Beta and Dirichlet sub-Gaussianity
Olivier Marchal, Julyan Arbel

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Optimal proxy variance for the Beta

Theorem Beta(α, β) is $\sigma_{\text{opt}}^2(\alpha, \beta)$-sub-Gaussian with:

$$\sigma_{\text{opt}}^2(\alpha, \beta) = \frac{\alpha}{(\alpha + \beta)\beta_0} - \left(\frac{1}{F_1(\alpha + 1; \alpha + \beta + 1; \beta_0)} - 1\right)$$

where $\beta_0$ is the unique solution of the equation

$$\log(F_1(\alpha + 1; \alpha + \beta + 1; \beta_0)) = \frac{\alpha \beta_0}{2(\alpha + \beta)} \left(1 + \frac{1}{F_1(\alpha + 1; \alpha + \beta + 1; \beta_0)}\right).$$

Simple explicit upper bound to $\sigma_{\text{opt}}^2(\alpha, \beta)$ is $\sigma_{\text{opt}}^2(\alpha, \beta) = \frac{1}{4\sqrt{\beta_0 + 1}}$.

Var[Beta(α, β)] $\leq \sigma_{\text{opt}}^2(\alpha, \beta) \leq \sigma_\alpha^2$ (strict when $\alpha \neq \beta$

Sketch of proof

- $\alpha = \beta$: direct coef. comparison of entire series representations of (1)
- $\alpha \neq \beta$: study the MGF aka confluent hypergeometric function or Kummer’s function

$$y(x) \overset{\text{def}}{=} \mathbb{E}[\exp(xX)] = \frac{\Gamma(\alpha + j\beta)(\alpha + \beta)/\Gamma(\alpha + j\beta)}{(\alpha + j\beta)/(\alpha + \beta)}x^j.$$ 

using the ordinary differential equation it satisfies

$$xy''(x) + (\alpha + \beta - x)y'(x) - ay(x) = 0$$

via the difference

$$u_i(x) \overset{\text{def}}{=} \exp(\mu x + \sigma_i^2 x^2/2) - \mathbb{E}[\exp(xX)]. \quad \sigma_i^2 = i \text{Var}[X] + (1 - i)\sigma_\alpha^2$$

- For $t = 0$ ($\sigma_0^2$), dotted black curve remains $> 0$
- For $t = t_{\text{opt}} (\sigma_{\text{opt}}^2)$, magenta curve has minimum $= 0$ at $\beta_0$
- For $t = 1$ (Var[X]), dashed green curve has negative second derivative at $x = 0$, directly negative around 0
- For $t_{\text{opt}}$ (in $t_{\text{opt}}$, 1), orange, dash and dots curve is first positive, then negative, and positive again

Looking for connections with literature

- Compare with Bernoulli setting (Berend and Kontorovich, 2013; Buldygin and Moskvichova, 2013)
- Possible links of our non-uniform sub-Gaussian result with
  - transportation inequalities
  - logarithmic Sobolev inequalities

References


