How do prospective teachers imagine mathematical discussions on fraction comparison?

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Prospective teachers plan a short mathematical discussion on comparing fractions by writing lesson plays. We analyse how their mathematical knowledge for teaching surfaces in their written scripts, using three dimensions of the Knowledge Quartet: foundation, transformation and connection. Our findings give insight into the prospective teachers' knowledge of fractions and comparison strategies, their perspectives on mathematics and mathematics teaching, and insight into how they transform their knowledge to make it accessible to middle school students.

Keywords: Mathematics knowledge for teaching, representations of teaching, classroom discussion, lesson planning, rational number sense.

Introduction

Several recent studies call for a practice-based approach to research on teacher education (e.g. Ball & Cohen, 1999; Grossman & McDonald, 2008). An integral part of teachers' work is planning for teaching. Recently, several researchers have advocated writing lesson plays as a means of learning how to plan for instruction (see Zazkis, Liljedahl, & Sinclair, 2009). Lesson plays are imagined mathematical discussions written verbatim. Zazkis et al. (2009) argue that lesson plays can give a "window" for researchers to investigate mathematical knowledge for teaching. We have collected and analysed prospective teachers' (PTs') planning documents for a practice assignment in their school placement. The assignment was to write a lesson play on fractions. Several studies show that many PTs struggle to understand fractions (Newton, 2008; Ma, 1999). Siegler et al. (2010) recommend, "[p]rofessional development programs should place a high priority on improving teachers' understanding of fractions and of how to teach them". With this in mind, our research focuses on the following question: *How does PTs' mathematical knowledge for teaching surface in their lesson plays on fraction comparison?*

Lesson plays

This study reports our efforts to develop practice assignments in our teacher education courses with respect to the notion of high-leverage practices. Ball, Sleep, Boerst, and Bass (2009) define "high-leverage practices" to be those that, when done well, are likely to lead to improved student learning. High-leverage practices are practices which novice teachers need to learn to do, and from which they will learn more about teaching (Lampert, 2009). Further, a high-leverage practice is such that novices can begin to master it (Grossman, Hammerness, & McDonald, 2009). Lampert, Beasley, Ghousseini, Kazemi, and Franke (2010) give several instructional activities that could be part of high-leverage practices in mathematics, which can be realised in a relatively short time, typically 10-15 minutes, through classroom discussions. The PTs used coursework literature to analyse and discuss a series of videos and transcripts of classroom discussions concerning computational strategies. We asked the PTs to plan a similar short mathematical discussion with a group of middle school students, age 9 to 13. This was part of a practice assignment, intended to be implemented during their school placement towards the end of the term. The PTs were asked to plan for the mathematical discussion by writing a *lesson play*.

Lesson plays, introduced by Zazkis et al. (2009), are proposed as a way to plan teaching by writing a script for (part of) a lesson. An envisaged interaction between a teacher and a group of students is given verbatim, as an alternative for the traditional lesson plan. Zazkis et al. (2009) argue that lesson plays can give an opportunity for in-depth discussions of crucial aspects of mathematics teaching *before* the lesson, while such discussions can only take place after the lesson if the lesson is planned using a traditional lesson plan. As such, lesson plays are not affected by John's (2006) claim that traditional lesson plans do not give insight into "the substance of the particular activity" (p. 487).

Zazkis et al. (2009) typically give their PTs a prompt representing a mathematical error or misinterpretation, and ask the PTs to write the script of a discussion, which resolves the prompt. We asked our PTs to plan all aspects of the discussion, including formulating a mathematical aim for the discussion and a task, or a sequence of tasks, to achieve this aim. Our requirements for the discussion were that the mathematical topic was fractions, and that the aim should be to discuss some calculation or reasoning strategy on fractions. Furthermore, the script should include argumentation and some type of generalisation of concepts and/or strategies. Generalisation, argumentation and reasoning was a major focus in the coursework. The duration of the discussion should be 10-15 minutes.

