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Small irreducible components of arc spaces in positive characteristic

Angélica Benito * Olivier Piltant † Ana J. Reguera ‡

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Abstract

We study the Irreducibility Problem for the arc space $X_\infty$ of an irreducible singular algebraic variety $X$ defined over a perfect field $k$ of characteristic $p > 0$. The existence of reducible such $X_\infty$ is related to the fact that Kolchin’s Irreducibility Theorem does not extend to positive characteristic. We obtain a complete characterization when $X$ is a surface. Section 6 introduces two main new problems in arbitrary dimension: in the line of O. Zariski and H. Hironaka, blowing up any $X$ to get $Y \to X$ with $Y_\infty$ irreducible; in the line of J. Nash’s work, characterizing irreducibility in terms of Resolution of Singularities.

1 Introduction

In 1968, J. Nash initiated the study of the space of arcs $X_\infty$ of a (singular) algebraic variety $X$ with the purpose of understanding the structure of the various resolutions of singularities of $X$. His work [10] was done shortly after

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Hironaka’s proof of Resolution of Singularities in characteristic zero [5].

Nash’s starting point was the following: let $X$ be a variety over a field $k$ of characteristic zero with a given resolution of singularities $\pi: Y \to X$. For every irreducible component $E$ of the exceptional locus of $\pi$, the Nash family of arcs $N_E \subset X_\infty$ is defined to be the Zariski closure of the image of the set of arcs on $Y$ which are centered at some point of $E$. He observed that each $N_E$ is irreducible and, moreover, $N_E$ only depends on the divisorial valuation computing the order along $E$. Due to the properness of $\pi$, every arc in $X_\infty \setminus (\text{Sing} X)_\infty$ which is centered at some point of the singular locus of $X$ belongs to some of the $N_E$’s since it lifts to $Y$. That is, the space of arcs $X_\infty^{\text{Sing}}$ centered in $\text{Sing} X$ decomposes as

$$X_\infty^{\text{Sing}} = \bigcup E N_E \cup (\text{Sing} X)_\infty.$$  \hfill (1.1)

From this, and arguing by induction on $\dim X$, one deduces that the number of irreducible components of $X_\infty^{\text{Sing}}$ is finite (see [10] or [7], [12]). It is in general not easy to deduce the decomposition of $X_\infty^{\text{Sing}}$ into its irreducible components from (1.1); the Nash problem consists precisely in characterizing these irreducible components.

This Nash program extends, with some important differences, to perfect ground fields $k$ of characteristic $p > 0$. A first obvious difference is that Resolution of Singularities is still an open problem if $\text{char} k = p > 0$ and $\dim X \geq 4$. Although Nash families $N_E$ can be defined only in terms of divisorial valuations, it is not known that the indexing set in (1.1) can be chosen to be a finite set. In particular, it is unknown if the number of irreducible components of $X_\infty^{\text{Sing}}$ is always finite [11].

Another difference is that, in contrast with characteristic zero, the right hand side term $(\text{Sing} X)_\infty$ in (1.1) may contain some of the irreducible components of $X_\infty^{\text{Sing}}$. Understanding these “small” components (called small because they consist of families of arcs concentrated inside the singular locus) is the main purpose of this article.

In the spirit of the Nash program, we propose characterizing all small irreducible components of $X_\infty^{\text{Sing}}$ in terms of birational morphisms $Y \to X$, or in terms of the various resolutions of singularities of $X$ whenever they exist. We obtain a satisfactory criterion for small components mapping in $X$ onto an irreducible component of $\text{Sing} X$ with codimension one in $X$. 

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In particular we identify all small components of $X_{\text{Sing}}$ when $X$ is any surface (corollary 5.9). In higher dimensions the question remains open except in special cases. Among them, an example which is extensively studied is that of the total space of a family of hypersurface cones

$$X_B := \text{Spec} \frac{\mathcal{O}_B[y_0, y_1, \ldots, y_n]}{(f)}$$

$$f := y_0^p + b_1 y_1^p + \cdots + b_n y_n^p, \quad b_1, \ldots, b_n \in \mathcal{O}_B,$$

where $\text{char} \, k = p > 0$ and the base $k$-variety $B$ is regular and irreducible. This subtle example is considered here because it conveys interesting difficulties showing up in the Resolution of Singularities Conjecture [6].

