Bayesian neural network priors at the level of units
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Introduction

We investigate deep Bayesian neural networks with Gaussian priors on the weights and ReLU-like nonlinearities. See Vladimirova et al. (2018).

\[
\begin{array}{ccccccc}
\text{input} & \text{1st hid.} & \text{2nd hid.} & \text{3rd hid.} & \text{layer} & \text{layer} & \text{layer} \\
\end{array}
\]

The \textit{covariance} between hidden units of the same layer is non-negative. Moreover, for any \(\ell\)-th hidden layer units \(h^{(\ell)}\) and \(h^{(\ell)}\), for \(s, t \in \mathbb{N}\) it holds

\[
\text{Cov} \left[ (h^{(\ell)}_s)^2, (h^{(\ell)}_t)^2 \right] \geq 0.
\]

\textbf{Theorem (Vladimirova et al., 2018)}

The \(\ell\)-th hidden layer units \(U^{(\ell)}\) (pre-activation \(g^{(\ell)}\) or post-activation \(h^{(\ell)}\)) of a feed-forward Bayesian neural network with:

- Gaussian priors on weights and
- extended envelope condition activation function \(\phi\)

have sub-Weibull marginal prior distribution with optimal tail parameter \(\theta = \ell/2\), conditional on the input \(x\):

\[
U^{(\ell)} \sim \text{subW}(\ell/2).
\]

\textbf{Assumptions}

- Gaussian prior on weights:
  \(W_{ij} \sim N(0, \sigma^2_w)\).
- A nonlinearity \(\phi : \mathbb{R} \rightarrow \mathbb{R}\) is said to obey the extended envelope property if there exist \(c_1, c_2, d_1 \geq 0\) such that
  \[
  |\phi(u)| \leq c_1 + d_1 |u| \quad \text{for } u \in \mathbb{R},
  \]
  \[
  |\phi(u)| \leq c_2 + d_2 |u| \quad \text{for } u \in \mathbb{R_+}.
  \]

\textbf{Sub-Weibull}

A random variable \(X\), such that

\[
\mathbb{P}(|X| \geq x) \leq \exp \left( -x^{3/2}/K \right)
\]

for all \(x > 0\) and for some \(K > 0\), is called a sub-Weibull random variable with tail parameter \(\theta > 0\):

\[
X \sim \text{subW}(\theta).
\]

\textbf{Moment property:}

\[
\mathbb{E}[X|X|^{3/2}] \leq K
\]

Meaning for all \(k \in \mathbb{N}\) and for some constants \(d, D > 0\),

\[
d < \|X\|_k^{3/2} < D.
\]

Proof sketch

\[
\text{Inductive step: suppose } \|h^{(\ell-1)}\|_k \leq k^{3/2}. \]

\[
\text{Lower bound: non-negative covariance theorem:}
\text{Cov} \left[ (h^{(\ell-1)})^2, (h^{(\ell-1)})^2 \right] \geq 0.
\]

\[
\text{Upper bound: Holder’s inequality:}
\|g^{(\ell)}\|_k = \sum_{j=1}^n W^{(\ell)}_{ij} h^{(\ell-1)}_j \leq k^{3/2}.
\]

Sparsity interpretation

\textbf{MAP on weights is L2-regularized Gaussian prior}

\[
\pi(W) \propto \prod_{t=1}^L \prod_{i,j} p\left( w^{(\ell)}_{ij} \right) C(F(U)),
\]

where \(C\) is the copula of \(U\) (characterizes all the dependence between the units), \(F\) is its cumulative distribution function. The penalty is the negative log-prior:

\[
\mathcal{L}(W) \propto \sum_{\ell=1}^L \sum_{i,j} (W^{(\ell)}_{ij})^2 = \|W\|_F^2.
\]

\textbf{MAP on units induces sparsity}

The joint prior distribution for all the units can be expressed by Sklar’s representation theorem as

\[
\pi(U) = \prod_{\ell=1}^L \prod_{i,j} p\left( u^{(\ell)}_{ij} \right) C(F(U)),
\]

where \(C\) is the copula of \(U\) (characterizes all the dependence between the units), \(F\) is its cumulative distribution function. The penalty is the negative log-prior:

\[
\mathcal{L}(U) \propto \|U^{(1)}\|_2 + \cdots + \|U^{(L)}\|_2
\]

Conclusion

We prove that the marginal prior unit distributions are heavier-tailed as depth increases. We further interpret this finding, showing that the units tend to be more sparsely represented as layers become deeper. This result provides new theoretical insight on deep Bayesian neural networks, underpinning their natural shrinkage properties and practical potential.

\textbf{References}