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We present a new tool – Realization Tree Assessment (RTA) for assessing the mathematical quality of lessons and the ways in which the whole classroom discussion expose students to mathematical concepts. The tool, built upon the commognitive framework, depicts the different realizations of a mathematical object treated in a lesson, and then uses different shades to signify who articulated the realization – the teacher or the students. We exemplify the tool on two lessons implementing an identical Hexagon pattern generalization task. The RTA visualizes the manner in which one lesson gave students sufficient opportunities to "same" different algebraic expressions, while the other lesson did not. We show how this visual presentation of the mathematical ideas complements existing assessment tools, particularly, the Instructional Quality Assessment and Accountable Talk. We conclude by discussing the potential of the tool as an aid for lesson planning.

Keywords: Teaching practices, Realization tree, commognition, cognitive demand, lesson assessment tools.

Introduction

Recent years have seen increasing efforts to train teachers to teach exploratively – provide students with opportunities to engage with cognitively demanding tasks, problem solve, and participate in rich mathematical discussions (Schoenfeld, 2014). Within such efforts an important role lies in the tools that are used to examine lessons enacted by the trained teachers (Boston & Smith, 2009; Schoenfeld, 2015). Scoring and evaluation tools (such as the Instructional Quality Assessment tool or TRU math) can be used both for evaluating lessons and thereby examining the effectiveness of the training program, as well as tools for teachers' professional development. A common difficulty with these tools, however, lies in operationalizing their criteria for evaluating the quality of mathematical ideas dealt with in the lesson. In this paper, we propose an analytical tool – the Realization Tree Assessment tool which is based on the "commognitive" framework (Nachlieli & Tabach, 2012; Sfard, 2008). This tool enables drawing a succinct yet sufficiently meaningful picture of the mathematical concepts surfaced in a specific lesson in such a way that lessons can be both compared with each other as well as planned ahead more accurately.

Theoretical background

Tools for the examination and evaluation of classroom instruction can be categorized into three types: scoring tools such as Instructional Quality Assessment, or IQA (Boston, 2012b) and the Teaching for Robust Understanding of Mathematics summary, or TRU math (Schoenfeld, 2014); "coding and counting" tools, such as Accountable Talk (O'Connor, Michaels, & Chapin, 2015); and qualitative analytical tools (e.g. commognition, Sfard (2008)). Scoring and coding measures have the benefit that they are quantifiable. They thus enable both the comparison of teachers with each other, as well as comparison of within-teacher change from lesson to lesson, for example, as a result of professional development. Scoring tools, however, have a drawback. They are heavily based on extensive training of scorers for the development of inter-rater reliability. This, because of their high-inference nature. Coding and counting tools, which are based on coding talk moves, necessitate lower inferences and are therefore easier for achieving reliability. However, these tools are mostly good for capturing non-mathematical aspects of the discourse.

The difficulty in assessing the surfacing of “important mathematical concepts” (Boston, 2012a) or “important content and practices” (Schoenfeld, 2015) in a lesson is not surprising, given that the definition of “mathematical concepts” has been under much dispute for decades (Sfard, 2008). To our aid, we draw on commognition (ibid), which we have extensively used in the past as a qualitative tool for describing learning-teaching processes. In the present work, we simplify this tool, to attune it with the demands of coding and scoring schemes that seek to evaluate lessons in a relatively short period of time, for the goal of comparing large sets of lessons.

