$ is essentially self adjoint on $\mathcal{C}_0^\infty(\mathbb{R}^d)$), the operator $2 - \Delta_q + R_V^{\geq 3}(q)^4$ is essentially self adjoint on $\mathcal{C}_0^\infty(\mathbb{R}^{2d})$ and hence E_1 corresponds to the spectrally defined subspace of $L^2(\mathbb{R}^{2d})$.

Given a partition of unity as in (2.1), define the linear map

$$T : E_0 \to (L^2(\mathbb{R}^{2d}))^{\mathbb{N}},\ u \mapsto (u_j)_{j\in\mathbb{N}} = (\chi_j u)_{j\in\mathbb{N}}\ ,$$

and denote $F_0 := \operatorname{Im} T$. Notice that $T : E_0 \to F_0$ is a unitary isometry. Indeed for all $u \in E_0$,

$$\|Tu\|_{F_0}^2 = \sum_{j\in\mathbb{N}} \|\chi_j u\|_{L^2}^2 = \|u\|_{L^2}^2 = \|u\|_{E_0}^2\ , \tag{2.25}$$

further the inverse map of T is well defined by

$$T^{-1} : F_0 \to E_0,\ (u_j)_{j\in\mathbb{N}} \mapsto u = \sum_{j\in\mathbb{N}} \chi_j u_j\ .$$

Now introduce the set

$$F_1 = \left\{ (u_j)_{j\in\mathbb{N}} \in F_0,\ \sum_{j\in\mathbb{N}} \langle u_j, (2 - \Delta_q + R_V^{\geq 3}(q_j')^4) u_j\rangle < +\infty \right\}\ ,$$

with its associated norm defined for all $(u_j)_{j\in\mathbb{N}} \in F_1$ by

$$\begin{aligned}\|(u_j)_{j\in\mathbb{N}}\|_{F_1}^2 &= \sum_{j\in\mathbb{N}} \left(2\|u_j\|_{L^2(\mathbb{R}^{2d})}^2 + \|D_q u_j\|_{L^2(\mathbb{R}^{2d})}^2 + \|R_V^{\geq 3}(q_j')^2 u_j\|_{L^2(\mathbb{R}^{2d})}^2 \right) \\ &= \sum_{j\in\mathbb{N}} \|(2 - \Delta_q + R_V^{\geq 3}(q_j')^4)^{1/2} u_j\|_{L^2(\mathbb{R}^{2d})}^2\ .\end{aligned}$$

Assume $u \in E_0$. For all $j \in \mathbb{N}$, let $q_j' \in \operatorname{supp} \chi_j$. Observe that

$$\begin{aligned}|\,\|Tu\|_{F_1}^2 - \|u\|_{E_1}^2| &= |\sum_{j\in\mathbb{N}} \langle u_j, (2 - \Delta_q + R_V^{\geq 3}(q_j')^4) u_j\rangle - \langle u, (2 - \Delta_q + R_V^{\geq 3}(q)^4) u\rangle| \\ &= |\sum_{j\in\mathbb{N}} \langle u_j, -\Delta_q u_j\rangle - \langle u, -\Delta_q u\rangle + \sum_{j\in\mathbb{N}} \langle u_j, (R_V^{\geq 3}(q_j')^4 - R_V^{\geq 3}(q)^4) u_j\rangle| \\ &\leq |\sum_{j\in\mathbb{N}} \langle u_j, -\Delta_q u_j\rangle - \langle u, -\Delta_q u\rangle| + \sum_{j\in\mathbb{N}} \langle u_j, |R_V^{\geq 3}(q_j')^4 - R_V^{\geq 3}(q)^4| u_j\rangle\ .\end{aligned} \tag{2.26}$$

Since we are dealing with cutoff functions satisfying $\sum_{j\in\mathbb{N}} |\nabla \chi_j|^2 \leq c\, R_V^{\geq 3}(q)^2$ and owning to the equivalence $R_V^{\geq 3}(q) \asymp R_V^{\geq 3}(q_j')$ for all $q \in \operatorname{supp} \chi_j$, it follows from (2.26)

$$|\,\|Tu\|_{F_1}^2 - \|u\|_{E_1}^2| \leq c_1 \sum_{j\in\mathbb{N}} \langle u_j, R_V^{\geq 3}(q_j')^4 u_j\rangle \leq c_1 \|Tu\|_{F_1}^2\ ,$$

and

$$|\,\|Tu\|_{F_1}^2 - \|u\|_{E_1}^2| \leq c_1' \langle u, R_V^{\geq 3}(q)^4 u\rangle \leq c_1' \|u\|_{E_1}^2\ ,$$

where c_1, c_1' are two strictly positive constants. As a result

$$\frac{1}{\sqrt{(c_1+1)}}\|u\|_{E_1} \leq \|Tu\|_{F_1} \leq \sqrt{(c_1'+1)}\|u\|_{E_1} \ . \tag{2.27}$$

In view of (2.25) and (2.27), we conclude by interpolation that for all $\alpha \in [0,1]$

$$T : E_\alpha \to F_\alpha,$$

verifies $\|T\|_{\mathcal{L}(E_\alpha, F_\alpha)} \leq c^\alpha$ and $\|T^{-1}\|_{\mathcal{L}(F_\alpha, E_\alpha)} \leq c^\alpha$, where E_α and F_α are two interpolated spaces endowed respectively with the norms

$$\|u\|_{E_\alpha} = \|(2 - \Delta_q + R_V^{\geq 3}(q)^4)^{\alpha/2} u\|_{L^2(\mathbb{R}^{2d})} \ ,$$

and

$$\|(v_j)_{j\in\mathbb{N}}\|_{F_\alpha} = \sum_{j\in\mathbb{N}} \|(2 - \Delta_q + R_V^{\geq 3}(q_j')^4)^{\alpha/2} u_j\|_{L^2(\mathbb{R}^{2d})} \ .$$

Hence there is a constant $c > 0$ so that

$$\langle u, (2 + D_q^2 + R_V^{\geq 3}(q)^4)^\alpha u\rangle \leq c \sum_{j\in\mathbb{N}} \langle u, \chi_j (2 + D_q^2 + R_V^{\geq 3}(q_j')^4)^\alpha \chi_j u\rangle \ , \tag{2.28}$$

holds for all $u \in \mathcal{C}_0^\infty(\mathbb{R}^{2d})$ and any $\alpha \in [0,1]$. In order to prove (2.24), repeat the same process as in Lemma 2.4. Starting with

$$2\|u\|^2 \leq \langle u, (2 + D_q^2 + R_V^{\geq 3}(q)^4)^\alpha u\rangle \leq c \sum_{j\in\mathbb{N}} \langle u, \chi_j (2 + D_q^2 + R_V^{\geq 3}(q_j')^4)^\alpha \chi_j u\rangle \ , \tag{2.29}$$

for all $u \in \mathcal{C}_0^\infty(\mathbb{R}^{2d})$ and any $\alpha \in [0,1]$, remark that when integrating over $\alpha \in [0, \frac{2}{3}]$ we can interchange the sum and the integral in the left hand side of (2.29) since the partition of unity is locally finite.

This completes the proof. □

3 Proof of Theorem 1.1

In this section we present the proof of Theorem 1.1. In the sequel for a given polynomial $V(q)$ with degree r greater than two, we always use a locally finite partition of unity

$$\sum_{j\in\mathbb{N}} \chi_j^2(q) = \sum_{j\in\mathbb{N}} \widetilde{\chi}_j^2\Big(R_V^{\geq 3}(q_j)^{-1}(q - q_j)\Big) = 1 \ ,$$

where

$$\text{supp } \widetilde{\chi}_j \subset B(q_j, a) \text{ and } \ \widetilde{\chi}_j \equiv 1 \text{ in } \ B(q_j, b)$$

for some $q_j \in \mathbb{R}^d$ with $0 < b < a$ independent of the natural numbers j, defined more specifically as in Lemma A.6 with $n = 3$.

Proof. Let $V(q)$ be a polynomial with degree larger than two that satisfies Assumption 1. Assume $u \in \mathcal{C}_0^\infty(\mathbb{R}^{2d})$. In the whole proof we denote $u_j = \chi_j u$ for all natural number j.

From Lemma 2.1 we get

$$\|K_V u\|^2_{L^2(\mathbb{R}^{2d})} \geq \sum_{j\in\mathbb{N}} \|K_V u_j\|^2_{L^2(\mathbb{R}^{2d})} - c_d R_V^{\geq 3}(q_j)^2 \|p u_j\|^2_{L^2(\mathbb{R}^{2d})} \ . \tag{3.1}$$

Given $\kappa \geq \kappa_0$, set

$$J(\kappa) = \Big\{ j \in \mathbb{N}, \text{ such that } \operatorname{supp} \chi_j \subset \Sigma(\kappa) \Big\} \ .$$

For all index $j \in \mathbb{N}$, let V_j^2 be the polynomial of degree less than three given by

$$V_j^2(q) = \sum_{0\leq|\alpha|\leq 2} \frac{\partial_q^\alpha V(q'_j)}{\alpha!}(q - q'_j)^\alpha \ ,$$

where

$$\begin{cases} q'_j = q_j & \text{if } j \in J(\kappa) \\ q'_j \in (\operatorname{supp} \chi_j) \cap \Big(\mathbb{R}^d \setminus \Sigma(\kappa)\Big) & \text{else.} \end{cases}$$

We associate with each polynomial V_j^2 the Kramers-Fokker-Planck operator $K_{V_j^2}$. Observe that using the parallelogram law $2(\|x\|^2 + \|y\|^2) - \|x+y\|^2 = \|x-y\|^2 \geq 0$,

$$\begin{aligned} \sum_{j\in\mathbb{N}} \|K_V u_j\|^2_{L^2(\mathbb{R}^{2d})} &= \sum_{j\in\mathbb{N}} \|K_{V_j^2} u_j + (K_V - K_{V_j^2}) u_j\|^2_{L^2(\mathbb{R}^{2d})} \\ &\geq \frac{1}{2} \sum_{j\in\mathbb{N}} \|K_{V_j^2} u_j\|^2_{L^2(\mathbb{R}^{2d})} - \|(\nabla V(q) - \nabla V_j^2(q))\partial_p u_j\|^2_{L^2(\mathbb{R}^{2d})} \end{aligned} \tag{3.2}$$

On the other hand, by (2.5) in Lemma 2.3

$$\sum_{j\in\mathbb{N}} \|(\nabla V(q) - \nabla V_j^2(q))\partial_p u_j\|^2_{L^2(\mathbb{R}^{2d})} \leq c_{1,d,r} \sum_{j\in\mathbb{N}} R_V^{\geq 3}(q'_j)^2 \|\partial_p u_j\|^2_{L^2(\mathbb{R}^{2d})} \ . \tag{3.3}$$

Combining (3.1), (3.2) and (3.3) we get immediately

$$\|K_V u\|^2_{L^2(\mathbb{R}^{2d})} \geq \frac{1}{2} \sum_{j\in\mathbb{N}} \|K_{V_j^2} u_j\|^2_{L^2(\mathbb{R}^{2d})} - c_{1,d,r} R_V^{\geq 3}(q'_j)^2 \|\partial_p u_j\|^2_{L^2(\mathbb{R}^{2d})} - c_d R_V^{\geq 3}(q_j)^2 \|p u_j\|^2_{L^2(\mathbb{R}^{2d})}$$

Therefore, making use of the equivalence $(A.5)$, it follows

$$\|K_V u\|^2_{L^2(\mathbb{R}^{2d})} \geq \frac{1}{2} \sum_{j\in\mathbb{N}} \|K_{V_j^2} u_j\|^2_{L^2(\mathbb{R}^{2d})} - c'_{d,r} R_V^{\geq 3}(q'_j)^2 \langle u_j, O_p u_j \rangle_{L^2(\mathbb{R}^{2d})} \ , \tag{3.4}$$

where $c'_{d,r} = 2(c_{1,d,r}^2 + c_d)$.

