The equivalence between many-to-one polygraphs and opetopic sets

Cédric Ho Thanh

THE EQUIVALENCE BETWEEN MANY-TO-ONE POLYGRAPHS AND OPETOPIC SETS

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Abstract. From the polynomial approach to the definition of opetopes of Kock et al., we derive a category of opetopes, and show that its set-valued presheaves, or opetopic sets, are equivalent to many-to-one polygraphs. As an immediate corollary, we establish that opetopic sets are equivalent to multitopic sets, introduced and studied by Harnick et al, and we also address an open question of Henry.

Contents

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1. Introduction

Opetopes were originally introduced by Baez and Dolan in [1] as an algebraic structure to describe compositions and coherence laws in weak higher dimensional categories. They differ from other shapes (such as globular or simplicial) by their (higher) tree structure, giving them the informal designation of "many-to-one". Pasting opetopes give rise to opetopes of higher dimension (it is in fact how they are defined!), and the analogy between opetopes and cells in a free higher category starts to emerge. On the other hand, polygraphs (also called computads) are higher dimensional directed graphs used to generate free higher categories by specifying generators and the way they may be pasted together (by means of source and targets).

In this paper, we relate opetopes and polygraphs in a direct way. Namely, we define a category $\mathbb{O}$ whose objects are opetopes, in such a way that the category of its $\mathcal{S}$et-valued presheaves, or opetopic sets, is equivalent to the category of many-to-one polygraphs. This equivalence was already known from [7, 8, 10], however the proof is very indirect. The recent work of Henry [9] showed the category of many-to-one polygraphs (among many others) to be a presheaf category, but left the equivalence between "opetopic plexes" (serving as shapes for many-to-one polygraphs in his paper) and opetopes open. We establish this in our present work.

The notion of multitope [11, 8] is related to that of opetope, and has been developed based on similar motivations. However the approaches used are different: *ope*topes are based on *ope*rads [14], while *multi*topes are based on *multi*categories. It is known that multitopic sets are equivalent to many-to-one polygraphs [8, 7], and thus together with our present contribution, we obtain an equivalence between multitopic sets and opetopic sets.

We begin by recalling elements of the theory of polygraphs and polynomial trees in section 2, and of the theory of polygraphs in section 3. We then give the definition of polynomial opetopes from [13] in section 4. Lastly, we outline the proof of the equivalence in section 5, by introducing the "opetal" functor $O[-] : \mathbb{O} \longrightarrow \mathcal{P}\mathrm{ol}^{\triangledown}$ from opetopes to many-to-one polygraphs, and the auxiliary notion of shape of a generator in a many-to-one polygraph.

2. Polynomial trees

We give elements of the theory of polynomial functors and polynomial trees, and point the reader to e.g. [12] for a more detailed reference.

2.1. Trees. A *polynomial endofunctor*[1] F is a $\mathcal{S}$et-diagram of the form:

$$I \xleftarrow{s} E \xrightarrow{p} B \xrightarrow{t} I. \tag{2.1}$$

Elements of B are called *nodes*, elements of the fiber $p^{-1}(b)$ are the *inputs* of b, and elements of I are *colors*. For $b \in B$, let $E(b) := p^{-1}b$, and if $e \in E(b)$, let $s_e(b) := s(e)$. We will sometimes refer to I, B and E as F_0, F_1, and F_2, respectively.

[1] The denomination "functor" comes from the fact that such a diagram induces a functor $\mathcal{S}\mathrm{et}/I \xrightarrow{s^*} \mathcal{S}\mathrm{et}/E \xrightarrow{\Pi_p} \mathcal{S}\mathrm{et}/B \xrightarrow{\Sigma_t} \mathcal{S}\mathrm{et}/I$ by composition of the pullback along s, dependent product along p, and dependent sum along t, respectively.

A morphism $f : F \longrightarrow F'$ of polynomial functors is a diagram of the form

$$\begin{tikzcd} & E \arrow[r] \arrow[dl] \arrow[d, "f_2"'] \arrow[dr, phantom, "\lrcorner" very near start] & B \arrow[d, "f_1"] \arrow[dr] & \\ I \arrow[d, "f_0"'] & E' \arrow[r] \arrow[dl] & B' \arrow[dr] & I \arrow[d, "f_0"] \\ I' & & & I', \end{tikzcd}$$

where the middle square is cartesian. We call $\mathcal{P}\mathrm{oly}\mathcal{E}\mathrm{nd}$ the category of polynomial endofunctors and morphisms.

A polynomial functor $T = \left(T_0 \xleftarrow{s} T_2 \xrightarrow{p} T_1 \xrightarrow{t} T_0\right)$ is a *polynomial tree* [12] if:

(1) the sets T_0, T_1 and T_2 are finite (in particular, each node has finitely many inputs);
(2) the map t is injective;
(3) the map s is injective, and the complement of its image $T_0 - \operatorname{im} s$ consists of a single element, called the *root*;
(4) let $T_0 = T_2 + \{r\}$, with r the root, and define the *walk-to-root* function σ by $\sigma(r) = r$, and otherwise $\sigma(e) = tp(e)$; we ask that for all $x \in T_0$, there exists $k \in \mathbb{N}$ such that $\sigma^k(x) = r$.

Let $\mathcal{T}\mathrm{ree}$ be the full subcategory of $\mathcal{P}\mathrm{oly}\mathcal{E}\mathrm{nd}$ consisting of trees. A *morphism of trees* is simply a morphism in $\mathcal{T}\mathrm{ree}$, i.e. a morphism of polynomial functor between two trees. We sometimes refer to the colors of a polynomial tree as *edges*.

Proposition 2.2 ([12]). *Morphisms in $\mathcal{T}\mathrm{ree}$ are embeddings, i.e. if $f : T \longrightarrow U$ is a tree morphism, then $f_i : T_i \longrightarrow U_i$ is injective, for $i = 0, 1, 2$.*

Let $F \in \mathcal{P}\mathrm{oly}\mathcal{E}\mathrm{nd}$. Define the category of F-trees $\operatorname{tr} F$ to be a chosen skeleton of the slice $\mathcal{T}\mathrm{ree}/F$. Then $T \in \operatorname{tr} F$ corresponds to a (isomorphism class of) morphism from a tree to F, and we shall denote that tree by $\langle T \rangle$ so that $T : \langle T \rangle \longrightarrow F$. We point out that in the latter case, $\langle T \rangle_1$ is the set of nodes of $\langle T \rangle$, while $T_1 : \langle T \rangle_1 \longrightarrow F_1$ is a map of sets, and likewise for $i = 0, 2$. Nodes of $\langle T \rangle$ are thought of as "decorated" by B via T, and likewise for edges.

If $f : F \longrightarrow G$ is a morphism of polynomial endofunctors, then it induces an obvious functor $f_* : \operatorname{tr} F \longrightarrow \operatorname{tr} G$ by postcomposition[2].

2.2. **Addresses.** Let $T \in \mathcal{T}\mathrm{ree}$ be a polynomial tree, and let σ be its walk-to-root function. We define the *address* function & on edges as follows:

(1) if r is the root edge, let $\& r := [\varepsilon]$,
(2) if $e \in T_0 - \{r\}$, write $\& \sigma(e) = [x]$, and define $\& e := [xe]$.

This extends to an address function on nodes: if $b \in T_1$, let its address be $\& b := \& t(b)$. Let $T^\bullet$ be the set of node addresses of T. A *leaf* is an edge $e \in T_0$ that is not the target of any node, i.e. there is no $b \in T_1$ such that $t(b) = e$, and let $T^|$ be the set of leaf addresses of T.

Assume now that $T : \langle T \rangle \longrightarrow F$ is an F-tree, for F a polynomial endofunctor as in equation (2.1). If $b \in \langle T \rangle_1$ has address $\& b = [p]$, write $\mathsf{s}_{[p]} T := T_1(b)$. For convenience, we let $T^\bullet := \langle T \rangle^\bullet$, and $T^| := \langle T \rangle^|$.

[2]This kind of functor between slices is often called a dependent sum.

The *prefix order* on $T^\bullet$ and $T^|$ is the minimal order such that $[\varepsilon]$ is the minimal element, and such that the right concatenation maps are increasing. If each fiber $\langle T \rangle_2 (b)$ is ordered (as will often be the case in the sequel), write $\leq$ the lexicographical order on $T^\bullet$ and $T^|$. If the fibers $\langle T \rangle_2 (b)$ do not have a preferred ordering, we let $\leq$ be the prefix order, which still gives a sense of lexicographical ordering, despite not being total.

2.3. **Grafting.** Let F be a polynomial endofunctor as in equation (2.1). For $i \in I$, define $\mathsf{I}_i \in \operatorname{tr} F$ as having underlying tree

$$\{*\} \longleftarrow \varnothing \longrightarrow \varnothing \longrightarrow \{*\},$$

and where I_i maps the unique edge $*$ to i. This corresponds to a tree with no node and a unique edge, decorated by i. Let now $b \in B$, and assume that its set of input $E(b)$ is implicitly totally ordered [3]. Write $n = \#E(b)$, and define $\mathsf{Y}_b \in \operatorname{tr} F$, the *corolla* at b, as having underlying tree

$$\mathbf{n} + \{*\} \longleftarrow \mathbf{n} \longrightarrow \{*\} \longrightarrow \mathbf{n} + \{*\},$$

where $\mathbf{n} := \{1, \ldots, n\}$, and where Y_b maps the only node $*$ to b, and is increasing on the set of inputs. This corresponds to a tree with a unique node, decorated by b.

For $T \in \operatorname{tr} F$, giving a morphism $\mathsf{I}_i \longrightarrow T$ is equivalent to specifying the address $[p]$ of an edge address of T decorated by i. Likewise, morphisms of the form $\mathsf{Y}_b \longrightarrow T$ are in bijection with addresses of nodes of T decorated by b.

For $S, T \in \operatorname{tr} F$, $[l] \in S^|$ such that the leaf of S at $[l]$ and the root of T are decorated by the same $i \in I$, define the *grafting* $S \circ_{[l]} T$ of S and T on $[l]$ by the following pushout:

$$\begin{tikzcd}
\mathsf{I}_i \arrow[r, "{[\varepsilon]}"] \arrow[d, "{[l]}"'] & T \arrow[d] \\
S \arrow[r] & S \underset{[l]}{\circ} T. \arrow[ul, phantom, "\ulcorner", very near start]
\end{tikzcd} \tag{2.3}$$

Proposition 2.4 ([12]). *(1) Every F-tree is either of the form I_i, for some $i \in I$, or obtained by iterated graftings of corollas.*
(2) If $f : F \longrightarrow G$ is a morphism of polynomial endofunctors, then $f_* : \operatorname{tr} F \longrightarrow \operatorname{tr} G$ preserves graftings.

We denote by $\operatorname{tr}^| F$ the set of F-trees with a marked leaf. Similarly, we denote by $\operatorname{tr}^\bullet F$ the set of F-trees with a marked node.

