Beta and Dirichlet sub-Gaussianity
Olivier Marchal, Julyan Arbel

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Abstract

- **Optimal proxy variance** $\sigma_{opt}^2$ for the sub-Gaussianity of Beta distribution, improves recent conjecture $\sigma_0^2$ by Elder (2016): $\sigma_{opt}^2 \leq \sigma_0^2 = \frac{1}{2(\alpha + 1)}$.

- Provide different proof techniques for (i) symmetrical case ($\alpha = \beta$): direct coef. comparison of entire series (ii) non-symmetrical case ($\alpha \neq \beta$): ordinary differential equation satisfied by moment-generating function, aka confluent hypergeometric function

- Derive optimal proxy variance for Dirichlet.

Sub-Gaussianity & optimal proxy variance

- A random variable $X$ with finite mean $\mu = \mathbb{E}[X]$ is **sub-Gaussian** if there is a positive number $\sigma^2$ (called a proxy variance) such that:

$$\mathbb{E}[\exp(x(X - \mu))] \leq \exp\left(\frac{x^2\sigma^2}{2}\right)$$

for all $x \in \mathbb{R}$. (1)

- If $X$ is sub-Gaussian, the **optimal proxy variance** is:

$$\sigma_{opt}^2(X) = \min(\sigma^2 \geq 0 \text{ such that } X \text{ is } \sigma^2\text{-sub-Gaussian}).$$

Variance lower bounds optimal proxy variance: $\text{Var}[X] \leq \sigma_{opt}^2(X)$. When $\sigma_{opt}^2(X) = \text{Var}[X]$, $X$ is said to be strictly sub-Gaussian.

Optimal proxy variance illustrated & compared

- Top: varying $\alpha$ on x-axis, $\alpha + \beta = 1$ (left) and $\alpha + \beta = 10$ (right).
- Left: $\alpha, \beta \in [0.2, 4]$.
  - Magenta: $\sigma_{opt}^2(\alpha, \beta)$
  - Green: $\text{Var}[\text{Beta}(\alpha, \beta)]$
  - Black (dots): $\sigma_0^2 = \frac{1}{2(\alpha + 1)}$

Looking for connections with literature

- Compare with **Bernoulli setting** (Berend and Kontorovich, 2013; Buldygin and Moskvichova, 2013)
- Possible links of our non-uniform sub-Gaussian result with
  - transportation inequalities
  - logarithmic Sobolev inequalities

Optimal proxy variance for the Beta

**Theorem** Beta($\alpha, \beta$) is $\sigma_{opt}^2(\alpha, \beta)$-sub-Gaussian with:

$$\sigma_{opt}^2(\alpha, \beta) = \frac{\alpha}{(\alpha + \beta)\alpha_0} \left( \frac{F_\alpha(\alpha + 1; \alpha + \beta + 1; x_0)}{F_\alpha(\alpha + 1; \alpha + \beta, x_0)} - 1 \right)$$

where $x_0$ is the unique solution of the equation

$$\log(F_\alpha(\alpha + 1; \alpha + \beta; x_0)) = \frac{\alpha x_0}{2(\alpha + \beta)} \left( 1 + \frac{F_\alpha(\alpha + 1; \alpha + \beta + 1; x_0)}{F_\alpha(\alpha + 1; \alpha + \beta, x_0)} \right).$$

Simple explicit upper bound to $\sigma_{opt}^2(\alpha, \beta)$ is $\sigma_{opt}^2(\alpha, \beta) = \frac{1}{4(\alpha + 1)}$.

$$\text{Var}[	ext{Beta}(\alpha, \beta)] \leq \sigma_{opt}^2(\alpha, \beta) \leq \sigma_0^2$$

strict when $\alpha \neq \beta$.

Sketch of proof

- $\alpha = \beta$: direct coef. comparison of entire series representations of (1)
- $\alpha \neq \beta$: study the MGF aka confluent hypergeometric function or Kummer’s function

$$y(x) \equiv \mathbb{E}[\exp(xX)] = \frac{\Gamma(\alpha + 1)\Gamma(\alpha + \beta)}{(\alpha + 1)\Gamma(\alpha + 1 + \beta)}x^\alpha(1-x)^{\beta-1} \quad \Rightarrow \quad y''(x) + (\alpha + \beta - 2x)y'(x) - \alpha y(x) = 0$$

using the ordinary differential equation it satisfies

$$u(x) \equiv \exp(\mu x + \sigma^2 x^2/2 - \mathbb{E}[\exp(xX)]) \sigma^2 = \text{Var}[X] + (1 - t)\sigma_0^2$$

$
\sigma_0^2 = \frac{1}{2(\alpha + 1)}$

- For $t = 0$ ($\sigma_0^2$), dotted black curve remains $> 0$
- For $t = t_{opt} (\sigma_{opt}^2)$, magenta curve has minimum $= 0$ at $x_0$
- For $t = 1$ (Var[$X$]), dashed green curve has negative second derivative at $x = 0$, directly negative around 0
- For $t_{too_{opt}} \in (t_{opt}, 1)$, orange, dash and dots curve is first positive, then negative, and positive again

References