Using respectively the Cauchy Schwarz inequality then the Cauchy inequality with epsilon (for any real numbers a, b and all $\epsilon > 0$, $ab \leq \epsilon a^2 + \frac{1}{4\epsilon} b^2$),

$$\begin{aligned} c'_{d,r} R_V^{\geq 3}(q'_j)^2 \langle u_j, O_p u_j \rangle &= c'_{d,r} R_V^{\geq 3}(q'_j)^2 \mathrm{Re}\langle u_j, K_{V_j^2} u_j \rangle \\ &\leq c'_{d,r} R_V^{\geq 3}(q'_j)^2 \|u_j\| \cdot \|K_{V_j^2} u_j\| \\ &\leq \left(c'_{d,r} R_V^{\geq 3}(q'_j)^2 \right)^2 \|u_j\|^2 + \frac{1}{4} \|K_{V_j^2} u_j\|^2 \ . \end{aligned}$$

Putting the above estimate and (3.4) together we obtain

$$\|K_V u\|^2 \geq \sum_{j \in \mathbb{N}} \frac{1}{4} \|K_{V_j^2} u_j\|^2 - (c'_{d,r})^2 R_V^{\geq 3}(q'_j)^4 \|u_j\|^2 \ . \tag{3.5}$$

From now on assume $\kappa \geq \kappa_1$, where $\kappa_1 \geq \kappa_0$ is introduced in Lemma 2.3. Remember as well that the constants $C_1, C_2(\kappa)$ are given respectively in Assumption 1 (see (1.7)) and Lemma 2.3 (see (2.7)). By introducing $C(\kappa) \geq \max(C_1, C_2(\kappa))$, which will be fixed later, we set for each κ,

$$I(\kappa) = \Big\{ j \in \mathbb{N}, \text{ such that } \operatorname{supp} \chi_j \subset \big\{ q \in \mathbb{R}^d, \ \ |q| \geq C(\kappa) \big\} \Big\} \ .$$

The rest of the proof is divided into three steps. The first one is devoted to the control of the terms in the the left hand side of (3.5) for which $j \in I(\kappa)$ for some large $\kappa \geq \kappa_0$ to be chosen. At the end of the first step the constants $\kappa > \kappa_1$ and $C(\kappa) \geq \max(C_1, C_2(\kappa))$ will be fixed. So on, the second step is concerned with the remaining terms for which the support of the cutoff functions χ_j are included in some closed ball $B(0, C'(\kappa))$. We finally sum up all the terms and refer to Lemma 2.5 after some elementary optimization trick in the last step.

Step 1, $\mathbf{j \in I(\kappa)}$, $\kappa \geq \kappa_1$ to be fixed: As proved in [BNV], there is a constant $c > 0$ such that for all $j \in I(\kappa)$

$$\|K_{V_j^2} u_j\|^2 + A_{V_j^2} \|u_j\|^2 \geq c \Big(\|O_p u_j\|^2 + \|\langle \partial_q V_j^2(q) \rangle^{2/3} u_j\|^2 + \|\langle D_q \rangle^{2/3} u_j\| \Big), \tag{3.6}$$

where

$$\begin{aligned} A_{V_j^2} &= \max\{(1 + \mathrm{Tr}_{+, V_j^2})^{2/3}, 1 + \mathrm{Tr}_{-, V_j^2}\} \\ &= \max\{(1 + \mathrm{Tr}_{+,V}(q'_j))^{2/3}, 1 + \mathrm{Tr}_{-,V}(q'_j)\} \ . \end{aligned}$$

Hence there is a constant $C > 0$ so that

$$\begin{aligned} \|K_{V_j^2} u_j\|^2 + (1 + 10C) t_j^4 \|u_j\|^2 \geq C \Big(&\|(1 + O_p) u_j\|^2 + \|\langle 1 + |\partial_q V_j^2(q)| \rangle^{2/3} u_j\|^2 \\ &+ \|\langle 1 + |D_q| \rangle^{2/3} u_j\| + 10 C t_j^4 \|u_j\|^2 \Big) \ , \end{aligned} \tag{3.7}$$

where we use the notation $t_j = \langle 1 + |\text{Hess } V(q'_j)|\rangle^{1/4}$ in the whole of the proof.

Recall that as mentioned in [BNV], the constant c in (3.6) does not depend on the polynomial V_j^2 and then so is the constant C in (3.7).
Now for all index $j \in I(\kappa)$ we distinguish two cases: either $j \in J(\kappa)$ or $j \notin J(\kappa)$.

Case 1. Assume $j \in J(\kappa)$. Then taking into account the inequality (2.6) in Lemma 2.3 and using the estimate (3.7) we obtain

$$\|K_{V_j^2} u_j\|^2 + (1+10C)t_j^4 \|u_j\|^2 \geq C\Big(\|(1+O_p)u_j\|^2 + \|\langle 1 + |\partial_q V(q)|\rangle^{2/3} u_j\|^2 + \|\langle 1 + |D_q|\rangle^{2/3} u_j\| + 10 t_j^4 \|u_j\|^2 \Big)\ , \quad (3.8)$$

Furthermore, since for all index $j \in \mathbb{N}$ the quantity $R_V^{\geq 2}(q'_j)^2$ is always greater than $|\text{Hess } V(q'_j)|$, there exists a constant $c_d > 0$ so that for every $j \in J(\kappa)$,

$$t_j^4 = \langle 1 + |\text{Hess } V(q'_j)|\rangle \leq c_d \langle 1 + R_V^{\geq 2}(q'_j)^2\rangle\ .$$

Using the fact that the metric $R_V^{\geq 2}(q)\, dq^2$ is $R_V^{\geq 3}(q)\, dq^2$ slow (see Definition (A.2) and Lemma A.4), it follows

$$t_j^4 \leq c_d \langle 1 + R_V^{\geq 2}(q)^2\rangle\ ,$$

for every $q \in \text{supp } \chi_j$. Hence there is a constant $c'_d > 0$ (depending on the dimension d) such that

$$\begin{aligned} t_j^4 &\leq c_d \langle 1 + \Big(\sum_{|\alpha|=2} |\partial_q^\alpha V(q)|^{\frac{1}{|\alpha|}} + R_V^{\geq 3}(q)\Big)^2\rangle \\ &\leq 3c_d \langle 1 + (\sum_{|\alpha|=2} |\partial_q^\alpha V(q)|^{\frac{1}{|\alpha|}})^2 + R_V^{\geq 3}(q)^2\rangle \\ &\leq c'_d \langle 1 + |\text{Hess } V(q)| + R_V^{\geq 3}(q)^2\rangle\ , \end{aligned}$$

holds for any $q \in \text{supp } \chi_j$. Or since for every $q \in \mathbb{R}^d$ on has $R_V^{\geq 3}(q) \geq R_V^{=r}(0)$, we derive from the previous estimate that for any $q \in \text{supp } \chi_j$,

$$\begin{aligned} t_j^4 &\leq c'_d \langle 1 + |\text{Hess } V(q)| + \frac{R_V^{\geq 3}(q)^4}{R_V^{=r}(0)^2}\rangle \\ &\leq \frac{c"_d}{\kappa} \max(1, R_V^{=r}(0)^{-2}) \langle 1 + |\partial_q V(q)|\rangle^{\frac{4}{3}}\ . \end{aligned} \quad (3.9)$$

Collecting the estimates (3.8) and (3.9), we get for $\kappa \geq \kappa_1$

$$\begin{aligned} &\|K_{V_j^2} u_j\|^2 + (1+10C)\frac{c"_d}{\kappa} \max(1, R_V^{=r}(0)^{-2}) \|\langle 1 + |\partial_q V(q)|\rangle^{\frac{2}{3}} u_j\|^2 \\ &\quad \geq C\Big(\|(1+O_p)u_j\|^2 + \|\langle 1 + |\partial_q V(q)|\rangle^{2/3} u_j\|^2 + \|\langle 1 + |D_q|\rangle^{2/3} u_j\|^2 + 10 t_j^4 \|u_j\|^2 \Big)\ . \end{aligned}$$

Choosing $\kappa_2 \geq \kappa_1$ so that

$$\frac{C}{2} \geq (1+10C)\frac{c"_d}{\kappa_2}\max(1, R_V^{=r}(0)^{-2}) \ ,$$

the following inequality

$$\|K_{V_j^2}u_j\|^2 \geq C\Big(\|(1+O_p)u_j\|^2 + \frac{1}{2}\|\langle 1+|\partial_q V(q)|\rangle^{2/3}u_j\|^2 + \|\langle 1+|D_q|\rangle^{2/3}u_j\|^2 + 10t_j^4\|u_j\|^2\Big) \ , \tag{3.10}$$

holds for all $j \in J(\kappa)$ with $\kappa \geq \kappa_2$.

Or since $j \in J(\kappa)$, there is a constant $c_1 > 0$ so that

$$\frac{1}{8}\langle 1+|\partial_q V(q)|\rangle^{\frac{4}{3}} \geq c_1\langle 1+|\mathrm{Hess}\ V(q)|\rangle \ , \tag{3.11}$$

holds for all $q \in \mathrm{supp}\ \chi_j$. In addition, using the equivalence (A.5) it follows

$$\frac{1}{8}\langle 1+|\partial_q V(q)|\rangle^{\frac{4}{3}} \geq c_2|\partial_q V(q)|^{\frac{4}{3}} \geq c_2\kappa\, R_V^{\geq 3}(q)^4 \geq c_2'\kappa\, R_V^{\geq 3}(q_j')^4 \ , \tag{3.12}$$

for any $q \in \mathrm{supp}\ \chi_j$.

Putting (3.10), (3.11) and (3.12) together,

$$\begin{aligned}\|K_{V_j^2}u_j\|^2 \geq & C\Big(\|(1+O_p)u_j\|^2 + \frac{1}{4}\|\langle 1+|\partial_q V(q)|\rangle^{2/3}u_j\|^2 + \|\langle 1+|D_q|\rangle^{2/3}u_j\|^2 \\ & + c_1\|\langle 1+|\mathrm{Hess}\ V(q)|\rangle^{1/2}u_j\|^2 + c_2'\kappa\, R_V^{\geq 3}(q_j')^4\|u_j\|^2 + 10\|t_j^2 u_j\|^2\Big) \ ,\end{aligned} \tag{3.13}$$

holds for all $\kappa \geq \kappa_2$.