Take $T, U_1, \ldots, U_k \in \operatorname{tr} F$, where the leaves of T are $[l_1], \ldots, [l_k]$, and assume the grafting $T \circ_{[l_i]} U_i$ is defined for all i. Then the *total grafting* will be denoted concisely by

$$T \underset{[l_i]}{\bigcirc} U_i = (\cdots (T \underset{[l_1]}{\circ} U_1) \underset{[l_2]}{\circ} U_2 \cdots) \underset{[l_k]}{\circ} U_k.$$

It is easy to see that the result does not depend on the order in which the graftings are performed.

[3] In the sequel, this order will be the lexicographical order $\leq$.

2.4. **Tree contexts.** For a polynomial endofunctor $F \in \mathcal{P}\mathrm{oly}\mathcal{E}\mathrm{nd}$ as in equation (2.1), a context over F is a tree $C = C[\square]$ over the extended functor

$$I \xleftarrow{s} E + E' \xrightarrow{p} B + \{\square\} \xrightarrow{t} I \tag{2.5}$$

for a chosen fiber $E' \xrightarrow{s} I$ of $\square$, and a value of $t\square \in I$, such that exactly one node of C is decorated by $\square$, i.e. such that there exists a unique $[p] \in C^\bullet$ with $\mathsf{s}_{[p]} C = \square$. Likewise, a bicontext $D[\square, \square]$ is a tree over (2.5) where the $\square$ decoration occurs exactly twice.

If T is a F-tree or another F-context (over a possibly different box symbol), parallel to $\square$ (i.e. endowed with a bijection $\wp$ over I between the leaves $[l_1], \ldots, [l_k]$ of T and $E(\square)$), then we define $C[T]$ to be C where $\square$ has been replaced by T: for C as on the left, the substitution $C[T]$ is given as on the right

$$C = A \underset{[p]}{\circ} \left(\square \bigcirc_{\wp[l_i]} B_i \right) \qquad \Longrightarrow \qquad C[T] \coloneqq A \underset{[p]}{\circ} \left(T \bigcirc_{[l_i]} B_i \right).$$

2.5. **The polynomial Baez–Dolan construction.**

2.5.1. *Free polynomial monads.* A polynomial monad is a strong cartesian monad whose underlying endofunctor is polynomial. Equivalently, a polynomial F as in (2.1) is a polynomial monad if it is endowed with a unit $\eta : B \longrightarrow \operatorname{tr} F$ and a partial law $\mu : E \times_I B \longrightarrow B$, subject to adequate laws [5, 13]. We shall write $\mathcal{P}\mathrm{oly}\mathcal{M}\mathrm{nd}$ the category of polynomial monads and morphisms of polynomial functors that are also morphisms of monads. Any polynomial endofunctor F as in equation (2.1) admits a free polynomial monad $F^\star$, whose underlying polynomial endofunctor is given by

$$I \xleftarrow{s} \operatorname{tr}^| F \xrightarrow{p} \operatorname{tr} F \xrightarrow{t} I \tag{2.6}$$

where s maps an F-tree with a marked leaf to the decoration of that leaf, p forgets the marking, and t maps a tree to the decoration of its root. Remark that for $T \in \operatorname{tr} F$ we have $p^{-1} T = T^|$.

Theorem 2.7 ([12], [13])**.** *The polynomial functor $F^\star$ has a canonical structure of polynomial monad. Moreover, the $(-)^\star$ construction extends as a functor that is left adjoint to the forgetful functor $\mathcal{P}\mathrm{oly}\mathcal{M}\mathrm{nd} \longrightarrow \mathcal{P}\mathrm{oly}\mathcal{E}\mathrm{nd}$.*

(sketch). The unit $\mathsf{Y}_{(-)} : F \longrightarrow F^\star$ maps $b \in B$ to the corolla $\mathsf{Y}_b \in \operatorname{tr} F$, and an element $e \in E(b)$ to the leaf $[e]$ of Y_b. Let $((T, [l]), S) \in \operatorname{tr}^| F \times_I \operatorname{tr} F$, so that the leaf of T at address $[l]$ has the same decoration as the root edge of S. The partial law $\circ$ of $F^\star$ maps $((T, [l]), S)$ to $T \circ_{[l]} S$. $\square$

The adjunction $(-)^\star : \mathcal{P}\mathrm{oly}\mathcal{E}\mathrm{nd} \overset{\dashv}{\longleftrightarrow} \mathcal{P}\mathrm{oly}\mathcal{M}\mathrm{nd} : U$ is monadic, and we abuse notation by letting $(-)^\star$ be the associated monad on $\mathcal{P}\mathrm{oly}\mathcal{E}\mathrm{nd}$. For a polynomial functor F as in (2.1), the unit $\mathsf{Y}_{(-)} : F \longrightarrow F^\star$ of $(-)^\star$ at F is given by $b \in B \longmapsto \mathsf{Y}_b \in \operatorname{tr} F$ (as in the proof of the previous theorem), and the multiplication $\circledcirc : F^{\star\star} \longrightarrow F^\star$ by

$$\circledcirc \mathsf{I}_i \coloneqq \mathsf{I}_i, \qquad \circledcirc \left(\mathsf{Y}_T \bigcirc_{[[l_i]]} X_i \right) \coloneqq T \bigcirc_{[l_i]} \circledcirc X_i,$$

for $i \in I$, $T \in \operatorname{tr} F$, $[l_1], \ldots, [l_k]$ the leaves of T, and $X_1, \ldots, X_k \in \operatorname{tr} F^\star$.

2.5.2. *The $(-)^+$ construction.* Let $M = (M, \mu, \eta) \in \mathcal{P}\mathrm{oly}\mathcal{M}\mathrm{nd}$ be a polynomial monad, where the underlying polynomial functor is as in equation (2.1). As such, it is a $(-)^\star$-algebra, and write its structure map $M^\star \longrightarrow M$ as

$$\begin{tikzcd}
 & \mathrm{tr}^{|} M \arrow[r] \arrow[dl] \arrow[d, "\wp"'] \arrow[dr, phantom, "\lrcorner", very near start] & \mathrm{tr}\, M \arrow[d, "\mathsf{t}"] \arrow[dr] & \\
I \arrow[d, equal] & E \arrow[r] \arrow[dl] & B \arrow[dr] & I \arrow[d, equal] \\
I & & & I
\end{tikzcd}$$

For $T \in \mathrm{tr}\, M$, we call $\wp_T : T^{|} \xrightarrow{\cong} E(\mathsf{t}T)$ the *reindexing* function of T, and the node $\mathsf{t}T \in B$ is called the *target* of T. If we think of the element of B as corollas, with leaves (or input edges) indexed in the relevant fiber in E, then M-trees are indeed trees obtained by coherent graftings of those corollas. The target map t then "contracts" a tree to a corolla, and since the middle square is cartesian, the number of leaves is preserved. The map $\wp$ establishes a coherent correspondence between the leaf addresses of a tree, and the node addresses of its target. The relevance of this map will show up in theorem 2.10.

Lemma 2.8. *For $T, U \in \mathrm{tr}\, M$, $[l]$ a leaf of T such that the grafting $T \circ_{[l]} U$ is defined, we have*

$$\mathsf{t}\left(T \underset{[l]}{\circ} U\right) = \mathsf{t}\left(\mathsf{Y}_{\mathsf{t}T} \underset{[\wp_T[l]]}{\circ} \mathsf{Y}_{\mathsf{t}U}\right).$$

Proof. This is a special case of the fact that the following square commutes

$$\begin{tikzcd}
M^{\star\star} \arrow[r, "\mathsf{t}^\star"] \arrow[d, "\circledcirc"'] & M^\star \arrow[d, "\mathsf{t}"] \\
M^\star \arrow[r, "\mathsf{t}"] & M
\end{tikzcd}$$

since M is a $(-)^\star$-algebra. $\square$

Define M^+ to be

$$B \xleftarrow{\mathsf{s}} \mathrm{tr}^\bullet M \xrightarrow{p} \mathrm{tr}\, M \xrightarrow{\mathsf{t}} B \tag{2.9}$$

where s maps an M-tree with a marked node to the label of that node, p forgets the marking, and t is the target map. If $T \in \mathrm{tr}\, M$, remark that $p^{-1}T = T^\bullet$ is set of nodes addresses of T. If $[p] \in T^\bullet$, then $\mathsf{s}[p] \coloneqq \mathsf{s}_{[p]} T$.

Theorem 2.10 ([13])**.** *The polynomial functor M^+ has a canonical structure of a polynomial monad.*

(sketch). The unit $\eta^+ : B \longrightarrow \mathrm{tr}\, M$ maps a node b to Y_b. The partial law $\mu^+ : \mathrm{tr}^\bullet M \times_B \mathrm{tr}\, M \longrightarrow \mathrm{tr}\, M$ is given by substitution as we now explain. Take $U \in \mathrm{tr}^\bullet M$, $T \in \mathrm{tr}\, M$ such that $\mathsf{s}U = b = \mathsf{t}T$, i.e. $(U, T) \in \mathrm{tr}^\bullet M \times_B \mathrm{tr}\, M$. We may think of U as a context corresponding to the selected node: $U = C[\mathsf{Y}_b]$, for some M-context $C[\square]$. The readdressing map $\wp_T$ of T gives a bijection between $\mathsf{Y}_b^{|}$ and $T^{|}$, and

thus specifies "rewiring instructions" to replace Y_b by T in U, i.e. evaluate C at T: $\mu^+(U, T) \coloneqq C[T]$. □

3. Polygraphs

3.1. **Reminders.** We review some elements of the theory of polygraphs. For a more complete introduction, we refer to [15] or [8].

A *polygraph* (also called a *computad*) P consists of a small ω-category P^* and sets $P_n \subseteq P^*_n$ for all $n \in \mathbb{N}$, such that P_0 is the set of objects of P^*, and such that the underlying $(n+1)$-category $P^*|_{n+1}$ is freely generated by P_{n+1} over its underlying n-category $P^*|_n$, for all $n \geq 1$. Write P^*_n the set of n-cells of P, and

$$\mathsf{s}, \mathsf{t} : P^*_{n+1} \longrightarrow P^*_n$$

the *source* and *target* maps, respectively. For $n \geq 1$, two n-cells $x, y \in P^*_n$ are *parallel*, denoted by $x \parallel y$, if $\mathsf{s}\,x = \mathsf{s}\,y$ and $\mathsf{t}\,x = \mathsf{t}\,y$. By convention, 0-cells are pairwise parallel. A morphism of polygraphs is an ω-functor mapping generators to generators. Let $\mathcal{P}\mathrm{ol}$ be the category of polygraphs and morphisms between them.

A polygraph P is an *n-polygraph* if $P_k = \varnothing$ whenever $k > n$. For $n \geq 1$, an n-cell $x \in P^*_n$ is said *many-to-one* of $\mathsf{t}\,x \in P_{n-1}$ (instead of P^*_{n-1}), and we write $P^{\triangledown}_n$ the set of many-to-one n-cells of P. By convention, all 0-cells are many-to-one. In turn, the polygraph P is *many-to-one* if all its generators are, or equivalently if the target of a generator is also a generator. Let $\mathcal{P}\mathrm{ol}^{\triangledown}$ be the corresponding full subcategory.

