Designing a research-based test for eliciting students' prior understanding on proportional reasoning

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Mathematics education in the Swedish prison education program is struggling with a high rate of students that fail to pass the basic mathematics courses. One of the main issues seems to be the challenge for the teachers to elicit students' widespread prior mathematical knowledge. The consequence of this is that the teachers cannot meet the students' educational needs with meaningful teaching activities. Focusing on the most pervasive mathematical idea in these courses, proportional reasoning, a test was designed that aimed to elicit students' mathematical reasoning. This paper illustrates that by making use of accumulated and selected research results and findings, we can gain valuable information on students' proportional reasoning competency. This information may be used as an access point for individualized instruction.

Keywords: Adults, individualized instruction, proportional reasoning, prison education.

Introduction

In the Swedish prison education program, only two out of ten students finish and pass their mathematics courses[1]. This is disturbing in itself, but particularly so given the resources available. The teachers are university trained upper secondary school mathematics teachers, and the students sign up voluntarily and typically are highly motivated. Moreover, all courses are individually designed for each student, which should ensure good teaching and learning conditions. However, a challenge is that the student group shows significant variation in age, ethnicity, socioeconomic background, school background and life experience in general. For the basic mathematics, the mathematics in compulsory school and the first course in upper secondary school, there exists no such thing as one course design that suits all students' different backgrounds. In 2015 the Swedish prison education program in mathematics had 728 students enrolled, spread across 47 prisons, and 80% of these were found in the basic mathematics courses

A plausible reason for the low pass rate in the basic courses is that the teachers fail to make proper use of the individualization possibilities. A prerequisite for actual individualization is that teachers have the opportunity to find out students' prior mathematical understanding and adapt the teaching accordingly. Realizing that this opportunity hinges on the teachers' competencies, e.g., they need to put their didactic and pedagogical *teaching competency* to play (Niss & Højgaard, 2011). But teachers' possibilities to individualize instruction might also depend on various forms of support. Inspired by Jankvist and Niss (2015), I report on a research-based effort to develop such support: a test for identifying beginner students' prior mathematical understanding. The test needs to provide information on students prior understanding in two ways: vertically, in relation to progression throughout school years, and horizontally, throughout taught topics in compulsory school. Hence, a major design decision was to focus the content on proportional reasoning. As will be argued below, proportional reasoning permeates the basic mathematics courses in a systematic way, which means that probing students' competencies in this area gives a good access point for individualized teaching. The foundation for the test is the accumulated and selected research results and findings related to proportional reasoning, since proportionality may be the most important, pervasive and powerful idea in elementary school mathematics (Behr, Harel, Post, & Lesh, 1992; Lamon, 2007).

Constructing such a test involves several design decisions involving the content and form of the test, as well as constructing, collecting and adapting test items that realize these design decisions. The

[1] Data from administrator Gunilla Jonsson, personal communication, July 12, 2016.

main question elaborated on in this paper is: *How can research findings inform the development of a test that elicits students' prior understanding on proportional reasoning so as to provide teachers with an access point for designing individualized teaching?*

Theoretical underpinnings for the development of the test

Mathematical reasoning is one of eight competencies for identifying and analyzing students' mathematical understanding, described in the Danish KOM-project (Niss & Højgaard, 2011). "The *mathematical reasoning competency* consists, first, of the ability to follow and assess mathematical reasoning, i.e., a chain of arguments put forward – orally or in writing – in support of a claim." (Jankvist & Niss, 2015,. p. 264). The kind of mathematical reasoning called proportional reasoning is a prerequisite for successful further studies in mathematics and science, since multiplicative relations underpin almost all number-related concepts studied in elementary school (Behr et al., 1992; Lamon, 2007). A proportion is defined as a statement of equity of two ratios a/b = c/d. Proportion can also be defined as a function with the isomorphic properties f(x+y) = f(x) + f(y) and f(ax) = af(x) (Vergnaud, 2009). A function, A(x,y), can also be linear with respect to several variables, (n-linear) functions. For example the area functions for a rectangle with sides x and y is bilinear (2-linear) since A(x,y) = xy and it is easy to check that this function is linear with respect to each of its variables when the other is considered constant.