Case 2. Assume $j \notin J(\kappa)$, with $\kappa \geq \kappa_2 \geq \kappa_1 \geq \kappa_0$. Hence by Assumption 1 (see (1.7)), one has

$$T_{-,V}(q) \neq 0 \ \text{ for all } \ q \in (\mathrm{supp}\ \chi_j) \cap (\mathbb{R}^d \setminus \Sigma(\kappa)) \ \text{ such that } \ |q| \geq C_1 \ .$$

In particular, since $|q_j'| \geq C(\kappa) \geq C_1$,

$$T_{-,V_j^2} = T_{-,V}(q_j') \neq 0 \ .$$

Then referring again to [BNV],

$$\|K_{V_j^2}u_j\|^2 \geq c\, B_{V_j^2}\|u_j\|^2 \ ,$$

where

$$\begin{aligned} B_{V_j^2} &= \max\Big(\min_{q\in\mathbb{R}^d}|\nabla V_j^2(q)|^{4/3}, \frac{1+T_{-,V_j^2}}{\log(2+T_{-,V_j^2})^2}\Big) \\ &= \max\Big(\min_{q\in\mathbb{R}^d}|\nabla V_j^2(q)|^{4/3}, \frac{1+T_{-,V}(q_j')}{\log(2+T_{-,V}(q_j'))^2}\Big) \neq 0 \ .\end{aligned}$$

Hence we get in particular

$$\|K_{V_j^2}u_j\|^2 \geq \frac{1+T_{-,V}(q_j')}{\log(2+T_{-,V}(q_j'))^2}\|u_j\|^2 \ . \tag{3.14}$$

Using again the condition (1.7) in Assumption 1, there is a constant $C_1 \geq 1$ so that

$$\frac{1}{2}T_{-,V}(q_j') \geq \frac{1}{2C_1}T_{+,V}(q_j') \ ,$$

holds, which in turn implies

$$T_{-,V}(q_j') \geq \frac{1}{2}T_{-,V}(q_j') + \frac{1}{2C_1}T_{+,V}(q_j') \geq \frac{1}{2C_1}(T_{-,V}(q_j') + T_{+,V}(q_j')) \ , \tag{3.15}$$

Then it follows from (3.14) and (3.15)

$$\|K_{V_j^2}u_j\|^2 \geq c'\|\frac{\sqrt{1+|\text{Hess } V(q_j')|}}{\log(2+|\text{Hess } V(q_j')|)}u_j\|^2 \ . \tag{3.16}$$

By Assumption 1 (see condition (1.8)) and (3.16), applying Lemma B.6, there is $\delta \in (0,1)$ and $\Lambda_\Sigma(\varrho)$, $\lim_{\varrho\to+\infty}\Lambda_{\Sigma(\kappa)}(\varrho) = +\infty$ such that

$$\begin{aligned}\frac{1+|\text{Hess } V(q_j')|}{\log(2+|\text{Hess } V(q_j')|)^2} &\geq \frac{1}{2^{2(1-\delta)}}(1+|\text{Hess } V(q_j')|)^{1-\delta}\\ &\geq \frac{1}{4}|\text{Hess } V(q_j')|^{1-\delta}\\ &\geq \frac{\Lambda_{\Sigma(\kappa)}(|q_j'|)}{4}R_V^{\geq 3}(q_j')^4 \geq \frac{\Lambda_{\Sigma(\kappa)}(C(\kappa))}{4}R_V^{\geq 3}(q_j')^4 \ .\end{aligned}$$

Therefore we get from the above inequality and (3.16),

$$\|K_{V_j^2}u_j\|^2 \geq \Lambda_{\Sigma(\kappa)}(C(\kappa))R_V^{\geq 3}(q_j')^4\|u_j\|^2 \ . \tag{3.17}$$

Next, remind that $t_j = \langle 1+|\text{Hess } V(q_j')|\rangle^{1/4}$. By (2.7) in Lemma 2.3, the equivalence

$$t_j \asymp \langle 1+|\text{Hess } V(q)|\rangle^{1/4} \ , \tag{3.18}$$

holds for any $q \in \text{supp } \chi_j$ with $|q| \geq C(\kappa) \geq C_2(\kappa)$. From (3.7) and (3.18) we see that

$$\begin{aligned}\|K_{V_j^2}u_j\|^2 + (1+10C)t_j^4\|u_j\|^2 \geq C\Big(&\|(1+O_p)u_j\|^2 + \|\langle 1+|\text{Hess } V(q)|\rangle^{1/2}u_j\|^2\\ &+ \|\langle 1+|D_q|\rangle^{2/3}u_j\| + 9Ct_j^4\|u_j\|^2\Big) \ ,\end{aligned} \tag{3.19}$$

Since $j \notin J(\kappa)$

$$|\text{Hess } V(q)| + R_V^{\geq 3}(q)^4 + 1 \geq \frac{1}{\kappa}|\nabla V(q)|^{\frac{4}{3}} \ , \tag{3.20}$$

for all $q \in \text{supp } \chi_j$. Furthermore, it results by Lemma B.6 that for all $q \in \text{supp } \chi_j$ (where $j \notin J(\kappa)$)

$$|\text{Hess } V(q)| + R_V^{\geq 3}(q)^4 + 1 \leq \frac{3}{2}|\text{Hess } V(q)| \ . \tag{3.21}$$

From (3.20) and (3.21) we get

$$|\text{Hess } V(q)| \geq \frac{1}{2\kappa}|\nabla V(q)|^{\frac{4}{3}} \quad , \quad |\text{Hess } V(q)| \geq \frac{2}{3} \geq \frac{1}{2\kappa} \ .$$

Hence there exists a constant $c'' > 0$ such that

$$\langle 1 + |\text{Hess } V(q)| \rangle \geq \frac{c''}{\kappa}\langle 1 + |\partial_q V(q)| \rangle^{4/3} \ ,$$

for any $q \in \text{supp } \chi_j$ with $|q| \geq C(\kappa) \geq C_2(\kappa)$.

The above inequality combined with (3.19) leads to

$$\begin{aligned} \|K_{V_j^2} u_j\|^2 + (1 + 10C)t_j^4\|u_j\|^2 \geq C\Big(&\|(1 + O_p)u_j\|^2 + \|\langle 1 + |D_q| \rangle^{2/3} u_j\|^2 \\ &+ 9t_j^4\|u_j\|^2 + \frac{c''}{\kappa}\|\langle 1 + |\partial_q V(q)| \rangle^{2/3} u_j\|^2 + \frac{1}{2}\|\langle 1 + |\text{Hess } V(q)| \rangle^{1/2} u_j\|^2\Big) \ , \end{aligned} \tag{3.22}$$

for all $\kappa \geq \kappa_2$

Collecting the estimates (3.22) and (3.16) we get

$$\begin{aligned} \log(t_j^4)^2\|K_{V_j^2} u_j\|^2 \geq C\text{''}\Big(&\|(1 + O_p)u_j\|^2 + \|\langle 1 + |D_q| \rangle^{2/3} u_j\|^2 \\ &+ 9t_j^4\|u_j\|^2 + \frac{c''}{\kappa}\|\langle 1 + |\partial_q V(q)| \rangle^{2/3} u_j\|^2 + \frac{1}{2}\|\langle 1 + |\text{Hess } V(q)| \rangle^{1/2} u_j\|^2\Big) \ . \end{aligned} \tag{3.23}$$

In order to reduce the written expressions we denote

$$\Lambda_{1,j} = \frac{1 + O_p}{\log(t_j^4)} \ , \quad \Lambda_{2,j} = \frac{\langle 1 + |\text{Hess } V(q)| \rangle^{1/2}}{\log(t_j^4)} \ , \quad \Lambda_{3,j} = \frac{\langle 1 + |\partial_q V(q)| \rangle^{2/3}}{\log(t_j^4)} \ , \quad \Lambda_{4,j} = \frac{t_j^2}{\log(t_j^4)} \ .$$

The estimate (3.23) can be rewritten as follows

$$\|K_{V_j^2} u_j\|^2 \geq C\text{''}\Big(\|\Lambda_{1,j}u_j\|^2 + \frac{1}{2}\|\Lambda_{2,j}u_j\|^2 + \frac{c''}{2\kappa}\|\Lambda_{3,j}u_j\|^2 + 9\|\Lambda_{4,j}u_j\|^2 + \|\frac{\langle 1 + |D_q| \rangle^{2/3}}{\log(t_j^4)}u_j\|^2\Big) \ . \tag{3.24}$$

Using (3.24) and (3.17) we obtain

$$(1+C")\|K_{V_j^2}u_j\|^2 \geq C"\Big(\|\Lambda_{1,j}u_j\|^2 + \frac{1}{2}\|\Lambda_{2,j}u_j\|^2 + \frac{c''}{2\kappa}\|\Lambda_{3,j}u_j\|^2 + 9\|\Lambda_{4,j}u_j\|^2 + \|\frac{\langle 1+|D_q|\rangle^{2/3}}{\log(t_j^4)}u_j\|^2 + \Lambda_{\Sigma(\kappa)}(C(\kappa))R_V^{\geq 3}(q_j')^4\|u_j\|^2\Big)\ .$$

Therefore in both cases, that is for all $j \in I(\kappa)$ where $\kappa \geq \kappa_2$

$$\|K_{V_j^2}u_j\|^2 \geq C^{(3)}\Big(\|\Lambda_{1,j}u_j\|^2 + \|\Lambda_{2,j}u_j\|^2 + \frac{1}{\kappa}\|\Lambda_{3,j}u_j\|^2 + \|\Lambda_{4,j}u_j\|^2 + \|\frac{\langle 1+|D_q|\rangle^{2/3}}{\log(t_j^4)}u_j\|^2 + \min\Big(\kappa, \Lambda_{\Sigma(\kappa)}(C(\kappa))\Big)R_V^{\geq 3}(q_j')^4\|u_j\|^2\Big)\ . \tag{3.25}$$

Due to the elementary inequality $u^{4/3}+v^4 \geq \frac{1}{c_0}(u^2+v^4)^{2/3}$ satisfied for all $u, v \geq 1$, we obtain for all $\kappa \geq \kappa_2$

$$\|\frac{\langle 1+|D_q|\rangle^{2/3}}{\log(t_j^4)}u_j\|^2 + \frac{1}{2}\min\Big(\kappa, \Lambda_{\Sigma(\kappa)}(C(\kappa))\Big)R_V^{\geq 3}(q_j')^4\|u_j\|^2 \geq \frac{1}{c_0}\|\Lambda_{5,j}u_j\|^2\ , \tag{3.26}$$

where

$$\Lambda_{5,j} = \frac{(2+D_q^2+R_V^{\geq 3}(q_j')^4)^{\frac{1}{3}}}{\log(t_j^4)}\ .$$