Lemma 3.1. *The category $\mathcal{P}\mathrm{ol}^{\triangledown}$ is cocomplete. Moreover, if $P = \operatorname{colim}_{i \in \mathcal{J}} P_i$, then $P_n = \operatorname{colim}_{i \in \mathcal{J}} P_{i,n}$.*

Let P be an arbitrary n-polygraph, and take $k \leq n$. Define a k-category $\mathbb{N}$ by

$$0 \xleftarrow{\mathsf{s},\mathsf{t}} \cdots \xleftarrow{\mathsf{s},\mathsf{t}} 0 \xleftarrow{\mathsf{s},\mathsf{t}} \mathbb{N},$$

where all compositions correspond to the addition of integers. For $x \in P_k$, define a function $\#_x : P_k \longrightarrow \mathbb{N}$ that maps x to 1, and all other generators to 0. This extends to a k-functor $P^* \longrightarrow \mathbb{N}$, and let $\#_x : P^*_k \longrightarrow \mathbb{N}$ be its k-th component. Similarly, let $\# : P_k \longrightarrow \mathbb{N}$ be the map sending all generators to 1, and extend it as $\# : P^*_k \longrightarrow \mathbb{N}$.

Recall the definition of the category of n-contexts $\mathcal{C}\mathrm{tx}_n Q$ of a n-polygraph Q from e.g. [6]: its objects are the n-cells of Q, and a morphism $C : x \longrightarrow y$ is an n-context $C = C[\square]$ such that $C[x] = y$. Let $\mathcal{C}\mathrm{tx}^{\triangledown}_n P$ be the full subcategory of $\mathcal{C}\mathrm{tx}_n P$ generated by many-to-one cells. Necessarily, the morphisms of $\mathcal{C}\mathrm{tx}^{\triangledown}_n P$ are many-to-one contexts.

Let P be a polygraph (not necessarily many-to-one), $x, y \in P^*_n$ be n-cells, and $C : \mathsf{t}\,y \longrightarrow \mathsf{s}\,x$ be a context. The partial composition $x \circ_D y$ is defined as follows:

$$x \underset{C}{\circ} y \coloneqq x \underset{n-1}{\circ} C[y],$$

where $\circ_{n-1}$ is the $(n-1)$-composition, and $C[y]$ is the C-whisker of y.

Lemma 3.2 ([8])**.** *With x, y, and C as above, we have*

$$\mathsf{s}\left(x \underset{C}{\circ} y\right) = C[\mathsf{s}\,y], \qquad \mathsf{t}\left(x \underset{C}{\circ} y\right) = \mathsf{t}\,x.$$

Assume now that P is many-to-one, and take $z \in P^{\triangledown}_n$. For $C_i : g_i \longrightarrow \mathsf{s}\,z$, $1 \leq i \leq k$, all the contexts from a generator to $\mathsf{s}\,z$, and $w_1, \ldots, w_k \in P^{\triangledown}_n$ cells such that $\mathsf{t}\, w_i = g_i$ (so that the partial composition $z \circ_{C_i} w_i$ is well-defined), define the *total composition*

$$z \bigcirc_{C_i} w_i = (\cdots(z \underset{C_1}{\circ} w_1) \underset{C_2}{\circ} w_2 \cdots) \underset{C_k}{\circ} w_k.$$

The result does not depend in the order in which the partial compositions are computed.

3.2. **The $\triangledown$ construction.** For $P \in \mathcal{P}\mathrm{ol}^{\triangledown}$, and $n \geq 1$, Let $\triangledown_n P$ be the following polynomial endofunctor:

$$P_{n-1} \xleftarrow{\mathsf{s}} P_n^{\bullet} \xrightarrow{p} P_n \xrightarrow{\mathsf{t}} P_{n-1},$$

where

$$P_n^{\bullet}(x) \coloneqq \bigsqcup_{a \in P_{n-1}} (\mathcal{C}\mathrm{tx}_{n-1}^{\triangledown} P)(a, \mathsf{s}\, x),$$

where for $C : a \longrightarrow \mathsf{s}\, x$ in $P_n^{\bullet}(x)$, $\mathsf{s}\, C \coloneqq a$, $pC \coloneqq x$, and t is the target map of P. Remark that $\# P_n^{\bullet}(x) = \# x$, and in particular, $P_n^{\bullet}(x)$ is finite.

Proposition 3.3. *A morphism of polygraphs $f : P \longrightarrow Q$ induces morphism $\triangledown_n f : \triangledown_n P \longrightarrow \triangledown_n Q$ for all $n \geq 1$, such that $(\triangledown_n f)_1 = f_n : P_n \longrightarrow Q_n$.*

Proof. Consider

$$\begin{tikzcd}
& P_n^{\bullet} \arrow[r, "p"] \arrow[dl, "\mathsf{s}"'] \arrow[d, "f_n^{\bullet}"'] & P_n \arrow[d, "f_n"'] \arrow[dr, "\mathsf{t}"] & \\
P_{n-1} \arrow[d, "f_{n-1}"'] & Q_n^{\bullet} \arrow[r, "p"'] \arrow[dl, "\mathsf{s}"] & Q_n \arrow[dr, "\mathsf{t}"'] & P_{n-1} \arrow[d, "f_{n-1}"] \\
Q_{n-2} & & & Q_{n-2},
\end{tikzcd}$$

where $f_n^{\bullet} : (C : a \to \mathsf{s}\, x) \longmapsto (f_{n-1} C : f_{n-1} a \longrightarrow f_{n-1}\, \mathsf{s}\, x)$. Clearly, all squares commute, and it remains to check that the middle one is cartesian, i.e. that $f_n^{\bullet}$ is a fiberwise bijection. Take $(D : b \to f_{n-1}\, \mathsf{s}\, x) \in Q_n^{\bullet}(f_n x)$. Then since $f_{n-1}\, \mathsf{s}\, x \in \operatorname{im} f$, all generators occurring in $f_{n-1}\, \mathsf{s}\, x$ are in the image too, so that b and generators in D are in the image of f. Hence $f_n^{\bullet}$ is surjective. Moreover, $\# P_n^{\bullet}(x) = \# x = \# f_n x = \# Q_n^{\bullet}(f_n x)$, so that $f_n^{\bullet}$ is injective as well. ☐

Thus, we have for each $n \geq 1$ a functor $\triangledown_n : \mathcal{P}\mathrm{ol}^{\triangledown} \longrightarrow \mathcal{P}\mathrm{oly}\mathcal{E}\mathrm{nd}$.

3.3. **The composition tree duality.** For $P \in \mathcal{P}\mathrm{ol}^{\triangledown}$, we define the *compositor* functor $(-)^{\circ} : \operatorname{tr} \triangledown_n P \longrightarrow \mathcal{C}\mathrm{tx}_n^{\triangledown} P$ inductively as follows.

(1) For $i \in P_{n-1}$, let $\mathsf{I}_i^{\circ} \coloneqq \mathrm{id}_i$.
(2) For $x \in P_n$, let $\mathsf{Y}_x^{\circ} \coloneqq x$.
(3) Let $T \in \operatorname{tr} \triangledown_n P$, and $[l] : \mathsf{I}_i \longrightarrow T$ be a leaf of T. We construct a $(n-1)$-context $[l]^{\circ} : i \longrightarrow \mathsf{s}(T^{\circ})$ inductively as follows.
   (a) If T is a trivial tree, then it is necessarily I_i and $[l] = [\varepsilon]$ is the identity, and we let $[l]^{\circ} \coloneqq \square : i \longrightarrow i$.

(b) If $T = \mathsf{Y}_x$ for $x \in P_n$, then the morphism $[l] : \langle \mathsf{I}_i \rangle \longrightarrow \langle \mathsf{Y}_x \rangle$ maps $* \in \langle \mathsf{I}_i \rangle_0$ to a non-root edge $e \in \langle \mathsf{Y}_x \rangle_0$, thus corresponding to an elements e' in $\langle \mathsf{Y}_x \rangle_2$, which in turn is mapped by Y_x to $C_{[l]} \in P_n^\bullet(x)$, which is necessarily of the form $C_{[l]} : i \longrightarrow \mathsf{s}\, x$. We let $[l]^\circ := C_{[l]}$.

(c) If T decomposes as $T = S \circ_{[k]} \mathsf{Y}_x$, for $[k] \in S^|$, then one of two cases occurs. If $[l] \in S^|$, then by induction we have a context $[l]^\circ : i \longrightarrow \mathsf{s}(S^\circ)$ and a context $[k]^\circ : \mathsf{t}\, x \longrightarrow \mathsf{s}(S^\circ)$. By construction, since $[l] \neq [k]$, we have $[l]^\circ \neq [k]^\circ$, and so there is a bicontext C such that $\mathsf{s}(S^\circ) = C[i, \mathsf{t}\, x]$. Let $l^\circ := C[\square, \mathsf{s}\, x]$. If $[l]$ decomposes as $[l] = [k[h]]$, then by induction, we have contexts $[[h]]^\circ : i \longrightarrow \mathsf{s}\, x$ and $[k]^\circ : \mathsf{t}\, x \longrightarrow \mathsf{s}(S^\circ)$. Finally, let $[k[h]]^\circ := ([k])^\circ[([[h]])^\circ[\square]]$.

We complete the definition of $(-)^\circ$ by letting $(T \circ_{[l]} S)^\circ := T^\circ \circ_{[l]^\circ} S^\circ$, for appropriate $S, T \in \operatorname{tr} \triangledown_n P$, and $[l] \in T^|$. It is straightforward to prove that for U a tree, U° does not depend on the choice of decomposition of U.

We now define $(-)^\circ$ on morphisms. Let $f : S \longrightarrow T$ be a $\triangledown_n P$-tree morphisms. Then in particular it is an embedding, hence T decomposes as $T = A \circ_{[a]} \big(S \bigcirc_{[b_i]} B_i\big)$, where $[a]$ is the address of the image of the root edge of S. Taking a $\square$ symbol parallel to S°, we obtain a context

$$f^\circ := A^\circ \underset{[a]^\circ}{\circ} \left(\square \bigcirc_{[b_i]^\circ} B_i^\circ \right),$$

and clearly, $f^\circ[S^\circ] = T^\circ$, whence $f^\circ : S^\circ \longrightarrow T^\circ$.

Conversely, we now define the *composition tree* functor $\operatorname{ct} : \mathcal{C}\mathrm{tx}_n^\triangledown P \longrightarrow \triangledown_n P$ inductively as follows.

