Beta and Dirichlet sub-Gaussianity
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**Abstract**

**Optimal proxy variance** $\sigma_{opt}^2$ for the sub-Gaussianity of Beta distribution, improves recent conjecture $\sigma_0^2$ by Elder (2016): $\sigma_{opt}^2 \leq \sigma_0^2 = \frac{x_0}{\alpha + \beta - 1}$.

- Provide different proof techniques for
  (i) symmetrical case ($\alpha = \beta$): direct coeff. comparison of entire series
  (ii) non-symmetrical case ($\alpha \neq \beta$): ordinary differential equation satisfied by moment-generating function, aka confluent hypergeometric function

- Derive optimal proxy variance for Dirichlet.

**Sub-Gaussianity & optimal proxy variance**

- A random variable $X$ with finite mean $\mu = \mathbb{E}[X]$ is **sub-Gaussian** if there is a positive number $\sigma^2$ (called a proxy variance) such that:

  \[
  \mathbb{E}[\exp(x(X - \mu))] \leq \exp\left(\frac{x^2\sigma^2}{2}\right) \quad \text{for all } x \in \mathbb{R}.
  \]

  \[
  (1)
  \]

- If $X$ is sub-Gaussian, the **optimal proxy variance** is:

  \[
  \sigma_{opt}^2(X) = \min\{\sigma^2 \geq 0 \text{ such that } X \text{ is } \sigma^2\text{-sub-Gaussian}\}.
  \]

- Variance lower bounds optimal proxy variance: $\text{Var}[X] \leq \sigma_{opt}^2(X)$. When $\sigma_{opt}^2(X) = \text{Var}[X]$, $X$ is said to be strictly sub-Gaussian.

**Optimal proxy variance illustrated & compared**

- Top: varying $\alpha$ on x-axis, $\alpha + \beta = 1$ (left) and $\alpha + \beta = 10$ (right).
- Left: $\alpha, \beta \in [0.2, 4]$.
  - Magenta: $\sigma_{opt}^2(\alpha, \beta)$
  - Green: $\text{Var}[$Beta$(\alpha, \beta)]$
  - Black (dots): $\sigma_0^2 = \frac{1}{\alpha + \beta - 1}$

**Optimal proxy variance for the Beta**

**Theorem** Beta$(\alpha, \beta)$ is $\sigma_{opt}^2(\alpha, \beta)$-sub-Gaussian with:

\[
\sigma_{opt}^2(\alpha, \beta) = \frac{\alpha}{(\alpha + \beta)\lambda_0} \left( \frac{F_1(\alpha + 1; \alpha + \beta + 1; \lambda_0)}{F_1(\alpha; \alpha + \beta, \lambda_0)} - 1 \right)
\]

where $\lambda_0$ is the unique solution of the equation

\[
\log(F_1(\alpha; \alpha + \beta; \lambda_0)) = \frac{\alpha \lambda_0}{2(\alpha + \beta)} \left( 1 + \frac{1}{F_1(\alpha + 1; \alpha + \beta + 1; \lambda_0)} \right).
\]

Simple explicit upper bound to $\sigma_{opt}^2(\alpha, \beta)$ is $\sigma_0^2(\alpha, \beta) = \frac{1}{\alpha + \beta - 1}$.

**Var[Beta$(\alpha, \beta)$] $\leq$ $\sigma_{opt}^2(\alpha, \beta) \leq \sigma_0^2$ (strict when $\alpha \neq \beta$)

**Sketch of proof**

- $\alpha = \beta$: direct coeff. comparison of entire series representations of (1)
- $\alpha \neq \beta$: study the MGF aka confluent hypergeometric function or Kummer’s function

\[
y(x) \overset{\text{def}}{=} \mathbb{E}[\exp(xX)] = \exp\left(\frac{\Gamma(\alpha + j)\Gamma(\alpha + \beta)}{\Gamma(\alpha + \beta + j)} x^j\right),
\]

using the ordinary differential equation it satisfies

\[
x^2(y') + (\alpha + \beta - x)y' - ay = 0
\]

via the difference

\[
u_t(x) \overset{\text{def}}{=} \exp\left(\mu x + \sigma_t^2 x^2/2 - \mathbb{E}[\exp(xX)]\right), \sigma_t^2 = t \text{Var}[X] + (1 - t)\sigma_0^2
\]

- For $t = 0$ ($\sigma_0^2$), dotted black curve remains $> 0$
- For $t = t_{opt}$ ($\sigma_{opt}^2$), magenta curve has minimum $= 0$ at $x_0$
- For $t = 1$ (Var[X]), dashed green curve has negative second derivative at $x = 0$, directly negative around 0
- For $t_{non_{opt}} \in (t_{opt}, 1)$, orange, dash and dots curve is first positive, then negative, and positive again

Looking for connections with literature

- Compare with Bernoulli setting (Berend and Kontorovich, 2013; Buldygin and Moskvichova, 2013)
- Possible links of our non-uniform sub-Gaussian result with
  - transportation inequalities
  - logarithmic Sobolev inequalities

**References**


