Chinese restaurant process from stick-breaking for Pitman-Yor
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INTRODUCTION

- The Chinese restaurant process and the stick-breaking process are the two most commonly used representations of the Pitman–Yor process.
- However, the usual proof of the connection between them is indirect.
- Miller (2018) proved directly that the stick-breaking process gives rise to the Chinese restaurant process representation of the Dirichlet process.
- The Dirichlet process is a special case of the Pitman–Yor process.
- We extend Miller’s proof to Pitman–Yor process random measures.

PITMAN–YOR & DIRICHLET PROCESSES

- The Dirichlet Process (DP) and the Pitman–Yor process (PY, Pitman and Yor, 1997) are discrete random probability measures.
- The PY is parametrized by $d \in (0, 1)$, $\alpha > -d$, and a base probability measure $P_0$. The DP is recovered by letting $d = 0$.
- The stick-breaking representation (Sethuraman, 1994) is given by
  
  $$ v_i \sim \text{Beta}(1, \alpha) \quad \text{for DP} $$
  
  $$ v_i \sim \text{Beta}(1+\alpha+d, \alpha+id) \quad \text{for PY} $$

  $$ \pi_k = v_1 \prod_{i=2}^{k-1} (1-v_i), \quad \phi_k \overset{iid}{\sim} P_0. $$

  We define the random process $P$ by
  
  $$ P = \sum_{i=1}^{\infty} \pi_k \delta_{\phi_k}. $$

  The Chinese restaurant process (Antoniak, 1974) is the distribution induced on random partitions $C$ given by
  
  $$ P(C = C) = \begin{cases} 
  \frac{\alpha^{c(C)}}{(\alpha)^{c(C)}} \prod_{c \in C} \Gamma(|c|) & \text{for DP} \\
  \frac{\alpha^{d(C)}}{(\alpha)^{d(C)}} \prod_{c \in C} \Gamma(|c|-1) & \text{for PY}. 
  \end{cases} $$

THEOREM

Suppose $\pi$ follows the PY stick-breaking, and

$$ z_1, \ldots, z_n | \pi \overset{iid}{\sim} \pi, \quad \text{that is, } P(z_i = k | \pi) = \pi_k, $$

and $C$ is the partition of $[n]$ induced by $z_1, \ldots, z_n$. Then $C$ follows the PY Chinese restaurant process.

TECHNICAL LEMMAS

Our proof relies on the following lemmas, which here we will state without proof. Let us abbreviate $z = (z_1, \ldots, z_n)$. Given $z \in \mathbb{N}^n$, let $C_z$ denote the partition $[n]$ induced by $z$. We define $m(z) = \max \{z_1, \ldots, z_n\}$, and $g_k(z) = \# \{i : z_i \geq k\}$. We define $\pi$ follows the PY stick-breaking, and

**Lemma 1** For any $z \in \mathbb{N}^n$,

$$ P(z = z) = \frac{1}{(\alpha)^{c(C_z)}} \prod_{c \in C_z} \Gamma(|c|+1-d) \prod_{k=1}^{m(z)} \frac{\alpha + (k-1)d}{g_k(z) + \alpha + (k-1)d}. $$

**Lemma 2** For any partition $C$ of $[n]$,

$$ \sum_{z \in \mathbb{N}^n} I(C_z = C) \prod_{k=1}^{m(z)} \frac{\alpha + (k-1)d}{g_k(z) + \alpha + (k-1)d} = \frac{d(\frac{\pi}{\alpha})}{\pi}. $$

PROOF OF THEOREM

$$ P(C = C) = \sum_{z \in \mathbb{N}^n} P(C = C | z = z) P(z = z) $$

$$ = (a) \sum_{z \in \mathbb{N}^n} 1(C_z = C) \frac{1}{(\alpha)^{c(C_z)}} \prod_{c \in C_z} \Gamma(|c|+1-d) \prod_{k=1}^{m(z)} \frac{\alpha + (k-1)d}{g_k(z) + \alpha + (k-1)d} $$

$$ = (b) \frac{1}{(\alpha)^{c(C)}} \prod_{c \in C} \Gamma(|c|+1-d) \prod_{z \in \mathbb{N}^n} \sum_{z \in \mathbb{N}^n} 1(C_z = C) \prod_{k=1}^{m(z)} \frac{\alpha + (k-1)d}{g_k(z) + \alpha + (k-1)d} $$

$$ = (c) \frac{1}{(\alpha)^{c(C)}} \prod_{c \in C} \Gamma(|c|+1-d) \prod_{z \in \mathbb{N}^n} \sum_{z \in \mathbb{N}^n} (1-d)_{[c]-1} \prod_{c \in C} (\frac{d(\frac{\pi}{\alpha})}{\pi}) $$

$$ = \frac{d(\frac{\pi}{\alpha})}{\pi} \prod_{j=1}^{m(C)} (1-d)_{[c]-1} $$

where $(a)$ is by Lemma 1, $(b)$ is by Lemma 2, and $(c)$ is since $\Gamma(|c|+1-d) = (|c|-d)\Gamma(|c|-d)$.

FURTHER RESEARCH

- The Dirichlet process and the Pitman–Yor process are only special cases of a broad class of random measures called Gibbs-type random measures.
- An interesting further study would be to investigate the possibility of extending this proof to Gibbs-type random measures.

REFERENCES


