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Oslo and Akershus University College of Applied Sciences, Norway; bodil.kleve@hioa.no

In a year 4 classroom, we studied students' presentations of their solutions of problem-solving tasks. Sitting in pairs (learning-partners), they solved the tasks before presenting their solutions orally in class. Based on transcripts from video recordings of the lesson, students' written notes and post interview with the teacher, the role of sociomathematical norms related to students' use of informal mental strategies and standard algorithm for subtraction is discussed. For students flexibly to carry out arithmetic operations, we suggest to develop switching between informal mental strategies and standard algorithm as a sociomathematical norm. In that respect, attention is put on mathematical knowledge for teaching (MKfT) and emphasis on place value system is suggested as amalgam between different strategies.

Keywords: Subtraction, place value system, learning-partners, mathematical knowledge for teaching,

> «Ah, mental (informal) algorithms are all very well, but they must learn the standard methods sooner or later» **Or must they?** Plunkett (1979, p 4).

Background and introduction

This paper is based on a study which purpose was to identify situations in a classroom where development of existing sociomathematical norms or establishing new norms may create potential for students' learning. A video research study was carried out in a year 4 classroom (9-10 years). An earlier publication reported situations in the classroom related to argumentations, and development of existing sociomathematical norms and establishing new norms were suggested in order to increase the potential for students' learning. (Kleve & Ånestad, 2016).

Based on students' (learning partners') written and oral presentations of a problem-solving task, where a three digits subtraction had to be carried out, sociomathematical norms (Yackel & Cobb, 1996) are identified and we discuss development of sociomathematical norms in light of mathematical knowledge for teaching (Ball, Thames, & Phelps, 2008; Rowland, Huckstep, & Thwaites, 2005). Our research question is: What role does mathematical knowledge for teaching play in order to develop sociomathematical norms, which can bridge the gap between informal mental strategies and standard algorithm for subtraction so the children can flexibly switch between different strategies?

Raveh, Koichu, Peled and Zaslavsky (2016) presented an integrative framework of knowledge for teaching the standard algorithm of four arithmetic operations. Studying connections between the four basic algorithms for arithmetic operations, they encouraged teaching the standard algorithm with emphasis on conceptual understanding, putting weight on connections between the four algorithms. In our study, the focus is on the relation between informal mental strategies and the standard algorithm for subtraction.

Torbyns and Verschaffel (2016) analyzed students' use of mental strategies and standard algorithm on subtraction. They found that when having been introduced for the standard algorithm:

> Children presumably became gradually more efficient in this algorithm, while their mastery of mental computation in general, and compensation in particular, may have stagnated or even *declined* (p. 112, italics in original).

Even when numbers involved were suitable for and "strongly invited to mental computation strategies", (ibid.) they found that students, when first having been introduced to the standard algorithm for subtraction, used this instead of mental strategies.

Torbyns and Verschaffel (2016) suggested that when the standard algorithm was introduced, the students would think that this newly introduced method was "the superior way to subtract larger number" (p.112). Referring to Yackel and Cobb (1996) they linked this to socio-cultural classroom norms in the classroom. Furthermore they referred to international efforts to reform elementary mathematics education which emphasizes children's abilities to flexibly apply informal mental strategies before they are introduced to the standard algorithm, and the claim that children then will continue to efficiently apply informal strategies. Torbyns and Verschaffel therefore encouraged research in more reform-oriented classrooms, comparing children from these with children taught in traditional oriented classrooms.

The classroom, in which our study took place, was reform oriented. There was a strong focus on children's use of own informal strategies and an extensive use of learning partners. The focus in this paper is on sociomathematical norms related to their use of informal mental strategies and/or use of standard algorithm for subtraction. Related to sociomathematical norms we also discuss what role mathematical knowledge for teaching play in the development of children's flexibility in using different mental strategies and the standard algorithm.