Proportionality is a key concept in mathematics and science education from elementary school to university (Lamon, 2007). Despite the pervasive nature of proportional reasoning throughout the school years it is well known that children around the world have considerable difficulty in developing the mathematical competency to reason about fractions, percentages, ratio, proportion, scaling, rates, similarity, trigonometry, and rates of change (Behr, Harel, Post, & Lesh, 1992; Lamon, 2007). Typically, proportional reasoning problems come in the shape of a missing value problems or comparison problems (Lamon, 2007). In the former, a multiplicative relation is present where three elements are provided and the fourth is to be found. The latter asks the student to compare which ratio is the bigger or smaller.

From accumulated research, some *key points* for the developing of proportional reasoning and the building of multiplicative structures can be identified (c.f. Behr, Harel, Post, & Lesh, 1992; Fernández et al., 2012; Lamon, 2007; Shield & Dole, 2013; Van Dooren, De Bock, Vleugels, & Verschaffel, 2010; Vergnaud, 1983). Students need to:

1. Be able to distinguish additive from multiplicative reasoning and recognize when a multiplicative relation is present;
2. Be able to draw connections to the algebraic rules for fractions when working with part/part ratios, part/whole fractions and proportions, a:b = c:d;
3. Recognize and use a range of concrete representations for proportions, e.g., tables, graphs, formulas and drawing pictures;
4. Acknowledge the properties of geometrical objects in two- and three dimensions for calculation of scaling and similarity.

Key point 1. Research studies and findings show that the ability to distinguish additive from multiplicative comparisons constitute a major stumbling block for students (Van Dooren et al., 2005). Students need to be able to recognize that a proportional situation exists when the comparison is multiplicative (Shield & Dole, 2013). In Sweden, students get acquainted with additive strategies for reasoning about quantities in grades 4 to 6. For example, an increase in price by 10% can be calculated in two steps. First, calculate how much 10% is and then add this to the original price. A transition from an additive to multiplicative thinking approach is introduced in grades 7 to 9. The new price can now be approached in one multiplicative step: the original price multiplied by the factor 1.1, to find

the new price. Far from all students embrace this new idea of approaching percentage change. The additive approach works well for calculating a single increase or decrease, while they may lack motivation to change strategy.
Fernández et al. (2012) found that the error of using additive strategies on proportional situations increased during primary school and decreased during secondary school. A desirable development in students' reasoning would be that they, after being introduced to multiplicative reasoning, still hold on to their ability to use additive strategies when appropriate. However, research findings show that once students have been introduced to multiplicative strategies they tend to overuse this approach on everything that resembles a proportional situation (Van Dooren et al., 2005). Further, non-integer ratios cause more errors than integer ratios (Fernandez et al., 2012; Gläser & Riegler, 2015), while the non-integer situations can be considered to require a more developed understanding of rational numbers.

Key point 2. Many situations require that students can relate to part/part ratios and part/whole fractions (Vergnaud, 1983). For example, if a company employs 11 women and 31 men, the part/whole fractions 11/42 and 31/42 represent the relation of women and men related to the whole. If asked to determine the company's gender distribution, it is instead the part/part ratio 11:31 between women and men that is relevant. When a ratio connects two parts of the same whole, students may not adequately recognize the difference between part/part and part/whole relationships (Clark, Berenson, & Cavey, 2003). It is not easy for students to approach situations that require shifting from part/part to part/whole situations. Moreover, students need to connect mathematical ideas. Since ratios can be written in fraction form, they obey the same mathematical laws as fractions (Shield & Dole, 2002).