Underlying the existence of small components when $\text{char} \, k = p > 0$ is the fact that Kolchin’s irreducibility theorem is not valid in positive characteristic. If $X = \text{Spec} R$ is an irreducible affine variety of characteristic zero, then $R_{\infty} := \mathcal{O}_{X_{\infty}}$ is isomorphic to the differential algebra associated with $R$ which Kolchin proved to be irreducible, i.e. the reduced ring $R_{\text{red}}$ is a domain. It was known to Kolchin that this statement does not hold in general when $\text{char} \, k = p > 0$; the most simple counterexample is the irreducible surface

$$X := V(y^p + zx^p) \subset \mathbb{A}^3_k = \text{Spec} k[x, y, z]$$

where Sing$X = V(x, y)$ is the $z$-axis and $X_{\infty}$ has two irreducible components, $(\text{Sing}X)_{\infty}$ and the Zariski closure of its complement in $X_{\infty}$. Obviously, any irreducible component of $X_{\infty}$ contained in $(\text{Sing}X)_{\infty}$ is also an irreducible component of $X_{\text{Sing}}$.

This article is organized as follows. In section two, we state and reprove Local Uniformization of varieties along formal arcs (proposition 2.3) which plays an important role in this article. It allows us to give a simple new proof of the fact that the arc space $X_\infty$ of a $k$-variety $X$ has finitely many irreducible components (proposition 2.1) in any characteristic. Every irreducible component $C$ of $X_\infty$ maps to an irreducible subvariety $Z_C$ and its generic point is the generic arc in Reg$Z_C$. Section 3 introduces basic valuation theory and the notion of Local Uniformization of valuations.

In section 4, we introduce the new terminology arc-sharp/arc-blunt. Given a scheme-theoretic point $\zeta \in X$, $X$ is said to be arc-sharp at $\zeta$ if $\zeta$ is a generic point of $X$, or if the generic arc in Reg$Z$, $Z := \overline{\{\zeta\}}$, is not a specialization of a generic arc in Reg$X$; otherwise, $X$ is said to be arc-blunt at $\zeta$. Using
Local Uniformization along formal arcs, we give a simple necessary condition for arc-sharpness in Theorem 4.4: if $X$ admits a resolution of singularities $\pi : Y \to X$, then $X$ is arc-sharp at $z$ only if $\pi^{-1}(\zeta)$ has no separable point over $k(\zeta)$. Since we work without the Resolution assumption, our criterion is formulated in terms of prime divisors centered at $z$, see Proposition 4.3 and Theorem 4.4 for a precise formulation. An interesting corollary is that $X$ is arc-sharp at $z$ only if its multiplicity at $\zeta$ is divisible by $p$ (Corollary 4.5).

Section 5 provides a sufficient condition for arc-sharpness, stated as Theorem 5.5. For a point $\zeta$ of codimension one in $X$, both necessary and sufficient conditions coincide and this proves our main result characterizing small irreducible components of $X_\infty$ whose generic point is the generic arc in $\text{Reg}_Z$, $Z := \{\zeta\}$. Our result is more general and is phrased in terms of radicial morphisms which are relevant here. The proof uses Frobenius techniques which are original in the context of arc spaces.

Section 6 contains several questions which came out of this work. Question 6.1 asks whether any small irreducible component of $X_\infty^{\text{Sing}}$ is necessarily an irreducible component of the whole space $X_\infty$. Question 6.2 asks whether our characterization of arc-sharpness for points $\zeta$ of codimension one in $X$ is also valid in higher codimension, at least for $p > \dim \mathcal{O}_{X, \zeta}$. Finally question 6.6 is a mild, but apparently challenging problem of Resolution of Singularities type: can small irreducible components be eliminated by blowing up? More precisely, we ask whether every irreducible proper variety $X$ is birationally equivalent to a proper variety $Y$ (resp. admits a proper birational morphism $Y \to X$) such that $Y_\infty$ is irreducible. Using a classical theorem of Albanese [9], we answer the “birationally equivalent” version in the affirmative for $X$ of arbitrary dimension and all big enough characteristics $p > (\dim X)!$. These questions are tested on a family of varieties with equation (1.2).

