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Abstract. Introducing the deformation theory of holomorphic Cartan geometries, we compute infinitesimal automorphisms and infinitesimal deformations. We also prove the existence of a semi-universal deformation of a holomorphic Cartan geometry.

1. Introduction

Let G be a complex Lie group and $H < G$ a complex Lie subgroup with Lie algebras $\mathfrak{g}$, and $\mathfrak{h}$ respectively. The quotient map $G \to G/H$ defines a holomorphic principal H-bundle. Moreover, on the total space of this principal bundle, namely G, we have a tautological $\mathfrak{g}$-valued holomorphic 1-form (the Maurer-Cartan form), constructed by identifying $\mathfrak{g}$ with the right-invariant vector fields. This 1-form is an isomorphism of vector bundles and its restriction to the fibers of $G \to G/H$ coincide with the Maurer-Cartan form of H.

A holomorphic Cartan geometry of type (G, H) on a compact complex manifold X is infinitesimally modeled on this. More precisely, on a holomorphic principal H-bundle E_H over X there is a holomorphic 1-form A with values in the Lie algebra $\mathfrak{g}$ of G that induces an isomorphism from the tangent bundle of E_H to the trivial bundle on E_H with fiber $\mathfrak{g}$. This isomorphism is required to be H-invariant and on each fiber of E_H it should be the Maurer-Cartan form [Sh]. A fundamental result of E. Cartan shows that the obstruction for A to satisfy the Maurer-Cartan equation of G is a curvature tensor which vanishes if and only if (E_H, A) is locally isomorphic to the H-principal bundle $G \to G/H$ endowed with the Maurer-Cartan form [Sh].

Our aim here is to introduce the deformation theory of holomorphic Cartan geometries on a compact complex manifold. We compute the tangent cohomology of a holomorphic Cartan geometry (E_H, A) in degree zero and one. Infinitesimal automorphisms of a holomorphic Cartan geometry (E_H, A) consist of all the $ad(E_H)$-valued holomorphic vector fields ϕ satisfying the condition that the Lie derivative $L_\phi(A)$ vanishes. The space of infinitesimal deformations of (E_H, A) fits into a short exact sequence that we construct in the fourth section. The construction of the semi-universal deformation is worked out in the last section. Using the same methods a semi-universal deformation can also be constructed when the principal H-bundle and the underlying compact complex manifold are both moving.

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2. Deformations of holomorphic Cartan geometries – Definition

We will denote by G a connected complex Lie group, and by $H < G$ a closed connected complex subgroup. Furthermore X is a compact complex manifold. Let E_H denote a holomorphic principal H-bundle over X. Let $E_G = E_H \times_H G$ be the holomorphic principal G-bundle on X obtained by extending the structure group of E_H using the inclusion of H in G. We denote by T_M the holomorphic tangent bundle of a complex manifold M.

The action of the group H on E_H produces an action of H on the tangent bundle T_{E_H}: for $p \in E_H$, a tangent vector $v \in T_p(E_H)$ and $h \in H$ we have $v \cdot h = R_{h*}v$, where R_h is the right multiplication by h. For $\gamma \in \mathfrak{g} = \mathrm{Lie}(G)$ we have $h \cdot \gamma = ad(h^{-1}(\gamma))$.

Let $\pi : E_H \to X$ be the projection, and $\pi_* : T_{E_H} \to T_X$ the induced map. The adjoint bundle $ad(E_H)$ is defined as $E \times_H \mathfrak{h}$, where $\mathfrak{h} = \mathrm{Lie}(H)$. The space of vertical tangent vectors $\ker \pi_*$ is invariant under the action of H, and we have

$$ad(E_H) = ker(\pi_*)/H. \tag{2.1}$$

Given a section of $ad(E_H)$ we will use the same notation for its pull-back to $\ker(\pi_*) \subset T_{E_H}$.

Definition 2.1. A *holomorphic Cartan geometry* of type (G, H) on X is a pair (E_H, A), where E_H is a holomorphic principal H-bundle on X, and A is a $\mathbb{C}$-linear holomorphic isomorphism of vector bundles

$$A : T_{E_H} \xrightarrow{\sim} E_H \times \mathfrak{g}$$

over E_H such that

(i) A is H-invariant, and
(ii) the restriction of A to fibers of $E_H \to X$ is equal to the Maurer-Cartan form.