Realization trees

Mathematical learning, says Sfard (2008), is a process whereby students gradually become able to communicate about *mathematical objects*. These objects are produced by discourse (or communication), and are made up of different “realizations” (ibid, p. 165). The term realization is used by Sfard instead of the more common term “representation”, to emphasize the fact that nothing is, in fact, “there” to be represented. All mathematical objects are products of human discourse and come to life by being different realizations being “samed” and alienated from human agency so that they are talked about as existing of themselves. For example, the signifier ½, the process of dividing a pizza into two pieces, and the process of shading 3 circles out of 6, are all samed into the object “one half”. Children often learn each of these realizations separately and only later come to relate to them all to one object. This is the heart of a process Sfard calls “objectification”. Objectification, or talking about mathematical signifiers as “standing for” mathematical objects that “exist” in the world, is a major and necessary accomplishment for advancing in the mathematical discourse. Sfard used the term “realization tree” to illustrate the fact that realizations are usually hierarchical. A half is made of different realizations (1/2, 0.5, 50%, 3/6 etc.) but the whole numbers making up these realizations also have endless realizations (3 apples, 3 fingers, etc.). Nachlieli & Tabach (2012) used realization trees to visually explain the complexity of the object *function* and to relate to the historical development of this object, as well as to make explicit students' development of the discourse of function. Before moving to explain our use of realization trees as tools for assessing the conceptual quality of a lesson, let us briefly describe the two other tools that have been serving us for quantifying and comparing mathematics lessons.

IQA

The IQA (Instructional Quality Assessment tool) has been designed by Boston and Smith (2009; Boston, 2012a) to evaluate the *cognitive demand* of mathematical lessons. This, based on the “task framework” put forward by Stein and her colleagues (1996), which differentiates between the cognitive demand of a task, the way it is presented to the classroom, and the way students eventually engage in it. Every rubric in the IQA is scored on a scale from 0 to 4. For reasons of space, we will concentrate here only on two rubrics: AR-2 (implementation) and AR-X (mathematical residue). Regarding the implementation rubric, 1 means students engage only in rote memorization and producing facts, 2 means they engage in the application of procedures explicitly taught, 3 means cognitive demand is not lowered but mathematical reasoning is not sufficiently explicated, and 4 means full engagement in a cognitively demanding mathematical task. 'Mathematical concepts' or 'ideas' are mentioned almost in every rubric in the IQA. For example, in the rubric that refers to the mathematical residue, the highest score should be given when: "The discussion following students' work on the task surfaces the important mathematical ideas, concepts, or connections embedded in the task" (Boston, 2012b, p. 20). However, IQA does not provide any clear guidance on this matter, besides giving a few examples of high and low level lessons.

Accountable Talk

Accountable Talk coding (Resnick, Michales, & O'connor, 2010) is a tool originating in socio-linguistic analysis of classroom talk (O'Connor & Michaels, 1993). It provides teachers with a set of specific *talk moves* they can make during whole classroom discussions, to hold students accountable to the community, to knowledge and to reasoning. Our version of Accountable Talk coding (Heyd-Metzuyanim, Smith, Bill, & Resnick, 2016) includes eight codes for teacher moves (e.g. press for reasoning, revoice, restate, agree/disagree) and four codes for students' moves (e.g. student-agree, student-justification). These moves track the amount in which teachers attempt to make students' thinking public, help students to reason mathematically, and hold them responsible for attending to the reasoning of others. Though the manual does contain examples of mathematical statements, Accountable Talk's basic framework does not deal specifically with content. It has no clear indicator of what consists as more important or "conceptual" reasoning, and what does not.

The study

In what follows, we first describe the setting of the study on which we developed the Realization Tree Assessment tool (RTA). We then describe the results of analysis using the IQA and AT, showing what could be achieved by them and what was missing or difficult to agree upon. We follow this by describing the RTA results for the data, showing where they agree, complement and elaborate on the findings obtained by the IQA and AT.

Setting

The study reported here was performed in the context of a project for training Israeli teachers to implement explorative instructional practices in middle school mathematics classrooms, using methods inspired by Smith & Stein's (2011) "Five Practices for Orchestrating Productive Mathematics Discussions". In this report, we focus on two teachers: Dani and Sivan. Dani was teaching a 7th grade classroom in a school serving a community of middle-high socio-economic background. Sivan was teaching an 8th grade classroom in a school serving a community mostly from a low-middle socio-economic background. Both teachers participated in training sessions where the instructor planned together with each of them separately a lesson according to the "5 Practices". In both cases, the lesson centered around an identical task: the Hexagon Task. The main session in the task was to write description that could be used to compute the perimeter of any train in the pattern of hexagons (See Figure 1):

