Emerging and developing multiplicative structure in students' visuospatial representations: Four key configuration types

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Visuospatial representations of quantities and their relations are widely used to support the understanding of basic arithmetic, including multiplicative relationships. These include drawn imagery and concrete manipulatives. This paper defines four particular configurations of nonstandard representation according to the spatial organization of their visual elements. These are: unit containers, unit arrays, array-container blends, and number containers, all of which have been observed to support developing multiplicative thinking, allowing low-attaining students to work with the equal-groups structures of natural number multiplication- and division-based tasks. Student-created examples are discussed, and pedagogical and diagnostic implications considered.

Keywords: Visuospatial representation, multiplicative thinking, arithmetic, low attainment.

In their early encounters with quantitative relationships, children become aware of concepts such as conservation of number, counting, etc., through interactions with collections of objects. For example, addition as the joining of collections and subtraction as removing a subset of objects from a collection – in which the ordering of individual objects is unimportant – can be considered conceptual 'grounding metaphors' (Lakoff & Núñez, 2000). Various models of children's arithmetical problem-solving development indicate a broadly similar progression from early concrete/enactive-based reasoning, to imagic/iconic, to abstract/symbolic reasoning (e.g. Bruner, 1974; Piaget, 1952). Within this broad outline, the actual external representations of learners' thinking during problem-solving include many possible sub-varieties (e.g., sets of actual objects, pictures of objects, tally marks in different configurations, dot arrays, etc.), and many possible categorizations of these for analytical purposes. The construction of appropriate analytical frameworks is necessary for the discerning of inter-individual differences and intra-individual trajectories (Meira, 1995; Voutsina, 2012). This is particularly the case when studying atypically-developing learners (Fletcher et al., 1998).

This aim of this paper is to share one aspect from the qualitative analytical framework for student- and co-created visuospatial data used in Finesilver (2014), delineating four particular types of visuospatial representation and demonstrating their use with selected examples. The project took an essentially grounded analytical approach, and so whilst this paper does not report results as such, a sample of research data is included with brief description of the process.

Theoretical background

To understand multiplication and division represents a significant qualitative change in learners' thinking compared to understanding addition and subtraction (Nunes & Bryant, 1996). These authors, amongst others, have recommended a *replications* model of multiplication, which is highly relevant both to counting-based strategies and to unitary drawn or modelled representations of multiplicative relationships. A central concept for considering this particular aspect of representation is *spatial structuring*:

> We define spatial structuring as the mental act of constructing an organization or form for an object or set of objects. The process [. . .] includes establishing units, establishing relationships between units [. . .] and recognizing that a subset of the objects, if repeated properly, can generate the whole set (the repeating subset forming a composite unit). (Battista & Clements, 1996, p.282)

There are two main forms of spatial structuring with which unitary visuospatial representations of multiplicative relationships emphasise their replicatory structure: by creating some kind of boundary to separate groups of units from each other, or by organising them in a pattern based on regular spacings. These two organisational strategies roughly correspond to Lakoff and Núñez's (2000) grounding metaphors *Arithmetic as Object Collection*/*Construction*, and to two of the common unitary configuration types I introduce below, *Unit containers* and *Unit arrays* (see Figures 1Figures 2).

Creating *container* configurations – i.e. visible boundaries within which the individual units of each group may be in any configuration – is particularly intuitive. Research that includes container representations (or equivalent) has been mainly focused on young children and their intuitive concrete models, such as sharing items (e.g. Carruthers & Worthington, 2006; Kouba, 1989). Rectangular *array* configurations, in which the groups are structured and defined by a configuration of all units in regular rows and columns – are also widely used in educational contexts. Research including array representations generally focuses on older children, grid arrays, and involves content such as rectangular area measurement; however, dot arrays have been shown as a powerful tool for supporting work in multiplication (Barmby et al., 2009; Harries & Barmby, 2007; Izsák, 2005; Matney & Daugherty, 2013), and, less frequently, division (Jacob & Mulligan, 2014). No prior studies were found that included both container and array representations, focused on the secondary age group and allowed freedom of representational strategy across multiple interviews and tasks.