Theoretical perspectives

"Sociomathematical norms are normative aspects of mathematical discussions that are specific to students' mathematical activity" (Yackel & Cobb, 1996, p. 458). Yackel and Cobb focused on sociomathematical norms when studying "how students develop specific mathematical beliefs and values and consequently become intellectually autonomous in mathematics [ ] how they come to develop mathematical dispositions" (p. 458). They distinguished sociomathematical norms from general classroom social norms in that they are specific to the mathematical activity carried out in a classroom. In our study, we focus on an episode where use of informal mental strategies and/or standard algorithm for subtraction as mathematical activity is discussed.

Sociomathematical norms can be what counts as an efficient mathematical solution, different mathematical solution and a sophisticated mathematical solution (Cobb, Stephan, McClain, & Gravemeijer, 2001). Cobb et al. emphasized that both social norms and sociomathematical norms are dealing with what is «Taken as shared» in the classroom.

Classroom norms are developed in collaboration between the teacher and students or between students. However, the teacher is the key person when norms are changing or new norms are established (McClain & Cobb, 2001). In our study, we consider that the teacher has great influence on sociomathematical norms, whether the norms are already established, in the process to be weakened or under development in the classroom. We therefore want to link development of sociomathematical norms to mathematical knowledge for teaching.

Ball, Thames and Phelps (2008) developed a framework for mathematical knowledge for teaching, MKfT. They distinguished between *Common content knowledge*, which is mathematical knowledge possessed not necessarily for teaching, and *Specialized content knowledge for teaching*, which is about the teacher's way of 'unpacking' mathematics, neither necessary nor desirable for others to do. They also included a third category, *Horizon content knowledge*, which is about making connections between different areas and topics. "Horizon knowledge is an awareness of how mathematical topics are related over the span of mathematics included in the curriculum [ ]. It also includes the vision useful in seeing connections to much later mathematics". (Ball et al. 2008, p. 403).

In order to investigate how aspects of the teacher's mathematical knowledge surfaced in the lesson observed, the Knowledge Quartet (KQ) developed by Rowland, Huckstep and Thwaites (2005) has been a valuable tool. The KQ has four dimensions: Foundation, Transformation, Connection, and Contingency. Foundation is informing the other three dimensions, and Connection is the dimension, which we see as linked to "Horizon content knowledge".

Plunkett (1979) discussed pros and cons with regard to use of informal mental strategies and standard algorithm in school, and questioned whether standard algorithms necessarily have to be taught and learned. He claimed that unlike standard algorithms, which only deal with separated digits, informal mental strategies are holistic in that they work with complete numbers and thus requires understanding. Liping Ma (2010) emphasized regrouping rather than technical use of standard algorithm. When regrouping, the subtraction algorithm will be understood on a holistic number level, rather than as separate digits. False mathematical statements as "we can't subtract a bigger number from a smaller" will be avoided. Such false statements are related to teachers' horizon content knowledge (Ball et al., 2008).

Anghileri, J., Beishuizen, M., & Putten, K. v. (2002) compared the Dutch approach to written division calculations in school, which extensively built on children's own informal strategies, to the English approach which was schematic and focused on separate digits. Based on the results from their study Anghileri et al. (2002) warned against replacing informal strategies with standard algorithms. Rather one should give support to structuring informal approaches in a written record.

Referring to Plunkett (1979) and that calculations are carried out in real life, Anghileri (2006) emphasized children's mental strategies as a starting point for developing more formal methods.

Based on the above, one can suggest that when introducing a standard algorithm, teachers should focus on regrouping numbers and take children's informal mental strategies as a starting point. This puts demands on the teacher's mathematical knowledge for teaching, which again will constrain the sociomathematical norms in the classroom.