In conclusion, we get by (3.25) and (3.26) for every $j \in I(\kappa)$ with $\kappa \geq \kappa_2$

$$\|K_{V_j^2}u_j\|^2 \geq C^{(3)}\Big(\|\Lambda_{1,j}u_j\|^2 + \|\Lambda_{2,j}u_j\|^2 + \frac{1}{\kappa}\|\Lambda_{3,j}u_j\|^2 + \|\Lambda_{4,j}u_j\|^2 + \frac{1}{c_0}\|\Lambda_{5,j}u_j\|^2 + \frac{1}{2}\min\Big(\kappa, \Lambda_{\Sigma(\kappa)}(C(\kappa))\Big)R_V^{\geq 3}(q_j')^4\|u_j\|^2\Big)\ .$$

We now fix the choice firstly of $C(\kappa)$ and secondly of κ. Because $\lim_{\varrho\to+\infty}\Lambda_{\Sigma(\kappa)}(\varrho) = +\infty$, we can choose for any $\kappa \geq \kappa_2$, $C(\kappa) \geq \max(C_1, C_2(\kappa))$ such that $\Lambda_{\Sigma(\kappa)}(C(\kappa)) \geq \kappa$. We then choose $\kappa = \kappa_3 \geq \kappa_2$ such that

$$\frac{C^{(3)}}{8}\min\Big(\kappa_3, \Lambda_{\Sigma(\kappa_3)}(\kappa_3)\Big) = \frac{C^{(3)}\kappa_3}{8} \geq (c_{d,r}')^2\ ,$$

where $c_{d,r}'$ is the constant in (3.5),

$$\sum_{j\in I(\kappa_3)}\frac{1}{4}\|K_{V_j^2}u\|^2 - (c_{d,r}')^2R_V^{\geq 3}(q_j')^4\|u_j\|^2 \geq \frac{C^{(3)}}{8}\sum_{j\in I(\kappa_3)}\Big(\|\Lambda_{1,j}u_j\|^2 + \|\Lambda_{2,j}u_j\|^2 + \frac{1}{\kappa_3}\|\Lambda_{3,j}u_j\|^2 + \|\Lambda_{4,j}u_j\|^2 + \frac{1}{c_0}\|\Lambda_{5,j}u_j\|^2\Big)\ . \tag{3.27}$$

Step 2, $\mathbf{j} \notin \mathbf{I}(\kappa_3)$: The set $\mathbb{N} \setminus I(\kappa_3)$ is now a fixed finite set and we can define

$$C^{(4)} = \max_{j \in \mathbb{N} \setminus I(\kappa_3)} A_{V_j^2} + \sup_{q \in \text{supp } \chi_j} \left[\langle 1 + |\text{Hess } V(q)| \rangle + \langle 1 + |\partial_q V(q)| \rangle^{4/3} \right] + \frac{t_j^4}{\log(t_j^4)^2} + (1 + (c'_{d,r})^2)(1 + R_V^{\geq 3}(q'_j))^4 \, .$$

From the lower bound (1.4) we deduce

$$\frac{1}{4}\|K_{V_j^2} u_j\| + C^{(4)}\|u_j\|^2 - (c'_{d,r})^2 R_V^{\geq 3}(q'_j)^4 \|u_j\|^2 \geq \frac{c}{4}\left[\|Ou_j\|^2 + \|\langle D_q \rangle^{2/3} u_j\|^2\right] + (1 + R_V^{\geq 3}(q'_j))^4 \|u_j\|^2 + \|\langle 1 + |\partial_q V(q)| \rangle^{2/3} u_j\|^2 + \|\langle 1 + |\text{Hess } V(q)| \rangle^{1/2} u_j\|^2 + \|\frac{t_j^2}{\log(t_j^4)} u_j\|^2$$

With the quantities

$$\Lambda_{1,j} = \frac{1 + O_p}{\log(t_j^4)} \, , \quad \Lambda_{2,j} = \frac{\langle 1 + |\text{Hess } V(q)| \rangle^{1/2}}{\log(t_j^4)} \, , \quad \Lambda_{3,j} = \frac{\langle 1 + |\partial_q V(q)| \rangle^{\frac{2}{3}}}{\log(t_j^4)} \, ,$$

$$\Lambda_{4,j} = \frac{t_j^2}{\log(t_j^4)} \, , \quad \Lambda_{5,j} = \frac{(2 + D_q^2 + R_V^{\geq 3}(q'_j)^4)^{\frac{1}{3}}}{\log(t_j^4)} \, .$$

where $t_j \geq 2$ we deduce

$$\sum_{j \notin I(\kappa_3)} \frac{1}{4}\|K_{V_j^2} u_j\|^2 - (c'_{d,r})^2 R_V^{\geq 3}(q'_j)^4 \|u_j\|^2 + C^{(4)}\|u_j\|^2 \geq C^{(5)} \sum_{j \notin I(\kappa_3)} \left(\|\Lambda_{1,j} u_j\|^2 + \|\Lambda_{2,j} u_j\|^2 + \frac{1}{\kappa_3}\|\Lambda_{3,j} u_j\|^2 + \|\Lambda_{4,j} u_j\|^2 + \frac{1}{c_0}\|\Lambda_{5,j} u_j\|^2 \right), \quad (3.28)$$

Collecting (3.5), (3.27) and (3.28), there exists a positive constant $C^{(6)} \geq 1$ depending on V such that

$$\|K_V u\|_{L^2}^2 + C^{(6)}\|u\|_{L^2}^2 \geq \frac{1}{C^{(6)}} \sum_{j \in \mathbb{N}} \left(\|\Lambda_{1,j} u_j\|^2 + \|\Lambda_{2,j} u_j\|^2 + \|\Lambda_{3,j} u_j\|^2 + \|\Lambda_{4,j} u_j\|^2 + \|\Lambda_{5,j} u_j\|^2 \right) \, . \quad (3.29)$$

Step 3. In this final step, set $L(s) = \frac{s}{\log(s)}$ for all $s \geq 2$. Notice that there exists a constant $c > 0$ such that for all $x \geq 2$,

$$\inf_{t \geq 2} \frac{x}{\log(t)} + t \geq \frac{1}{c} L(x) \, .$$

In view of the above estimate,

$$\|\Lambda_{1,j}u_j\|^2 + \frac{1}{4}\|\Lambda_{4,j}u_j\|^2 \geq \frac{1}{4}\int \frac{\lambda^2}{(\log(t_j^4))^2} + t_j^2 \, d\mu_{u_j}(\lambda)$$
$$\geq \frac{1}{8}\int (\frac{\lambda}{\log(t_j)} + t_j)^2 d\mu_{u_j}(\lambda)$$
$$\geq \frac{1}{c_3}\|L(1+O_p)u_j\|^2 \ .$$

Summing over j, we obtain the first term in the desired estimation (1.9). Likewise for the second term

$$\|\Lambda_{3,j}u_j\|^2 + \frac{1}{4}\|\Lambda_{4,j}u_j\|^2 \geq \frac{1}{c_4}\|L(\langle 1+|\partial_q V(q)|\rangle^{2/3})u_j\|^2 \ ,$$

with

$$\sum_{j\in\mathbb{N}} \|L(\langle 1+|\partial_q V(q)|\rangle^{2/3})u_j\|^2 = \|L(\langle 1+|\partial_q V(q)|\rangle^{2/3})u\|^2 \ .$$

To obtain the third term in (1.9) write samely

$$\|\Lambda_{2,j}u_j\|^2 + \frac{1}{4}\|\Lambda_{4,j}u_j\|^2 \geq \frac{1}{c_5}\|L(\langle 1+|\text{Hess }V(q)|\rangle^{1/2})u_j\|^2 \ ,$$

with

$$\sum_{j\in\mathbb{N}} \|L(\langle 1+|\text{Hess }V(q)|\rangle^{1/2})u_j\|^2 = \|L(\langle 1+|\text{Hess }V(q)|\rangle^{1/2})u\|^2 \ .$$

Doing similarly again

$$\|\Lambda_{5,j}u_j\|^2 + \frac{1}{4}\|\Lambda_{4,j}u_j\|^2 \geq \frac{1}{c_6}\|L((2+D_q^2+R_V^{\geq 3}(q_j')^4)^{\frac{1}{3}})u_j\|^2 \ .$$

By Lemma 2.5 we get

$$\sum_{j\in\mathbb{N}} \|L((2+D_q^2+R_V^{\geq 3}(q_j')^4)^{\frac{1}{3}})u_j\|^2 \geq \frac{1}{c_6}\|L((2+D_q^2+R_V^{\geq 3}(q)^4)^{\frac{1}{3}})u\|^2 \ ,$$

To conclude, just remark that

$$\langle u, (2+D_q^2+R_V^{\geq 3}(q)^4)u\rangle \geq \langle u, (2+D_q^2)u\rangle \geq \langle u, \langle 1+D_q^2\rangle u\rangle \geq 2\|u\|^2$$

holds for all $u \in \mathcal{C}_0^\infty(\mathbb{R}^{2d})$, then applying (2.15) in Lemma 2.4 with $A = (2+D_q^2+R_V^{\geq 3}(q)^4)$, $B = \langle 1+D_q^2\rangle$, $\alpha_0 = \frac{2}{3}$ and $k=2$ we obtain

$$\|L((2+D_q^2+R_V^{\geq 3}(q)^4)^{\frac{1}{3}})u\|^2 \geq \|L(\langle 1+D_q^2\rangle^{\frac{1}{3}})u\|^2 \geq \frac{1}{c_7}\|L(\langle 1+|D_q|\rangle^{2/3})u\|^3$$

for all $u \in \mathcal{C}_0^\infty(\mathbb{R}^{2d})$.

Finally collecting all terms, we have found $C_V \geq 1$ such that

$$\|K_V u\|_{L^2}^2 + C_V\|u\|_{L^2}^2 \geq \frac{1}{C_V}\Big(\|L(1+O_p)u\|_{L^2}^2 + \|L(\langle 1+|\nabla V(q)|\rangle^{\frac{2}{3}})u\|_{L^2}^2 + \|L(\langle 1+|\text{Hess }V(q)|\rangle^{\frac{1}{2}})u\|_{L^2}^2 + \|L(\langle 1+|D_q|\rangle^{\frac{2}{3}})u\|_{L^2}^2\Big) \tag{3.30}$$

holds for all $u \in \mathcal{C}_0^\infty(\mathbb{R}^{2d})$. Because $\mathcal{C}_0^\infty(\mathbb{R}^{2d})$ is dense in $D(K_V)$ endowed with the graph norm, the result extends to any $u \in D(K_V)$. □

4 Applications

This section is devoted to some applications of Theorem 1.1. In each of the following examples we examine that the Assumption 1 is well fulfilled.