2 The irreducibility problem on arc spaces.

Let $k$ be a perfect field. By a variety over $k$, we mean a reduced separated $k$-scheme of finite type. Given a variety $X/k$, let $X_\infty/k$ denote the space of arcs of $X$. It represents the functor on $k$-algebras $A \mapsto X(A[[t]])$. If $X \subset \mathbb{A}^N_k = \text{Spec } k[x_1, \ldots, x_N]$ is affine with ideal $I_X = (f_1, \ldots, f_r)$, we pick infinitely many variables $X_n = (X_{1,n}, \ldots, X_{N,n})$, $n \geq 0$. For $1 \leq j \leq r$,
\[ f_j(\sum_n X_n t^n) = \sum_{n=0}^{\infty} F_{j,n} t^n, \]  
so we get a description of \( X_\infty \) as 
\[ X_\infty = \text{Spec} \left( \frac{k[\{X_n\}_{n \geq 0}]}{(\{F_{1,n}, \ldots, F_{r,n}\}_{n \geq 0})} \right). \]

There is a natural map \( j : X_\infty \to X \). Given a subvariety \( Z \subseteq X \), we denote by \( X_Z \) the subscheme \( j^{-1}(Z) \) of \( X_\infty \). As a subset, \( X_Z \) consists of all arcs with center inside \( Z \). There is an inclusion \( Z_\infty \subseteq X_Z \) which is strict if \( Z \neq X \).

The scheme \( X_\infty \) is not of finite type over \( k \) if \( \dim X > 0 \). However it satisfies several finiteness properties. The following result refers to one of them. It is proved in [8] chap. IV, prop. 10 if \( \text{char } k = 0 \) and in [2] cor. 1.28, [12] th. 2.9 for any perfect field \( k \).

**Proposition 2.1.** ([8] and [2], [12]) Let \( X \) be a variety over a perfect field \( k \) with irreducible components \( X_1, \ldots, X_c \). Let \( Z \subseteq X \) be a nowhere dense subvariety such that 
\[ \bigcup_{i=1}^c \text{Sing} X_i \subseteq Z. \]

The natural map \( X_\infty \setminus Z_\infty \to X \) induces a bijection on irreducible components. In particular \( X_\infty \) has finitely many irreducible components.

The last part of the statement follows immediately from the first one: apply Noetherian induction on \( X \), changing \( X \) by its (reduced) singular locus \( \text{Sing} X \). We will give below an alternate proof of this statement as an application of uniformization along arcs (proposition 2.3 below). Uniformization also plays an important role in the next sections. First we consider the simpler case of a regular variety.

**Lemma 2.2.** Let \( Y \) be a regular \( k \)-variety. The following holds:

1. For every subvariety \( Z \subseteq Y \), the induced map \( Y_\infty^Z \to Z \) induces a bijection on irreducible components. More precisely, if \( Z = E_1 \cup \cdots \cup E_r \) is the decomposition of \( Z \) into irreducible components, then 
\[ Y_\infty^Z = Y_\infty^{E_1} \cup \cdots \cup Y_\infty^{E_r} \]
is the decomposition of \( Y_\infty^Z \) into irreducible components.
(2) If furthermore $Z \subseteq Y$ is nowhere dense, then $Y_\infty \setminus Y^Z_\infty = (Y \setminus Z)_\infty$ is Zariski dense in $Y_\infty$.

**Proof.** We may suppose that $Y$ is affine, let $Y \subseteq \mathbb{A}^m$. Picking a finite open covering of $Y$, we may assume that there exists an étale morphism from $Y$ to a subset of $\mathbb{A}^d$ where $d = \dim Y$. Then we have

$$O_{Y_\infty} \cong O_Y[X_1, X_2, \ldots, X_n, \ldots]$$

where, for $n \geq 0$, $X_n = (X_{1,n}, \ldots, X_{m,n})$ is a $m$-uple of variables. Then, (1) follows from the induced isomorphisms

$$O_{Y^Z_\infty} \cong O_Z[X_1, \ldots, X_n, \ldots] \quad \text{and} \quad O_{Y^i_\infty} \cong O_{E_i}[X_1, \ldots, X_n, \ldots] \quad 1 \leq i \leq r$$

and the fact that, for $1 \leq i \leq r$, $E_i$ is irreducible hence $O_{E_i}$ is a domain. Analogously, (2) follows from the fact that $Y \setminus Z$ is Zariski dense in $Y$. 