Referring to among others Ball et al. (2008), Raveh et al. (2016) proposed a framework consisting of four components: Procedural Knowledge (PK), Knowledge of Underlying Concepts (KC), Knowledge of Similarity between the algorithms (KS) and Knowledge of Representations (KR). In our analysis, we will use components from this framework, mainly PK and KC, and some of KR. PK is about carrying out the steps correctly in the (subtraction) algorithm, while KC refers to knowledge about mathematical concepts underlying different algorithms as place-value and number regrouping. Rather than analyzing different representations of the subtraction algorithm as KR refers to in Raveh et al's framework, we emphasize the relationship between informal mental strategies, and the standard algorithm for subtraction.

In our study, we will not argue for not introducing the standard algorithm. However, we consider bridging between informal mental strategies and standard algorithm as valuable features of mathematics, which may influence students' mathematical beliefs. Development of flexibility and children's ability to switch between informal strategies and standard algorithm are linked to sociomathematical norms and to the aspects pf teachers' mathematical knowledge for teaching.

Methods

We observed two mathematics lessons in a 4th grade (mixed ability, 9-10 years old) classroom. Prior to the classroom observations, we had come to know the teacher. Her educational background was a pre-school teacher. She described her teaching as being reform oriented and that her students performed well on "transition tests". She had established an extensive use of learning partners in her classroom. She put her students together in pairs at random, same partner in all subjects, and changing partners every week. According to the teacher, the students never complained or protested against whom they received as their partners. This was established as a social norm in the class. With regard to Eli's view on mathematics teaching and learning, she emphasized the process rather than the product, saying: "For me it is not so important if the answer is correct. I am more interested in the strategy they use, that they have understood the principle behind solving such tasks". She also told us that she encouraged students to develop their own strategies in solving arithmetic problems and to discuss their strategies with their learning partner. Against this background, we wanted to study sociomathematical norms in Eli's class.

During our first visit in the class, we observed and wrote field notes. The second time we video recorded a 90 mins mathematics lesson. We used two cameras. The students were sitting in pairs and should solve different tasks, which were written on the board. After having solved the tasks, and written down their solutions, they presented their solutions orally. Towards the end of the lesson, they had some ('warm ups') whole class activities where they "worked with concepts" (teacher's expression). One of the activities was linked to the place value system.

Our analysis here is based on transcriptions of video recordings of their oral presentations and on their written work, which we collected. We also studied the video recordings together with the teacher several weeks afterwards. We interviewed the teacher and asked her to comment the different pairs' written sketches and oral presentations.

The task, on which we base our analysis, was written on the board:

> Tobias has two 200 NOK notes and six 10 NOK coins. He spends half on a gift, 142 NOK on a book and then he buys "pig ears" to Doffen. One ear costs 11 NOK. How many ears can he buy?

Analysis

The subtraction 230-142 had to be carried out to solve the task. Eleven out of twelve pairs of students had used the standard algorithm, however with different degree of detail in their oral presentations. The standard algorithm had become the "taken as shared" method (Cobb et al., 2001). First, we discuss four pairs' written calculations (sketches) together with their oral presentations, illuminated with the teacher' comments to the presentations, then we go further into the teacher's comments from the interview. The teacher did not comment on any of the presentations in the lesson.

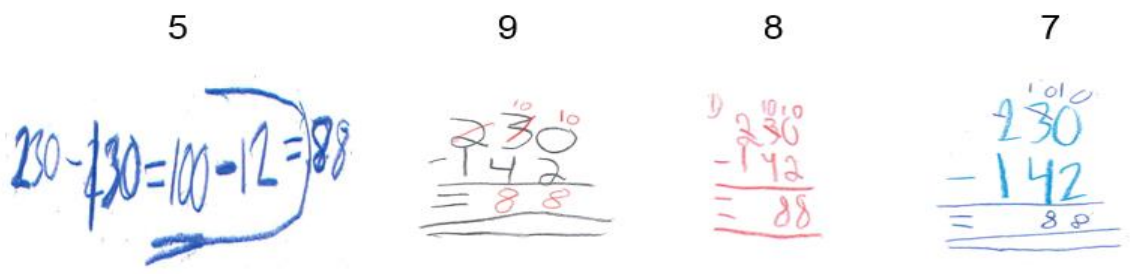

Figure 1: Four pairs' written calculations

The pairs' oral presentations:

Pair 5: 230, it was easier to do 230 minus 130 equals 100 and then we did minus 12 because 130 plus 12 is 142, and that made 88

Pair 9: 230 – 142 is zero minus two, doesn't work. We have to exchange from the three, and ten minus two is eight and then we have two left there. So then we take, however four minus two doesn't work either, so then we will have to exchange from the two. Ten minus four is six plus two is eight and then we have only one left, which makes one minus one is zero. So then it is 88".

