Beta and Dirichlet sub-Gaussianity
Olivier Marchal, Julyan Arbel

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Optimal proxy variance for the Beta

**Theorem** Beta(α, β) is \( \sigma^2_{\text{opt}}(α, β) \)-sub-Gaussian with:

\[
\sigma^2_{\text{opt}}(α, β) = \frac{α}{(α + β) x_0} \left( F_1(α + 1; α + β + 1; x_0) - 1 \right)
\]

where \( x_0 \) is the unique solution of the equation

\[
\log(F_1(α; α + 1; α + β + 1; x_0)) = \frac{α x_0}{2(α + β)} \left( 1 + \frac{1}{2} F_1(α + 1; α + β + 1; x_0) \right).
\]

Simple explicit upper bound to \( \sigma^2_{\text{opt}}(α, β) \) is \( \frac{1}{4(α + β + 1)} \):

\[
\text{Var}[	ext{Beta}(α, β)] \leq \sigma^2_{\text{opt}}(α, β) = \frac{1}{4(α + β + 1)}.
\]

**Optimal proxy variance illustrated & compared**

- **Top**: varying \( α \) on x-axis, \( α + β = 1 \) (left) and \( α + β = 10 \) (right).
- **Left**: \( α, β \in [0, 2, 4] \).
  - Magenta: \( \sigma^2_{\text{opt}}(α, β) \)
  - Green: \( \text{Var}[	ext{Beta}(α, β)] \)
  - Black (dots): \( \sigma^2_0 = \frac{1}{4(α + β + 1)} \)

**Sketch of proof**

- \( α = β \): direct coef. comparison of entire series representations of (1)
- \( α ≠ β \): study the MGF aka confluent hypergeometric function or Kummer’s function

\[
y(x) \overset{def}{=} \mathbb{E}[\exp(x X)] = \Gamma(μ + j) / \Gamma(μ + j + x)
\]

using the ordinary differential equation it satisfies

\[
x y''(x) + (α + β - x)y'(x) - ay(x) = 0
\]

via the difference

\[
u_t(x) \overset{def}{=} \exp(μx + σ^2tx^2/2 - \mathbb{E}[\exp(x X)]) - \text{Var}[X] (t - 1)σ^2_0
\]

- For \( t = 0 \) (\( σ^2_0 \)), dotted black curve remains > 0
- For \( t = t_{\text{opt}}(σ^2_{\text{opt}}) \), magenta curve has minimum = 0 at \( x_0 \)
- For \( t = 1 \) (\( \text{Var}[X] \)), dashed green curve has negative second derivative at \( x = 0 \), directly negative around 0
- For \( t_{\text{noon opt}} \in (t_{\text{opt}}, 1) \), orange, dash and dots curve is first positive, then negative, and positive again

**References**


**Abstract**

- **Optimal proxy variance** \( σ^2_{\text{opt}} \) for the sub-Gaussianity of Beta distribution, improves recent conjecture \( σ^2_0 \) by Elder (2016): \( σ^2_{\text{opt}} \leq σ^2_0 = \frac{1}{4(α + β + 1)} \).

- Provide different proof techniques for
  - (i) symmetrical case (\( α = β \)): direct coef. comparison of entire series
  - (ii) non-symmetrical case (\( α ≠ β \)): ordinary differential equation satisfied by moment-generating function, aka confluent hypergeometric function

- Derive optimal proxy variance for Dirichlet.

**Sub-Gaussianity & optimal proxy variance**

- A random variable \( X \) with finite mean \( μ = \mathbb{E}[X] \) is **sub-Gaussian** if there is a positive number \( σ^2 \) (called a proxy variance) such that:

\[
\mathbb{E}[\exp(x(X - μ))] \leq \exp\left(\frac{x^2σ^2}{2}\right) \text{ for all } x \in \mathbb{R}.
\]

- If \( X \) is sub-Gaussian, the **optimal proxy variance** is:

\[
σ^2_{\text{opt}}(X) = \min\{σ^2 ≥ 0 \text{ such that } X \text{ is } σ^2\text{-sub-Gaussian}\}.
\]

Variance lower bounds optimal proxy variance: \( \text{Var}[X] \leq σ^2_{\text{opt}}(X) \). When \( σ^2_{\text{opt}}(X) = \text{Var}[X] \), \( X \) is said to be strictly sub-Gaussian.

**Looking for connections with literature**

- Compare with Bernoulli setting (Berend and Kontorovich, 2013; Buldygin and Moskvichova, 2013)
- Possible links of our non-uniform sub-Gaussian result with
  - transportation inequalities
  - logarithmic Sobolev inequalities

**Beta and Dirichlet sub-Gaussianity**

Olivier Marchal, Univ. Lyon, Univ. Monnet, Institut Camille Jordan, France

Julyan Arbel, Inria, Mistis, Grenoble, France, [www.julyanarbel.com](http://www.julyanarbel.com)