Mathematics teachers' assessment of accounts of problem solving

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In mathematics education it has been argued that traditional assessment provides insufficient evidence of students' overall achievements. Assessment of problem solving has been put forward as a more comprehensive form of assessment. This however entails a subjectivity which raises concerns regarding the reliability. This study aims to investigate mathematics teachers' assessment of mathematical problem solving. Nineteen teachers have been interviewed in five groups and asked to discuss a sample of 16 accounts of problem solving by 10-year-old students. The analysis focused on examining how five mathematical abilities, described in the Swedish mathematics syllabus, were addressed and discussed by the teachers. Preliminary findings indicate that the accounts provide teachers with very little evidence of students' mathematical abilities. One of the reasons for this appears to be that the accounts do not offer clear descriptions of the problem-solving process.

Keywords: Assessment, mathematical problem solving, abilities, teachers.

Introduction

Assessment forms a large part of teachers' practice yet studies indicate that teachers feel inadequately prepared for the task of judging students' performances, skills and understandings. (Cumming & Wyatt-Smith, 2009; Mertler, 2004). Research that has investigated teachers' assessment practices has also criticized such practices for failing to meet standards of reliability, objectivity and validity (Allal, 2012). Assessment is inherently a process of professional judgment in which the element of interpretation is salient. In mathematics education Morgan (1998) has shown that teachers can interpret the meaning of the same passages of texts, produced by students in mathematics, very differently. When teachers interpret observed test results or other types of information to come to a conclusion about a student's level of knowledge or skill such a conclusion may be referred to as inference, and although some inferences can be made with more confidence than others, no conclusion about a particular student's knowledge or skill can ever be made with certainty (Cizek, 2009). Assessment in school mathematics has always relied heavily on students' written work (Morgan, 2001b). Written responses to mathematical tasks, such as problem solving, require that students explain both their thinking and the proposed solution. For such written material to act as valid evidence, from which judgements regarding students' problem-solving processes and mathematical abilities may be inferred, it has to be clear and comprehensive. There is reason to believe that not all students possess the ability to produce clear and comprehensive accounts of their mathematical problem solving (Monaghan, Pool, Roper, & Threlfall, 2009).

This study represents a microstudy on groups of teachers' assessment of a specific set of accounts of solving mathematical problems. The accounts were collected from two classes of 10-year old students. The aim of the study is to investigate the aspects of mathematical problem solving which are addressed and discussed by the teachers and to relate these to the five mathematical abilities set by the Swedish mathematics syllabus. These abilities are related to: *problem solving*, *mathematical concepts, mathematical methods, mathematical reasoning* and *communication*.

Students' writing in mathematics

It can be argued that the written mathematical work of students in school mathematics typically serves two very different functions. It can be seen as a part of a learning process in which writing is used to record and perhaps reflect on various mathematical ideas; hence, the text is written by and for the student herself. It can also however, be seen as a product for the purpose of assessment; hence, written for a teacher or examiner. Unlike the work of professional mathematicians, which is often thought to be the model for school mathematics, the work in school mathematics often serves these two functions at the same time (Morgan, 2001a). When problem solving is viewed as an individual cognitive activity, students use their writing to understand, explore, record, and monitor their own problem solving (Stylianou, 2011). Several studies indicate that writing poses problems for students. Evidence suggests, for example, that it is far more common for children to experience problems with semantic structure, vocabulary and mathematical symbolism than they do with, for example, standard algorithms (Ellerton & Clarkson, 1996).