An *isomorphism* of holomorphic Cartan geometries $\Phi : (E_H, A) \to (F_H, B)$ is given by a holomorphic isomorphism $\phi : E_H \to F_H$ of principal bundles that takes A to B so that

$$\begin{tikzcd} & T_{E_H} \arrow[r, "A"] \arrow[dl] \arrow[dd, "\phi_*"'] & E_H \times \mathfrak{g} \arrow[dd, "\phi \times id_{\mathfrak{g}}"] \\ X & & \\ & T_{F_H} \arrow[r, "B"] \arrow[ul] & F_H \times \mathfrak{g} \end{tikzcd} \tag{2.2}$$

is a commutative diagram.

We note that the above isomorphism A induces an isomorphism

$$A_H : T_{E_H}/H \xrightarrow{\sim} (E_H \times \mathfrak{g})/H, \tag{2.3}$$

where T_{E_H}/H is, by definition, the Atiyah bundle $At(E_H)$, which fits into the Atiyah sequence

$$0 \to ad(E_H) \to At(E_H) \to T_X \to 0$$

(see [At]). We also have $(E_H \times \mathfrak{g})/H \simeq ad(E_G)$, where the quotient is for the conjugation action mentioned earlier.

The isomorphism A_H in (2.3) induces a holomorphic connection on E_G [Sh], [BD, (2.8)], which in turn produces a holomorphic connection on $ad(E_G)$. Therefore, we have a holomorphic differential operator

$$D : ad(E_G) \longrightarrow \Omega^1_X \otimes ad(E_G) \tag{2.4}$$

of order one.

Let us define now deformations of holomorphic Cartan geometries.

Definition 2.2. Let (E_H, A) be a principal H-bundle together with a holomorphic Cartan geometry.

(i) Let S be a complex space, and let $\mathcal{E}_H$ be a holomorphic principal H-bundle on $X \times S$. Let $\mathcal{E}_{H,s} = \mathcal{E}|X \times \{s\}$ for $s \in S$. A holomorphic family of principal H-bundles with holomorphic Cartan geometries over S consists of a principal H-bundle $\mathcal{E}_H$ over $X \times S$ together with a linear isomorphism $\mathcal{A}$ over S

$$\begin{tikzcd} T_{\mathcal{E}_H} \arrow[r, "\mathcal{A}", "\sim"'] \arrow[dr] & \mathcal{E}_H \times \mathfrak{g} \arrow[d] \\ & S \end{tikzcd} \tag{2.5}$$

such that the restrictions $\mathcal{A}_s$ of $\mathcal{A}$ to fibers $T_{\mathcal{E}_{H,s}}$ over $s \in S$ define holomorphic Cartan geometries.

(ii) Let (S, s_0) be a complex space with a distinguished point (or a germ of a complex space). A deformation of (E, A) over a space (S, s_0) consists of
  (a) a holomorphic family $(\mathcal{E}_H, \mathcal{A})$ of holomorphic Cartan geometries over S, and
  (b) an isomorphism $\Phi : (E_H, A) \xrightarrow{\sim} (\mathcal{E}_{H,s_0}, \mathcal{A}_{s_0})$.

(iii) Deformations are identified with their restrictions to any neighborhood of s_0 in S.

(iv) Two deformations of (E_H, A) over a space (S, s_0) are called isomorphic, if (after replacing S by a neighborhood of s_0, if necessary) there is an isomorphism of the respective families that induces the identity of (E_H, A) via the maps Φ of the above type.

3. Deformations of holomorphic Cartan geometries

The tangent cohomology of E_H is equal to $H^\bullet(X, ad(E_H))$, where $ad(E_H)$ is the adjoint bundle of E_H with fiber $\mathfrak{h}$ [Don]. In particular in degrees 0, 1, and 2 we have infinitesimal automorphisms, infinitesimal deformations, and the space containing obstructions respectively.

3.1. **Infinitesimal automorphisms of holomorphic Cartan geometries.** We begin with tangent cohomology $T^0(E_H, A)$ of degree zero for holomorphic Cartan geometries. Let (E_H, A) be a holomorphic Cartan geometry with A taking values in $\mathfrak{g}$. From (2.1) it follows that holomorphic section ψ of $ad(E_H)$ over $U \subset X$ gives a H–invariant holomorphic vector field $\widetilde{\psi}$ over $E_H|_U$ which is vertical for the projection π. We will identify ψ with $\widetilde{\psi}$. By L_ψ we denote the Lie derivative with respect to this vector field $\widetilde{\psi}$.