Pair 8: We did 230 minus 142 using the standard algorithm. We got 88

Pair 7: 230 minus 142 is 88

Studying Pair 5's calculation, both their written work and oral presentation, we see that they did not use a standard algorithm for subtraction. The students regrouped the subtrahend 142 into 130 and 12. This way of calculating is flexible and requires understanding and a holistic way of thinking. They treated the complete numbers rather than separated digits (Plunkett, 1979).

The interview with the teacher Eli, with regard to Pair 5, revealed that she did not see the way of solving and presenting this as a potential for further development. The teacher expressed her acknowledgement of different ways of doing subtractions, but that this was a cumbersome and much lengthier way. She considered one of the students in the pair as a "funny one", and that "you need to be much sharpened to follow his thoughts. However, I keep telling him that he ought to start using another strategy in order to make things go faster. So after a while you' will have to do that". This is in line with Plunkett's (1979) characterization of informal mental strategies: "often difficult to catch hold of "(p. 3). This comment revealed that the teacher did not see the potential in her students' mental calculations for further development. She now looked upon the standard algorithm as the most efficient and sophisticated way of carrying out subtraction, and using the standard algorithm was in the process of being established as a new sociomathematical norm.

As we can see from figure 1, the three other pairs used the standard algorithm for subtraction. All correctly written out, displaying decomposition (exchange or borrowing). This can also be interpreted as regrouping of the minuend based on the place value system. However, the students did not express any regrouping. Studying their oral presentations reveals that the students were on different levels in using the algorithm. "Standard algorithms are *not easily internalized*. They do not correspond to ways in which people tend to think about numbers" (Plunkett, 1979, p. 3, italics in original). The pairs only referred to digits between 0 and 10. We suggest that this is why Pair 9 presented a detailed explanation of the algorithm as such. Their claims: "zero minus two, doesn't work" and "four minus two doesn't work reveal either", reveal a misconception or a "false mathematical statement" (Ma, 2010, p. 3). These students have not been presented for negative numbers yet, and false mathematical statements like these may lead to later misconceptions about negative numbers. The students in this class (except Pair 5) used the term "exchange" when regrouping the minuend 230 into 220 + 10, and when they later regrouped 220 into 120+100.

A question here is whether the students know what they are doing. According to Plunkett, use of standard algorithms encourage suspended understanding. Pair 9's explanation reflects procedural knowledge in carrying out an algorithm rather than conceptual understanding. Ma (2010) encourages regrouping rather than exchanging or borrowing (decomposing) when introducing the standard algorithm. Because then they will be working on a holistic number level rather than with separate digits.

With regard to Pair 8, they only said they had used the standard algorithm, whereas Pair 7 only presented the answer. They can be considered as having internalized the algorithm, and as Plunkett (1979) puts it: "While the calculation is being carried out, one does not think much about why one does it in that way" (p. 3).