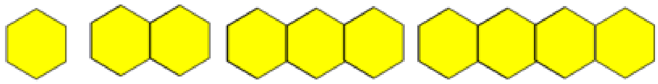


Figure 1: The Hexagons Pattern

The reason this task was used, was that it has proved in a previous study (Heyd-Metzuyanim et al., 2016) to be very productive for teachers who are beginning to implement the "5 Practices". We observed, video recorded, and transcribed both lessons. In addition, Dani and Sivan were both interviewed before and after the lessons, and their lesson planning sessions were recorded. In what follows, we present the IQA and AT measures of the two lessons, as well as what was still missing from them for a full understanding of the task implementation.

Findings

Accountable Talk in the two lessons. Both Dani and Sivan's lessons were conducted over a double period (90 Minutes) and both included work in groups (or pairs) where the teacher was walking between the groups, followed by a whole classroom discussion. The two whole classroom discussions took similar time (in Dani's classroom 28 minutes and in Sivan's lesson 26 minutes).

Overall, there were many more AT moves in Dani's lesson (98) than Sivan's (46). In particular, Dani's lesson had much more student talk moves coded as AT moves, either as student agree/disagree (N_Dani=22, N_Sivan=0), or as student justifications (N_Dani=20, N_Sivan=11). Dani was also higher than Sivan in pressing for students' reasoning (N_Dani=23, N_Sivan = 14). The overall picture drawn from the AT measure is, thus, that Dani's lesson had more accountability to reasoning and to the community than Sivan's lesson. Using the AT measure alone, however, does not enable learning about what mathematical concepts were dealt with, and which mathematical ideas surfaced through the discussion.

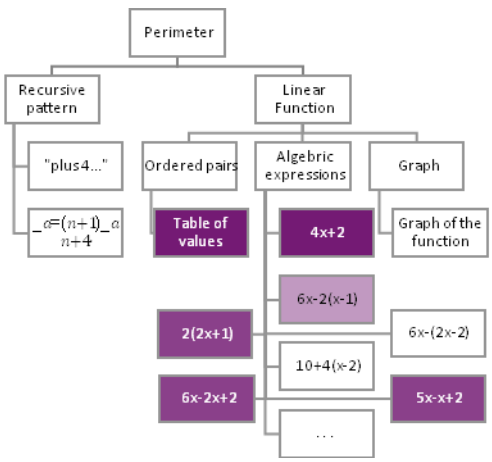
IQA scoring of two lessons. According to the IQA, Dani's lesson got higher scores then Sivan's lesson on all the rubrics, except the potential of the task, which was given in both cases by the teachers' trainer. In the Implementation rubric, we scored Dani's lesson as a 4, since multiple solutions were found and presented by the students; the teacher did not lead the students towards any particular solution; solutions were linked to each other both by the teacher and by the students; and there was no proceduralization of the task. In contrast, Sivan's implementation scored a 2. Though students generalized the Hexagon pattern into a expression, this was not done through the visual Hexagon's representation, only through the table; connections were not made with other algebraic expressions; in particular, students seemed to be well rehearsed in producing a table, algebraic expression from it and a graph of that expression, thus the task was proceduralized.

In the mathematical residue rubric, the results of the scoring were similar. Dani's lesson received a 4 since: the mathematical idea of equivalence of algebraic expressions was driven through the different algebraic solutions student presented. Evidence for students' understanding could be seen in one of the girls' exclamation "so they're all the same!" In contrast, Sivan's lesson scored a "2" on the mathematical residue rubric, since although the discussion dealt with some mathematical ideas, it did not touch upon the main idea behind the Hexagon task. The teacher did not focus on the different algebraic expressions but rather on the different representations of a linear function (graph, table and algebraic expression). However, as will be shown later, even this idea was not treated fully and appropriately.