Proposition 3.1. *The space of infinitesimal automorphisms is equal to*

$$T^0(E_H, A) = H^0(X, ad(E_H))_A = \{\psi \in H^0(X, ad(E_H)) \mid L_\psi(A) = 0\}. \tag{3.1}$$

Proof. We consider (2.2) for $F_H = E_H$. Now the infinitesimal action of the group of vertical automorphisms of the given principal bundle gives rise to the following diagram. We denote by $\psi \in H^0(X, ad(E_H))$ an infinitesimal automorphism of E_H. We have the following diagram of homomorphisms on E_H:

$$\begin{tikzcd}
TE_H \arrow[r, "A"] \arrow[d, "L_\psi"'] & E_H \times \mathfrak{g} \arrow[d, "\psi"] \\
TE_H \arrow[r, "A"] & E_H \times \mathfrak{g}
\end{tikzcd} \tag{3.2}$$

Diagram (3.2) is interpreted as follows. As mentioned before, holomorphic sections of $ad(E_H)$ are H-invariant, vertical sections of the holomorphic tangent bundle T_{E_H}. We apply A to a holomorphic section v of T_{E_H}; then ψ is applied to $A(v)$, which is simply the Lie bracket on $\mathfrak{g}$ because ψ is a function on E_H with values in $\mathfrak{h}$. On the other hand, ψ acts on holomorphic sections of $ad(E_H)$ by applying the Lie derivative L_ψ, which is the infinitesimal version of the adjoint action.

The infinitesimal automorphism ψ is compatible with the holomorphic Cartan geometry, if (3.1) commutes for all sections v of $ad(E_H)$, i.e.

$$A(L_\psi(v)) = \psi(A(v)) = L_\psi(A)(v) + A(L_\psi(v))$$

for all v. Therefore, (3.1) commutes if and only if $L_\psi(A) = 0$. ☐

3.2. **Infinitesimal deformations of holomorphic Cartan geometries.** Let $\mathbb{C}[\epsilon] = \mathbb{C}[t]/(t^2)$, so that $\mathbb{C}[\epsilon] = \mathbb{C} \oplus \epsilon \cdot \mathbb{C}$ holds with $\epsilon^2 = 0$. The space $D = (\{0\}, \mathbb{C}[\epsilon])$ is also called double point with $\mathcal{O}_D = \mathbb{C}[\epsilon]$. The tangent space T_{S,s_0} of an arbitrary complex space S at a point $s_0 \in S$ can be identified with the space of all holomorphic mappings $D \to S$ such that the underlying point 0 is mapped to s_0. A deformation of H-principal bundles (with holomorphic Cartan geometries) over $(D, 0)$ is called an infinitesimal deformation of principal bundles (of holomorphic Cartan geometries).

We assume that an (isomorphism class of an) infinitesimal deformation $\mathcal{E}_H$ of E_H is given, corresponding to an element of $H^1(X, ad(E_H))$ [Don]. This gives rise to exact sequences

$$\begin{tikzcd}
0 \arrow[r] & \epsilon \cdot \mathbb{C} \arrow[r] \arrow[d, hook] & \mathbb{C} \oplus \epsilon \cdot \mathbb{C} \arrow[r] \arrow[d, hook] & \mathbb{C} \arrow[r] \arrow[d, hook] & 0 \\
0 \arrow[r] & T_{E_H} \arrow[r] \arrow[d, "\sim"', "A"] & T_{\mathcal{E}_H} \arrow[r] \arrow[d, dashed, "\mathcal{A}"] & T_{E_H} \arrow[r] \arrow[d, "\sim"', "A"] \arrow[dll, dashed, "\mathcal{A}_{ij}"'] & 0 \\
0 \arrow[r] & E_H \times \mathfrak{g} \arrow[r] & \mathcal{E}_H \times \mathfrak{g} \arrow[r] & E_H \times \mathfrak{g} \arrow[r] & 0\,,
\end{tikzcd} \tag{3.3}$$

where the isomorphism $\mathcal{A}$ is still to be constructed.

Let us cover X by a system $\mathfrak{U}$ of contractible, Stein, open subsets U_i such that over any U_i an isomorphism $\mathcal{A}_i$ does exist. The difference of any two maps $\mathcal{A}_i$ and $\mathcal{A}_j$ determines a cocycle $\mathcal{A}_{ij} \in Z^1(\mathfrak{U}, Hom(T_{E_H}, E_H \times \mathfrak{g})^H)$ in the space of H-invariant homomorphisms. Suppose that $\mathcal{A}_{ij}$ is a coboundary. Then the morphisms $\mathcal{A}_i$ can be changed to define the desired (global) map $\mathcal{A}$. It can be verified immediately that the cohomology class of $\mathcal{A}_{ij}$ is uniquely determined by the infinitesimal deformation of the principal H-bundle E_H and the holomorphic Cartan geometry A.