**Definition 2.1.** Given $P \in X_\infty$, with residue field $\kappa(P)$, we denote by

$$h_p : \text{Spec} \kappa(P)[[t]] \to X$$

the induced $\kappa(P)$-arc on $X$. We denote by $0$ and $\eta$ respectively the closed point and the generic point of $\text{Spec} \kappa(P)[[t]]$. The point $h_p(0) = j(P) \in X$ is called the center of $h_p$. The closure of $h_p(\eta)$ is called the support of $h_p$ and denote by $\Sigma(h_p)$. 

We denote by $v_p$ the order function $\text{ord}_t h_p^* : O_{X,h_p(0)} \to \mathbb{N} \cup \{\infty\}$. The arc $h_p$ is said to be nonconstant if $h_p(\eta) \neq h_p(0)$. 

Of course, there is a specialization $h_p(\eta) \sim h_p(0)$. Specialization in $X$ and $X_\infty$ will play an important role in this article.

Let $P \in X_\infty$, and $\pi : X' \to X$ be a blowing up along a subscheme $Y \subset X$ such that $h_p(\eta) \notin Y$. There exists a unique lifting

$$h'_p : \text{Spec} \kappa(P)[[t]] \to X'.$$

We have $x' := h'_p(0) \in \Sigma(h'_p)$ and $\pi$ is an isomorphism at $h'_p(\eta)$. Iterating, let

$$X \leftarrow X' \leftarrow \cdots \leftarrow X^{(r)} \leftarrow \cdots$$

be the resulting sequence of blowing ups and centers with

$$x^{(r)} \in \Sigma(h^{(r)}_p), \quad x^{(r)} \in Y^{(r)} \not\in \Sigma(h^{(r)}_p). \quad (2.2)$$
Note that the local ring $\mathcal{O}_{X^{(r)},h_P^{(r)}(\eta)}$ is independent of $r \geq 0$. An important case of such sequences is when taking $Y^{(r)} = \{x^{(r)}\}$ for every $r \geq 0$; then (2.2) is called the quadratic sequence along $h_P$.

**Proposition 2.3. (Uniformization along arcs)** Let $P \in X_\infty$. Assume that $h_P$ is a nonconstant arc and $h_P(\eta) \notin \text{Sing } X$. Consider the quadratic sequence (2.2) and let

$$\pi^{(r)} : X^{(r)} \to X, \text{ and } h_P^{(r)} : \text{Spec}(P)[[t]] \to X^{(r)}$$

be the corresponding morphisms for $r \geq 0$. Then both $\Sigma(h_P^{(r)})$ and $X^{(r)}$ are regular at $x^{(r)}$ for every $r >> 0$.

**Proof.** To prove the statement for $\Sigma(h_P^{(r)})$, it can be assumed without loss of generality that $(h_P) = X$. We build up a discrete invariant

$$i(x) := (a(x), b(x), e(x), \delta(x)) \in \mathbb{N}^4$$

which decreases for the lexicographical ordering by blowing up along $x$ provided $\mathcal{O}_{X,x}$ is not regular.

Let $v = v_P$ be the discrete valuation associated to $h_P$ and fix an isomorphism $v(k(h_P(\eta))\{0\}) \simeq \mathbb{Z}$. Consider the value semigroup

$$S(x) := \{v(f), \ 0 \neq f \in \mathcal{O}_{X,x}\} \subseteq \mathbb{N}.$$ 

Let $a(x) := \min\{S(x)\{0\}\}$. If $S(x) \neq \mathbb{N}$, we denote $b(x) := \min\{S(x)\setminus a(x)\mathbb{N}\}$. Note that we have

$$(a(x'), b(x')) < (a(x), b(x))$$

for the lexicographical ordering if $S(x) \neq \mathbb{N}$. In other terms, it can be assumed that