Of all the Academic Rigor rubrics, we found the “Mathematical Residue[1]” most difficult to operationalize. It appeared Dani and Sivan had different ideas regarding the mathematical goals of their lessons and this had consequence for the way they led the lesson. While Dani seemed to be well aligned with the goal of showing the equivalence of algebraic expressions, Sivan seemed as though she was mostly aiming at ideas related to linear functions (which are, indeed, part of the 8th grade curriculum). We therefore searched for a tool that would aid in explicating the mathematical ideas explored in the two lessons. For this end, we developed the RTA.

Realization Tree Assessment tool

The first step in RTA is examining the task and explicating the *mathematical object(s)* that can be surfaced through engagement with the task. This includes the different realizations that are reasonable to expect from students at a certain grade level. In our case, we built our realization tree based on a lesson plan provided by the Institute for Learning (http://ifl.pitt.edu/index.php/educator_resources), where the different solutions, expected from middle schoolers for this task were drawn out. This produced a “blank” tree, with nodes as seen in Figures 2 and 3. We then proceeded to shade the tree nodes with four different colors, as follows: Shade no 4: the student's explanation was complete and accurate; Shade no. 3: the student’s explanation was not complete and accurate but the teacher helped explicating the idea; Shade no. 2: the student did not articulate the realization, but the teacher did; Shade no. 1: The realization was partially mentioned, but neither the student nor the teacher explained it fully.



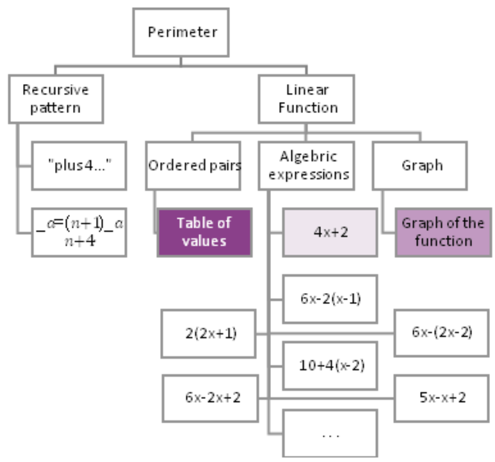


Figure 2: The RTA of Dani's lesson

Figure 3: The RTA of Sivan's lesson

[1] The Mathematical Residue rubric appears in our manual as “under development”.

Finally, if the realization was not mentioned at all, but was hypothesized to be relevant to the lesson and the grade level according to the lesson plan, it was shaded white (no. 0).

As can be seen in Figures 2 and 3, the main branch of our realization tree ("algebraic expression"), branches out on the multiple realizations of the algebraic expression . This, in accordance with the potential of the task to explore the different ways in which the visual representation of the hexagon sides can be generalized into a pattern and expressed algebraically.

Figure 2 describes the RTA for Dani's lesson. It shows that three realizations were explained fully and completely by students, three were explained by students, but the teacher filled in some gaps in these explanations, and one realization was explained only by the teacher. This full treatment of the "algebraic expressions" branch led students to endorsement of the narrative that "they all (all the algebraic expressions describing the pattern) equal to ", thus to the saming of different realizations, which was the goal of the lesson, as expressed both by Dani and by the teacher trainer.

In contrast, the RTA for Sivan's lesson (see Figure 3) is much lighter and sparser. It shows that only three realizations were treated in the lesson, and none of them was fully explained by the students. Moreover, the main branch of the tree – the "algebraic expressions" branch, is particularly empty. Only the realization was treated, and even that one was not explained accurately by the teacher or the students. The relative "emptiness" of Sivan's RTA corresponds well with the relatively low IQA and AT scores her lesson received. Still, it puzzled us, since Sivan was prepared in the PD very specifically for a lesson that was envisioned as similar to that of Dani. "What went wrong?", we asked ourselves. In order to answer that, we went back to the planning session, as well as to the post-lesson interview with Sivan, conducted right after the lesson. We found that, despite the PD instructor's conviction that she and Sivan were "on the same page", Sivan, in fact, had different goals for the lesson. She was focused on connecting the lesson to the previously learned unit on linear functions, where she had taught students to connect the concept of "slope" with the term "m" in $y = mx + n$, as well as connect it with the visual slope of a linear graph:

> "I wanted the students to see that every time it rises by four so that they will connect it with the slope that we have done with functions… I deliberately divided the board into three sections, to show the different stages in reaching the function itself - the graph that combines all the various representations of the function". (Sivan, Post-lesson interview)

It appears, then, that Sivan had a *different* mathematical object in mind (though probably only tacitly) when she planned the lesson – the "linear function" object[2]. Within the linear function object, the "slope" attribute of that object was her focus of attention. This could have been an appropriate goal for the lesson, had it been explicated and thought through. In particular, the following realization tree (see Figure 4) could have been appropriate for discussing slope and linear functions.

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[2] Though she named it inaccurately simply a "function", we understood from the context and from the curriculum she was referring to linear functions.

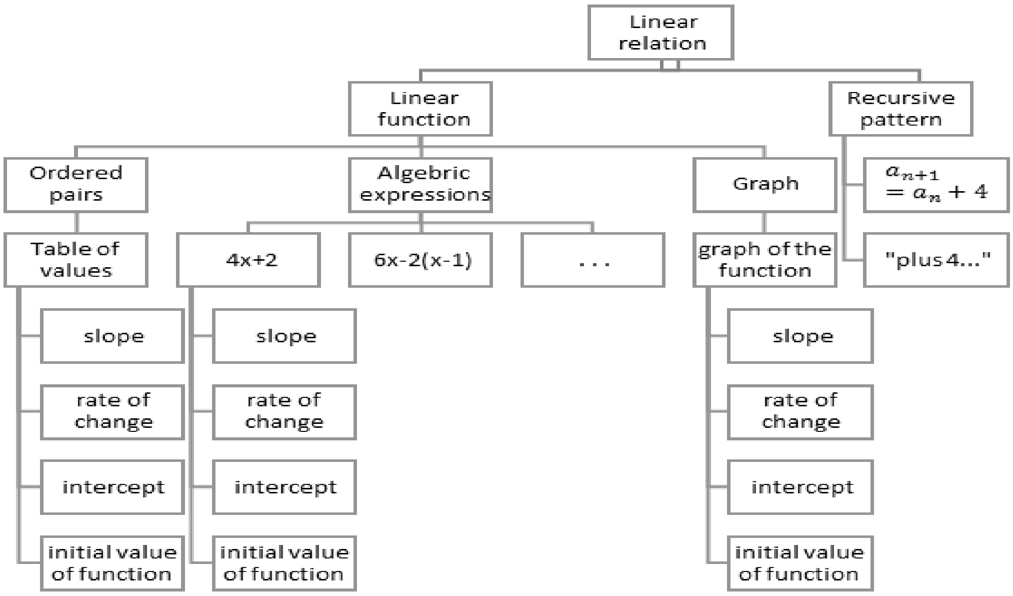

Figure 4: Alternative Realization Tree for discussing slopes and rate of change

However, the Hexagon task, especially as written for this lesson, was probably not the optimal task for talking about “slope”. This, since it depicts a situation where the function is discreet and cannot be described using a linear line. In practice, Sivan neglected very early the connection to the Hexagons drawing. Thus even the “rate of change” (which could have been visualized as the addition of four sides with the addition of each hexagon) was not connected to the “slope” on the graph.

Discussion

Our goal in the present report was to present a new analytical tool for the evaluation of mathematical lessons – the RTA. Though this tool does not give a numerical value such as scoring and “coding and counting” tools do, it still enables relatively easy qualitative comparisons between lessons. We have used this tool to enable comparison between two more lessons that were performed on the Hexagon task, and the results give a quick overview of the mathematical opportunities to learn in each lesson. The RTA can also serve as an aid for determining the quality of mathematical content (or “mathematical residue”) that is sought after in coarser grained assessment tools such as the IQA. In addition, the RTA can give us information about the potential of the task to engage students in explorative mathematical learning and about the relation between this potential and the actual implementation of the task in the classroom.