Mathematical knowledge for teaching

We are interested in how PTs' mathematical knowledge for teaching surfaces in their lesson plays, and we use the Knowledge Quartet (KQ) (Rowland, Huckstep, & Thwaites, 2005) as a framework for our analysis. The KQ consists of four dimensions, three of them resting on the first, named *Foundation*. Foundation concerns the teacher's or PT's knowledge of mathematics and mathematics teaching as acquired in their education. It underpins a teacher's ability to make rational, reasoned choices and decisions about instruction based on knowledge of mathematics and mathematics pedagogy. The second dimension of the KQ, *Transformation,* is about how the teacher transforms her own subject matter knowledge of mathematics into forms which enable others to learn it. Such transformation is informed by the teacher's choice of examples and representations and how these support learning of the intended mathematical topic. *Connection* is about the choices a teacher makes in order to ensure the consistency of planning and teaching a topic or concept through a lesson or lessons. As such it concerns anticipation of what students will find problematic, and decisions about sequencing. Crucial is the teacher's understanding of connections between mathematical concepts and between concepts and procedures as well as anticipation of complexity when planning and teaching a topic. *Contingency* concerns a teacher's responses to events that were not anticipated or planned for. Since we are considering planning for teaching this dimension will not be relevant to our analysis.

As the lesson plays analysed concern comparing fractions, it is useful to clarify the implications this has for the foundation dimension of the KQ. When describing number sense, researchers state that it manifests in flexible mental computation, understanding number magnitude, making judgements about calculations, using benchmarks, and having an inclination to use and develop understanding of numbers and operations (McIntosh, Reys, & Reys, 1992; Sowder, 1992). Researchers have identified several strategies for comparing fractions based on number sense (see Yang, 2007). The *parts* strategy can be used when comparing fractions with the same numerator or denominator. The *benchmark* strategy refers to comparing two fractions to some well-known third fraction, typically $\frac{1}{2}$ or $\frac{3}{4}$. When using *residual thinking* one builds up the fractions to 1. There are also “standard” ways of comparing fractions, which do not overtly depend on number sense, such as finding a common denominator or converting to decimals. Since the task for the PTs was to write a lesson play about reasoning with fractions, we expected the planned discussions to contain more than performing an algorithm. Using visual models of fractions can be a legitimate strategy, but researchers also warn about the limitations of relying on a visual strategy alone (Petit, Laird, Mardsen, & Ebby, 2015; Lamon, 2012). Thus, we expected strategies beyond visual strategies in the planned discussions.

Method

The participants in the study were PTs following a 4-year teacher education programme, for age 6-13, at a university in Norway. The data were collected from their responses to a coursework assignment given during their first mathematics education module, in their first year of study. Of the 178 PTs in this cohort, 32 had chosen tasks on comparing fractions for their written classroom discussion. These 32 scripts are the data analysed in this paper, in particular they were chosen for analysis because fraction comparison has good potential for reasoning based on number sense (Yang, 2007). The excerpts presented in this paper are chosen to exemplify general trends identified in the 32 lesson plays.

All four authors conducted the analysis. Together, we first analysed in detail two lesson plays, using the descriptions of the dimensions from the KQ in our analysis of the two scripts. We then individually analysed the rest of the lesson plays looking for occurrences of similar and contrasting forms of mathematical teacher knowledge related to the KQ. After this independent analysis, we compared similarities and differences in our analyses, and agreed on an interpretation of different aspects of the lesson plays, using notions included in the KQ framework.

Analysis

Of the 32 scripts, 6 used no strategies based on number sense, instead relying on visual “parts of shapes” strategies, or on algorithms such as finding a common denominator or converting to decimal numbers. In the remaining 26 scripts, the PTs used a number sense-based strategy at least once. 9 PTs used *benchmarking,* 13 used *parts* and 17 used *residual thinking*. In the following, we analyse some examples from the scripts in light of the mathematical content.

Anne was one of the PTs who based most of the imagined discussion on number sense-based strategies. Her stated goal of the discussion is to build understanding of the strategies of benchmarking and residual thinking. She gives two tasks designed to encourage the students to utilise these strategies, and we quote here[1] two excerpts from the imagined discussion between Anne's teacher and her students while discussing the sorting of $\frac{7}{8}, \frac{3}{4}, \frac{2}{10}$ and $\frac{3}{6}$ by magnitude:

Ola: Yes, at least you see that $\frac{3}{6}$ is the same as the half of something, that was where I started.

Teacher: OK, so you believe that it is one half. But how can that help us?

Ola: Well, since it is one half, we also see that $\frac{2}{10}$ is less than one half.

Teacher: Per, can you try to elaborate Ola's thinking?

Per: $\frac{2}{10}$ is sort of lacking 3 parts to become one half. Because 5 parts is half of 10 parts. So then $\frac{2}{10}$ is less than $\frac{3}{6}$

Teacher: Right, do the rest of you agree? Yes. OK, what do we do next?