All the above morphisms $\mathcal{A}_i$, $\mathcal{A}_{ij}$ can be chosen as H-invariant. Then the obstructions are cohomology classes of $Hom(At(E_H), ad(E_G))$.

Proposition 3.2. *Let (E_H, A) be a holomorphic Cartan geometry of type (G, H) on X. Let $\mathcal{E}_H$ be an isomorphism class of infinitesimal deformations of E_H. Then the obstructions to extend the holomorphic Cartan geometry A to $\mathcal{E}_H$ are in*

$$H^1(E_H, Hom(T_{E_H}, E_H \times \mathfrak{g})^H)$$

or equivalently in

$$H^1(X, Hom(At(E_H), ad(E_G)).$$

We denote by $H^1(X, ad(E_H))_A$ the space of isomorphism classes of infinitesimal deformations of E_H such that the Cartan geometry A of the central fiber can be extended, so $H^1(X, ad(E_H))_A \subset H^1(X, ad(E_H))$. Furthermore we denote by $T^1(E_H, A)$ the space of isomorphism classes of infinitesimal deformations of (E_H, A).

If we assume that a holomorphic deformation $(\mathcal{E}_H, \mathcal{A})$ of the holomorphic Cartan geometry (E_H, A) does exist, then $\mathcal{A}$ is unique up to an H-invariant morphism $T_{E_H} \to E_H \times \mathfrak{g}$. This implies the following result.

Theorem 3.3 (Infinitesimal deformations)**.** *Let (E_H, A) be a holomorphic Cartan geometry. Then there is an exact sequence*

$$0 \to H^0(E_H, Hom(T_{E_H}, E_H \times \mathfrak{g})^H) \to T^1(E_H, A) \to H^1(X, ad(E_H))_A \to 0,$$

where $H^0(E_H, Hom(T_{E_H}, E_H \times \mathfrak{g})^H)$ can be identified with $H^0(X, Hom(At(E_H), ad(E_G)))$.

3.3. **Semi-universal deformation of principal bundles.** We begin the construction with a semi-universal deformation of E_H. A deformation over a complex space S with base point s_0 is given by a holomorphic family $\mathcal{E}_H$ of principal H bundles over $X \times S$, together with an isomorphism $\Xi : E_H \to \mathcal{E}_H | X \times \{s_0\}$. Semi-universality amounts to the following conditions (cf. also the more general Definition 2.2):

(i) *(Completeness)* for any deformation $\underline{\xi}$ of E_H over a space (W, w_0), given by a holomorphic family $\mathcal{E}_{H,W} \to X \times W$ together with an isomorphism of the above type, there is a base change morphism $f : (W, w_0) \to (S, s_0)$ such that (after replacing the base space with open neighborhoods of the base points, if necessary) the pull back $f^*\underline{\xi} = (id_X \times f)^*(\mathcal{E}_H)$ and $\mathcal{E}_{H,W}$ are isomorphic with isomorphism inducing the identity map over $X \times \{w_0\}$.

(ii) Let $(W, w_0) = (D, 0)$ be as in (i). Then any such base change f is uniquely determined by the given deformation. The set of isomorphism classes of deformations over $(D, 0)$ is called "tangent space of the deformation functor" or first tangent cohomology associated to the given deformation problem. For deformations of a holomorphic principal H-bundle on X this space is known to be equal to $H^1(X, ad(E_H))$. (See [Don]; see also [BHH].)

For the sake of completeness we mention the Kodaira-Spencer map:

Given a deformation $\underline{\xi}$ over (W, w_0) we identify a tangent vector v of W at w_0 with a holomorphic map $f : (D, 0) \to (W, w_0)$. Then the Kodaira-Spencer map $\rho : T_{W,w_0} \to H^1(X, ad(E_H))$ maps $\underline{\xi}$ to the isomorphism class of $f^*\underline{\xi}$. We already computed the tangent cohomologies of order zero and one.

4. Semi-universal deformation for holomorphic Cartan geometries

4.1. **Morphisms of vector bundles.** Let again X be a compact complex manifold, S a complex space and E_1, E_2 holomorphic vector bundles on $X \times S$.

Definition 4.1. Let $\mathbf{An}_S$ be the category of complex analytic spaces over S and **Sets** the category of sets. The objects of $\mathbf{An}_S$ are complex space $W \to S$ and the morphisms compatible holomorphic mappings.