Example 1: Let us consider as a first example of application the case

$$V(q_1,q_2) = -q_1^2q_2^2,\ \text{with } q=(q_1,q_2)\in\mathbb{R}^2\ ,$$

By direct computation

$$\partial_q V(q) = \begin{pmatrix} -2q_1q_2^2 \\ -2q_2q_1^2 \end{pmatrix}\ ,\ |\partial_q V(q)| = 2|q_1q_2||q|\ ,$$

$$\text{Hess }V(q) = \begin{pmatrix} -2q_2^2 & -4q_1q_2 \\ -4q_1q_2 & -2q_1^2 \end{pmatrix}\ ,\ |\text{Hess }V(q)| = 2\sqrt{|q|^4+6q_1^2q_2^2} \asymp |q|^2\ ,$$

$$R_V^{\geq 3}(q) = |4q_2|^{1/3} + |4q_1|^{1/3} + 2\times 4^{1/4}\ .$$

It is clear that the trace of Hess $V(q)$ given by $-2|q|^2$ is stricly negative for all $q \in \mathbb{R}^2$. Hence

$$\text{Tr}_{-,V}(q) \geq \text{Tr}_{+,V}(q) \quad \text{for all}\ \ q\in\mathbb{R}^2\ .$$

Moreover, for all $\kappa > 0$ the algebraic set $\mathbb{R}^2 \setminus \Sigma(\kappa)$ is not bounded since $(0,q_2) \in \mathbb{R}^2\setminus\Sigma(\kappa)$ for all $q_2 \in \mathbb{R}$. Furthermore for $\kappa > 1$ chosen as we want

$$\lim_{\substack{q\to\infty \\ q\in\mathbb{R}^2\setminus\Sigma(\kappa)}} \frac{R_V^{\geq 3}(q)^4}{|\text{Hess }V(q)|} = \lim_{\substack{q\to\infty \\ q\in\mathbb{R}^2\setminus\Sigma(\kappa)}} \frac{|q|^{4/3}}{|q|^2} = 0\ ,$$

since $R_V^{\geq 3}(q)^4 \leq |q|^{4/3}$ when $|q| \geq 2^3 \times 4^{3/4}$.

Below we sketch as example $\Sigma(800)$ in a blue color.

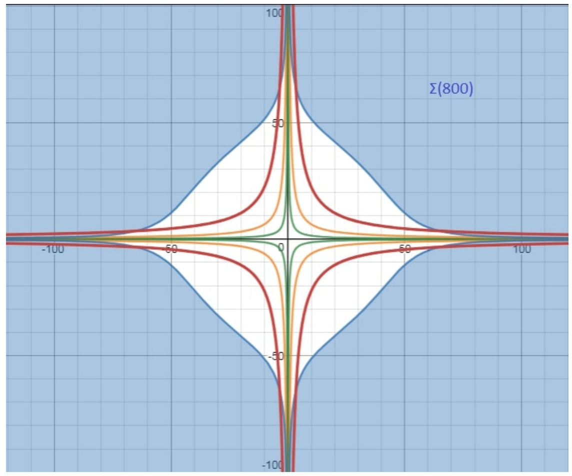

Figure 1: Contour lines of $V(q_1, q_2) = -q_1^2 q_2^2$

Example 2: Let $n \in \mathbb{N}$. The polynomial $V(q) = -q_1^2(q_1^2 + q_2^2)^n$ verifies the Assumption 1 only for $n = 1$.

A straight forward computation shows that

$$\partial_q V(q) = -\begin{pmatrix} 2q_1(|q|^{2n} + nq_1^2|q|^{2(n-1)}) \\ 2nq_2 q_1^2 |q|^{2(n-1)} \end{pmatrix} ,$$

$$\text{Hess } V(q) = -2|q|^{2(n-2)} \begin{pmatrix} |q|^4 + 5nq_1^2|q|^2 + 2n(n-1)q_1^4 & 2nq_1q_2|q|^2 + 2n(n-1)q_1^3 q_2 \\ 2nq_1q_2|q|^2 + 2n(n-1)q_1^3q_2 & nq_1^2|q|^2 + 2n(n-1)q_1^2q_2^2 \end{pmatrix} .$$

Notice that the trace of Hess $V(q)$ equals

$$-2|q|^{2(n-2)}\Big(|q|^4 + 5nq_1^2|q|^2 + 2n(n-1)q_1^4 + nq_1^2|q|^2 + 2n(n-1)q_1^2q_2^2\Big) \leq 0 ,$$

for all $q \in \mathbb{R}^2$. Hence

$$-\mathrm{Tr}_{-,V}(q) + \mathrm{Tr}_{+,V}(q) \leq 0, \quad \text{for any} \quad q \in \mathbb{R}^2 .$$

In addition for all $\kappa > 0$ the set $\mathbb{R}^2 \setminus \Sigma(\kappa)$ is not bounded since $(0, q_2) \in \mathbb{R}^2 \setminus \Sigma(\kappa)$ for all $q_2 \in \mathbb{R}$.

For q large enough $|\text{Hess } V(q)| \asymp |q|^{2n}$ and $|D^3V(q)| \asymp |q|^{2n-1}$ then

$$\frac{R_V^{\geq 3}(q)^4}{|\text{Hess } V(q)|} \asymp \frac{(|q|^{2n-1})^{4/3}}{|q|^{2n}} .$$

Hence

$$\lim_{\substack{q\to\infty \\ q\in\mathbb{R}^2\setminus\Sigma(\kappa)}} \frac{R_V^{\geq 3}(q)^4}{|\text{Hess } V(q)|} = 0 \quad \text{if and only if} \quad n < 2 .$$

Taking as example $\kappa = 800$, we get the following shape of $\Sigma(800)$ colored in blue.

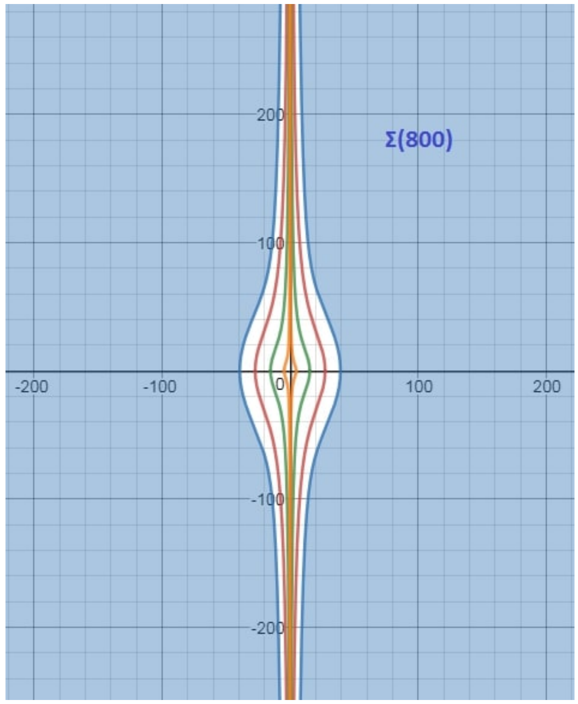

Figure 2: Contour lines of $V(q_1, q_2) = -q_1^2(q_1^2 + q_2^2)$

Example 3: For $\epsilon \in \mathbb{R} \setminus \{0, -1\}$, we consider $V(q_1, q_2) = (q_1^2 - q_2)^2 + \epsilon q_2^2$. For all $q \in \mathbb{R}^2$ one has

$$\partial_q V(q) = \begin{pmatrix} 4q_1(q_1^2 - q_2) \\ -2(q_1^2 - q_2) + 2\epsilon q_2 \end{pmatrix} \ , \ |\partial_q V(q)| = 4|q_1(q_1^2 - q_2)| + |-2(q_1^2 - q_2) + 2\epsilon q_2| \ ,$$

$$\text{Hess } V(q) = \begin{pmatrix} 12q_1^2 - 4q_2 & -4q_1 \\ -4q_1 & 2(1+\epsilon) \end{pmatrix} \ , \ |\text{Hess } V(q)| = |12q_1^2 - 4q_2| + 8|q_1| + 4|1+\epsilon| \ ,$$

$$R_V^{\geq 3}(q) = (24|q_1|)^{1/3} + 3 \times 4^{1/3} + 24^{1/4} \ .$$

In this case, we are going to show that for all $\kappa > 0$ the algebraic set $\mathbb{R}^2 \setminus \Sigma(\kappa)$ is bounded. Let $(q_1, q_2) \in \mathbb{R}^2 \setminus \Sigma(\kappa)$ then

$$\Big(|\text{Hess } V(q)| + R_V^{\geq 3}(q)^4 + 1\Big) \geq \frac{1}{\kappa} |\nabla V(q)|^{\frac{4}{3}} \ .$$

Up to a change of coordinates $X_1 = q_1, \ X_2 = q_1^2 - q_2$ the above inequality is equivalent to

$$\Big(4|2X_1^2 + X_2| + 8|X_1| + 4|1+\epsilon| + \Big((24|X_1|)^{1/3} + 3 \times 4^{1/3} + 24^{1/4}\Big)^4 + 1\Big) \\ \geq \frac{1}{\kappa} \Big(4|X_1 X_2| + |-2(1+\epsilon)X_2 + 2\epsilon X_1^2|\Big)^{\frac{4}{3}} \ .$$

Using the triange inequality in the right hand side and the reverse triangle inequality with the elementary inequality $(u+v)^{\frac{4}{3}} \geq u^{\frac{4}{3}} + v^{\frac{4}{3}}$ satisfied for all $u, v \geq 0$, it follows that

$$|X_1|^2 + |X_2| + |X_1| + \left(|X_1|^{\frac{1}{3}} + c\right)^4 \geq \frac{c'}{\kappa}\left(\Big||2(1+\epsilon)X_2| - |2\epsilon X_1^2|\Big|^{\frac{4}{3}} + |X_1X_2|^{\frac{4}{3}}\right) . \quad (4.1)$$

Suppose first that $|X_1| \leq 1$. The inequality (4.1) implies

$$|X_2| + c_1 \geq \frac{c'}{\kappa}\Big||2(1+\epsilon)X_2| - |2\epsilon X_1^2|\Big|^{\frac{4}{3}} . \quad (4.2)$$

The right hand part in the above inequality is upper bounded by $|X_2| + c_1$ where c_1 is some positive constant. Now we distinguish two case:
Case 1: If $\frac{1}{2}|2(1+\epsilon)X_2| \leq |2\epsilon X_1^2|$ or equivalently $|X_2| \leq |\frac{2\epsilon}{1+\epsilon}||X_1^2|$ then $|X_2| \leq |\frac{2\epsilon}{1+\epsilon}|$.
Case 2: Else if $\frac{1}{2}|2(1+\epsilon)X_2| \geq |2\epsilon X_1^2|$ then we get

$$|X_2| + c_1 \geq \frac{c'}{\kappa}|1+\epsilon||X_2|^{4/3} .$$

Using the fact that $\epsilon \neq -1$, we deduce that X_2 must be also bounded.