Key point 3. Another stumbling block for students is that they tend to apply linear proportional reasoning on scaling, without considering the nature of the item. Van Dooren et al. (2010) found that students tend to use linear proportional reasoning even when it is inappropriate e.g., in word problems where a real word context is required to solve the problem. For example: *Farmer Gus needs 8 hours to fertilize a square pasture with sides of 200 meters. Approximately how much time will he need to fertilize a square pasture with sides of 600 meters?* Recognizing this as a missing value problem i.e., three values given and one unknown, this problem will trigger a cross-multiplication type solution which gives the wrong answer of 24 hours. Since scale is one of the major themes that span mathematics, chemistry, physics, earth/space science and biology it is crucial for students to gain understanding of the concept of scale. Scale in one, two, and three dimensions is a central unifying concept that crosses the science domains, crucial for understanding science phenomena (Taylor & Jones, 2009).

Key point 4. Proportionalities can be represented in different ways, e.g., with words, pictures, algebraically, with graphs or tables. Shield and Dole (2013) enhance the use of a range of representations to promote students' learning. If students are given the opportunity to work with graphs, tables and other diagrams that illustrate the proportional situation present in the mathematical task, their conceptual understanding is promoted (Vergnaud, 2009). Further, their ability to see connections between problems that are based on the same mathematical idea is enhanced, e.g. to see that missing value problems on similarity, proportional functions and speed problems can be illustrated with different representations but approached with the same mathematical idea.

Several concepts are in play when students reason with proportional quantities. The intertwined concepts required for the development of proportional reasoning makes up a *conceptual field* (Vergnaud, 2009). A conceptual field is a set of situations and concepts tied together. As the theory of conceptual fields show, together with other well-known theoretical frameworks for conceptual understanding, the meaning of a single concept does not come from one situation only (Sfard, 1991; Tall & Vinner, 1981; Vergnaud, 2009) but from a variety of situations demanding mathematical reasoning related to the concept in question. The conceptual field of intertwined concepts in play in proportional reasoning cover at the least “linear and n-linear functions, vector spaces, dimensional analysis, fraction, ratio, rate, rational number, and multiplication and division” (Vergnaud, 1983, p. 141). It is the complexity of the concepts in play together with the pervasive nature of proportional reasoning from elementary school to university that makes proportional reasoning suitable for the design of the test.

Design of the test on proportional reasoning

An important design choice for the test was to use a multiple-choice design. Even though open response tests are a powerful method to elicit students’ understanding, the advantages of multiple-choice tests were in this case considered to be the best option. An open response test can be a negative experience for students with low prior understanding, since they may be unable to supply any answers. Since the students often have bad experiences from school mathematics, we want to avoid negative experiences in the beginning of a mathematics course. A multiple-choice test, on the other hand, is easy to take for the students. Even when they do not have the mathematical competencies to reason and solve an item, they can still provide an answer by intuition or chance. The test is designed to be followed up with student interviews. This is an important step since many students do not have Swedish as their mother tongue, which of course may cloud their interpretation of the items. Many of the students also have concentration difficulties, so a written test may not give a satisfactory picture of students’ prior understanding.

A downside of multiple-choice is the possibility to choose the right answer by chance. For this reason, a two-tier design was chosen (see examples below) yielding only 0.125 probability to pick both the right *true* or *false* value and the right claim. A pilot version of the test, consisting of 22 items, was tried out in April 2016. Feedback from the participants informed me that the test was too long and that some of the items were difficult to interpret. After revision and further testing, the resulting test consists of 16 proportional reasoning items. The final version of the test takes about 20 to 40 minutes to complete, without any time pressure.

The items in the test were chosen from published research papers, with the intention to draw on knowledge from the research field on proportional reasoning. The rationale for my choices is as follows: a) the items have already been proved to work well for giving information on students’ understanding, and b) extensive background information of the nature of the mathematical reasoning in play are provided as well as analyzes of students results. Referring to the *key points* presented in the theory section, the potential reasoning related to each item involves several concepts and abilities, yet the items can still be categorized as referring mainly to one or two of the four presented key points:

Key point 1. Students’ ability to distinguish additive from multiplicative reasoning and recognize when a multiplicative relation is present, and is always required for carrying out proportional reasoning, however mainly tested by items 1, 5, 6, 7, 11 and 16.

