Pre-service elementary teachers' knowledge of comparing decimals based on the anthropological theory of the didactic

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University of Copenhagen, Department of Science Education, Copenhagen, Denmark

University of Riau, Faculty of Education and Teacher Training, Pekanbaru, Indonesia; zetra.putra@ind.ku.dk

In this paper, the idea of hypothetical teacher task (HTT), designed and analysed using the anthropological theory of the didactic (ATD), was presented to study pre-service elementary teachers' (PsETs) mathematical and didactical knowledge of comparing decimals. This study is part of the author's PhD project about PsETs' knowledge of rational numbers. The subjects for this study were 32 fourth year PsETs from University of Riau, Indonesia. The study illustrates how HTT can be useful as an alternative method to investigate PsETs' knowledge through praxeological reference models.

Keywords: Anthropological theory of the didactic, praxeologies, hypothetical teacher tasks, mathematical and didactical knowledge.

Introduction

The results from the Programme for International Student Assessment (PISA) in 2015 ranked the performance of Indonesian pupils 62 out of 70 countries (OECD, 2015). Most pupils were only able to solve problems directly related to the routine procedures (mostly at level 1 and 2 in the PISA framework). These results reflect how they learned mathematics at schools, and this situation raises a question about the knowledge of teachers as the main support for the success of pupils' learning: namely, how teachers' pedagogical content knowledge (Kuntur et al., 2013) and mathematical knowledge for teaching (Hill, Rowan, & Ball, 2005) significantly affect pupils' achievement.

Many studies have been conducted on teachers' knowledge concerning specific mathematical topics (Ma, 1999), including international comparative studies of teachers' knowledge (Tatto et al., 2008). Ma (1999) studied teachers' performance about rational numbers, especially on calculations and representations of division of fractions. She evaluated teachers' knowledge through posing two tasks: to compute and to represent meaning for the resulting mathematical sentences. Meanwhile, the Teacher Education and Development Study in Mathematics (TEDS-M) studied teachers' knowledge through questionnaires (Tatto et al., 2008). TEDS-M used three question formats: multiple-choice, complex multiple-choice, and open constructed-response. TEDS-M argued that only the third format allows teachers to demonstrate the depth of their thinking on mathematical knowledge and mathematical teaching knowledge. However, both studies share the focus on individual teachers' knowledge through written tests. This method is commonly used by other studies and sometimes followed by an individual interview of selected teachers.

Teachers' knowledge can also be studied through different approaches or methods. One possible approach is to design open constructed tasks based on pupils' difficulties and misconceptions. The tasks can be proposed to teachers both individually and collectively. The focus of this study is on designing a model for teachers' shared mathematical and didactical knowledge of rational numbers based on the *anthropological theory of the didactic* (ATD), specifically on the notion of *praxeology* (Chevallard, 2006). I focus on rational numbers because they constitute one of the most difficult topics for elementary and secondary teachers (Depaepe et al., 2015). Teaching this topic requires relevant knowledge of teachers to properly deal with pupils' difficulties. I use the notion of praxeology to model teachers' knowledge. Durand-Guerrier, Winsløw, and Yoshida (2010), and Winsløw and Durand-Guerrier (2007) have developed a tool based on this notion to investigate teachers' specific mathematical and didactical knowledge that is known as *hypothetical teacher task* (HTT). In my larger study, the focus is on designing HTT about rational numbers that can investigate not only pre-service elementary teachers' (PsETs) individual knowledge but also the collective one. This paper presents a case study of comparing decimals as a part of my PhD project about PsETs' knowledge of rational numbers. The research questions that drive this paper are: how can HTT on comparing decimals function to study PsETs' mathematical and didactical knowledge? What praxeologies, specifically mathematical and didactical techniques, are shared by Indonesian PsETs related to comparing decimals?

Teachers' knowledge and the anthropological theory of the didactic (ATD)

Many studies about teachers' knowledge refer to content knowledge and pedagogy content knowledge introduced by Shulman (1986). These notions also have influenced several later studies on mathematics teacher education (Hill, et al., 2005; Ma, 1999; Winsløw & Durand-Guerrier, 2007). Winsløw and Durand-Guerrier (2007) identified three components of teachers' knowledge: content knowledge (mathematical techniques, theories etc.), pedagogical knowledge (concerning education, learning and teaching in general), and didactical knowledge (regarding the conditions and mechanisms of mathematics teaching and learning, often quite specific to the content taught).

To study teachers' knowledge, ATD provides an epistemological tool to describe and analyse mathematical and didactical knowledge as human activities among others (Chevallard, 2006). In fact, ATD holds, as a central assumption, that any knowledge, including teachers' knowledge, can be investigated in term of a praxeology. I use this notion as a framework to study teachers' mathematical and didactical knowledge of comparing decimals.