Now if $|X_1| \geq 1$, we derive from (4.1) the following esimates

$$|X_1|^2 + |X_2| + c_3 \geq \frac{c_4}{\kappa}\Big||2(1+\epsilon)X_2| - |2\epsilon X_1^2|\Big|^{\frac{4}{3}} , \quad (4.3)$$

$$|X_1|^2 + |X_2| + c_3 \geq \frac{c_4}{\kappa}|X_1X_2|^{\frac{4}{3}} . \quad (4.4)$$

Here we study three cases.
• Firstly if $\frac{1}{2}|2(1+\epsilon)X_2| \geq |2\epsilon X_1^2|$ or equivalently $|X_1| \leq |\frac{1+\epsilon}{2\epsilon}||X_2|$ then (4.3) gives

$$(1 + |\frac{1+\epsilon}{\epsilon}|)|X_2| + c_3 \geq \frac{c_4}{\kappa}|(1+\epsilon)X_2|^{\frac{4}{3}}$$

Since $\epsilon \neq -1$, it follows that X_2 is bounded and so is X_1.
• Now if $2|2(1+\epsilon)X_2| \leq |2\epsilon X_1^2|$ or samely $|X_2| \leq |\frac{\epsilon}{2(1+\epsilon)}||X_1^2|$ the estimates (4.3) leads to

$$(1 + |\frac{\epsilon}{2(1+\epsilon)}|)|X_1|^2 + c_3 \geq \frac{c_4}{\kappa}|\epsilon X_1|^{\frac{8}{3}}$$

Since $\epsilon \neq 0$, it follows that X_1 is bounded and so is X_2.
• Finally if $\frac{1}{2}|2(1+\epsilon)X_2| \leq |2\epsilon X_1^2| \leq 2|2(1+\epsilon)X_2|$, then by (4.4)

$$(1 + |\frac{2\epsilon}{1+\epsilon}|)|X_1|^2 + c_3 \geq \frac{c_4}{\kappa}\left(|X_1||\frac{\epsilon}{2(1+\epsilon)}|X_1^2|\right)^{\frac{4}{3}}$$

Hence since $\epsilon \neq 0$, X_1 is bounded and then X_2 is so.

Below we sketch as example $\Sigma(2)$ in a blue color.

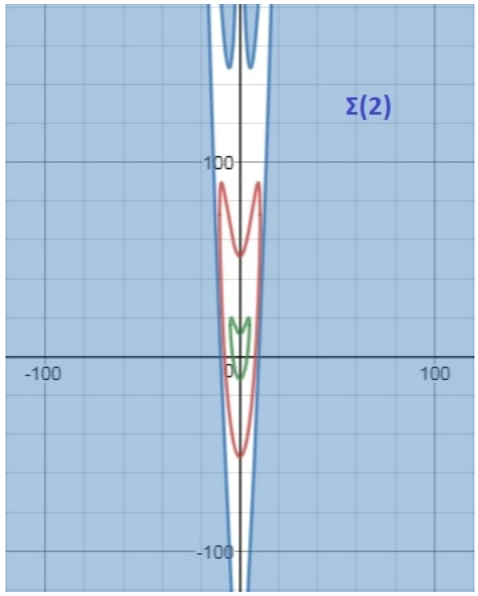

Figure 3: Contour lines of $V(q_1, q_2) = (q_1^2 - q_2)^2 + 0.5q_2^2$.

For $\epsilon = 0$, thanks to [HeNi] we know that the Witten Laplacien defined by

$$\Delta_V^{(0)} = -\Delta_q + |\nabla V(q)|^2 - \Delta V(q) \ , \quad q = (x_1, x_2) \in \mathbb{R}^2$$

has no compact resolvent and then the Kramers-Fokker-Planck operator K_V has no compact resolvent.

This example was studied in the case of the Witten Laplacien operator by B.Helffer and F.Nier in their book [HeNi]. A small mistake was done in [HeNi] in Proposition 10.20. In fact the equations $l_{11} = l_{12} = l_{111} = 0$ should be replaced by $(1+\epsilon)l_{11} = l_{12} = l_{111} = 0$. When $\epsilon = -1$, we can eventually construct a Weyl sequence for the Witten Laplacien operator in the following way. In this case the potential $V(q_1, q_2) = (q_1^2 - q_2)^2 - q_2^2$ is equal to $-2q_2q_1^2 + q_1^4$.

In order to construct a Weyl sequence for $\Delta_V^{(0)}$, it is sufficient to take $\chi(\frac{(q_2+n^2)}{n})$ where χ is a cutoff function supported in $[-1, 1]$ and then consider the sequence

$$u_n(q_1, q_2) = \chi(\frac{(q_2 + n^2)}{n}) \exp(-V(q_1, q_2)) \ .$$

The support of u_n is then included in $-n^2 - n \leq q_2 \leq -n^2 + n$. Hence the u_n's have disjoint supports for large n.

Therefore we have

$$-2n^2 \leq q_2 \leq -\frac{n^2}{2} \quad \text{and} \quad -4n^2q_1^2 - q_1^4 \leq -V(q_1, q_2) \leq -n^2q_1^2 - q_1^4 \leq -n^2q_1^2 \ .$$

As a result, we get for n large

$$\frac{\langle u_n, \Delta_V^{(0)} u_n\rangle}{\|u_n\|^2} = \frac{\|(\partial_q + \partial_q V(q))(u_n)\|^2}{\|u_n\|^2}$$
$$= \frac{\|(\partial_q \chi) e^{-V}\|^2}{\|u_n\|^2} = O(\frac{1}{n^2}) \ .$$

Here to get the lower bound of the the above quantity we restrict the integral in $q_1 = O(\frac{1}{n})$. As a conclusion, for $\epsilon = -1$ the Witten Laplacian attached to $V(q_1, q_2) = q_1^2 q_2^2 + \epsilon(q_1^2 + q_2^2)$ has no compact resolvent and then the Kramers-Fokker-Planck operator K_V has no compact resolvent.

A Slow metric, partition of unity

The purpose of this appendix is to state with references or proofs the facts concerning metrics which are needed in the article. We first remind the following definitions.

Definitions A.1. *A metric g on $\mathbb{R}^m$ is called a slowly varying metric if there exists a constant $C \geq 1$ such that for all $x, y \in \mathbb{R}^m$ satisfying $g_x(x-y, x-y) \leq C^{-1}$ it follows that*

$$C^{-1} g_x(z,z) \leq g_y(z,z) \leq C g_x(z,z) \tag{A.1}$$

holds for all $z \in \mathbb{R}^m$.

Let g^1 and g^2 be two metrics. We say that g^1 is g^2 slow if there is a constant $c \geq 1$ such that for all $x, y \in \mathbb{R}^m$

$$g_x^2(x-y, x-y) \leq c^{-1} \Rightarrow c^{-1} g_x^1(z,z) \leq g_y^1(z,z) \leq c g_x^1(z,z) \ . \tag{A.2}$$

holds for all $z \in \mathbb{R}^m$.

Remark A.2. *The property A.1 will be satisfied if we ask only that*

$$\exists C \geq 1, \forall x, y, z \in \mathbb{R}^m,\ g_x(x-y) \leq C^{-1} \Longrightarrow g_y(z) \leq C g_x(z) \ . \tag{A.3}$$

Indeed, assuming (A.3) gives that wherever $g_x(x-y) \leq C^{-1}$ (which is less than or equal to one since $C \geq 1$ from (A.3) with x=y) this implies $g_y(y-x) \leq C^{-1}$ and then $g_x(z) \leq C g_y(z)$, so that (A.1) is well satisfied.

Notations A.3. *For $r \in \mathbb{N}$, let E_r denote the set of polynomials with degree not greater than r:*

$$E_r = \{P \in \mathbb{R}[X_1, ..., X_d],\ d^\circ P \leq r\} \ .$$

For a polynomial $P \in E_r$ and $n \in \{1, ..., r\}$, the function $R_P^{\geq n} : \mathbb{R}^d \to \mathbb{R}$ is defined by

$$R_P^{\geq n}(q) = \sum_{n \leq |\alpha| \leq r} |\partial_q^\alpha P(q)|^{\frac{1}{|\alpha|}} \ . \tag{A.4}$$

In the present article we are mainly concerned with the metric $g^n = R_P^{\geq n}(q)^2\, dq^2$ where $n \in \{1, ..., r\}$ which satisfies the following properties.

Lemma A.4. *Let n a natural number in $\{1, ..., r\}$.*
1) The metric g^n is slow: There exists a uniform $C = C(n, r, d) \geq 1$ such that

$$R_P^{\geq n}(q)|q - q'| \leq C^{-1} \Longrightarrow \Big(\frac{R_P^{\geq n}(q)}{R_P^{\geq n}(q')}\Big)^{\pm 1} \leq C \tag{A.5}$$

2) The metric g^{n-1} is g^n slow: There is a constant $C' = C'(n, r, d) \geq 1$ so that

$$R_P^{\geq n}(q)|q - q'| \leq C'^{-1} \Longrightarrow \Big(\frac{R_P^{\geq n-1}(q)}{R_P^{\geq n-1}(q')}\Big)^{\pm 1} \leq C' \tag{A.6}$$

Proof. Assume $n, r \in \mathbb{N}^*$ with $n \geq r$. Consider the map

$$\begin{aligned} f :& E_r \to E_r/E_n; \\ & P \mapsto \overline{P} : \mathbb{R}^d \to \mathbb{R} \\ & \qquad x \mapsto \overline{P}(x) \sum_{n \leq |\alpha| \leq r} \frac{\partial_x^\alpha P(0)}{\alpha!} \Big(\frac{x}{R_P^{\geq n}(0)}\Big)^\alpha . \end{aligned}$$

Set $K_{n,r} := f(E_r) = \Big\{\overline{P} \in E_r/E_n,\ R_{\overline{P}}^{\geq n}(0) = 1\Big\}$. Assume $\overline{P} \in K_{n,r}$ and $\beta \in \mathbb{N}^d$ with $|\beta| \geq n$. Notice that there is a constant $c \geq 1$ (uniform with respect to $\overline{P}$ and β) such that for $|t| \leq c^{-1}$,

$$\begin{aligned} |\partial_t^\beta \overline{P}(t) - \partial_t^\beta \overline{P}(0)| &= |\sum_{|\alpha|=1} \frac{\partial_t^{\alpha+\beta} \overline{P}(0)}{\alpha!} t^\alpha| \\ &\leq \sum_{|\alpha|=1} |\frac{\partial_t^{\alpha+\beta} \overline{P}(0)}{\alpha!}|\ |t| \leq d|t| . \end{aligned} \tag{A.7}$$

On the other hand, the application

$$\mathbb{R}^{\mathbb{N}} \to \mathbb{R}^{\mathbb{N}},\ (u_\beta)_{n \leq |\beta| \leq r} \mapsto (|u_\beta|^{\frac{1}{|\beta|}})_{n \leq |\beta| \leq r}$$

is continuous. Then for all $\delta > 0$ there exists $\eta = \eta(n, r) > 0$ so that

$$\max_{n \leq |\beta| \leq r} |u_\beta - v_\beta| \leq \eta \Longrightarrow \sum_{n \leq |\beta| \leq r} \Big| |u_\beta|^{\frac{1}{|\beta|}} - |v_\beta|^{\frac{1}{|\beta|}} \Big| \leq \delta . \tag{A.8}$$

Thus for all $\delta > 0$ there is a strictly positive constant $C_1 = C_1(n, r, d) = \min(\frac{\eta}{d}, c^{-1}) \leq 1$ so that

$$|R_{\overline{P}}^{\geq n}(t) - R_{\overline{P}}^{\geq n}(0)| \leq \delta = \delta R_{\overline{P}}^{\geq n}(0) , \tag{A.9}$$

holds when $|t| \leq C_1$.