Data

The data discussed below, including all examples, derive from a larger research project using microgenetic methodology to elicit and study emerging and developing multiplicative structure in low-attaining students' visuospatial representations within a flexible context (Finesilver, 2014).

There were thirteen participants, aged 11-15, attending mainstream schools in London, and identified by their teachers, educational histories, and initial sifting assessments as particularly numerically weak compared to their peers. Although having complex individual etiologies and patterns of arithmetical issues, they had in common difficulties experienced at the particular stage of moving from additive to multiplicative thinking (as highlighted by Nunes and Bryant, above).

The representations were produced during individual or paired problem-solving interviews carried out by the author (four per participant). Participants worked on tasks based within two multiplicative scenarios chosen for their ease and likelihood of visuospatial representation. These were 'Biscuits' (numbers of biscuits shared between numbers of children) and 'Passengers' (numbers of different-sized vehicles required to transport numbers of passengers). There were also some calculations presented symbolically with no scenario. The representational media available were multilink cubes, coloured pens and paper. Some representations were co-created by student and researcher at 'cognitive snapshot' points (Schoenfeld, Smith, & Arcavi, 1993), i.e. when a participant was unable to proceed further independently, and support was given in the form of a minimal 'nudge' prompt;

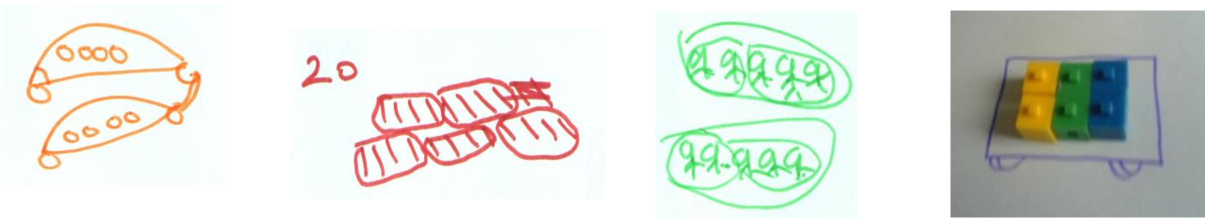
(e.g. ringing or counting a group aloud). Due to project methodology, support cannot be easily quantified (especially gestural interaction) and is not attempted in this paper. Documentation was via audio recording, photographs, scans of students’ papers, and field notes.

Four key types of representational configuration

Over 200 visuospatial representations were collected (exact figures cannot be given as participants re-appropriated whole and parts of prior representations for subsequent tasks and expansions). The great majority were found to group into four types; inclusion criteria, as defined below, were allowed to emerge, then refined, as part of a grounded analytical process. The most common types, *(unit) containers* and *arrays*, will be familiar. A smaller substantial proportion combined both container and array elements, and a further type emerged which I call *Number Containers*. (There is only space to include a few examples here; more will be included in this paper’s accompanying presentation, or see Finesilver (2014) for a complete set.)

Unit Containers (UC)

Criteria: Groups of two or more units enclosed by visible boundaries. Includes representations where units are aligned in rows and/or columns, but these do not represent divisor/quotient or multiplier/multiplicand.