A praxeology consists of two main interrelated components: *praxis* (practical block) and *logos* (theoretical block). Both the practical and theoretical block of a praxeology are divided into two elements. The practical block is made of a *type of tasks* (**T**) and corresponding *techniques* ($\boldsymbol{\tau}$) which apply to accomplish tasks of type **T**. An example of a type of mathematical tasks (**T**) is to compare two given decimal numbers. To solve this task, a technique ($\boldsymbol{\tau}$) is needed; for instance, one can change both decimals into fractions with a common denominator, and then compare numerators. The theoretical block is made of *technologies* (θ) and *theories* (Θ). A technology (θ) is a discourse used to explain and justify the techniques ($\boldsymbol{\tau}$), while a theory (Θ) explains and justifies the technology (θ). An example of technologies is an explanation of available methods to decide which of two different given decimals is greater, when the methods work or are more efficient, etc. The order structure of rational numbers is a mathematical theory (Θ) which can be used to justify and explain the technology (θ).

A praxeology is not only used to describe mathematical knowledge but also didactical knowledge (i.e. knowledge about teaching that depends on what is taught). The praxeology used to describe didactical knowledge is known as a *didactical praxeology*. Like a mathematical praxeology, didactical praxeology includes a type of didactical tasks, didactical techniques, didactical technologies and theories (Rodríguez, Bosch & Gascón, 2008). The didactical praxeology is thus closely related to the mathematical praxeology because didactical praxeology is about tasks related to the teaching of the mathematical praxeology. An example of a type of didactical tasks is to teach pupils how to compare two decimals. A didactical technique is to present directly a mathematical technique for comparing two decimals and then ask pupils to apply this technique for other similar mathematical tasks. A technological discourse to justify this didactical technique is an assumption that pupils might learn better if they get the correct method from the teacher. This may even derive from a more general didactic theory, favouring direct instruction in general.

Methodology: Design of hypothetical teacher task (HTT)

The notion of HTT was introduced by Durand-Guerrier et al. (2010) and Winsløw and Durrand-Guerrier (2007) to investigate pre-service lower secondary teachers' knowledge. HTT consists of mathematical and didactical tasks for teachers. The mathematical task is one that is problematic to pupils in the hypothetical situation, often related to some common misconceptions. Teachers have to analyse this task and provide some mathematical techniques. They work individually for this task and then share their ideas for the discussion on the didactical task. The didactical task asks, with variations depending on the situation described, what could be done to further pupils' overcoming of particular difficulties with the mathematical task. So the didactical task strongly relates to the mathematical task.

The HTT about comparing decimals was designed based on known misconceptions related to place value (Irwin, 2001). As an example, pupils may argue that 0.15 is greater than 0.2 because 0.15 is longer than 0.2 or 15 is greater than 2. Beginning with a situation where pupils hold such views, the HTT reads as follows:

Fifth grade pupils are asked to compare the size of 0.5 and 0.45. Some pupils answer that 0.45 is greater than 0.5, while others say that 0.5 is greater than 0.45.

a. Analyse the pupils' answers. Explain your ideas to handle the situation in this class? (*to be solved individually in 3 minutes*)

b. How do you use this situation to further the pupils' learning? (*to be discussed and solved in pairs within 5 minutes*)

Figure 1: HTT about comparing two decimals

The HTT was originally written by the author in English, and then it was translated into Indonesian. Two Indonesian researchers checked the translations for consistency. The HTT was also piloted with a pair of recently graduated students from the Elementary School Teacher Education (ESTE) study program at University of Riau, Indonesia. I asked for the students' comments and used them to revise the HTT. The data consist of PsETs' written answers for the first question and video recording of the discussion for the second question. I transcribed the video recording for all groups using the NVivo computer program. Then, the written answers and video transcripts were analysed based on the mathematical and didactical praxeologies, to identify the techniques produced. The subjects for the implementation of HTT were 32 (16 pairs) fourth year PsETs from the ESTE study program, and the data were collected in March 2016. All participants wrote their answers on the worksheets for the individual question a, and then they used their answers to support a common discussion for the question b. A more comprehensive analysis of these data was based on the techniques identified among individual pairs.

Praxeological reference models

In the first phase of analysis, I focus on the practical blocks (i.e. types of tasks and techniques). The mathematical task ($\mathbf{T_m}$) contained in the HTT (Figure 1) can be stated as follows:
$\mathbf{T_m}$: given two different decimal numbers, $0 < a < 1$, and $0 < b < 1$, decide if $a > b$ or $a < b$.