Now given a polynomial $V \in E_r$ and $q \in \mathbb{R}^d$ define

$$\overline{P}_q(t) = V(q + R_V^{\geq n}(q)^{-1}t) \ .$$

Writting

$$R_{\overline{P}_q}^{\geq n}(t) = \sum_{n \leq |\alpha| \leq r} |\partial_t^\alpha \overline{P}_q(t)|^{\frac{1}{|\alpha|}} = R_V^{\geq n}(q)^{-1} R_V^{\geq n}(q + R_V^{\geq n}(q)^{-1}t) \ , \tag{A.10}$$

clearly the polynomial $\overline{P}_q$ belongs to the set K. Hence for $\delta = \frac{1}{2}$ we get by (A.9)

$$\frac{1}{2} R_{\overline{P}_q}^{\geq n}(0) \leq R_{\overline{P}_q}^{\geq n}(t) \leq 2 R_{\overline{P}_q}^{\geq n}(0) \ , \tag{A.11}$$

when $|t| \leq C_1$.

It follows from (A.11) and (A.11),

$$\Big(\frac{R_V^{\geq n}(q + R_V^{\geq n}(q)^{-1}t)}{R_V^{\geq n}(q)} \Big)^{\pm 1} \leq 2 \ , \tag{A.12}$$

for $|t| \leq C_1$.

Therefore by the above inequality there is a constant $C_1 \leq 1$ (chosen uniformly with respect to q, V once r, n and d are fixed) so that for all $q, q' \in \mathbb{R}^d$ such that $R_V^{\geq n}(q)|q' - q| \leq C_1$,

$$\Big(\frac{R_V^{\geq n}(q')}{R_V^{\geq n}(q)} \Big)^{\pm 1} \leq 2 \ . \tag{A.13}$$

It remains now to prove that for every $n \in \{1, ..., r\}$, the metric g^{n-1} is g^n slow. Assuming the slowlness of g^n, the inequality

$$\Big(\frac{R_V^{\geq n}(q')}{R_V^{\geq n}(q)} \Big)^{\pm 1} \leq 2 \ . \tag{A.14}$$

holds when $R_V^{\geq n}(q)|q' - q| \leq C_1^{-1}$.

Denote as before $t = R_V^{\geq n}(q)(q' - q)$ and $\overline{P}_q(t) = V(q + R_V^{\geq n}(q)^{-1}t)$. Taking into account (A.14) and (A.10) it results

$$|t| \leq C_1^{-1} \Longrightarrow |\partial_t^\alpha \overline{P}_q(t)|^{\frac{1}{|\alpha|}} \leq R_{\overline{P}_q}^{\geq n}(t) \leq 2 \ ,$$

for all $\alpha \in \mathbb{N}^d$ with $|\alpha| \geq n$. Consequently,

$$|t| \leq C_1^{-1} \Longrightarrow |\partial_t^\alpha \overline{P}_q(t)| \leq 2^r \ , \tag{A.15}$$

for all $\alpha \in \mathbb{N}^d, \ |\alpha| \geq n$.

Using (A.15), one has when $|\alpha| = n-1$ and $|t| \leq C_1^{-1}$

$$|\partial_t^\alpha \overline{P}_q(t) - \partial_t^\alpha \overline{P}_q(0)| = |\sum_{|\beta|=1} \frac{\partial_t^\beta[\partial_t^\alpha \overline{P}_q](0)}{\beta!} t^\beta| = |\sum_{|\beta|=1} \frac{\partial_t^{\alpha+\beta} \overline{P}_q(0)}{\beta!} t^\beta|$$

$$\leq \sum_{|\beta|=1} |\frac{\partial_t^{\alpha+\beta} \overline{P}_q(0)}{\beta!}| \; |t| \leq d2^r |t| \;, \tag{A.16}$$

since $|\alpha + \beta| \geq n$. On the other hand, the application

$$\mathbb{R}^{\mathbb{N}} \to \mathbb{R}^{\mathbb{N}}, \; (u_\beta)_{|\beta|=n-1} \mapsto (|u_\beta|^{\frac{1}{|\beta|}})_{|\beta|=n-1}$$

is continuous. Then for all $\delta > 0$ there exists $\eta' = \eta'(n) > 0$ so that

$$\max_{|\beta|=n-1} |u_\beta - v_\beta| \leq \eta' \Longrightarrow \sum_{|\beta|=n-1} \left| |u_\beta|^{\frac{1}{|\beta|}} - |v_\beta|^{\frac{1}{|\beta|}} \right| \leq \delta \;. \tag{A.17}$$

Hence for $\delta = 1$ there is a strictly positive constant $C_1' = C_1'(n, r, d) = \min(\frac{\eta'}{d2^r}, C_1^{-1}) \leq 1$ so that

$$\left| \sum_{|\alpha|=n-1} |\partial_t^\alpha \overline{P}_q(t)|^{\frac{1}{|\alpha|}} - |\partial_t^\alpha \overline{P}_q(0)|^{\frac{1}{|\alpha|}} \right| \leq 1 \;, \tag{A.18}$$

holds when $|t| \leq C_1'$.

Using respectively Peetre's inequality ($\left(\frac{\langle X' \rangle}{\langle X \rangle}\right)^s \leq 2^{\frac{|s|}{2}} \langle X - X' \rangle$ for all $s \in \mathbb{R}, X, X' \in \mathbb{R}$) then (A.18) yields when $|t| \leq C_1'$

$$\left(\frac{\langle \sum_{|\alpha|=n-1} |\partial_t^\alpha \overline{P}_q(t)|^{\frac{1}{|\alpha|}} \rangle}{\langle \sum_{|\alpha|=n-1} |\partial_t^\alpha \overline{P}_q(0)|^{\frac{1}{|\alpha|}} \rangle} \right)^{\pm 1} \leq \sqrt{2} \langle \sum_{|\alpha|=n-1} |\partial_t^\alpha \overline{P}_q(t)|^{\frac{1}{|\alpha|}} - |\partial_t^\alpha \overline{P}_q(0)|^{\frac{1}{|\alpha|}} \rangle \leq 2 \;. \tag{A.19}$$

Remember that for any sequence $(a_i)_{1 \leq i \leq N}$ of positive numbers

$$\Big(\sum_1^N a_i^p \Big)^{\frac{1}{p}} \leq \sum_1^N a_i \leq N^{\frac{1}{q}} \Big(\sum_1^N a_i^p \Big)^{\frac{1}{p}} \;, \tag{A.20}$$

where the two real numbers $p, q > 1$ are conjugate indices. In particular for any real numbers a, b

$$(a^2 + b^2)^{\frac{1}{2}} \leq (|a| + |b|) \leq 2^2 (a^2 + b^2)^{\frac{1}{2}}$$

It results from the elementary above inequality

$$\langle \sum_{|\alpha|=n-1} |\partial_t^\alpha \overline{P}_q(t)|^{\frac{1}{|\alpha|}} \rangle \leq (1 + \sum_{|\alpha|=n-1} |\partial_t^\alpha \overline{P}_q(t)|^{\frac{1}{|\alpha|}}) \leq 4 \langle \sum_{|\alpha|=n-1} |\partial_t^\alpha \overline{P}_q(t)|^{\frac{1}{|\alpha|}} \rangle \;,$$

and

$$\langle \sum_{|\alpha|=n-1} |\partial_t^\alpha \overline{P}_q(0)|^{\frac{1}{|\alpha|}} \rangle \leq (1 + \sum_{|\alpha|=n-1} |\partial_t^\alpha \overline{P}_q(0)|^{\frac{1}{|\alpha|}}) \leq 4 \langle \sum_{|\alpha|=n-1} |\partial_t^\alpha \overline{P}_q(0)|^{\frac{1}{|\alpha|}} \rangle \ .$$

Using the above two estimates with (A.19) we immediately get for $|t| \leq C_1'$

$$\left(\frac{1 + \sum_{|\alpha|=n-1} |\partial_t^\alpha \overline{P}_q(t)|^{\frac{1}{|\alpha|}}}{1 + \sum_{|\alpha|=n-1} |\partial_t^\alpha \overline{P}_q(0)|^{\frac{1}{|\alpha|}}} \right)^{\pm 1} \leq 8 \ . \tag{A.21}$$

Notice that by (A.14)

$$\left(\frac{1 + \sum_{|\alpha|=n-1} |\partial_t^\alpha \overline{P}_q(t)|^{\frac{1}{|\alpha|}}}{1 + \sum_{|\alpha|=n-1} |\partial_t^\alpha \overline{P}_q(0)|^{\frac{1}{|\alpha|}}} \right)^{\pm 1} = \left(\frac{1 + \sum_{|\alpha|=n-1} \left(\frac{|\partial_q^\alpha V(q')|}{R_V^{\geq n}(q)^{|\alpha|}} \right)^{\frac{1}{|\alpha|}}}{1 + \sum_{|\alpha|=n-1} \left(\frac{|\partial_q^\alpha V(q)|}{R_V^{\geq n}(q)^{|\alpha|}} \right)^{\frac{1}{|\alpha|}}} \right)^{\pm 1} \geq \frac{1}{2} \left(\frac{R_V^{\geq n-1}(q')}{R_V^{\geq n-1}(q)} \right)^{\pm 1} \ . \tag{A.22}$$

In conclusion, from (A.21) and (A.22) there is a constant $C_1' = \min(\frac{\eta'}{d2^r}, C_1^{-1}) \leq 1$ so that

$$R_V^{\geq n}(q)|q - q'| \leq C_1' \Longrightarrow \left(\frac{R_V^{\geq n-1}(q')}{R_V^{\geq n-1}(q)} \right)^{\pm 1} \leq 16 \ . \tag{A.23}$$

□

The main feature of a slow varying metric is that it is possible to introduce some partitions of unity related to the metric in a way made precise in the following theorem. For more details and proof see [Hor1].

Theorem A.5. *[Hor1] For any slowly varying metric g in $\mathbb{R}^m$ one can choose a sequence $x_\nu \in \mathbb{R}^m$ such that the balls*

$$B_\nu = \{x;\ g_{x_\nu}(x - x_\nu) < 1\}$$

form a covering of $\mathbb{R}^m$ for which the intersection of more than $N = (4C^3 + 1)^m$ balls B_ν is always empty (C is the constant in (A.1)). In addition, for any decreasing sequence d_i with $\sum_j d_j = 1$ one can choose non negative $\phi_\nu \in \mathcal{C}_0^\infty(B_\nu)$ with $\sum \phi_\nu = 1$ in $\mathbb{R}^m$ so that for all k

$$|\phi_\nu^{(k)}(x; y_1, ..., y_k)| \leq (NCC_1)^k g_x(y_1, 0)...g_x(y_k, 0)/d_1...d_k$$

where C is the constant in (A.1) and C_1 is a constant that depends only on m.

Regarding the above Theorem we have the following result.