Assessing mathematical problem solving

Assessment of students' mathematical problem solving is complex. There are different definitions of what mathematical problem solving is and what constitutes a problem. A generally accepted definition suggests that problem solving can be seen as a response to a question for which one does not already know or have access to a method (Monaghan et al., 2009). This understanding is also used by OECD in the PISA 2012 Assessment and Analytical Framework (OECD, 2013). Problem solving can be seen as a goal, a process, and a skill and problem-solving activities are thought to engage students in a number of different processes such as reasoning, communication and connections (Rosli, Goldsby, & Capraro, 2013). In a situation where traditional assessment in mathematics is increasingly seen as providing insufficient evidence of mathematical knowledge and abilities beyond routine skills and algorithms there are high hopes for alternative forms of assessment of which problem solving is one (Jones & Inglis, 2015; Rosli et al., 2013). Despite its power to engage students however, problem solving has been problematic to use as a source from which to make inferences about students' mathematical achievement. Reliance on the traditional mathematics test has often been justified on the grounds of reliability and comparability, but this has often been at the expense of validity (Watt, 2005). The challenges to assessment of problem solving are several. The first is that it requires access to evidence of the process. Most test situations do not include the option of observation to provide such evidence but rather require students to produce an extended written account which includes an explanation of both their problem-solving process and their proposed solution(s). This is problematic because considerable skill is required to produce clear and comprehensive accounts of problem-solving processes, a skill that students may or may not have (Monaghan et al., 2009). The second challenge is the element of interpretation and, thus, subjectivity. As teachers read and assess students' texts, their professional judgment is formed by a set of resources which varies with their personal, social and cultural history as well as their relation to the particular discourse. These resources are individual, as well as collective, and they include: personal knowledge of mathematics and the curriculum, beliefs about the nature of mathematics and how these relate to assessment, expectations about how mathematical knowledge can be communicated, experience and expectations of students and classrooms in general, and experience, impressions, and expectations of individual students (Morgan & Watson, 2002). Individual teachers may also have particular preferences for particular modes of communication as indicators of understanding. A study from Australia has also indicated that teachers themselves object to the use of alternative assessment methods such as problem solving on the grounds that it is perceived as too subjective (Watt, 2005). In Sweden there have been calls for national tests to be assessed and graded externally instead of by the teachers who already know the students. External grading is seen as a way to secure objectivity and fairness.

Mathematical abilities

Assessment in mathematics has many concerns, of which perhaps the most important one is: what is it that is being assessed? This issue has been dealt with and given many names throughout the history of mathematics education including numeracy, mathematical proficiency, mathemacy, matheracy and quantitative literacy, to name a few (Wedege, 2010). Competency frameworks in mathematics are constructs that build on the assumption that mathematics is a domain in which it is possible to provide a generic set of mathematical practices (Säfström, 2013). Given that mathematical activities have to be about something, arriving at a common and generic set of such skills and abilities proves a challenging task, as has been pointed out by many (see for example Jablonka, 2003; Kanes, 2002; Kilpatrick, 2001; Wedege, 1999). Some frameworks have focused on this 'something' whereas others have focused on the mental processes that are associated with mathematical activities in general. Influential examples of the latter include the five strands of mathematical proficiency introduced by the Mathematical Learning Study of the NCTM in the US (Kilpatrick, Swafford, & Findell, 2001) and the KOM project in Denmark (Niss, 2003; Niss & Højgaard Jensen, 2002).

One of the motives behind the above referenced frameworks is the clear intention to break with a traditional teaching of mathematics associated with rote learning and procedures and instead promote a more dynamic view of what it means to do mathematics (Boesen et al., 2013). In Sweden the Swedish national curricula has been influenced by the ideas from these frameworks and in the Swedish syllabus in mathematics, introduced in 2011, five different abilities which the teaching in mathematics should provide the students the opportunity to develop, are described. These include the ability to:

- formulate and solve problems using mathematics and also assess selected strategies and methods,
- use and analyse mathematical concepts and their interrelationships,
- choose and use appropriate mathematical methods to perform calculations and solve routine tasks,
- apply and follow mathematical reasoning, and
- use mathematical forms of expression to discuss, reason and give an account of questions, calculations and conclusions. (SNAE, 2011, pp. 59-60)

In the syllabus the abilities, described above, are actualized in a set of knowledge requirements which define what constitutes an acceptable level of knowledge for the grades E, C, and A, where A represents the most advanced. In the results section the five knowledge requirements are shortened to: *problem solving, mathematical concepts, mathematical methods, mathematical reasoning and communication*.