There are many possible mathematical techniques to solve a mathematical task of type $\mathbf{T_m}$ which could be developed by the PsETs individually, or during their discussion. I describe some of them in the following table:

Code of techniques	General description of techniques
$\boldsymbol{\tau_1}$	Change a and b into integers, multiplying by an appropriate power of ten.
$\boldsymbol{\tau_2}$	Use lexicographical orders to compare the decimals.
$\boldsymbol{\tau_3}$	Add 0 digits where required to get the same number of digits in both decimals.
$\boldsymbol{\tau_4}$	Change decimals into fractions with a common denominator and compare the numerators.
$\boldsymbol{\tau_5}$	Subtract b from a or divide a by b. When the result is less than 0 (for subtraction) or less than 1 (for division), $a < b$, otherwise $a > b$.

Table 1: Mathematical techniques for a mathematical task of type $\mathbf{T_m}$

In addition, there are several possible mathematical techniques based on diagrammatical representations and number lines. For instance, one can represent both decimals by a rectangle or a circle diagram and then compare areas or sizes ($\boldsymbol{\tau_6}$), or locate both decimals on a number line and compare the positions ($\boldsymbol{\tau_7}$). Furthermore, to each correct mathematical technique, one might associate with one or more incorrect mathematical techniques. For example, when someone multiplies both decimals with different powers of ten, one does a similar but an incorrect mathematical technique of $\boldsymbol{\tau_1}$. This mathematical technique is denoted as $\boldsymbol{\tau_1}^-$, where the minus means "incorrect variation of $\boldsymbol{\tau_1}$". Hence, there will be at least a similar number of incorrect mathematical techniques to the correct ones.

The question b and also part of question a contain a didactical task ($\mathbf{T_d}$) as follows:
$\mathbf{T_d}$: given that pupils' answers as stated to a task of type $\mathbf{T_m}$, determine what to do as a teacher to facilitate pupils' learning.

Most didactical techniques to solve $\mathbf{T_d}$ relate to the mathematical techniques proposed to solve the task of type $\mathbf{T_m}$. When PsETs recommend teaching pupils by simply explaining a mathematical technique, for instance $\boldsymbol{\tau_1}$, this technique is coded as $\boldsymbol{\tau_1}^*$, so similar numbers of didactical techniques can be derived from the previous mathematical techniques. In addition, some didactical techniques can be variants of those didactical techniques. For instance, PsETs provide pupils with similar problems, such as comparing 0.5 and 0.25, they choose these decimals because pupils might simply recognise both decimals as a half and a quarter, and may then realise their original mistake. Many other possible didactical techniques might appear during the discussion, but space does not allow me to describe them in detail here. One common didactical technique is to build the mathematical task into a real word problem. PsETs may even say that the mathematical task presented in the HTT is too abstract to pupils, so they need to present it within a more familiar situation. Such a justification furnishes a technological discourse for the didactical technique, could conceivably even invoke a didactic theory.

Results

The analysis of answers to the task of type $\mathbf{T_m}$ was mainly based on the PsETs' written solutions, but I also looked at the video transcripts when I found some difficulties in categorising the mathematical techniques from the written solutions. In general, almost all mathematical techniques described in the reference models appeared in PsETs' written answers, but some techniques were more common than others. The mathematical techniques presented by PsETs are summarised in the following table:

Mathematical Techniques	τ_1	τ_1^-	τ_2	τ_3	τ_4	τ_4^-	τ_5	τ_6	τ_6^-	τ_7	τ_7^-	**N/A**	**Total**
Number of Answers	2	1	2	10	6	5	1	1	1	3	2	1	35

Table 2: A summary of PsETs' mathematical techniques for the task of type $\mathbf{T_m}$

The most common mathematical techniques were adding 0s to equalise the number digits after the decimal point (τ_3) and changing decimals into fractions (τ_4) (Table 2). But when changing decimals into fractions, five PsETs could not change 0.45 into a fraction. One PsET said during the discussion: "*We can change decimals into fractions, but I do not know how to change 0.45 into a fraction*". Among six PsETs who gave a correct mathematical technique of τ_4, only two PsETs changed the fractions to have a common denominator and then compared numerators, whereas the others presented both decimals into simple fractions and compared intuitively. Five PsETs also provided the mathematical technique of representing decimals on a number line, but two of them placed the numbers in incorrect positions on the number line. One of these PsETs stated on her worksheet that she agreed 0.5 was greater than 0.45, but still represented the decimal numbers incorrectly on the number line (Figure 1). Another finding was that a PsET answered that 0.5 is greater than 0.45, but she could not represent 0.45 correctly as a shaded portion of a circle. Overall, only 71% of the mathematical techniques presented by the PsETs are correct.