Lemma A.6. *Let $P \in E_r$ and $n \in \{1,...,r\}$, then there exists a partition of unity* $\sum_{j\in\mathbb{N}} \psi_j(q)^2 \equiv 1$ *in $\mathbb{R}^d$ such that:*
1) For all $q \in \mathbb{R}^d$, the cardinality of the set $\{j, \Psi_j(q) \neq 0\}$ is uniformely bounded.
2) For any natural number $j \in \mathbb{N}$,

$$\text{supp } \Psi_j \subset B(q_j, aR_P^{\geq n}(q_j)^{-1}) \quad \text{and} \quad \Psi_j \equiv 1 \ \text{ in } B(q_j, bR_P^{\geq n}(q_j)^{-1}) \ ,$$

for some $q_j \in \mathbb{R}^d$ with $0 < b < a$ independent of $j \in \mathbb{N}$.
3) For all $\alpha \in \mathbb{N}^d \setminus \{0\}$, there exists $c_\alpha > 0$ such that

$$\sum_{j\in\mathbb{N}} |\partial_q^\alpha \Psi_j|^2 \leq c_\alpha R_P^{\geq n}(q)^{2|\alpha|} \ .$$

Moreover the constants a, b et c_α can be chosen uniformly with respect to $P \in E_r$, once the degree $r \in \mathbb{N}$ and the dimension $d \in \mathbb{N}$ are fixed.

B Around Tarski-Seidenberg theorem

In this appendix we give an application of the Tarski-Seidemberg theorem [Hor2], which we state in the following geometric form. We first introduce a few basic concepts which are needed for the state.

Definition B.1. *A subset of $\mathbb{R}^n$ is called semi-algebraic if it is a finite union of finite intersections of sets defined by polynomial equations or inequalities.*

Definition B.2. *Let $A \subset \mathbb{R}^n$ and $B \subset \mathbb{R}^m$ be two sub-algebraic sets. The function $f : A \to B$ is said to be semi-algebraic if its graph $\Gamma_f = \{(x, y) \in A \times B;\ y = f(x)\}$ is a semi-algebraic set of $\mathbb{R}^n \times \mathbb{R}^m$.*

Theorem B.3. *[Hor2](Tarski-Seidenberg) If A is a semi-algebraic subset of $\mathbb{R}^{n+m} = \mathbb{R}^n \oplus \mathbb{R}^m$, then the projection A' of A in $\mathbb{R}^m$ is also semi-algebraic.*

Proposition B.4. *[Hor2] If E is a semi-algebraic set on $\mathbb{R}^{2+n}$, and*

$$f(x) = \inf \{y \in \mathbb{R};\ \exists z \in \mathbb{R}^n, (x, y, z) \in E\}$$

is defined and finite for large positive x, then f is identically 0 for lage x or else

$$f(x) = Ax^a(1 + o(1)) \ , \quad x \to +\infty$$

where $A \neq 0$ and a is a rational number.

We refer to [Hor2] for detailed proofs of Theorem B.3 and Proposition B.4.
In the final part of this section we list and recall the following notations.

Notation B.5. *Let P be a polynomial of degree r. For all natural number $n \in \{0, \cdots r\}$ and every $q \in \mathbb{R}^d$*

$$R_P^{\geq n}(q) = \sum_{n \leq |\alpha| \leq r} |\partial_q^\alpha P(q)|^{\frac{1}{|\alpha|}} , \tag{B.1}$$

$$R_P^{=n}(q) = \sum_{|\alpha|=n} |\partial_q^\alpha P(q)|^{\frac{1}{|\alpha|}} . \tag{B.2}$$

Lemma B.6. *Let Σ be an unbounded semialgebraic set and V a polynomial of degree r satisfying the following assumption*

$$\lim_{\substack{q \to \infty \\ q \in \Sigma}} \frac{R_V^{\geq n}(q)^\alpha}{R_V^{=m}(q)^2} = 0 , \tag{B.3}$$

where $\alpha \in \mathbb{Q}, n, m \in \{0, 1, \cdots, r-1\}$ are fixed numbers.

Then there exist $\delta \in (0,1)$ and a positive function $\Lambda_\Sigma : (0, +\infty) \to [0, +\infty)$ so that

$$\begin{aligned} &\forall q \in \Sigma , |q| \geq \varrho , \quad \Lambda_\Sigma(\varrho) R_V^{\geq n}(q)^\alpha \leq R_V^{=m}(q)^{2(1-\delta)} \\ \textit{and} \qquad &\lim_{\varrho \to +\infty} \Lambda_\Sigma(\varrho) = +\infty . \end{aligned}$$

Proof. Suppose that there are $\alpha \in \mathbb{Q}, n, m \in \{0, 1, \cdots, r-1\}$ such that

$$\lim_{\substack{q \to \infty \\ q \in \Sigma}} \frac{R_V^{\geq n}(q)^\alpha}{R_V^{=m}(q)^2} = 0 , \tag{B.4}$$

where Σ is a given unbounded semialgebraic set.

After setting $\tau = \mathrm{ppcm}\Big(|\beta|, \ \min(n, m) \leq |\beta| \leq r\Big)$, define the functions $\widetilde{R}_V^{\geq n}$ and $\widetilde{R}_V^{=m}$, for all $q \in \mathbb{R}^d$ by

$$\widetilde{R}_V^{\geq n}(q) = \sum_{n \leq |\alpha| \leq r} |\partial_q^\alpha V(q)|^{\frac{\tau}{|\alpha|}}$$

and

$$\widetilde{R}_V^{=m}(q) = \sum_{|\alpha|=m} |\partial_q^\alpha V(q)|^{\frac{\tau}{|\alpha|}} .$$

Notice that one has the equivalences $R_V^{\geq n}(q) \asymp \Big(\widetilde{R}_V^{\geq n}(q)\Big)^{\frac{1}{\tau}}$ and $R_V^{=m}(q) \asymp \Big(\widetilde{R}_V^{=m}(q)\Big)^{\frac{1}{\tau}}$ for all $q \in \mathbb{R}^d$ where the functions $R_V^{\geq n}$ and $R_V^{=m}$ are defined respectively as in (B.1) and (B.2). Clearly the Assumption (B.4) is equivalent to

$$\lim_{\substack{q \to \infty \\ q \in \Sigma}} \frac{\widetilde{R}_V^{\geq n}(q)^\alpha}{\widetilde{R}_V^{=m}(q)^2} = 0 . \tag{B.5}$$

Remark here that $\widetilde{R}_V^{\geq n}(q)$ and $\widetilde{R}_V^{=m}(q)$ are polynomials in $q \in \mathbb{R}^d$ variable. Furthermore, the Assumption (B.5) can be written as follows

$$\widetilde{R}_V^{\geq n}(q)^\alpha \leq \epsilon(q) \widetilde{R}_V^{=m}(q)^2 \,,$$

for all $q \in \Sigma$ where

$$\epsilon(q) = \inf \left\{ \epsilon > 0,\ \epsilon \widetilde{R}_V^{=m}(q)^2 - \widetilde{R}_V^{\geq n}(q)^\alpha > 0 \right\} \,, \qquad \lim_{\substack{q \to \infty \\ q \in \Sigma}} \epsilon(q) = 0 \,. \tag{B.6}$$

Now, following the notations of Proposition B.4, we introduce the set

$$E = \left\{ (q, \varrho, \epsilon) \in \mathbb{R}^{d+2} \text{ such that } \epsilon \widetilde{R}_V^{=m}(q)^2 - \widetilde{R}_V^{\geq n}(q)^\alpha > 0 \text{ and } |q|^2 \geq \varrho^2 \right\} \,,$$

and the function f defined in $\mathbb{R}_+$ by

$$f(\varrho) = \inf \left\{ \epsilon > 0, \text{ such that } (q, \varrho, \epsilon) \in E \right\} \,. \tag{B.7}$$

By Tarski-Seidenberg theorem (see Theorem B.3), the function f is semialgebraic in R. Moreover f is defined, finite and not identically zero. Then by Proposition B.4, there exist a constant $A > 0$ and a rational number γ such that

$$f(\varrho) = A \varrho^\gamma + o_{\varrho \to +\infty}(\varrho^\gamma) \,.$$

By the definition (B.7) and (B.6), $\lim_{\varrho \to +\infty} f(\varrho) = 0$ and then $\gamma < 0$. Hence for $\varrho \geq 1$, we know $f(\varrho) \leq \frac{2A}{\varrho^{|\gamma|}}$. We deduce for $|q| \geq 1$,

$$\widetilde{R}_V^{\geq n}(q)^\alpha \leq f(|q|) \widetilde{R}_V^{=m}(q)^2 \leq \frac{2A}{|q|^{|\gamma|}} \widetilde{R}_V^{=m}(q)^2 \tag{B.8}$$

and

$$\frac{|q|^{|\gamma|/2}}{2A} \widetilde{R}_V^{\geq n}(q)^\alpha \leq \frac{1}{|q|^{\frac{|\gamma|}{2}}} \widetilde{R}_V^{=m}(q)^2 \,. \tag{B.9}$$

In particular, since $\widetilde{R}_V^{\geq n}(q) \geq \widetilde{R}_V^{=r}(0) > 0$, $\widetilde{R}_V^{=m}(q)$ does not vanish for $q \in \Sigma$ with $|q| \geq 1$.

On the other hand, notice

$$\forall q \in \Sigma \,,\ |q| \geq 1 \,, \quad \widetilde{R}_V^{=m}(q) \leq c|q|^{\tau r} \,. \tag{B.10}$$

The inequalities (B.8) and (B.10) lead to

$$\widetilde{R}_V^{\geq n}(q)^\alpha \leq C|q|^{2\tau r - |\gamma|}$$

for every $q \in \Sigma$ with $|q| \geq \rho \geq 1$. Therefore since $\widetilde{R}_V^{\geq n}(q) \geq \widetilde{R}_V^{=r}(0) > 0$ we deduce $|\gamma| \leq 2\tau r$. Using again (B.10) we get

$$\frac{1}{|q|^{\frac{|\gamma|}{2}}} \leq \frac{c^{\frac{|\gamma|}{2\tau r}}}{\widetilde{R}_V^{=m}(q)^{\frac{|\gamma|}{2\tau r}}} \,, \tag{B.11}$$

for any $q \in \Sigma$ with $|q| \geq 1$.

From (B.9) and (B.11), we deduce

$$\forall q \in \Sigma\,, |q| \geq \varrho \geq 1\,, \quad \frac{\varrho^{|\gamma|/2}}{2A} \widetilde{R}_V^{\geq n}(q)^{\alpha} \leq \frac{|q|^{|\gamma|/2}}{2A} \widetilde{R}_V^{\geq n}(q)^{\alpha} \leq c^{\frac{|\gamma|}{2\tau r}}\, \widetilde{R}_V^{=m}(q)^{2(1-\frac{|\gamma|}{4\tau r})}\,. \tag{B.12}$$

We now take $\delta = \frac{|\gamma|}{4\tau r} \in (0,1)$ and

$$\Lambda_\Sigma(\varrho) = \begin{cases} \frac{\varrho^{|\gamma|/2}}{2Ac^{\frac{|\gamma|}{2\tau r}}} & \text{if} \quad \varrho \geq 1 \\ 0 & \text{else}\,. \end{cases}$$

□

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