In the two cases reported here, the application of the RTA was done post-hoc, after the lessons were planned, implemented and recorded. However, we believe there is much potential for using this tool as an aid for planning lessons and training teachers for explorative mathematics instruction. Such a tool is particularly needed in light of previous findings which point to the difficulty of teachers to explicate to themselves the *mathematical goals* of the lesson (Heyd-Metzuyanim, Smith, Bill, & Resnick, submitted). We also believe that drawing realization trees with teachers will help them plan tasks and whole classroom discussions that provide sufficient opportunities for explorative participation. Often, when teachers talk about explorative instruction, their focus lies on the social or socio-mathematical norms of the classroom, such as students talking and listening to each other

(Heyd-Metzuyanim, Munter, & Greeno, submitted). We believe no less emphasis should be put on the nature of mathematical objects that students get exposed to, and on the paths for objectification that are opened through sufficiently rich mathematical discussions.

References

Boston, M. D. (2012a). Assessing Instructional Quality in Mathematics. *The Elementary School Journal*, *113*(1), 76–104.

Boston, M. D. (2012b). IQA Mathematics Lesson Observation Rubrics and Checklists.

Boston, M. D., & Smith, M. S. (2009). Transforming secondary mathematics teaching: Increasing the cognitive demands of instructional tasks used in teachers' classrooms. *Journal for Research in Mathematics Education*, *40*(2), 119–156.

Heyd-Metzuyanim, E., Smith, M., Bill, V. L., & Resnick, L. B. (2016). Change in Teachers' Practices Towards Explorative Instruction. In *In Proceedings of the 40th Conference of the International Group for the Psychology of Mathematics Education* (pp. 393–400).

Heyd-Metzuyanim, E., Smith, M., Bill, V. L., & Resnick, L. B. (submitted). Ritual to explorative participation: a lens for examining change in teachers' instructional practices over time.

Heyd-Metzuyanim , E., Munter, C., & Greeno, J. (submitted). Conflicting frames: A case of misalignment between professional development efforts and a teacher's practice in a high-school mathematics classroom.

Nachlieli, T., & Tabach, M. (2012). Growing mathematical objects in the classroom – The case of function. *International Journal of Educational Research*, *51-52*, 10–27.

O'Connor, M. C., & Michaels, S. (1993). Aligning Academic Task and Participation Status through Revoicing : Analysis of a Classroom Discourse Strategy. *Anthropology & Education Quarterly*, *24*(4), 318–335.

O'Connor, M. C., Michaels, S., & Chapin, S. H. (2015). "Scaling down" to explore the role of talk in learning: From district intervention to controlled classroom study. In L. B. Resnick, C. S. C. Asterhan, & S. N. Clarke (Eds.), *Socializing Intelligence Through Academic Talk and Dialogue* (pp. 111–126). AERA.

Resnick, L. B., Michales, S., & O'connor, M. C. (2010). How (well structured) talk builds the mind. In R. Stermberg & D. Preiss (Eds.), *Innovations in Educational Psychology: Perspectives on Learning, Teaching, and Human Development* (pp. 163–194). New York: Springer.

Schoenfeld, A. H. (2014). What Makes for Powerful Classrooms, and How Can We Support Teachers in Creating Them? A Story of Research and Practice, Productively Intertwined. *Educational Researcher*, *43*(8), 404–412.

Schoenfeld, A. H. (2015). Thoughts on scale. *ZDM Mathematics Education*, *47*(1), 161–169.

Sfard, A. (2008). *Thinking as communicating*. New York: Cambridge University Press.

Stein, M. K., Grover, B. W., & Henningsen, M. (1996). Building Student Capacity for Mathematical Thinking and Reasoning: An Analysis of Mathematical Tasks Used in Reform Classrooms. *American Educational Research Journal*, *33*(2), 455–488.

Stein, M. K., & Smith, M. S. (2011). *5 Practices for Orchestrating Productive Mathematics Discussions*. National Council of Teachers of Mathematics. Reston, VA: NCTM.