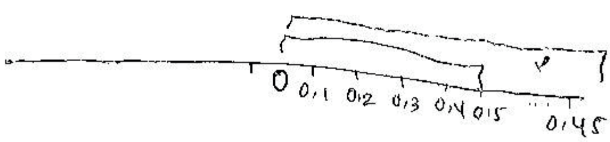

Figure 1: A PsET's incorrect number line representation of decimals

The total number of didactical techniques proposed by PsETs is greater that the number of those mathematical techniques because some pairs presented more than one didactical technique during their discussion. The most common didactical technique was a direct instruction of pupils based on how PsETs themselves solved the pupils' task of type $\mathbf{T_m}$. For instance, eight pairs said that they would instruct pupils to add 0 after 0.5 and compare the result to 0.45 ($\boldsymbol{\tau_3}^*$), and seven pairs discussed direct instruction of the mathematic technique $\boldsymbol{\tau_4}$, while three of these pairs could not change 0.45 into a fraction. The didactical technique related to number line representations was also discussed by eight pairs of PsETs, but two of them placed 0.45 incorrectly in relation to 0.5. For example, the following discussion shows how two PsETs shared their incorrect mathematical techniques $\boldsymbol{\tau_4}^-$ and $\boldsymbol{\tau_7}^-$ in order to produce possible didactical techniques.

PsET A: Let's use a number line. Here is 0, and here is 0.1; 0.2. (She explained her drawing presented in Figure 1.)

PsET B: And so on.

PsET A: So, 0.5 is greater than 0.45.

PsET B: How can we know that 0.5 is greater than 0.45? I thought, using your number line, that one is greater than the other.

PsET A: How do you think?

PsET B: I am confused. I change them into fractions. From fractions, they can be represented in rectangle diagrams, so we can see them. For instance, we know that 0.5 is equal to a half.

PsET A: Hmm.

PsET B: If this is 0.45, what fraction is it? Later, it is drawn. From the drawing, pupils can compare, to see which one is greater.

From the discussion, PsET B might realise that her partner placed the two decimals incorrectly on the number line, but she did not have any idea on how to fix it. Instead, she proposed to change decimals into fractions and then suggested to represent the fractions into rectangle diagrams. However, it turned out that they could not change 0.45 into a fraction or represent it by a correct rectangle diagram. They appeared to lack a general technique to convert decimals into fractions.

In addition, five pairs suggested explaining to pupils how to change decimals into percentages, but three of them were in fact unable to do so correctly. For example, one PsET presented to his partner the mathematical technique of changing decimals into fractions. He changed 0.5 into 5/100 or 500%, but no-one realised the mistake. Furthermore, some PsETs also considered presenting the mathematical task into a contextual or real life problem, providing other decimal comparison problems, or giving some technological elements, such as writing 0s after the decimal point is rarely written but may be useful. In general, twelve pairs suggested reasonable didactical techniques, most of the techniques being classified as direct instruction of mathematical techniques. Two pairs suggested both reasonable and unreasonable didactical techniques, and the other two totally could not recommend any didactical technique.

Discussion and further remarks

An important point for this study is to explore the idea of HTT as an alternative method to investigate PsETs' mathematical and didactical knowledge of comparing decimals. This method asks PsETs to demonstrate their collective development of mathematical and didactical knowledge as they solve the task because the design of tasks involves open constructed-responses and conversations among pairs of informants. This situation challenges PsETs to produce more than a single technique for each task. They shared their mathematical knowledge to provide didactical techniques for further pupil learning through a collaborative effort (Question b). This method is quite different from a diagnostic test in which PsETs' knowledge is measured through a single correct answer, such as multiple-choice or complex multiple-choice questions in the TEDS-M study (Tatto et al., 2008). It is also different in that teachers' didactical logos is developed in discussion with a peer.

The most common mathematical technique shared by PsETs was to put 0s after numbers behind the comma to equalise the number of digits for both decimals ($\boldsymbol{\tau}_3$). This mathematical technique can be simply applied by PsETs because it reduces the comparison to the more familiar task of comparing two integers. The technique is valid for comparing two decimal numbers in [0,1], but it does not work as immediately in other cases; so it is a more limited technique than, for instance, $\boldsymbol{\tau}_4$.

When PsETs discuss how they might handle the didactical task, they tend to just explain, based on their mathematical techniques, how to solve the mathematical task. In fact, when they have an inappropriate mathematical technique for the mathematical task, they then struggle to provide an appropriate didactical technique during the discussion. With subtle didactical techniques in mind, they could conceivably realise their mathematical mistake; unfortunately, this was not observed in any case.

Finally, I conclude this study with two remarks. First, the mathematical task designed in the HTT did not involve a contextual or real life situation. Such a situation could both facilitate and add to the difficulty of the HTT, and variations of this type would be interesting to investigate. The second one is related to the PsETs' collective discussion on didactical techniques. I expected that they could resolve their difficulties in constructing didactical techniques during their discussion in pairs, but some could not do that because none of them had an adequate mathematical technique for the first part. Therefore, the such problematic HTT may become a useful subject for a classroom discussion in the teacher education program in order to overcome both the PsETs' own mathematical misconception and construct didactical techniques for their future tasks as teachers.

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