Key point 2. Students’ ability to draw connections to the algebraic rules for fractions when working on part/part ratios, part/whole fractions and proportions, a:b = c:d, is mainly tested by items 2, 4, 12, and 13.

Key point 3. Students’ ability to recognize and use a range of concrete representations for proportions, e.g., tables, graphs, formulas and drawing pictures is mainly tested by items 3, 8, 9 and 15.

Key point 4. Students’ ability to acknowledge the properties of geometrical objects in two- and three dimensions for calculation of scaling and similarity is mainly tested by items 8, 10, 13 and 14.

Several errors on items referring to the same *key point* indicate a lack of understanding that should be investigated further in the following student interview. The test items are also adapted to mirror the progression throughout the basic course. Items 1 and 4 refer to content taught in part two of the basic course. Items 2, 3, 6 and 10 deal with content from part three and part four is reflected in items 7, 8, 9 and 11-16.

Key points / Basic course	Distinguish additive from multiplicative reasoning	Draw connections to the algebraic rules for fractions	Recognize and use a range of concrete representations	Acknowledge the properties of geometrical objects
Part 2	Item 1	Item 4		
Part 3	Item 5 Item 6	Item 2	Item 3	Item 10
Part4	Item 7 Item 11 Item 15 Item 16	Item 12 Item 13 Item 16	Item 8 Item 9 Item 15	Item 8 Item 13 Item 14

Table 1. Schema over items in relation to key points and progression in the basic courses

The sources for the test items are: Hilton, Hilton, Dole, and Goos (2013); Fernadéz et al. (2012); Niss and Jankvist (2013a; 2013b); and Gläser and Riegler (2015). The items from Hilton et al. were already designed as two tier multiple test items. The other items were adapted from their original design to a multiple-choice design, using erroneous answer alternatives either reported in the original studies or answer alternatives recalled from my experience from teaching.

Examples of test items

In what follows, I will exemplify how research results on common difficulties on proportional reasoning are guiding the choice of the test items. To illustrate, items included to elicit students' difficulties to discriminate *additive* from *multiplicative* situations and difficulties with *scaling* are displayed below.

Consider this item, adapted from Fernadez et al. (2012):

> Loading boxes: Petra and Tina are loading boxes in a truck. They started together but Tina loads faster. When Petra has loaded 40 boxes, Tina has loaded 160 boxes. ***When Petra has loaded 80 boxes, Tina has loaded 200 boxes.***
>
> **True** or **False** because (choose the best reason)
>
> a) Tina will always be 120 boxes ahead of Petra.
> b) Petra loads faster than Tina.
> c) Tina loads 4 times faster than Petra.
> d) Tina loads with double speed.

This is a proportional situation where Tina is loading 4 times faster than Petra, so the claim "***When Petra has loaded 80 boxes, Tina has loaded 200 boxes.***" is false. Students should consider whether it is appropriate to use additive reasoning, that is, if Tina has still loaded 120 boxes more than Petra. If the students answer *a) Tina is always 120 boxes ahead of Petra*; further investigation of their reasoning strategies is required, though the answer indicates that there may be a lack of transition from additive to multiplicative thinking. This suspicion is further strengthened if the student is successful in items requiring additive reasoning, like in the item below, from Hilton, et al. (2013):

Running laps: Sara and Johan runs equally fast around a track. Johan starts first. When Johan has run 4 laps, Sara has run 2 laps. ***When Sara has completed 6 laps, Johan has run 12 laps.***

True or **False** because (choose the best reason)

a) The further they run; the further Johan will get ahead Sara.
b) Johan is always 2 laps ahead of Sara.
c) Johan completes double the laps of Sara.
d) Sara has run 3 lots of 2 laps to make a total of 6 laps, so Johan must have run 3 lots of 4 laps to make a total of 12 laps.