(1) For $i \in P_{n-1}$, let $\operatorname{ct} \mathrm{id}_i := \mathsf{I}_i$.
(2) For $x \in P_n$, let $\operatorname{ct} x := \mathsf{Y}_x$.
(3) Let $\alpha \in P_n^\triangledown$, $i \in P_{n-1}$, and $C : i \longrightarrow \mathsf{s}\,\alpha$. We construct a leaf $\operatorname{ct}^| C : \mathsf{I}_i \longrightarrow \operatorname{ct} \alpha$ inductively as follows:
   (a) If α is an identity, then it is necessarily id_i, and $C = \square$. We let $\operatorname{ct}^| C := \mathrm{id}_{\mathsf{I}_i}$.
   (b) If $\alpha \in P_n$ is a generator, then $C \in P_n^\bullet(\alpha)$, so that $e = (\mathsf{Y}_\alpha)_2^{-1}(C) \in \langle \mathsf{Y}_\alpha \rangle_2$, and let $\operatorname{ct}^| C$ be the address of the corresponding edge, which is necessarily a leaf.
   (c) If α decomposes as $\alpha = \beta \circ_D x$, for $x \in P_n$ and $D : \mathsf{t}\, x \longrightarrow \mathsf{s}\,\beta$, then $\mathsf{s}\,\alpha = D[\mathsf{s}\, x]$, and one of two cases occurs. If there is a bicontext E such that $C[\square] = E[\square, \mathsf{s}\, x]$ and $D[\square] = E[i, \square]$, then let $\operatorname{ct}^| C := (\operatorname{ct} E[\square, \mathsf{t}\, x]) \circ_{\operatorname{ct} D} \mathsf{Y}_x$. If not, then there is a context $F : i \longrightarrow \mathsf{s}\, x$, and $\mathsf{s}\,\alpha = D[F[i]]$. Let then $\operatorname{ct}^| C := (\operatorname{ct} \beta) \circ_{\operatorname{ct} D} (\operatorname{ct}^| F)$.

We complete the definition of ct by letting $\operatorname{ct}(\alpha \circ_C \beta) := (\operatorname{ct} \alpha) \circ_{\operatorname{ct}^| C} (\operatorname{ct} \beta)$, for $\alpha, \beta \in P_n^\triangledown$ and $C : \mathsf{t}\,\beta \longrightarrow \mathsf{s}\,\alpha$. It is straightforward to prove that for γ a many-to-one cell, $\operatorname{ct} \gamma$ does not depend on the choice of decomposition of γ.

We now define ct on morphisms. Let $C : \alpha \longrightarrow \beta$ be a context. Then C decomposes as $C = u \circ_U (\square \bigcirc_{V_i} v_i)$, and we set

$$\operatorname{ct} C := (\operatorname{ct} u) \underset{\operatorname{ct} U}{\circ} \left(\square \bigcirc_{\operatorname{ct} V_i} (\operatorname{ct} v_i) \right).$$

Proposition 3.4. *The functors* $(-)^\circ$ *and* ct *are mutually inverse isomorphisms of categories.*

Corollary 3.5. *For* $n \geq 2$ *and* $x \in P_n$*, the functor* ct *induces a natural bijection over* P_{n-1}*:*

$$P_n^\bullet(x) \cong \coprod_{i \in P_{n-1}} (\operatorname{tr} \triangledown_{n-1} P)(\mathsf{Y}_i, \operatorname{ct} \mathsf{s}\, x).$$

Notation 3.6. Let $x \in P_n$, and $[p]$ an address of $\operatorname{ct} \mathsf{s}\, x$. Then we write $\mathsf{s}_{[p]}\, x := \mathsf{s}_{[p]} \operatorname{ct} \mathsf{s}\, x \in P_{n-1}$.

4. Opetopes

4.1. **Polynomial approach.** We make use of the polynomial functor approach to the definition of opetopes as presented in [13]: let $\mathfrak{Z}^0$ be the identity polynomial monad on $\mathcal{S}\mathrm{et} = \mathcal{S}\mathrm{et}/\{*\}$, and $\mathfrak{Z}^n = (\mathfrak{Z}^{n-1})^+$. Expand $\mathfrak{Z}^n$ as

$$\mathbb{O}_n \xleftarrow{\mathsf{s}} \mathbb{O}_{n+1}^\bullet \xrightarrow{p} \mathbb{O}_{n+1} \xrightarrow{\mathsf{t}} \mathbb{O}_n.$$

An n-opetope is by definition an element of $\mathbb{O}_n$, or equivalently a $\mathfrak{Z}^{n-2}$-tree, if $n \geq 2$. In the latter case, an n-opetope is then a tree whose nodes are (labeled by) $(n-1)$-opetopes, and edges are (labeled by) $(n-2)$-opetopes. Note that for $\omega \in \mathbb{O}_n$ with $n \geq 2$, an element of $\mathbb{O}_n^\bullet(\omega)$ is a morphism of $\mathfrak{Z}^{n-2}$-trees of the form $\mathsf{Y}_\psi \longrightarrow \omega$, where $\psi \in \mathbb{O}_{n-1}$.

Let $\omega \in \mathbb{O}_n$ with $n \geq 2$, $[p] \in \omega^\bullet$, and $\psi = \mathsf{s}_{[p]}\, \omega \in \mathbb{O}_{n-1}$. Then by construction, there is a bijection between the input edges of the node at address $[p]$ in ω and $\psi^\bullet$. If $[q] \in \psi^\bullet$, we call $[q]$ the associated input edge, so that the address of that specific edge in ω is $[p[q]]$. Moreover, the $(n-2)$-opetope decorating that edge is by construction $\mathsf{s}_{[q]}\, \mathsf{s}_{[p]}\, \omega = \mathsf{s}_{[q]}\, \psi$.

An opetope $\omega \in \mathbb{O}_n$ with $n \geq 2$ is called *degenerate* if it is of the form $\omega = \mathsf{I}_\phi$ for some $\phi \in \mathbb{O}_{n-2}$. We call an edge *inner* if it is neither the root nor a leaf. Inner edges of ω are exactly those whose address is of the form $[p[q]]$, with $[p] \in \omega^\bullet$, $[q] \in (\mathsf{s}_{[p]}\, \omega)^\bullet$, and $[p[q]] \in \omega^\bullet$.

4.2. **The category of opetopes.** Akin to the work of Cheng [2], we define a category of opetopes by means of generators and relations. The difference with the aforementioned reference is our use of polynomial opetopes (also equivalent to Leinster's definition [14, 13]), while Cheng uses an approach by multicategorical slicing, yielding "symmetric" opetopes.

Theorem 4.1 (Opetopic identities)**.** *Let* $\omega \in \mathbb{O}_n$ *with* $n \geq 2$.

(1) *(Inner edge) For* $[p[q]] \in \omega^\bullet$ *we have*

$$\mathsf{t}\, \mathsf{s}_{[p[q]]}\, \omega = \mathsf{s}_{[q]}\, \mathsf{s}_{[p]}\, \omega. \tag{4.2}$$

(2) *(Globularity 1) If* ω *is non degenerate, we have*

$$\mathsf{t}\, \mathsf{s}_{[\varepsilon]}\, \omega = \mathsf{t}\, \mathsf{t}\, \omega. \tag{4.3}$$

(3) *(Globularity 2) If* ω *is non degenerate, and* $[p[q]] \in \omega^|$*, we have*

$$\mathsf{s}_{[q]}\, \mathsf{s}_{[p]}\, \omega = \mathsf{s}_{\wp_\omega [p[q]]}\, \mathsf{t}\, \omega. \tag{4.4}$$

(4) *(Degeneracy) If* ω *is degenerate, we have*

$$\mathsf{s}_{[\varepsilon]}\, \mathsf{t}\, \omega = \mathsf{t}\, \mathsf{t}\, \omega. \tag{4.5}$$

Proof. (1) By definition of a $\mathfrak{Z}^{n-2}$-tree.

(2) The monad structure on $\mathfrak{Z}^{n-2}$ amounts to a structure map $(\mathfrak{Z}^{n-2})^\star \longrightarrow \mathfrak{Z}^{n-2}$ which gives the following commutative square:

$$\begin{tikzcd} \operatorname{tr} \mathfrak{Z}^{n-2} \arrow[r, "r"] \arrow[d, "\mathsf{t}"'] & \mathbb{O}_{n-2} \arrow[d, equal] \\ \mathbb{O}_{n-1} \arrow[r, "\mathsf{t}"] & \mathbb{O}_{n-2}, \end{tikzcd}$$

where for a tree $T \in \operatorname{tr} \mathfrak{Z}^{n-2}$, the opetope rT is the decoration of the root edge of T, i.e. $\mathsf{s}_{[\varepsilon]} \mathsf{t} T$.

(3) By definition, $\wp_\omega$ is a bijection $\omega^| \longrightarrow (\mathsf{t}\,\omega)^\bullet$ over $\mathbb{O}_{n-2}$.

(4) Let $\omega = \mathsf{I}_\phi$, for $\phi \in \mathbb{O}_{n-2}$. Then $\mathsf{t}\,\omega = \mathsf{Y}_\phi$, and $\mathsf{s}_{[\varepsilon]} \mathsf{Y}_\phi = \mathsf{s}_{[\varepsilon]} \mu \mathsf{Y}_\phi = \mathsf{t}\,\mathsf{Y}_\phi$ where μ is the monad law of $\mathfrak{Z}^{n-2}$. □

With those identities in mind, we define the category $\mathbb{O}$ of opetopes by generators and relations as follows.

(1) Objects: We set $\operatorname{ob} \mathbb{O} = \bigsqcup_{n \in \mathbb{N}} \mathbb{O}_n$.

(2) Generators: Let $\omega \in \mathbb{O}_n$ with $n \geq 1$. We introduce a generator, called *target embedding*: $\mathsf{t} : \mathsf{t}\,\omega \longrightarrow \omega$. If $[p] \in \omega^\bullet$, then we introduce a generator, called *source embedding*: $\mathsf{s}_{[p]} : \mathsf{s}_{[p]}\,\omega \longrightarrow \omega$. A *face embedding* is either a source or target embedding.