We have the following functor

$$F : Hom(E_1, E_2)_S : \mathbf{An}_S \to \mathbf{Sets}$$

(i) Any $f : W \to S$ is assigned to the set

$$F(f) = F(f : W \to S) = Hom((id_X \times f)^* E_1, (id_X \times f)^* E_2),$$

(ii) given a holomorphic map α

$$\begin{tikzcd} W_1 \arrow[r, "\alpha"] \arrow[dr, "f_1"'] & W_2 \arrow[d, "f_2"] \\ & S \end{tikzcd}$$

we set $E_{i,W_i} = (id_X \times f_i)^* E_i$ for $i = 1, 2$. Then $F(\alpha)$ assigns to

$$\kappa : \quad \begin{tikzcd} E_{1,W_2} \arrow[r, "\beta"] \arrow[dr] & E_{2,W_2} \arrow[d] \\ & X \times W_2 \end{tikzcd}$$

the pull back $g^*\kappa$ via g of κ:

$$g^*\kappa : \quad \begin{tikzcd} E_{1,W_1} \arrow[r, "(id_X \times \alpha)^* \beta"] \arrow[dr] & E_{2,W_1} \arrow[d] \\ & X \times W_1 \end{tikzcd}$$

We will use the representability of the morphism functor. In the algebraic case it was shown by Grothendieck in [Gr, EGAIII 7.7.8 and 7.7.9]. In the analytic case the corresponding theorem for complex spaces is due to Douady [Dou, 10.1 and 10.2]. For coherent (locally free sheaves) $\mathcal{M}$, and $\mathcal{N}$ over a space $\mathcal{X} \to S$, one considers morphisms of the simple extensions $\mathcal{X}[\mathcal{M}] \to \mathcal{X}[\mathcal{N}]$.

Fact. The functor F is representable by a complex space $g : R \to S$. There is a universal object in $F(g)$

$$\upsilon : \quad \begin{tikzcd} E_{1,R} \arrow[r, "\gamma"] \arrow[dr] & E_{2,R} \arrow[d] \\ & X \times R \end{tikzcd}$$

such that any other object over $\widetilde{R} \to S$ is isomorphic to $\widetilde{g}^* \upsilon$, where $\widetilde{g} : \widetilde{R} \to R$ is a holomorphic map over S.

4.2. Application to holomorphic Cartan geometries.

Theorem 4.2. *Let X be a compact complex manifold, $H < G$ connected complex Lie groups, and (E_H, A) a a holomorphic Cartan geometry on X of type (G, H). Then (E_H, A) possesses a semi-universal deformation.*

Proof. We begin with a semi-universal deformation of E_H:

$$\begin{tikzcd}
E_H \arrow[r, "\sim"] \arrow[dr] & \mathcal{E}_H|X \times \{s_0\} \arrow[r, hook] \arrow[d] & \mathcal{E}_H \arrow[d] \\
& X \times \{s_0\} \arrow[r, hook] & X \times S ,
\end{tikzcd}$$

and look at the holomorphic vector bundles

$$\begin{tikzcd}
At(\mathcal{E}_H) \arrow[dr] & & ad(\mathcal{E}_G) \arrow[dl] \\
& X \times S &
\end{tikzcd}$$

to which we apply the representability of the homomorphism functor. We have the following universal homomorphism η. (It can be checked easily that At and ad commute with base change).

$$\begin{tikzcd}
At(\mathcal{E}_{H,R}) \arrow[rr, "\eta"] \arrow[dr] \arrow[rrr, bend left] & & ad(\mathcal{E}_{G,R}) \arrow[dl] \arrow[rrr, bend left] & At(\mathcal{E}_H) \arrow[dr] & & ad(\mathcal{E}_G) \arrow[dl] \\
& X \times R \arrow[rrr, "id_X \times h"'] \arrow[d] & & & X \times S \arrow[d] & \\
& R \arrow[rrr, "h"'] & & & S &
\end{tikzcd}$$

Now the holomorphic Cartan geometry $A_H : At(E_H) \xrightarrow{\sim} ad(E_G)$ amounts to a point $r_0 \in R$, which is mapped to s_0 under $R \to S$ (observing the deformation theoretic isomorphisms of the given objects and distinguished fibers of the semi-universal families). We take the connected component of R through r_0 and restrict it to an open neighborhood, where the universal morphism is an isomorphism. Going through the construction, we see that the restricted family yields a semi-universal deformation of the holomorphic Cartan geometry (E_H, A). □

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