This is an additive situation where Sara and Johan run at the same speed. Students should consider whether it is appropriate to use multiplicative reasoning, that is, if Johan runs 3 times faster than Sara. If the students answer d) *Sara has run 3 lots of 2 laps to make a total of 6 laps, so Johan must have run 3 lots of 4 laps to make a total of 12 laps*, further investigation of their reasoning strategies is required though the answer indicates that a difficulty to discriminate multiplicative from additive situations exists.

The two examples above illustrate how research findings on proportional reasoning have been used in the design of the test. By including items requiring multiplicative reasoning as well as items requiring additive reasoning you may elicit the students' ability to discern when a multiplicative situation is present.

The Dice- and the Circle item below are adapted from Niss and Jankvist (2013b), The Dice item is originally phrased: *A cube of wood with all edges 2 cm weighs 4.8 grams. What weighs a cube of wood, where all edges are 4 cm? Justify your answer.* [En terning af træ med alle kanter lik 2 cm vejer 4.8 gram. Hvad vejer en terning af træ, hvor alle kanterne er 4 cm? Begrund dit svar.] I added the claim: **“*A wooden dice where all edges are 4 cm weight 19.2 g.”***, and the response alternatives.

Dice: A wooden dice where all edges are 2 cm weighs 4.8 g. ***A wooden dice where all edges are 4 cm weight 19.2 g.***

True or **False** because (choose the best reason)

a) The weight increases 4 times if the edge doubles.
b) The weight increases 6 times if the edge doubles.
c) The weight increases 8 times if the edge doubles.
d) The weight doubles if the edge doubles.

Circle: ***Simon says that if you draw a new circle with half the diameter of another circle, the new circle will have half the perimeter and half the area of the other circle.***

True or **False** because (choose the best reason)

e) If the diameter is halved, the perimeter and area is halved.
f) The area will be ¼ and the perimeter ½ of the original.
g) You cannot know without knowing the length of the diameter in the new circle.
h) You cannot know without knowing the length of the diameter in the original circle.

Students may fail to interpret the effects on volume from a doubling of the edges, while further investigation on the students’ conceptualization of geometrical objects needs to be undertaken. To reason about the circle item, the students need to consider the conjunction that both the perimeter and the area are halved. Since (area scale) = (length scale)^2; a halving of diameter will result in a ¼ size of area while the perimeter halves. An error on these items may indicate difficulties to acknowledge the properties of geometrical objects in two- and three dimensions for calculation of scaling and similarity.

Reflection

There are many reasons why educational research tends to be isolated from practice. Research results and findings need to undergo a number of transformations from theory to practice, before they can be adapted to teaching practice, as illustrated in the design of the discussed in this paper. The test was designed with considerations to a special prison context and early results from using the test shows that it provides valuable support for the teacher when eliciting students' prior understanding of mathematics. Although, the test focuses on the *mathematical reasoning competency* it also informs us of students' *mathematical thinking competency*, *problem-handling competency* and *modeling competency* since these competencies are intertwined and overlapping. Together these four competencies create one out of two overall competences associated with mathematics: *The ability to ask and answer questions in and with mathematics* (Niss & Højgaard, 2011). The other overall competence: *The ability to deal with mathematical language and tools*, covers the intertwined competencies *representing competency, symbol and formalism competency, communication competency* and *aids and tools competency*. The scope of the test does not cover *the ability to deal with mathematical language and tools*. These competencies are left to be tackled within the course design, as well as the further development of the students' *ability to ask and answer questions in and with mathematics*.

A fundamental idea of educational research is that research findings should be put in play in teaching practice to help students to succeed with their studies in mathematics. I have discussed the design of a test for supporting teachers when pursuing the goal of finding an access point for individualized instruction. Through making use of accumulated and selected research results in the area of proportional reasoning in the design of the test, we gain a more thoughtful idea of the students' prior understanding.

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