(3) Relations: We impose 4 relations described by commutative squares, that are well defined thanks to theorem 4.1. Let $\omega \in \mathbb{O}_n$ with $n \geq 2$

(a) **[Inner]** for $[p[q]] \in \omega^\bullet$ (forcing ω to be non degenerate), the following square must commute:

$$\begin{tikzcd} \mathsf{s}_{[q]}\,\mathsf{s}_{[p]}\,\omega \arrow[r, "\mathsf{s}_{[q]}"] \arrow[d, "\mathsf{t}"'] & \mathsf{s}_{[p]}\,\omega \arrow[d, "\mathsf{s}_{[p]}"] \\ \mathsf{s}_{[p[q]]}\,\omega \arrow[r, "\mathsf{s}_{[p[q]]}"] & \omega \end{tikzcd}$$

(b) **[Glob1]** if ω is non degenerate, then the following square must commute:

$$\begin{tikzcd} \mathsf{t}\,\mathsf{t}\,\omega \arrow[r, "\mathsf{t}"] \arrow[d, "\mathsf{t}"'] & \mathsf{t}\,\omega \arrow[d, "\mathsf{t}"] \\ \mathsf{s}_{[\varepsilon]}\,\omega \arrow[r, "\mathsf{s}_{[\varepsilon]}"] & \omega. \end{tikzcd}$$

(c) **[Glob2]** if ω is non degenerate, and for $[p[q]] \in \omega^|$, then the following square must commute:

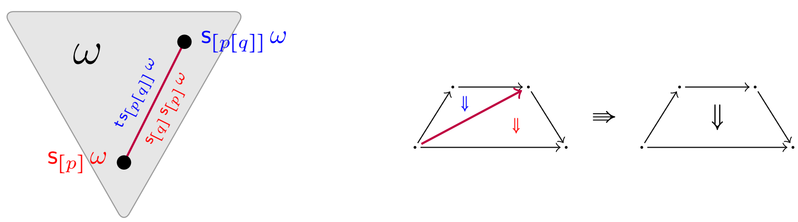
$$\begin{tikzcd} \mathsf{s}_{\wp_\omega [p[q]]}\,\mathsf{t}\,\omega \arrow[r, "\mathsf{s}_{\wp_\omega [p[q]]}"] \arrow[d, "\mathsf{s}_{[q]}"'] & \mathsf{t}\,\omega \arrow[d, "\mathsf{t}"] \\ \mathsf{s}_{[p]}\,\omega \arrow[r, "\mathsf{s}_{[p]}"] & \omega. \end{tikzcd}$$

(d) **[Degen]** if ω is degenerate, then the following square must commute:

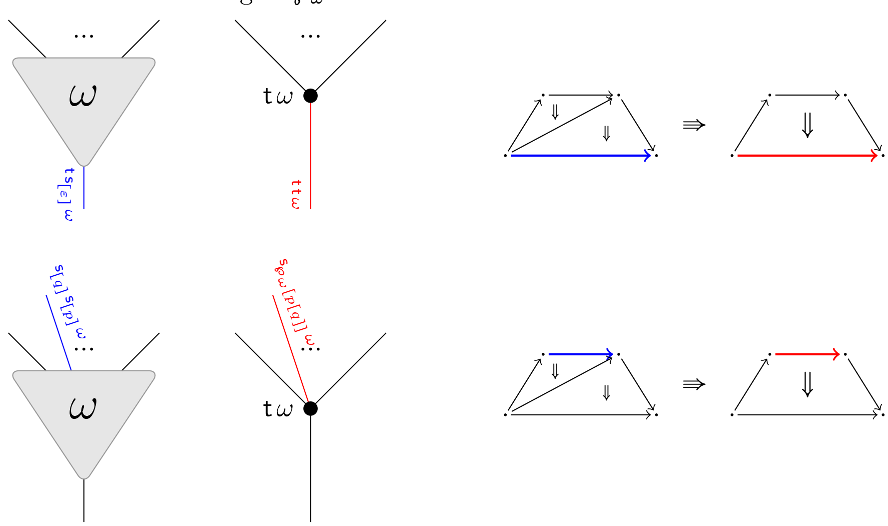
$$\begin{tikzcd} \mathsf{t}\,\mathsf{t}\,\omega \arrow[r, "\mathsf{t}"] \arrow[d, "\mathsf{s}_{[\varepsilon]}"'] & \mathsf{t}\,\omega \arrow[d, "\mathsf{t}"] \\ \mathsf{t}\,\omega \arrow[r, "\mathsf{t}"] & \omega. \end{tikzcd}$$

Let us explain this definition a little more. Opetopes are trees whose nodes (and edges) are decorated by opetopes. The decoration is now interpreted as a geometrical feature, namely as an embedding of a lower dimensional opetope. Further, the target of an opetope, while not an intrinsic data, is also represented as an embedding. The relations can be understood as follows.

(1) **[Inner]** The inner edge at $[p[q]] \in \omega^\bullet$ is decorated by the target of the decoration of the node "above" it (here $\mathsf{s}_{[p[q]]}\,\omega$), and in the $[q]$-source of the node "below" it (here $\mathsf{s}_{[p]}\,\omega$). By construction, those two decorations match, and this relation makes the two corresponding embeddings $\mathsf{s}_{[q]}\,\mathsf{s}_{[p]}\,\omega \longrightarrow \omega$ match as well. On the left is an informal diagram about ω as a tree (reversed gray triangle), and on the right is an example of pasting diagram represented by an opetope, with the relevant features of the **[Inner]** relation colored or thickened.



(2) **[Glob1-2]** If we consider the underlying tree of ω (which really is ω itself) as its "geometrical source", and the corolla $\mathsf{Y}_{\mathsf{t}\,\omega}$ as its "geometrical target", then they should be parallel. The relation **[Glob1]** expresses this idea by "gluing" the root edges of ω and $\mathsf{Y}_{\mathsf{t}\,\omega}$ together, while **[Glob2]** glues the leaves according to $\wp_\omega$.



(3) **[Degen]** If ω is a degenerate opetope, depicted as on the right, then its target should be a "loop", i.e. its only source and target should be glued together.

5. The equivalence between many-to-one polygraphs and opetopic sets

We now aim to prove that the category of opetopic sets, i.e. $\mathcal{S}$et-presheaves over the category $\mathbb{O}$ defined previously, is equivalent to the category of many-to-one polygraphs $\mathcal{P}\mathrm{ol}^{\triangledown}$. We achieve this by first constructing the *opetal*[4] functor $O[-] : \mathbb{O} \longrightarrow \mathcal{P}\mathrm{ol}^{\triangledown}$ in subsection 5.1. This functor "realizes" an opetope as a polygraph, in that it freely implements all its tree structure by means of adequately chosen generators in each dimension. Secondly, writing $\hat{\mathbb{O}} = \mathcal{S}\mathrm{et}^{\mathbb{O}^{\mathrm{op}}}$ (as per French tradition), we consider the "polygraphic realization" $|-| : \hat{\mathbb{O}} \longrightarrow \mathcal{P}\mathrm{ol}^{\triangledown}$, which is the left Kan extension of $O[-]$ along the Yoneda embedding. This realization has a right adjoint, the "opetopic nerve" $N : \mathcal{P}\mathrm{ol}^{\triangledown} \longrightarrow \hat{\mathbb{O}}$, and we prove this adjunction to be an adjoint equivalence. This is done using the shape function, defined in subsection 5.2, which to any generator x of a many-to-one polygraph P associates an opetope $x^{\natural}$ along with a canonical morphism $\tilde{x} : O[x^{\natural}] \longrightarrow P$.

5.1. **The opetal functor.** An opetope $\omega \in \mathbb{O}_n$, with $n \geq 1$, has one target $\mathsf{t}\,\omega$, and sources $\mathsf{s}_{[p]}\,\omega$ laid out in a tree. If the sources $\mathsf{s}_{[p]}\,\omega$ happened to be generators in some polygraph, then that tree would describe a way to compose them. With this in mind, we define a many-to-one polygraph $O[\omega]$, whose generators are essentially iterated faces (i.e. sources or targets) of ω (hypothesis **[IND1]** below). Moreover, $O[\omega]$ will be "maximally unfolded" (or "free") in that two (iterated) faces that are the same opetope, but located at different addresses, will correspond to distinct generators. The opetal functor $O[-]$ is defined inductively, together with its boundary $\partial O[-]$.

For $\blacklozenge$ the unique 0-opetope, let $\partial O[\blacklozenge]$ be the polygraph with no generator in any dimension, and $O[\blacklozenge]$ be the polygraph with a unique generator in dimension 0, which we denote by $\blacklozenge$.

For $\blacksquare$ the unique 1-opetope, let $\partial O[\blacksquare] := O[\blacklozenge] \sqcup O[\blacklozenge]$, and let $O[\blacksquare]$ be the cellular extension $\left(\partial O[\blacksquare] \xleftarrow{\mathsf{s},\mathsf{t}} \{\blacksquare\}\right)$, where s and t map $\blacksquare$ to distinct 0-generators. There are obvious functors $O[\mathsf{s}_{[\varepsilon]}], O[\mathsf{t}] : O[\blacklozenge] \longrightarrow O[\blacksquare]$, mapping $\blacklozenge$ to $\mathsf{s}\,\blacksquare$ and $\mathsf{t}\,\blacksquare$, respectively.

Let $n \geq 2$ and assume by induction that $\partial O[-]$ and $O[-]$ are defined on $\mathbb{O}_{<n}$, the full subcategory of $\mathbb{O}$ spanned by opetopes of dimension strictly less than n. Assume further that the following induction hypothesis hold.

(1) **[IND1]** For $k < n$, $\psi \in \mathbb{O}_k$, and $l \in \mathbb{N}$, we have $O[\psi]_l \cong \mathbb{O}_l/\psi$.
(2) **[IND2]** For $k < n$ and $\psi \in \mathbb{O}_k$, we have that in $O[\psi]$, $\langle \mathrm{ct}\,\mathsf{s}\,\psi \rangle = \langle \psi \rangle$.

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[4] The name intends to follow the unofficial "-al" convention e.g. *cubical*, *dentroidal*, *oriental*, *simplicial*, etc.

Let $\omega \in \mathbb{O}_n$ and start by defining

$$\partial O[\omega] := \operatorname*{colim}_{\mathbb{O}_{<n}/\omega} O[-].$$

This extends as a functor $\partial O[-] : \mathbb{O}_{\leq n} \longrightarrow \mathcal{P}\mathrm{ol}^{\triangledown}$, mapping a k-opetope to a $(k-1)$-polygraph, for $k \leq n$. By hypothesis [**IND1**], for $k < n$, we have $\partial O[\omega]_k \cong \mathbb{O}_k/\omega$.

We now take a break to explain the subsequent developments of this subsection. In $\partial O[\omega]$, the target $\mathsf{t}\,\omega$ and all sources $\mathsf{s}_{[p]}\,\omega$ are $(n-1)$-generators. On the other hand, ω itself is a tree whose nodes are its sources. Thus ω should correspond to the composition tree of some cell in $\partial O[\omega]^{\triangledown}_{n-1}$ (this holds for lower-dimensional opetopes by hypothesis [**IND2**]), which will be denoted $\bar{\omega}^{\circ}$. Then, in proposition 5.2, it is shown that this cell is parallel to the generator corresponding to the target $\mathsf{t}\,\omega$. The subsection concludes by defining $O[\omega]$ as the induced cellular extension of $\partial O[\omega]$.