In the interview with the teacher, Eli said that when starting a new arithmetic operation, she encouraged everyone to do it his or her own way, and that she used to present all students' different informal strategies on the board. She expressed a great concern about these differences when a new arithmetic operation was being introduced. Thus, we consider that use of mental strategies for subtraction was earlier established as a sociomathematical norm. However, after having introduced the standard algorithm, this sociomathematical norm was in the process of disappearing, or at least fading, and a new sociomathematical norm was about to be developed. About the introduction of the standard algorithm, Eli said, "We practiced memory numbers and exchange in detail". Consequently, we consider such detailed explanation as a new sociomathematical norm. This sociomathematical norm is also in the process of disappearing. Everybody was now expected to use the algorithm without further explanations or comments. As we see from our data, some students used the standard algorithm naturally without further explanation, while others still explained the procedure in detail. Only one pair (5) explained the subtraction as use of mental strategies without mentioning the standard algorithm. Hence, eleven out of twelve pairs looked upon the newly learnt algorithm "as a superior way" (Torbeyns & Verschaffel, 2016, p. 112).

Discussion

Our findings suggest that although Pair 5's way of doing subtraction was not acknowledged ("he ought to start using another strategy in order to make things go faster"), the students displayed both number sense and a well-developed subtraction concept. Of those who had used the standard algorithm, some explained the procedure in detail, Pair 9, while others just referred to the algorithm. Although they might have had conceptual understanding, they did not display it. Their focus was on the skill carrying out the subtraction *procedure*. According to Eli, the students who used the standard algorithm had developed a more mature number sense than those still using mental strategies. Although being influenced by reform-oriented working methods, Eli expressed the necessity of learning the standard algorithm, both as a tool, an assurance to always have a method to use, and as an efficient way of doing subtraction. She looked upon standard algorithms as a supplement to informal mental strategies. However, she was not aware what research has shown; when first have been introduced to the standard algorithm for subtraction and exposed to instruction emphasizing mastery of the standard algorithm, children will gradually become more efficient in using the standard algorithm, while their use of informal and mental strategies will fade (Torbeyns & Verschaffel, 2016). The challenge is to bridge or close this gap. A goal must be to develop sociomathematical norms where students are able to switch between different strategies dependent on the nature of the numbers involved. This puts demands on the teacher's MKfT, and especially the Horizon knowledge.

There was no indication in what the teacher said in the interview that the informal mental strategies the children earlier had used had been taken as a starting point when introducing the standard algorithm. The teacher's mathematical knowledge for teaching seemed too fragile to give her courage to rely on students' mental strategies, and thus to bridge the gap. The mathematics presented for her students seemed to be fragmented. During the place value activity towards the end of the lesson students should answer questions as "what value does 1 have in 5129?" If this had been linked to the standard algorithm for subtraction, regrouping, based on place value system, could serve as an amalgam between informal strategies and the formal algorithm. This refers to Raveh et al.'s (2016) KR, which we see as knowledge about connections between informal strategies and mental strategies. Attention could here be brought to the foundation and connection aspects of the teacher's knowledge (Rowland et al., 2005). Knowledge of the mathematical concepts underlying the algorithm, KC (Raveh et al., 2016) did not surface in what she said. However, she demonstrated procedural knowledge (PK) related to correct computations and the steps in the algorithm.

Our findings suggest that the teacher did not see the potential in taking earlier established sociomathematical norms about students' use of mental strategies as a starting point. We claim that linking informal mental strategies to the place value system in introducing the standard algorithm for subtraction would enhance students' ability to switch between informal strategies and the standard algorithm, dependent on the numbers involved. Regrouping based on the place value system could then serve as an amalgam between informal mental strategies and the standard algorithm.

Based on our data, we cannot say anything about the sociomathematical norms related to other areas of the mathematics in this classroom. However, the students had not yet been introduced to the standard algorithm for division. We saw that in carrying out the necessary division operation to solve the task (how many ears can he buy?) 88:11, the students used either repeated subtraction or repeated addition as (informal) mental strategies.

Encouraging teachers to rely on and see the potential in earlier established sociomathematical norms where students use informal mental strategies, are important issues for further research. In that respect, attention must be directed towards to teachers' mathematical knowledge for teaching, with special focus on Horizon knowledge and the connection dimension of the teacher's mathematical knowledge (Ball et al., 2008; Ma, 2010; Raveh et al., 2016; Rowland et al., 2005).

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