We see that Anne, in Ola's words, uses one half as a benchmark when comparing $\frac{2}{10}$ and $\frac{3}{6}$. We note that Ola's explanation is incomplete; it does not state why $\frac{2}{10}$ is less than one half. The PT seems to be aware of this, as she asks Per to *elaborate* Ola's thinking, from whom she receives the completed reasoning. In the next sample from Anne's script, a residual argument is pursued:

Teacher: All right, so now we know that $\frac{2}{10}$ is the smallest, and then comes $\frac{3}{6}$, but which is the biggest of $\frac{7}{8}$ and $\frac{3}{4}$?

Per: Since $\frac{1}{8}$ is smaller than $\frac{1}{4}$ then $\frac{7}{8}$ is the most

Mia: But why is that when $\frac{1}{4}$ is a lot bigger than $\frac{1}{8}$?

Teacher: Good question Mia, does anyone want to explain?

Ola: When we looked at $\frac{1}{8}$ and $\frac{1}{4}$ we were looking at how much was missing to fill one whole. The one that miss the biggest part is then missing the most, and therefore that fraction is the smallest. Because $\frac{7}{8}$ is only missing a small $\frac{1}{8}$ to become one whole.

Mia: Oh yes, now I understand, because we are looking at what is missing.

Anne does not acknowledge any students' claim without a justification. Through the whole of Anne's script, the teacher is encouraging the students to utilise their number sense when reasoning about the tasks. Similar approaches are also apparent in the rest of the 26 scripts, but not always as comprehensive as in Anne's case. One typical feature is that even though a mathematically correct conclusion is reached, there is no valid argument given by the students, and the PTs tend to accept this without comment. This is evident in Alice's script, when the students are comparing $\frac{4}{8}$ and $\frac{3}{4}$.

[1] The original scripts were written in Norwegian, with translation to English by the authors of this paper. In the translation we have retained linguistic inaccuracies and imprecise use of terms, as in the original Norwegian.

Fredrik: Yes, me and my group decided that $\frac{3}{4}$ is the biggest. I think that 3 is closer to 4, than 4 is to 8.

Teacher: That was good thinking. [Proceeds with a different task]

Our analysis shows a general tendency in the scripts that strategies based on number sense have some kind of justification, while strategies based on algorithms and rules are more likely accepted without justification. The above excerpt from Alice's script is one of few exceptions, where she gives a correct conclusion with an attempted justification that is not valid as an argument. Judging by the teacher's response, it seems that Alice regards Fredrik's argument as valid.

The following excerpt from Christine's script shows another problem.

Sindre: $\frac{3}{6}$ must be the biggest, because that fraction is only missing 3 parts to become one whole, while $\frac{3}{8}$ is missing 5 parts to become one whole.

Teacher: Yes, that's right Sindre. Did the rest of you understand what Sindre was thinking? Nina, can you explain what Sindre meant?

Nina: Yes, you can also say that $\frac{3}{6}$ misses one half to be whole, while $\frac{3}{8}$ lacks more than half to be whole, since it is lacking 5, and half of $\frac{8}{8}$ is $\frac{4}{8}$.

Teacher: That was a good explanation, Nina. Did the rest of you also understand what Nina meant? (The class agrees.)

In this excerpt, we notice that Sindre's argument is wrong even though the conclusion is correct. Christine (in the role of the teacher in the discussion) does not comment upon this, instead simply accepting Sindre's argument. Interestingly, Nina subsequently gives a valid residual argument, but Christine does not draw attention to the difference in the two arguments in her script. Our analysis shows that similar arguments that "work" on the fractions in question, but where a counterexample would prove the argument not generally valid, are typical for many of the lesson plays.

Another aspect of our findings was the PTs' choice of tasks used in the discussions. Returning to Anne's script, she uses only two tasks. They both underpin the strategy in focus, and have a natural progression in complexity: The fractions involved seem to be carefully chosen to make her target strategy suitable, and residual thinking is further highlighted by Anne asking the question "Which of the fractions are missing the most to become one whole?" at the start of the discussion. In contrast to this, Molly's choice of tasks and the sequencing chosen, seems less appropriate: Compare $\frac{3}{5}$ and $\frac{3}{7}$; $\frac{7}{7}$ and $\frac{7}{9}$; $\frac{2}{4}$ and $\frac{3}{4}$; $\frac{2}{4}$ and $\frac{4}{8}$; and $\frac{1}{2}$ and $\frac{3}{6}$. Molly does not state explicitly a mathematical goal for the planned discussion, and her imagined discussion covers several ideas in a brief way. Moreover, the progression of difficulty in the sequence of tasks does not seem to be well thought through: in the lesson script on the first tasks, Molly's fictive students use benchmarking with one half, indicating that one half is a well-known concept for them. To then proceed with the final three tasks focusing on equivalent fractions to $\frac{1}{2}$, seems exaggerated.