Data collection

The study sets out to investigate teachers' assessment of a specific set of accounts of mathematical problem solving and aims to identify the aspects of mathematical problem solving which are addressed and discussed by the teachers. Nineteen elementary school teachers from four schools in a middle-sized town in mid-Sweden were interviewed in groups. There were five groups of 3, 4 or 5 teachers respectively. At the time of the interview all nineteen teachers were teaching mathematics. They were initially chosen by their principals and asked to participate based on their own interest. The interviews were all recorded on video and an additional audio recorder. The teachers were presented with 10-16 accounts of problem solving produced by students, aged 10. The problem-solving was centered on two specific problems. They were both Diophantine equations involving the identification of a number of ways to distribute: a) 30 legs on 12 animals or b) 36 wheels on 11 vehicles (see figure 1). This type of problem can be formulated in this way where there is only one possible combination or as an open problem to which there are many solutions. A small number of legs or wheels also results in a small number of combinations; the problem can therefore be adapted to fit different students or age groups. The students can also be asked to demonstrate that they have found all possible combinations and explain how they know this. The problem offers opportunities to adopt a more or less systematic trial-and-error strategy, but there are also other ways to solve the problem. Given that the problem involves concrete objects it also offers students opportunities to draw. All these properties contributed to the choice of the problem type.

The teachers in the interviews were given information on the problems but very little information on the situation in which the texts were created. Being faced with an account of mathematical problem whose origin you know very little about forces a teacher to focus on the account itself and the interpretations derive to a larger extent from the account than it would had the teacher been asked to comment on their own students' written material. The teachers were asked to discuss the different accounts from an assessment perspective and to provide arguments for their reasoning. The group interview was chosen so as to create room for discussions but also for eliciting the teachers' idea of possible 'common grounds' in evaluating students' accounts. The interviews, which amounted to a total of 4 hours 26 minutes, were transcribed.

Analysis

The analysis was performed in two steps. In the first step the transcribed interviews were analyzed with the intention of identifying instances in which the teachers discussed what the students seemed to be **doing**. This focus was inspired by the understanding that knowing mathematics is doing mathematics, as described above. This analysis included identifying verbs connected to instances of action such as *understand, know, think, draw, calculate, see* and *show*.

The second step in the analysis was focused on relating the identified instances to the different abilities described in the syllabus. The five abilities *problem solving, methods, concepts, reasoning,* and *communication,* did not have to be mentioned specifically. A discussion regarding a method such as trial-and-error was considered as relating to method even if the term *method* was not used. Discussions about failed attempts or deficiencies were also considered as belonging to the category of the ability in question. Examples of quotes from the teachers are shown below together with the abilities they were thought to relate to. One quote can be related to several of the listed abilities.

Teacher: Here they have really tried…drawn all the tires… (problem solving, method, communication)

Teacher: He has counted the number of fours he has taken away and those are plus signs… it is plus 7… (problem solving, method)

Teacher: There is no reasoning to show that this is correct… (reasoning, communication)

Teacher: They cannot reason without explaining a little bit more… she has not used any concepts for example… (concepts, reasoning, communication)

Teacher: It is not enough to just write an answer…you have to be able to show in writing how you arrived at this… (communication)

Teacher: Yes but she…she does know how to solve the problem… (problem solving, communication)

Teacher: And then you try different numbers… that is how they have done it… you can see that they have erased… (problem solving, method, communication)

Preliminary results

The preliminary results are presented under headlines which are consistent with the five abilities described in the syllabus. In some cases the teachers' discussions are covering two abilities at the same time and in these cases they are either presented under both headlines or presented as a compound ability which is treated under one headline.