Let us resume. There is an obvious "forgetful morphism" $u : \triangledown_{n-1}\partial O[\omega] \longrightarrow \mathfrak{Z}^{n-2}$, mapping an $(n-1)$-cell $(\psi \to \omega)$ to ψ. We now construct a factorization $\bar{\omega}$ of ω along u, so that $\bar{\omega}$ really is a composition tree in $\partial O[\omega]$ whose underlying tree is $\langle\omega\rangle$:

$$\begin{tikzcd} & \langle\omega\rangle \arrow[dl, "\bar{\omega}"'] \arrow[dr, "\omega"] & \\ \triangledown_{n-1}\partial O[\omega] \arrow[rr, "u"] & & \mathfrak{Z}^{n-2}. \end{tikzcd}$$

(1) Let $i \in \langle\omega\rangle_0$ be an edge.
   (a) If i is not a leaf, let $\bar{\omega}_0(i) := \left(\mathsf{t}\,\mathsf{s}_{\&i}\,\omega \xrightarrow{\mathsf{t}\,\mathsf{s}_{\&i}} \omega\right) \in \mathbb{O}_{n-2}/\omega = (\triangledown_{n-1}\partial O[\omega])_0$.
   (b) If i is not the root, then $\&i = [p[q]]$ for some p and q, and let $\bar{\omega}_0(i) := \left(\mathsf{s}_{[q]}\,\mathsf{s}_{[p]}\,\omega \xrightarrow{\mathsf{s}_{[q]}\,\mathsf{s}_{[p]}} \omega\right)$.

   Remark that if i is neither a leaf nor the root (i.e. an inner edge), then the two definitions of $\bar{\omega}_0(i)$ agree by relation [**Inner**].
(2) For $b \in \langle\omega\rangle_1$, let $\bar{\omega}_1(b) := \left(\mathsf{s}_{\&b}\,\omega \xrightarrow{\mathsf{s}_{\&b}} \omega\right)$.
(3) Take $b \in \langle\omega\rangle_1$ and let $\bar{\omega}_1(b) := \left(\psi \xrightarrow{\mathsf{j}} \omega\right)$, for some face embedding j. We know that ω_2 is a fiberwise isomorphism, in this case $\omega_2 : \langle\omega\rangle_2(b) \xrightarrow{\cong} \mathbb{O}^{\bullet}_{n-1}(\psi)$. Let $C \in \langle\omega\rangle_2(b)$. Then it corresponds to an address in $\langle\psi\rangle = \langle \mathrm{ct}\,\mathsf{s}\,\psi\rangle = \left\langle \mathrm{ct}\,\mathsf{s}\left(\psi \xrightarrow{\mathsf{j}} \omega\right)\right\rangle$, where the first equality comes from [**IND2**], and we let $\bar{\omega}_2(C) := C$ be that same address.

Proposition 5.1. *The following displays a morphism of polynomial functors $\bar{\omega} : \langle\omega\rangle \longrightarrow \triangledown_{n-1}\partial O[\omega]$:*

$$\begin{tikzcd}
 & \langle\omega\rangle_2 \arrow[r, "p"] \arrow[ld, "s"'] \arrow[d, "\bar{\omega}_2"] & \langle\omega\rangle_1 \arrow[d, "\bar{\omega}_1"] \arrow[rd, "t"] & \\
\langle\omega\rangle_0 \arrow[d, "\bar{\omega}_0"'] & \partial O[\omega]^\bullet_{n-1} \arrow[r, "p"'] \arrow[ld, "\mathsf{s}"] & \mathbb{O}_{n-1}/\omega \arrow[rd, "\mathsf{t}"'] & \langle\omega\rangle_0 \arrow[d, "\bar{\omega}_0"] \\
\mathbb{O}_{n-2}/\omega & & & \mathbb{O}_{n-2}/\omega.
\end{tikzcd}$$

Proof. (1) We show that the left square commutes. For $b \in \langle\omega\rangle_1$ and $C \in \langle\omega\rangle_2(b)$, we have $\bar{\omega}_0(sC) = \left(\mathsf{s}_C\,\mathsf{s}_{\&b}\,\omega \xrightarrow{\mathsf{s}_C\,\mathsf{s}_{\&b}} \omega\right) = \mathsf{s}_C\,\bar{\omega}_1(b) = \mathsf{s}\,\bar{\omega}_2(C)$.

(2) We show that the middle square commutes. For $b \in \langle\omega\rangle_1$ and $C \in \langle\omega\rangle_2(b)$, we have $p\bar{\omega}_2(C) = pC = \bar{\omega}_1(b) = \bar{\omega}_1 p(C)$.

(3) We show that the middle square is cartesian. By definition, $\bar{\omega}_2$ maps an address of a polynomial tree to the same address of the same polynomial tree, and is thus a fiberwise isomorphism.

(4) We show that the right square commutes. For $b \in \langle\omega\rangle_1$, we have that $\bar{\omega}_0(tb) = \left(\mathsf{t}\,\mathsf{s}_{\&tb}\,\omega \xrightarrow{\mathsf{t}\,\mathsf{s}_{\&tb}} \omega\right) = \mathsf{t}\left(\mathsf{s}_{\&tb}\,\omega \xrightarrow{\mathsf{s}_{\&tb}} \omega\right) = \mathsf{t}\,\bar{\omega}_1(b)$. □

Thus, $\bar{\omega}$ is a $\triangledown_{n-1}\partial O[\omega]$-tree, and so by applying the compositor we obtain a many-to-one cell $\bar{\omega}^\circ \in \partial O[\omega]^\triangledown_{n-1}$.

Proposition 5.2. *In $\partial O[\omega]$ we have $\bar{\omega}^\circ \parallel \left(\mathsf{t}\,\omega \xrightarrow{\mathsf{t}} \omega\right)$.*

Proof. (1) If $\omega = \mathsf{I}_\phi$ is degenerate, for $\phi \in \mathbb{O}_{n-2}$, then $\bar{\omega}^\circ = \mathrm{id}_{(\phi \xrightarrow{\mathsf{t}\mathsf{t}} \omega)}$, while $\left(\mathsf{t}\,\omega \xrightarrow{\mathsf{t}} \omega\right) = \left(\mathsf{Y}_\phi \xrightarrow{\mathsf{t}} \omega\right)$. By **[Degen]**, those two cells are parallel.

(2) For the rest of the proof, we assume that ω is not degenerate. We have $\mathsf{t}\,\bar{\omega}^\circ = \mathsf{t}\,\mathsf{s}_{[\varepsilon]}\,\bar{\omega} = \mathsf{t}\left(\mathsf{t}\,\mathsf{s}_{[\varepsilon]}\,\omega \xrightarrow{\mathsf{t}\,\mathsf{s}_{[\varepsilon]}} \omega\right) = \left(\mathsf{t}\,\mathsf{t}\,\omega \xrightarrow{\mathsf{t}\mathsf{t}} \omega\right) = \mathsf{t}\left(\mathsf{t}\,\omega \xrightarrow{\mathsf{t}} \omega\right)$.

Then, in order to show that $\mathsf{s}\,\bar{\omega}^\circ = \mathsf{s}(\mathsf{t}\,\omega \xrightarrow{\mathsf{t}} \omega)$, we show that the $(n-2)$-generators occurring both sides are the same, and that the way to compose them is unique.

(a) Generators in $\mathsf{s}\,\bar{\omega}^\circ$ are of the form $\left(\phi \xrightarrow{[q]} \psi \xrightarrow{[p]} \omega\right)$, for $[p[q]]$ a leaf of ω. By **[Glob2]**, those are equal to $\left(\phi \xrightarrow{\wp_\omega[p[q]]} \mathsf{t}\,\omega \xrightarrow{\mathsf{t}} \omega\right)$, which are exactly the generators in $\mathsf{s}\left(\mathsf{t}\,\omega \xrightarrow{\mathsf{t}} \omega\right)$.

(b) To show that the composite of all $(n-2)$-generators of the form $(\phi \xrightarrow{\mathsf{s}_{[q]}} \psi \xrightarrow{\mathsf{s}_{[p]}} \omega)$, for $[p[q]] \in \langle\omega\rangle^|$, is unique, it is enough to show that no two have the same target. Assume $(\phi_i \xrightarrow{\mathsf{s}_{[q_i]}} \psi_i \xrightarrow{\mathsf{s}_{[p_i]}} \omega)$, with $i = 1, 2$, are $(n-2)$-generators in $\mathsf{s}\,\bar{\omega}^\circ$ with the same target. Consider the following diagram:

$$\begin{tikzcd}
& \phi_1 \arrow[r, "\mathsf{s}_{[q_1]}"] \arrow[dr, "\mathsf{s}_{\wp_\omega[p_1[q_1]]}"] & \psi_1 \arrow[dr, "\mathsf{s}_{[p_1]}"] & \\
\rho \arrow[ur, "\mathsf{t}"] \arrow[dr, "\mathsf{t}"'] & & \mathsf{t}\,\omega \arrow[r, "\mathsf{t}"] & \omega \\
& \phi_2 \arrow[r, "\mathsf{s}_{[q_2]}"'] \arrow[ur, "\mathsf{s}_{\wp_\omega[p_2[q_2]]}"] & \psi_2 \arrow[ur, "\mathsf{s}_{[p_2]}"'] &
\end{tikzcd}$$

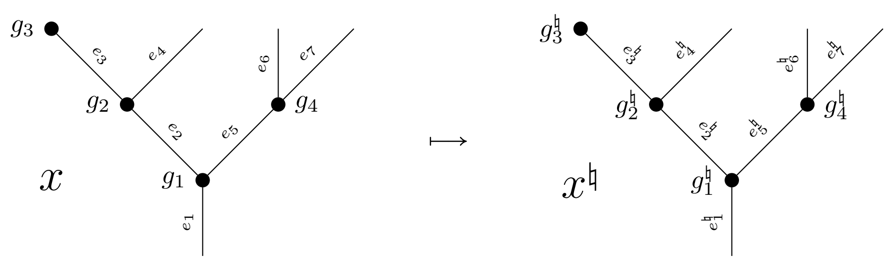
The outer hexagon commutes by assumption, the two squares on the right are instances of **[Glob2]**, and the left square commutes as $\mathsf{t} : \mathsf{t}\,\omega \longrightarrow \omega$ is a mono, since ω is non degenerate. By inspection of the opetopic identities, the only way for the left square to commute is the trivial way, i.e. $\wp_\omega[p_1[q_1]] = \wp_\omega[p_2[q_2]]$. Since $\wp_\omega$ is a bijection, we have $[p_1[q_1]] = [p_2[q_2]]$, thus $[p_1] = [p_2]$ and $[q_1] = [q_2]$. □

By the previous proposition, there is a well defined cellular extension

$$O[\omega] = \left(\partial O[\omega] \xleftarrow{\mathsf{s},\mathsf{t}} \{\omega\} \right)$$

where s and t map ω to $\bar{\omega}^\circ$ and $(\mathsf{t}\,\omega \xrightarrow{\mathsf{t}} \omega)$, respectively. The induction hypothesis holds by definition.