Discussion

We now relate the findings presented in the analysis to the dimensions of the Knowledge Quartet.

Foundation

For the foundation dimension, the most visible aspects are the PTs' mathematical knowledge of fractions and comparison strategies, as well as their beliefs about mathematics itself, and about mathematics teaching. In general, the PTs try to use strategies relying on number sense to compare fractions. This could indicate that the PTs value developing understanding rather than focusing on an algorithmic approach. The strategies attempted in the scripts are not always followed through in a mathematically valid argument, and some of the PTs fail to recognise the difference between valid and invalid arguments. Christine's script is an example of this.

Another finding is that very few scripts contain any attempt at discussing the *generality* of the strategies used. This indicates that the PTs' beliefs about mathematics might not include this as an important aspect of doing mathematics. Instead, the PTs seem to be satisfied as soon as the problem at hand is solved, as in Christine's script when neither Sindre's or Nina's arguments are investigated further from a general point of view. Recall that the task given to the PTs particularly required them to emphasise the development of their students' understanding and reasoning.

Transformation

The PTs' scripts afford good insight into their choice of examples to elicit an idea. With very few exceptions, the tasks chosen by the PTs are suitable comparison tasks where it is clear that there is at least one number sense-based strategy that could be applied.

We proceed to consider the PTs' use of questions. In the context of PTs writing an imagined discussion, we regard this as a form of teacher demonstration, and thus consider it a part of transformation. We find in most scripts a use of certain techniques and types of questions known from their coursework literature on orchestrating mathematical discussions. For example, in Anne's script, the teacher's questions structure what her students have discovered and then seek to develop their ideas further. When Anne's teacher asks Per to elaborate Ola's thinking, she succeeds in bringing to light an argument. In other scripts, the PTs seem to emphasise the use of discussion techniques in itself to such an extent that it suppresses the attention on connecting the mathematical ideas. This can be seen e.g. in the excerpt from Christine's script above, where the teacher asks a student to repeat another student's reasoning without connecting the different explanations. Sometimes the PTs fail to notice when a clarifying question is needed. This can be seen in Alice's script above, where Fredrik's attempted justification is an invalid argument in general, and yet the teacher accepts it and proceeds without further enquiry.

Connection

The *sequencing of tasks*, how one task should connect to the previous task, and the anticipation of what students will find problematic, is part of the connection dimension. We find that in most scripts, the sequencing of tasks is appropriate. However, we find examples of situations where the PTs do not seem to anticipate the complexity of the sequence of tasks. An example is Molly's script as discussed above. Other scripts seem to have too many tasks, given the time allotted. In these scripts the discussion moves forward smoothly with students giving the desired response quickly and effortlessly. This may indicate that these PTs do not anticipate complexity in the discussion and that the conceptual challenge for the students is underestimated. Thus, these discussions take more the form of numerous repetitions of the same procedure, which relates to the foundation dimension of the KQ and perspectives on how mathematics is learned: These PTs seem to emphasize procedural repetition as an important aspect of learning mathematics, perhaps on behalf of unpacking the mathematics of the procedures. However, some scripts include deliberate mistakes and misconceptions made by the students, which are then discussed. We see this as an anticipation of complexity.

Conclusions

Following the discussion above, we claim that lesson plays encourage the PTs to use and develop several aspects of their mathematical knowledge for teaching. For the foundation dimension, we claim that the insight we get from the scripts, is more than what we would get from simply assigning the PTs fraction comparison problems for them to solve. We note that several PTs write discussions including both valid and invalid arguments and both are accepted without further probing. For instance, Christine knows what a valid argument for comparing fractions looks like, but at the same time she accepts an invalid one. Such inconsistency in the PTs' thinking might become more visible when they plan teaching by imagining a detailed mathematical discussion.

We also claim that our findings show the importance of emphasis in mathematics teacher education on generalisation and argumentation, and how classroom discussions concerning generalisations could play out. Our PTs were asked to have those aspects in mind when writing their lesson plays, and yet it is rarely found in the scripts. How to develop the PTs' ability to emphasise this aspect more needs to be studied further. Managing classroom discussions is a complicated task for novice teachers. However, due to its high leverage on students' development of mathematical understanding it is a critical factor in mathematics teaching, and thus in teacher education.

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