Problem solving

Many of the teachers' discussions are focused on the students' choice of method or strategy for solving the problem and the teachers spend considerable time trying to identify the specific method of each student. Once this has been identified however, the discussions tend to turn to other issues. A problem solving strategy is seldom judged based on its appropriateness or sophistication. Other aspects of problem solving that are addressed by the teachers include the ability to describe a problem-solving approach. The ability to describe a method, strategy or problem-solving approach can be seen as part of a problem-solving ability and this aspect is also mentioned in the knowledge requirements. This aspect however is very difficult to distinguish from the ability to account for and communicate a method, strategy or problem-solving approach. The teachers' discussions on students' ability to communicate are treated under this headline below. The ability to reason about the plausibility of results of the problem solving, or to propose alternative approaches, which is mentioned in the knowledge requirements, is not discussed.

Mathematical concepts

Very few discussions deal with mathematical concepts. The four operations are mentioned but they are referred to as calculations which illustrate the process rather than as concepts. One student is identified by several teachers as having used the equals sign in a non-standard way which can be interpreted as relating to the concept of equality but this can also be connected to the way students choose to present their calculations.

Mathematical methods

As was presented above this is the ability which many of the teachers' discussions are focused on. The method that most teachers identify is the trial-and-error method. Several teachers claim that this is the method that all students have used. There are several accounts which show different ways in which the students have carried out and represented this method but these differences are most often referred to as relating to the ability to communicate. There are examples of accounts where the trial-and-error method is not used systematically and other examples where the representation indicates a calculation that precedes the trial-and-error since the account either contains no errors at all, or displays errors that have been erased but which are still traceable. This difference stirs many discussions among all the teacher groups. They are discussing whether they can tell if a student has tried different combinations and ruled some out or if the student came by the right combination by chance or by doing mental calculations that are not represented in writing. Sometimes they agree that they cannot tell and that this is due to students' lack of ability to communicate their problem-solving processes, other times they have different opinions regarding what can be inferred.

Mathematical reasoning

There is only one teacher who addresses the students' mathematical reasoning. This teacher argues that any account of problem solving which describes a method or strategy constitutes evidence of some form of mathematical reasoning. The rest of the teachers in this group are not questioning her but they are not offering her support and the issue of mathematical reasoning does not come up again.

Communication

There are very few discussions that do not involve students' ability to account for and describe their approaches to solving the problem. Practically every instance involves a question from the teachers regarding what the students have done or what they mean. Even in cases where the teachers have identified a successful strategy along with a correct answer to the problem they still raise questions regarding the clarity and coherence of the account. The discussions on presentation are focused on the students' [lack of] logic, neatness, clearness, abstraction, accuracy, appropriateness, and comprehensiveness. When discussing students' choice and employment of different method a typical comment from the teachers is "if she had only shown…". This fictional comment summarizes the teachers' frustration with what they perceive as lack of evidence on which they can base their judgements regarding other abilities.

Discussion

The preliminary results presented above can be used in response to the calls for external grading of national tests in mathematics in Sweden, and elsewhere, as a way to ensure objectivity and fairness. The results indicate that students' lack of communicative skill makes it difficult for the teachers to use these written accounts to assess other mathematical abilities. The study thus confirms Monaghan et al's (2009) claim that students' ability to communicate, to describe and to account for, their processes or their thinking, is crucial for teachers as well as for students. In order for teachers to evaluate students' abilities, they need to understand what the students have done and why. In order for students to write in a way that reflects their mathematical knowledge they need to know how to represent their problem-solving process along with explanations and arguments for their various choices. The fact that there are different ways to interpret what the students have written, further strengthens the conclusion that using this writing, to assess other mathematical abilities, may be problematic. The results should not be interpreted as suggesting that problem-solving should not be used to assess students' mathematical abilities but rather that both teachers and students need to know more about different ways to clearly and comprehensively account for problem-solving processes.

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