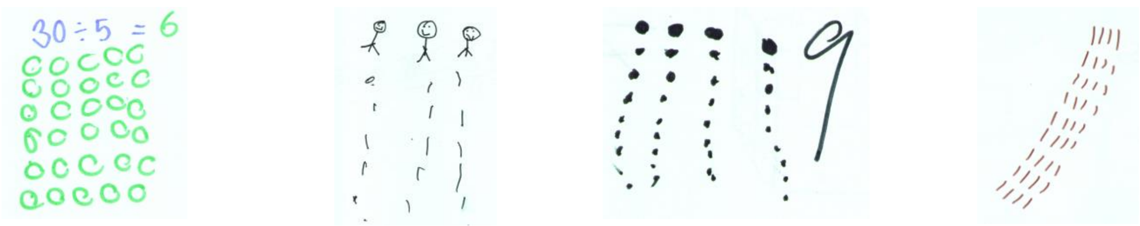


Figures 1(a-d): Examples of Unit Containers

Overall, this was the most common type (106 instances); eleven of the cohort chose to draw unit containers at some point while working on a task, although some much more frequently, and even the least able could sometimes use them independently. For the students with the severest arithmetical difficulties (e.g. dyscalculia), who could not make any start independently, visuospatial prompts were provided, e.g. drawing a set of circles (“plates”) for ‘Biscuits’. UCs were for the most part drawn, often with various scenario-based decorative elements, but some made use of mixed-mode, mixed-media representations with cubes or other physical units placed in drawn containers (see Figure 1d).

Unit Array (UA)

Criteria: Groups of two or more units aligned in rows and columns, where number of units in the rows/columns represents divisor/quotient or multiplier/multiplicand.



Figures 2(a-d): Examples of Unit Arrays

Plain unit arrays (of dots, tally marks, etc.) were used frequently (47 instances), the majority being produced independently by nine of the cohort, and an almost exclusive choice for three participants. All were drawn, and none constructed with cubes. (This may be surprising, as it is easy and visually effective to produce cube arrays. However, in general it was the arithmetically weakest students who made greatest use of concrete media, and that group also tended to prefer container representations.)

With a shift of perspective between vertical and horizontal structure, a learner may see that both rows and columns are formed of a set of equal groups, which underlies the commutative principle. This was independently noticed by some participants; e.g. on being asked to work out 28 biscuits shared between four people followed by 28 shared between seven, some re-used the same array, while others produced both 4×7 and 7×4, only realizing the equivalence after completion.

Array-Container Blend (ACB)

Criteria: Unit array representation with additional containing rings, where number of units in each row/column/container represents divisor/quotient or multiplier/multiplicand.

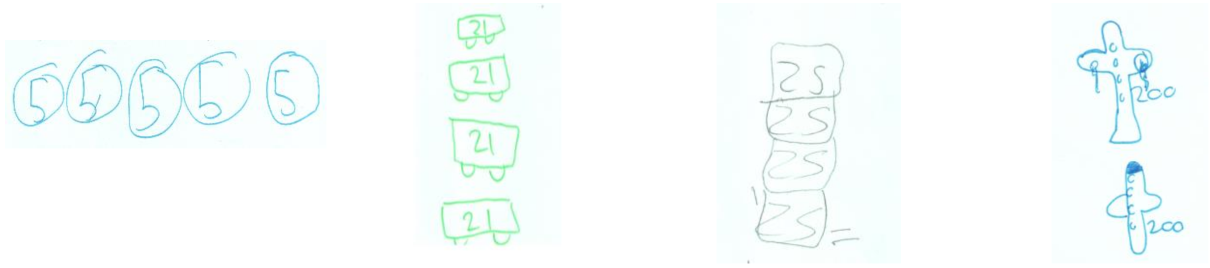
Figures 3(a-d): Examples of Array-Container Blends

While 47 instances of successful ACB use were collected, many of these were co-created and/or drawn during one particular task (see below); however, 27 were otherwise produced independently by participants. These were used mainly in ‘Passengers’ and the bare tasks, usually (although not always) with each row or column being counted out then ringed before proceeding to the next. Taking the additional time and effort to superimpose rings onto an array was thus clearly considered advantageous for certain participants on certain tasks. One student in particular began with a strong preference for plain dot arrays, but once she had seen an ACB, switched almost exclusively to that representation type for subsequent tasks.

In one particular (and uncharacteristic) task on multiplicative relationships, students were directly encouraged to produce an ACB which had both rows and columns ringed. A certain behaviour was observed with this representation type alone: some students independently looked back at it during later tasks and interviews for reference, in some cases ‘bookmarking’ it. As the numbers involved were different to those in their current task, and they only took a brief look, I suggest the images were functioning as an instant visual reminder of the commutative property of multiplicative structures.

