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Caroline Lawless, Julyan Arbel

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Chinese restaurant process from stick-breaking for Pitman–Yor
CAROLINE LAWLESS AND JULYAN ARBEL
Univ. Grenoble Alpes, Inria, CNRS, LJK, 38000 Grenoble, France

REFERENCES

INTRODUCTION

• The Chinese restaurant process and the stick-breaking process are the two most commonly used representations of the Pitman–Yor process.
• However, the usual proof of the connection between them is indirect.
• Miller (2018) proved directly that the stick-breaking process gives rise to the Chinese restaurant process representation of the Dirichlet process.
• The Dirichlet process is a special case of the Pitman–Yor process.
• We extend Miller’s proof to Pitman–Yor process random measures.

PITMAN–YOR & DIRICHLET PROCESSES

• The Dirichlet process (DP) and the Pitman–Yor process (PY, Pitman and Yor, 1997) are discrete random probability measures.
• The PY is parametrized by \( d \in (0,1), \alpha > -d \), and a base probability measure \( P_0 \). The DP is recovered by letting \( d = 0 \).
• The stick-breaking representation (Sethuraman, 1994) is given by
  \[
  v_i \sim \text{Beta}(1, \alpha) \quad \text{for DP}
  \]
  \[
  v_i \sim \text{Beta}(1 + d, \alpha + id) \quad \text{for PY}
  \]
  \[
  \pi_k = v_1 \prod_{i=2}^{k-1} (1 - v_i), \phi_k \equiv \frac{d}{\alpha + (k-1)d} P_b.
  \]
We define the random process \( P \) by
  \[
  P = \sum_{k=1}^{\infty} \pi_k \delta_{\phi_k}.
  \]
• The Chinese restaurant process (Antoniak, 1974) is the distribution induced on random partitions \( C \) given by
  \[
  P(C = C) = \begin{cases}
  \frac{\alpha^{\|C\|}}{\Gamma(\alpha)} \prod_{c \in C} \Gamma(\|c\|) & \text{for DP} \\
  \frac{\alpha^{\|C\|}}{\Gamma(\alpha)} \prod_{c \in C} (1 - \|c\| + \|c\| - 1) & \text{for PY}
  \end{cases}
  \]

THEOREM

Suppose \( \pi \) follows the PY stick-breaking, and
  \[
  z_1, \ldots, z_n | \pi = \pi \sim \pi, \quad \text{that is, } P(z_i = k | \pi) = \pi_k,
  \]
and \( C \) is the partition of \( \{n\} \) induced by \( z_1, \ldots, z_n \).
Then \( C \) follows the PY Chinese restaurant process.

TECHNICAL LEMMAS

Our proof relies on the following lemmas, which here we will state without proof. Let us abbreviate \( z = (z_1, \ldots, z_n) \). Given \( n \in \mathbb{N} \), let \( C_n \) denote the partition \( [n] \) induced by \( z \). We define \( \mu(z) = \max\{z_1, \ldots, z_n\} \), and \( g_k(z) = \#\{i : z_i \geq k\} \).

Lemma 1 For any \( z \in \mathbb{N}^n \),
  \[
  P(z = z) = \frac{1}{\Gamma(n)} \prod_{c \in C_n} \Gamma(\|c\| + 1 - d) \prod_{k=1}^{\mu(z)} \frac{\alpha + (k-1)d}{g_k(z) + \alpha + (k-1)d}.
  \]

Lemma 2 For any partition \( C \) of \( [n] \),
  \[
  \sum_{z \in \mathbb{N}^n} I(C_z = C) \prod_{k=1}^{\mu(z)} \frac{\alpha + (k-1)d}{g_k(z) + \alpha + (k-1)d} = \frac{d^\alpha(\frac{\pi}{\alpha\gamma})}{\Gamma(d)}.
  \]

PROOF OF THEOREM

\[
\begin{align*}
P(C = C) &= \sum_{z \in \mathbb{N}^n} P(C = C | z = z) P(z = z) \\
&= \sum_{z \in \mathbb{N}^n} (a) \prod_{c \in C_n} \Gamma(\|c\| + 1 - d) \prod_{k=1}^{\mu(z)} \frac{\alpha + (k-1)d}{g_k(z) + \alpha + (k-1)d} \\
&= \sum_{z \in \mathbb{N}^n} (b) \prod_{c \in C_n} \Gamma(\|c\| + 1 - d) \prod_{k=1}^{\mu(z)} \frac{\alpha + (k-1)d}{g_k(z) + \alpha + (k-1)d} \\
&= \sum_{z \in \mathbb{N}^n} (c) \prod_{c \in C_n} \Gamma(\|c\| + 1 - d) \frac{d^\alpha(\frac{\pi}{\alpha\gamma})}{\Gamma(d)}
\end{align*}
\]
where \( (a) \) is by Lemma 1, \( (b) \) is by Lemma 2, and \( (c) \) is since \( \Gamma(\|c\| + 1 - d) = (\|c\| - d) \Gamma(\|c\| - d) \).

FURTHER RESEARCH

• The Dirichlet process and the Pitman–Yor process are only special cases of a broad class of random measures called Gibbs-type random measures.
• An interesting further study would be to investigate the possibility of extending this proof to Gibbs-type random measures.