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# Chinese restaurant process from stick-breaking for Pitman–Yor

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## INTRODUCTION

- The Chinese restaurant process and the stick-breaking process are the two most commonly used representations of the Pitman–Yor process.
- However, the usual proof of the connection between them is indirect.
- Miller (2018) proved directly that the stick-breaking process gives rise to the Chinese restaurant process representation of the Dirichlet process.
- The Dirichlet process is a special case of the Pitman–Yor process.
- We extend Miller’s proof to Pitman–Yor process random measures.

## PITMAN–YOR & DIRICHLET PROCESSES

- The Dirichlet Process (DP) and the Pitman–Yor process (PY, Pitman and Yor, 1997) are discrete random probability measures.
- The PY is parametrized by  $d \in (0, 1)$ ,  $\alpha > -d$ , and a base probability measure  $P_0$ . The DP is recovered by letting  $d = 0$ .
- The stick-breaking representation (Sethuraman, 1994) is given by

$$v_i \sim \begin{cases} \text{Beta}(1, \alpha) & \text{for DP} \\ \text{Beta}(1 + d, \alpha + id) & \text{for PY} \end{cases}$$

$$\pi_k = v_k \prod_{i=1}^{k-1} (1 - v_i), \phi_k \stackrel{\text{iid}}{\sim} P_0.$$

We define the random process  $P$  by

$$P = \sum_{i=1}^{\infty} \pi_i \delta_{\phi_i}.$$

- The Chinese restaurant process (Antoniak, 1974) is the distribution induced on random partitions  $\mathcal{C}$  given by

$$P(\mathcal{C} = C) = \begin{cases} \frac{\alpha^{|\mathcal{C}|} \Gamma(\alpha)}{\Gamma(n + \alpha)} \prod_{c \in C} \Gamma(|c|) & \text{for DP} \\ \frac{d^t (\frac{\alpha}{d})_t}{(\alpha)_n} \prod_{j=1}^t (1 - d)_{(|c_j| - 1)} & \text{for PY.} \end{cases}$$

## THEOREM

Suppose  $\pi$  follows the PY stick-breaking, and

$$z_1, \dots, z_n | \pi = \pi \stackrel{\text{iid}}{\sim} \pi, \text{ that is, } \mathbb{P}(z_i = k | \pi) = \pi_k,$$

and  $\mathcal{C}$  is the partition of  $[n]$  induced by  $z_1, \dots, z_n$ . Then  $\mathcal{C}$  follows the PY Chinese restaurant process.

## TECHNICAL LEMMAS

Our proof relies on the following lemmas, which here we will state without proof. Let us abbreviate  $z = (z_1, \dots, z_n)$ . Given  $z \in \mathbb{N}^n$ , let  $C_z$  denote the partition  $[n]$  induced by  $z$ . We define  $m(z) = \max\{z_1, \dots, z_n\}$ , and  $g_k(z) = \#\{i: z_i \geq k\}$ .

**Lemma 1** For any  $z \in \mathbb{N}^n$ ,

$$\mathbb{P}(z = z) = \frac{1}{(\alpha)_n} \prod_{c \in C_z} \frac{\Gamma(|c| + 1 - d)}{\Gamma(1 - d)} \prod_{k=1}^{m(z)} \frac{\alpha + (k - 1)d}{g_k(z) + \alpha + (k - 1)d}.$$

**Lemma 2** For any partition  $C$  of  $[n]$ ,

$$\sum_{z \in \mathbb{N}^n} \mathbb{1}(C_z = C) \prod_{k=1}^{m(z)} \frac{\alpha + (k - 1)d}{g_k(z) + \alpha + (k - 1)d} = \frac{d^t (\frac{\alpha}{d})_t}{\prod_{c \in C} (|c| - d)}.$$

## PROOF OF THEOREM

$$\mathbb{P}(\mathcal{C} = C) = \sum_{z \in \mathbb{N}^n} \mathbb{P}(\mathcal{C} = C | z) \mathbb{P}(z = z)$$

$$\stackrel{(a)}{=} \sum_{z \in \mathbb{N}^n} \mathbb{1}(C_z = C) \frac{1}{(\alpha)_n} \prod_{c \in C_z} \frac{\Gamma(|c| + 1 - d)}{\Gamma(1 - d)} \prod_{k=1}^{m(z)} \frac{\alpha + (k - 1)d}{g_k(z) + \alpha + (k - 1)d}$$

$$= \frac{1}{(\alpha)_n} \prod_{c \in C} \frac{\Gamma(|c| + 1 - d)}{\Gamma(1 - d)} \sum_{z \in \mathbb{N}^n} \mathbb{1}(C_z = C) \prod_{k=1}^{m(z)} \frac{\alpha + (k - 1)d}{g_k(z) + \alpha + (k - 1)d}$$

$$\stackrel{(b)}{=} \frac{1}{(\alpha)_n} \prod_{c \in C} \frac{\Gamma(|c| + 1 - d)}{\Gamma(1 - d)} \frac{d^t (\frac{\alpha}{d})_t}{\prod_{c \in C} (|c| - d)}$$

$$\stackrel{(c)}{=} \frac{1}{(\alpha)_n} \prod_{c \in C} (1 - d)_{(|c| - 1)} \prod_{c \in C} (|c| - d) \frac{d^t (\frac{\alpha}{d})_t}{\prod_{c \in C} (|c| - d)}$$

$$= \frac{d^t (\frac{\alpha}{d})_t}{(\alpha)_n} \prod_{j=1}^t (1 - d)_{(|c_j| - 1)}$$

where (a) is by Lemma 1, (b) is by Lemma 2, and (c) is since  $\Gamma(|c| + 1 - d) = (|c| - d)\Gamma(|c| - d)$ .

## FURTHER RESEARCH

- The Dirichlet process and the Pitman–Yor process are only special cases of a broad class of random measures called Gibbs-type random measures.
- An interesting further study would be to investigate the possibility of extending this proof to Gibbs-type random measures.

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