Number Containers (NC)

Criteria: Container representation with numerals (rather than unit marks) representing the number in each group written inside, or close by, each container.



Figures 4(a-d): Examples of Number Containers

Unlike the previous three configuration types, NCs were not found in the literature or theorized prior to fieldwork, and some students introduced them spontaneously. Having observed their successful use, I included them in some later interactive support occasions, but of the 30 instances collected (from 9 participants), 22 were entirely independent. This change from unitary (iconic) to non-unitary (partially symbolic) representation is very significant cognitive step. Note, however, that some participants still chose to incorporate decorative elements from the task scenario (i.e. the vehicles were still depicted, although individual passengers were not).

Discussion

Students' use of the four types of representational configuration

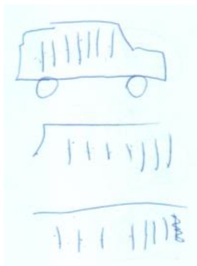


Figure 5: Transitional representation

Unit container representations allowed those students with the greatest arithmetical difficulties to create manipulable simulacra of imaginable scenarios, with as much visual resemblance as they preferred, to carry out organized sharing and grouping distributions and record their thinking. Unit array representations (with or without rings) allowed those students with a grasp of equal-groups structures, but who were not yet confident working symbolically, to perceive and make use of replicatory patterns spatially structured along two dimensions. However, the split between participants choosing to include container and/or array structuring elements also indicated personal preferences as a separate factor to arithmetical ability. (This has potential for further investigation, involving testing participants' visual pattern recognition).

While some individuals displayed firm preferences for container- or array-based forms throughout, others' representational strategy choices changed over the course of interviews, and sometimes intra-task. For example, Figure 5 shows a student's representation for calculating the number of 7-seater vehicles needed for 21 passengers, starting with a container resembling a car, then immediately discarding decorative elements and containers, in transition towards an array format.

Increasing the quantities within tasks (for those students judged likely to cope with the challenge) sometimes resulted in strategic change, in particular the introduction of number symbols. However, the general persistence of container elements surrounding those symbols (i.e. Number Containers) is striking. As seen in Figures 4b and 4d, non-mathematically-functional decorative elements (bus wheels, aeroplane wings) were included inconsistently. From a purely calculation-based viewpoint, students using NCs might as well be using plain columns of numbers – therefore the container elements clearly fulfil some other, non-enumerative, yet important, function. I suggest containers forms are a powerful visuospatial/perceptual phenomenon relating to equal-groups number structures and relationships, which persists later than might be expected. It is reasonable to expect that as confidence is gained, the containers begin to disappear (but could be retrieved as a reassuring strategy at times of low confidence – for example, when tasks increase in difficulty).

Obviously, all types of representational configuration were used to a great extent for the enumeration of quantities, and for the visuospatial organization of these quantities so that the correct set of objects (units or groups) could be enumerated. However, it is worth noting that the representations created were not immediately rendered useless once a task solution was found. Students completed visuospatial patterns when an incomplete pattern would have been sufficient to obtain an answer; they sometimes added further organizational (or decorative) detail after giving an answer. Occasionally they even created a whole new representation to record their working retrospectively, or to help them explain an exciting discovery they had just made about numerical relationships (e.g. the commutative principle). The fact that these representational activities were important to the students for their own sake (i.e. not just for obtaining the answer in a single task) suggests that they can be an important part of these students' developing arithmetical reasoning, and their real and perceived agency in this development.

Representational configurations and developing multiplicative thinking

> Representations of mathematical objects […] can be seen as concretizations of abstract mathematical concepts and at the same time as representations of real objects. (Wittmann, 2005, p.18)

The four related types of representational configuration defined and discussed above integrate numerical and spatial concepts to form visuospatial mathematical objects that allow such a dual role: concretizing numerical relationships and representing real-life objects referred to in scenario tasks.