5.2. **The shape function.** This subsection is devoted to define the *shape function* $(-)^\natural$. We first sketch the idea. Take $P \in \mathcal{P}\mathrm{ol}^\triangledown$ and define $(-)^\natural : P_n \longrightarrow \mathbb{O}_n$ by induction. The cases $n = 0, 1$ are trivial, since there is a unique 0-opetope and a unique 1-opetope. Assume $n \geq 2$, and take $x \in P_n$. Then the composition tree of $\mathsf{s}\,x$ is a coherent tree whose nodes are $(n-1)$-generators, and edges are $(n-2)$-generators. Replacing those $(n-1)$ and $(n-2)$-generators by their respective shape, we obtain a coherent tree whose nodes are $(n-1)$-opetopes, and edges are $(n-2)$-opetopes, in other words, we obtain an n-opetope, which we shall denote by $x^\natural$.



The fact that $x^\natural$ corresponds to the intuitive notion of "shape" of x is justified by theorem 5.6. The rest of this subsection makes this sketch formal. We first define a many-to-one polygraph $\mathtt{1}$, that will turn out in proposition 5.4 to be terminal in $\mathcal{P}\mathrm{ol}^\triangledown$. We then proceed to define the shape function for $\mathtt{1}$, before stating the general case.

We set $\mathtt{1}_0 := \{\blacklozenge\}$, and $\mathtt{1}_{n+1} := \left\{ (u, v) \in \mathtt{1}_n^\triangledown \times \mathtt{1}_n \mid u \parallel v \right\}$, with $\mathsf{s}(u,v) := u$ and $\mathsf{t}(u,v) := v$.

Lemma 5.3. *If $x, y \in \mathtt{1}_n$ are two parallel generators, then they are equal.*

Proof. We have $x = (\mathsf{s}\,x, \mathsf{t}\,x) = (\mathsf{s}\,y, \mathsf{t}\,y) = y$. □

Proposition 5.4. *The polygraph $\mathbb{1}$ is terminal in $\mathcal{P}\mathrm{ol}^{\triangledown}$.*

Proof. Let $P \in \mathcal{P}\mathrm{ol}^{\triangledown}$, we show that there exists a unique $! : P \longrightarrow \mathbb{1}$.

(1) (Existence) For $x \in P_0$ let $!_0 x = \blacklozenge$, and for $x \in P_n$ with $n \geq 1$, let $!_n x = (!_{n-1} \mathsf{s}\, x, !_{n-1} \mathsf{t}\, x)$. The source and target compatibility is trivial.

(2) (Uniqueness) Let $f : P \longrightarrow \mathbb{1}$ be another functor. Then necessarily $f_0 =!_0$ as $\mathbb{1}_0$ is a singleton. Let $x \in P_n$ be such that $f_n x \neq !_n x$, with n minimal. By previous remark, $n \geq 1$, and we have $!_n x = (!_{n-1} \mathsf{s}\, x, !_{n-1} \mathsf{t}\, x) = (f_{n-1} \mathsf{s}\, x, f_{n-1} \mathsf{t}\, x) \parallel f_n x$. Hence by previous lemma, $!_n x = f_n x$, a contradiction. ∎

Proposition 5.5. *For $x \in \mathbb{1}_n$ there exists a unique $x^\natural \in \mathbb{O}_n$ such that the terminal morphism $!^{x^\natural} : O[x^\natural] \longrightarrow \mathbb{1}$ maps $x^\natural$ to x.*

Proof. (1) (Uniqueness) Assume $\phi, \phi' \in \mathbb{O}_k$, $\phi \neq \phi'$, are such that $!^{\phi}_k(\phi) =!^{\phi'}_k(\phi')$, with k minimal for this property. Then necessarily, $k \geq 2$. On the one hand, we have $\langle \phi \rangle = \langle \mathrm{ct}\, \mathsf{s}\, \phi \rangle = \langle \mathrm{ct}\, \mathsf{s}\, \phi' \rangle = \langle \phi' \rangle$. On the other hand, for $[p] \in \phi^\bullet = (\phi')^\bullet$, we have $!^{\mathsf{s}_{[p]}\, \phi}_{k-1} \mathsf{s}_{[p]}\, \phi =!^{\phi}_{k-1} \mathsf{s}_{[p]}\, \phi = \mathsf{s}_{[p]} !^{\phi}_k \phi = \mathsf{s}_{[p]} !^{\phi'}_k \phi' =!^{\phi'}_{k-1} \mathsf{s}_{[p]}\, \phi' = !^{\mathsf{s}_{[p]}\, \phi'}_{k-1} \mathsf{s}_{[p]}\, \phi'$, and by minimality of k, we have $\mathsf{s}_{[p]}\, \phi = \mathsf{s}_{[p]}\, \phi'$, for all address $[p]$. Consequently, $\phi = \phi'$, a contradiction.

(2) (Existence) The cases $n = 0, 1$ are trivial, so assume $n \geq 2$, and that by induction, the result holds for all $k < n$. For $g \in P_k$, there is a unique opetope $g^\natural \in \mathbb{O}_k$ such that $!^{g^\natural}_k g^\natural = g$. In particular the following two triangles commute:

$$
\begin{tikzcd}
O[\mathsf{s}_{[p]}\, g^\natural] \arrow[r, "O[\mathsf{s}_{[p]}]"] \arrow[dr, "!^{\mathsf{s}_{[p]}\, g^\natural}"'] & O[g^\natural] \arrow[d, "!^{g^\natural}"] \\
& P,
\end{tikzcd}
\qquad
\begin{tikzcd}
O[\mathsf{t}\, g^\natural] \arrow[r, "O[\mathsf{t}]"] \arrow[dr, "!^{\mathsf{t}\, g^\natural}"'] & O[g^\natural] \arrow[d, "!^{g^\natural}"] \\
& P,
\end{tikzcd}
$$

where $[p] \in (g^\natural)^\bullet$. Consequently, $(\mathsf{s}_{[p]}\, g)^\natural = \mathsf{s}_{[p]}\, g^\natural$ and $(\mathsf{t}\, g)^\natural = \mathsf{t}\, g^\natural$, and the following displays an isomorphism $\triangledown_{n-1} \mathbb{1} \xrightarrow{\cong} \mathfrak{Z}^{n-2}$:

$$
\begin{tikzcd}
& \mathbb{1}^\bullet_{n-1} \arrow[r, "p"] \arrow[dl, "\mathsf{s}"'] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathbb{1}_{n-1} \arrow[d, "(-)^\natural"'] \arrow[dr, "\mathsf{t}"] & \\
\mathbb{1}_{n-2} \arrow[d, "(-)^\natural"'] & \mathbb{O}^\bullet_{n-1} \arrow[r, "p"'] \arrow[dl, "\mathsf{s}"] & \mathbb{O}_{n-1} \arrow[dr, "\mathsf{t}"'] & \mathbb{1}_{n-2} \arrow[d, "(-)^\natural"] \\
\mathbb{O}_{n-2} & & & \mathbb{O}_{n-2}.
\end{tikzcd}
$$

Hence, the composite $x^\natural = \left(\langle \mathrm{ct}\, \mathsf{s}\, x \rangle \xrightarrow{\mathrm{ct}\, \mathsf{s}\, x} \triangledown_{n-1} \mathbb{1} \xrightarrow{(-)^\natural} \mathfrak{Z}^{n-2} \right)$ defines an n-opetope with $\langle x^\natural \rangle = \langle \mathrm{ct}\, \mathsf{s}\, x \rangle$.

We claim that $!^{x^\natural}_n x^\natural = x$. We first show that $!^{x^\natural}_{n-1} \mathsf{s}\, x^\natural = \mathsf{s}\, x$. We have $\langle \mathrm{ct}\, \mathsf{s}\, x \rangle = \langle x^\natural \rangle = \langle \mathrm{ct}\, \mathsf{s}\, x^\natural \rangle = \langle \mathrm{ct} !^{x^\natural}_{n-1} \mathsf{s}\, x^\natural \rangle$. Then, for any address $[p]$ in $\langle \mathrm{ct}\, \mathsf{s}\, x \rangle$, we have $\mathsf{s}_{[p]}\, x = !^{x^\natural}_{n-1} (\mathsf{s}_{[p]}\, x)^\natural = !^{x^\natural}_{n-1} \mathsf{s}_{[p]}\, x^\natural = \mathsf{s}_{[p]} !^{x^\natural}_n x^\natural$. Then $\mathsf{t} !^{x^\natural}_n x^\natural = !^{x^\natural}_{n-1} \mathsf{t}\, x^\natural \parallel !^{x^\natural}_{n-1} \mathsf{s}\, x^\natural = \mathsf{s}\, x \parallel \mathsf{t}\, x$, thus $\mathsf{t} !^{x^\natural}_n x^\natural = \mathsf{t}\, x$, and finally, as $!^{x^\natural}_n x^\natural \parallel x$, we have $!^{x^\natural}_n x^\natural = x$. $\square$

In the light of this proposition, we identify $\imath_n = \mathbb{O}_n$. This identification is compatible with faces, i.e. $\mathsf{s}_{[p]}$ and t. Then, $!^\omega : O[\omega] \longrightarrow \imath$ maps a generator $(\phi \to \omega)$ to ϕ.

Theorem 5.6. *For $P \in \mathcal{P}\mathrm{ol}^\triangledown$ and $x \in P_n$, there exists a unique pair*

$$\left(x^\natural, O[x^\natural] \xrightarrow{\tilde{x}} P \right) \in O[-]/P$$

such that $\tilde{x}_n (x^\natural) = x$. Moreover, $x^\natural = !^P_n x$. Thus, the map

$$\widetilde{(-)} : P_n \longrightarrow \coprod_{\omega \in \mathbb{O}_n} \mathcal{P}\mathrm{ol}^\triangledown (O[\omega], P)$$

is an isomorphism.

Proof. (1) (Uniqueness) Assume $O[\omega] \xrightarrow{f} P \xleftarrow{f'} O[\omega']$ are such that $f_n(\omega) = x = f'_n(\omega')$. Then $!^\omega_n(\omega) = !^P f_n(\omega) = !^P f'_n(\omega') = !^{\omega'}(\omega')$, hence $\omega = \omega'$. Let $\left(\phi \xrightarrow{a} \omega\right) \in O[\omega]_k$ be such that $f_k \left(\phi \xrightarrow{a} \omega\right) \neq f'_k \left(\phi \xrightarrow{a} \omega\right)$, with k minimal for this property. Then $k < n$, and a factorizes as $\left(\phi \xrightarrow{\mathsf{j}} \psi \xrightarrow{b} \omega\right)$, where j is a face embedding, i.e. either t or $\mathsf{s}_{[p]}$ for some p. Then by assumption, $f_k \left(\phi \xrightarrow{a} \omega\right) = \mathsf{j}\, f_{k+1} \left(\psi \xrightarrow{b} \omega\right) = \mathsf{j}\, f'_{k+1} \left(\psi \xrightarrow{b} \omega\right) = f'_k \left(\phi \xrightarrow{a} \omega\right)$, a contradiction.

