Beta and Dirichlet sub-Gaussianity
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Abstract

Optimal proxy variance $\sigma_{opt}^2$ for the sub-Gaussianity of Beta distribution, improves recent conjecture $\sigma_0^2$ by Elder (2016): $\sigma_{opt}^2 \leq \sigma_0^2 = \frac{1}{4\alpha\beta+1}$.

- Provide different proof techniques for
  (i) symmetrical case ($\alpha = \beta$): direct coef. comparison of entire series
  (ii) non-symmetrical case ($\alpha \neq \beta$): ordinary differential equation satisfied by moment-generating function, aka confluent hypergeometric function

- Derive optimal proxy variance for Dirichlet.

Optimal proxy variance for the Beta

Theorem Beta($\alpha, \beta$) is $\sigma_{opt}^2(\alpha, \beta)$-sub-Gaussian with:

$$\sigma_{opt}^2(\alpha, \beta) = \frac{\alpha}{(\alpha + \beta)\lambda_0} \left( \frac{F_1(\alpha + 1; \alpha + \beta + 1; \lambda_0) - 1}{F_1(\alpha; \alpha + \beta, \lambda_0)} \right)$$

where $\lambda_0$ is the unique solution of the equation

$$\log(\frac{F_1(\alpha + 1; \alpha + \beta; \lambda_0) + 1}{F_1(\alpha; \alpha + \beta, \lambda_0)}) = \frac{\alpha}{2(\alpha + \beta)} \left( \frac{1}{F_1(\alpha; \alpha + \beta, \lambda_0)} - 1 \right).$$

Simple explicit upper bound to $\sigma_{opt}^2(\alpha, \beta)$ is $\sigma_0^2(\alpha, \beta) = \frac{1}{4\alpha\beta+1}$.

$$\text{Var}[\text{Beta}(\alpha, \beta)] \leq \sigma_{opt}^2(\alpha, \beta) \leq \sigma_0^2(\alpha, \beta) \text{ (strict when } \alpha \neq \beta)$$

Sketch of proof

- $\alpha = \beta$: direct coef. comparison of entire series representations of (1)
- $\alpha \neq \beta$: study the MGF aka confluent hypergeometric function or Kummer’s function

$$y(x) \overset{\text{def}}{=} \mathbb{E}[\exp(xX)] = \frac{\Gamma(\alpha + j\lambda/\beta)(\alpha + \beta + j\lambda^2)x^j}{\Gamma(\alpha + j)}.$$ 

using the ordinary differential equation it satisfies

$$xy''(x) + (\alpha + \beta - x)y'(x) - ay(x) = 0$$

via the difference

$$u_j(x) \overset{\text{def}}{=} \exp(\mu x + \sigma^2 x^2/2) - \mathbb{E}[\exp(xX)], \sigma_j^2 = t \text{Var}[X] + (1 - t)\sigma_0^2$$

- For $t = 0$ ($\sigma_0^2$), dotted black curve remains $> 0$
- For $t = t_{opt}(\sigma_{opt}^2)$, magenta curve has minimum $= 0$ at $\lambda_0$
- For $t = 1$ (Var[X]), dashed green curve has negative second derivative at $x = 0$, directly negative around 0
- For $t = t_{non-opt}(\sigma_{non-opt}^2)$, orange, dash and dots curve is first positive, then negative, and positive again

Looking for connections with literature

- Compare with Bernoulli setting (Berend and Kontorovich, 2013; Buldygin and Moskvichova, 2013)
- Possible links of our non-uniform sub-Gaussian result with
  - transportation inequalities
  - logarithmic Sobolev inequalities

References