Whilst all four types represent equal-groups arithmetical structures, they do not fall along a single line of progression (see Figure 6, below). In the same way that concrete representations (e.g. modelled with cubes) are not necessarily less mature than iconic ones (e.g. drawn images), different types of configuration have different affordances which may be relevant at certain points. Number Containers, being non-unitary, are a clear progression from Unit Containers in terms of calculation, by requiring step-counting or repeated addition rather than unitary counting. However, Unit Arrays better instantiate the two-dimensional, reversible, nature of multiplicative relationships, whilst the ringing of rows or columns in ACBs could link procedural and static conceptions of multiplication/division.

The analysis of a set of relatively open-ended, student-generated, qualitative data based on their use of four key types of representational configuration highlighted a particular aspect of these students' late- and slow-developing multiplicative thinking: the many small adjustments that together can indicate a gradual change of focus of attention from units to groups, all happening within what is often considered to be a single stage of 'counting-based strategies'. Whether a task is multiplication- or division-based, there is a total quantity which is made up of, or can be separated into, equal groups. In terms of enumeration, the most basic strategies involve counting without any awareness of the repeating structure, while the more advanced ones make use of it. In terms of representational strategy, the most basic involve manipulating concrete or drawn units individually, to seeing and using visuospatial repeating patterns of units, to manipulating component groups as though they were units, to – eventually – focusing on these groups as new, composite units.

An individual's progress in this move from units to groups as main focus may be diagnosable via their representational strategic choices, along various possible trajectories (see Figure 6). (The bracketed items are likely or potential subsequent steps which, however, did not feature in the project from which this data derives.)

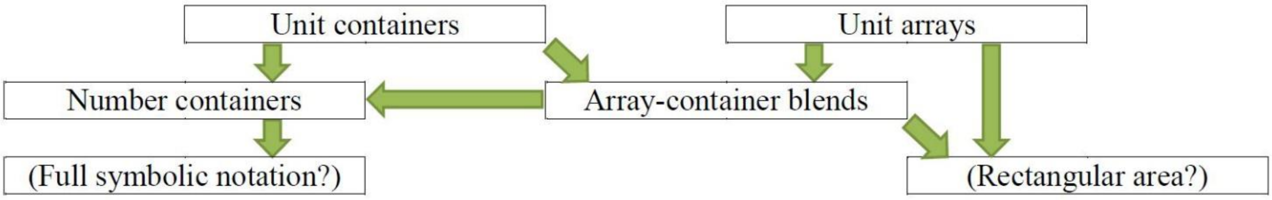


Figure 6: Potential developmental trajectories through representation types

Regarding this change of focus, there is a particular point of interest in ACBs: although they are still unitary representations (i.e. every unit is visibly present and countable), the visual and enactive emphasis on ringed subgroups serves to shift the student's level of visual focus, drawing attention away from the units and towards the groups. Thus, it encourages the possibility of seeing containers (enclosing well-aligned sets) as the new 'units' for manipulation. Meanwhile, with NCs, the replacing of (iconic) units with (symbolic) numbers is not only important for its progression toward standard notation, but as another part of this change of focus from units to groups – the change from using one mark to stand for one thing, to using one mark to stand for a collection of many things.

Even from a small sample of students it is clear that their patterns of capability, difficulty, and the representations which work best for them, are complex, interrelating, and individual. There is no single ideal path through from, for example, dealing out a pile of physical items to a set of actual present people, and carrying out a fully symbolic division calculation. However, from a teaching/learning perspective it appears important that at no stage is the leap too wide or too hasty, and that there are visual links when moving from more intuitive to more abstract representational strategies. From an analytical perspective, I suggest that tracking students' use of these four key representational configuration types in their arithmetical problem-solving (both in their initial choice of type, and in the emerging and developing spatial organization of elements within representations) may be beneficial in further study of the progression from additive to multiplicative thinking.

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