(2) (Existence) The cases $n = 0, 1$ are trivial, so assume $n \geq 2$, and that by induction, the result holds for all $k < n$. Let $x^\natural = !^P_n x \in \mathbb{O}_n$. We wish to construct a morphism $O[x^\natural] \xrightarrow{\tilde{x}} P$ having x in its image. For $\left(\psi \xrightarrow{\mathsf{j}} x^\natural\right)$ a face of $x^\natural$ (i.e. $\mathsf{s}_{[p]}$ for some $[p]$, or t), we have $(\mathsf{j}\, x)^\natural = \psi$, so that by induction, there exists a morphism $O[\psi] \xrightarrow{\widetilde{\mathsf{j}x}} P$ having $\mathsf{j}\, x$ in its image, yielding a commutative square

$$\begin{tikzcd} O[\psi] \arrow[r, "\widetilde{\mathsf{j}x}"] \arrow[d, "{O[\mathsf{j}]}"'] & P \arrow[d, "!^P"] \\ O[x^\natural] \arrow[r, "!^{x^\natural}"] & \imath. \end{tikzcd}$$

To alleviate upcoming notations, write $\bar{\mathsf{j}} := \widetilde{\mathsf{j}x}$. Let $\left(\phi \xrightarrow{a} x^\natural\right) \in \mathbb{O}_{<n}/x^\natural$. If a is a face embedding, define $\bar{a}$ as before. If not, then it factors through a face embedding as $\left(\phi \xrightarrow{\mathsf{j}} \psi \xrightarrow{b} \omega\right)$, and let $\bar{a} := \bar{b} \circ O[\mathsf{j}]$. Then the left square

commutes, and passing to the colimit $\mathbb{O}_{<n}/x^\natural$ we obtain the right square:

$$\begin{tikzcd} O[\phi] \arrow[r, "\bar{a}"] \arrow[d, "O[a]"'] & P \arrow[d, "!^P"] \\ O[x^\natural] \arrow[r, "!^{x^\natural}"] & 1, \end{tikzcd} \qquad \begin{tikzcd} \partial O[x^\natural] \arrow[r, "f"] \arrow[d] & P \arrow[d, "!^P"] \\ O[x^\natural] \arrow[r, "!^{x^\natural}"] & 1. \end{tikzcd}$$

We want a lift of the right square, and by the universal property of the cellular extension, it is enough to check that $f_{n-1}\,\mathsf{s}\,x^\natural = \mathsf{s}\,x$, and $f_{n-1}\,\mathsf{t}\,x^\natural = \mathsf{t}\,x$. The second equality is clear, as f extends $\bar{\mathsf{t}}$, and $f_{n-1}\,\mathsf{t}\,x^\natural = \bar{\mathsf{t}}_{n-1}\,\mathsf{t}\,x^\natural = \mathsf{t}\,x$ by definition. We now proceed to prove the first one. First, $\left\langle \mathrm{ct}\,\mathsf{s}\,x^\natural \right\rangle = \langle \mathrm{ct}\,\mathsf{s}\,x\rangle$ since both are mapped to the same element of 1_n. Then, for $[p]$ an address in $\mathrm{ct}\,\mathsf{s}\,x^\natural$, we have $f_{n-1}\,\mathsf{s}_{[p]}\,x^\natural = \overline{\mathsf{s}_{[p]}}_{n-1}\,\mathsf{s}_{[p]}\,x^\natural = \mathsf{s}_{[p]}\,x$. Hence $f_{n-1}\,\mathsf{s}\,x^\natural = \mathsf{s}\,x$. $\square$

5.3. **The adjoint equivalence.** We have the opetal functor $O[-] : \mathbb{O} \longrightarrow \mathcal{P}\mathrm{ol}^\triangledown$. This gives rise to an adjunction

$$|-| : \hat{\mathbb{O}} \overset{\dashv}{\longleftrightarrow} \mathcal{P}\mathrm{ol}^\triangledown : N,$$

where $|-| := \mathrm{Lan}_{\mathsf{y}}\, O[-]$ is the left Kan extension of $O[-]$ along the Yoneda embedding $\mathsf{y} : \mathbb{O} \longrightarrow \hat{\mathbb{O}}$, and N is given by

$$NP := \mathcal{P}\mathrm{ol}^\triangledown (O[-], P) : \mathbb{O}^{\mathrm{op}} \longrightarrow \mathcal{S}\mathrm{et},$$

for $P \in \mathcal{P}\mathrm{ol}^\triangledown$. We note $\eta : \mathrm{id}_{\hat{\mathbb{O}}} \longrightarrow N|-|$ the unit, $\varepsilon : |N| \longrightarrow \mathrm{id}_{\mathcal{P}\mathrm{ol}^\triangledown}$ the counit, and $\Phi : \hat{\mathbb{O}}(-, N) \xrightarrow{\cong} \mathcal{P}\mathrm{ol}^\triangledown(|-|, -)$ the natural hom-set isomorphism.

For $X \in \hat{\mathbb{O}}$, its realization can be written as

$$|X| = \frac{\coprod_{\omega \in \mathbb{O}} X_\omega \times O[\omega]}{\begin{pmatrix} f^* x \\ \phi \xrightarrow{h} \psi \end{pmatrix} \sim \begin{pmatrix} x \\ \phi \xrightarrow{h} \psi \xrightarrow{f} \omega \end{pmatrix} \text{ with } x \in X_\omega, h : \phi \to \psi, f : \psi \to \omega}.$$

In particular, all classes have a representative of the form $\left[\begin{smallmatrix} y \\ \mathrm{id}_\theta \end{smallmatrix}\right]$, for some $y \in X_\theta$.

Proposition 5.7. *Take $X \in \hat{\mathbb{O}}$, $P \in \mathcal{P}\mathrm{ol}^\triangledown$, and $f : X \longrightarrow NP$. The unit η at X, the transpose Φf of f, and the counit ε at P are respectively given by:*

$$\begin{aligned} \eta : X_\omega &\longrightarrow N|X|_\omega \\ x &\longmapsto \widetilde{\begin{bmatrix} x \\ \mathrm{id}_\omega \end{bmatrix}}, \end{aligned} \qquad \begin{aligned} \Phi f : |X|_\omega &\longrightarrow P_\omega \\ \begin{bmatrix} x \\ \mathrm{id}_\omega \end{bmatrix} &\longmapsto f(x)(\omega), \end{aligned} \qquad \begin{aligned} \varepsilon : |NP|_\omega &\longrightarrow P_\omega \\ \begin{bmatrix} \tilde{x} \\ \mathrm{id}_\omega \end{bmatrix} &\longmapsto x. \end{aligned}$$

Proof. (1) (Unit and transpose) We have to check that the following diagram commutes

$$\begin{tikzcd} X \arrow[r, "\eta"] \arrow[d, "f"'] & N|X| \arrow[dl, "N\Phi f"] \\ NP, & \end{tikzcd}$$

and that f is unique for that property. For $x \in X_\omega$ we have

$$(N\Phi f)\eta(x) = (N\Phi f)\left(\widetilde{\begin{bmatrix} x \\ \mathrm{id}_\omega \end{bmatrix}}\right) = (\Phi f)\widetilde{\begin{bmatrix} x \\ \mathrm{id}_\omega \end{bmatrix}},$$

which maps ω to $f(x)(\omega)$. Since a map $O[\omega] \longrightarrow P$ is uniquely determined by the image of ω, we have $(N\Phi f)\eta = f$. Let $g : |X| \longrightarrow P$ be another morphism such that $(N\Phi g)\eta = f$. Then for $x \in X_\omega$ we have

$$g(x)(\omega) = (\Phi g)\left(\begin{bmatrix} x \\ \mathrm{id}_\omega \end{bmatrix}\right) = (\Phi g)\widetilde{\begin{bmatrix} x \\ \mathrm{id}_\omega \end{bmatrix}}(\omega) = f(x)(\omega)$$

whence $f = g$.

(2) (Counit) The counit is given by $\varepsilon = \Phi(\mathrm{id}_{NP})$, so that

$$\varepsilon\begin{bmatrix} \tilde{x} \\ \mathrm{id}_\omega \end{bmatrix} = (\Phi\, \mathrm{id}_{NP})\begin{bmatrix} \tilde{x} \\ \mathrm{id}_\omega \end{bmatrix} = \tilde{x}(\omega) = x.$$

□

Theorem 5.8. *The unit and counit are natural isomorphisms. Consequently, the opetopic "nerve – realization" adjunction displays an adjoint equivalence between* $\hat{\mathbb{O}}$ *and* $\mathcal{P}\mathrm{ol}^{\triangledown}$.

Proof. (1) (Unit) Remark that for $x, y \in X_\omega$, if $\left[\begin{smallmatrix} x \\ \mathrm{id}_\omega \end{smallmatrix}\right] = \left[\begin{smallmatrix} y \\ \mathrm{id}_\omega \end{smallmatrix}\right]$, then $x = y$, which shows that η is injective. Take $f \in N|X|_\omega$. Then $f(\omega)$ is of the form $\left[\begin{smallmatrix} x \\ \mathrm{id}_\omega \end{smallmatrix}\right]$, hence $f = \eta(x)$, and η is surjective.

(2) (Counit) The following triangle identity shows that $N\varepsilon$ is a natural isomorphism:

$$\begin{tikzcd} N \arrow[rr, equal] \arrow[dr, "\eta N"'] & & N \\ & N|N-|. \arrow[ur, "N\varepsilon"'] & \end{tikzcd}$$

It is easy to check that the following square commutes, and since $\widetilde{(-)}$ is a bijection by theorem 5.6, ε is a natural isomorphism:

$$\begin{tikzcd} {|NP|} \arrow[r, "\varepsilon"] \arrow[d, "\widetilde{(-)}"'] & P \arrow[d, "\widetilde{(-)}"] \\ N|NP| \arrow[r, "N\varepsilon"] & NP. \end{tikzcd}$$

□

Many-to-one polygraphs have been the subject of other work [7, 8], and proved to be equivalent to the notion of *multitopic sets*. This, together with our present contribution, proves the following:

Corollary 5.9. *The category* $\hat{\mathbb{O}}$ *of opetopic sets is equivalent to the category of multitopic sets.*

In [9], Henry shows that $\mathcal{P}\mathrm{ol}^{\triangledown}$ a presheaf category: $\mathcal{P}\mathrm{ol}^{\triangledown} \simeq \widehat{\mathbb{O}\mathrm{plex}}$, where $\mathbb{O}\mathrm{plex}$ is the category of "opetopic plexes".

Corollary 5.10. *The category* $\mathbb{O}\mathrm{plex}$ *of opetopic plexes is equivalent to* $\mathbb{O}$.

Proof. Opetopic plexes are proved to be generators of the terminal many-to-one polygraph $\imath$ in [9, Proposition 2.2.3], and so together with proposition 5.5, we have that opetopic plexes are exactly opetopes. On the other hand, morphisms of opetopic plexes are by definition morphisms of polygraphs between the representables they induce, which by the Yoneda lemma are exactly morphisms of opetopes. $\square$

6. Conclusion

We proved the equivalence between opetopic sets (where "opetope" is understood in the sense of Leinster [14, 13]) and many-to-one polygraphs. Along the way, we introduced formal tools and notations to ease the manipulation of opetopes.

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