$$S(x^{(r)}) = \mathbb{N}, \ r \geq 0. \quad (2.4)$$

We denote:

$$e(x) := \text{emb.dim}_x X.$$ 

Since $e(x') \leq e(x)$, we may also assume that $e(x^{(r)}) = e := e(x)$ for every $r \geq 0$. In particular, we have

$$d(x^{(r)}) := \dim\mathcal{O}_{X^{(r)},x^{(r)}} = d$$
is constant and \( k(x^{(r)})/k(x) \) is finite algebraic. Let \( t := \text{tr.deg}_k k(x) \geq 0 \). We write

\[
\mathcal{O}_{X,x} = R/P,
\]

where \((R, M, k(x))\) is a regular local ring of dimension \( e \). There is nothing to prove if \( P = 0 \), so we assume that \( P \neq 0 \) from now on.

For \( f \in R \), we denote by \( \overline{f} \) its image in \( R/P \). Let \( (u_1, \ldots, u_e) \) be a r.s.p. of \( R \), with the convention that \( v(\overline{v_1}) = 1 \), viz. (2.4). Let

\[
\Omega := \Omega_{R/k}(\log u_1)
\]

be the module of Kähler differentials over \( k \) with logarithmic pole along \( u_1 \). Then \( \Omega \) is a free \( R \)-module with basis

\[
B = \left( \frac{du_1}{u_1}, \{du_j\}_{2 \leq j \leq e}, \{d\lambda_s\}_{1 \leq s \leq t} \right),
\]

where the images of \( \lambda_1, \ldots, \lambda_e \in R_x \) in \( k(x) \) form a separating transcendence basis of \( k(x)/k \). Let

\[
\Delta(x) \subseteq \text{Der}_{R/k}
\]

be the dual module of \( \Omega \) and define:

\[
\mathcal{D}(x) := \{ D \in \Delta(x), D \cdot M \subseteq M \}.
\]

The \( R \)-module \( \mathcal{D}(x) \) is generated by the family

\[
< u_1 \frac{\partial}{\partial u_1}, \{ M \frac{\partial}{\partial u_j} \}_{2 \leq j \leq e}, \{ \frac{\partial}{\partial \lambda_s} \}_{1 \leq s \leq t} > .
\]

We define

\[
\delta(x) := \min \{ v(D \cdot f), D \in \Delta(x), f \in P, D \cdot f \notin P \}.
\]

Note that since \( k \) is perfect, \( X \) is generically smooth over \( k \), so there exists \( f \in P \) and \( D \in \Delta(x) \) such that \( D \cdot f \notin P \). In particular, we have \( \delta(x) < +\infty \). Pick then \( D_0 \in \Delta(x), f \in P \) such that

\[
v(D_0 \cdot f) = \delta(x).
\]

W.l.o.g. it can be assumed that \( D_0 \) belongs to the dual basis \( B' \). Let \( m := \text{ord}_x f \geq 2 \). We now compute how \( \delta(x) \) transforms by blowing up along \( x \). To begin with, let

\[
\mathcal{O}_{X',x'} = R'/P', \quad R' = R[\frac{u_2}{u_1}, \ldots, \frac{u_e}{u_1}]_{x'}.
\]
We have \( f' := u_1^{-m} f \in P' \). Note that any \( D \in \mathcal{D}(x) \) extends to a derivation \( D' \in \Delta(x') \). On the other hand, we have
\[
(D' \cdot f') = u_1^{-m} \left( (D \cdot f) - m \frac{(D \cdot u_1)}{u_1} f \right)
\]
and we deduce that
\[
v(D' \cdot f') = v(D \cdot f) - m.
\]
Since \( u_1 \Delta(x) \subseteq \mathcal{D}(x) \), we apply the previous equality to \( D := u_1 D_0 \) to get
\[
\delta(x') \leq v(u_1 D_0 \cdot f) - m \leq \delta(x) - (m - 1) < \delta(x)
\]
and the conclusion follows. The proof of the proposition for \( X^{(r)} \) is similar.

\[ \square \]

**Proof of proposition 2.1:** there is a commutative diagram
\[
\begin{array}{ccc}
X \setminus Z & \xrightarrow{j_1} & X_i \setminus Z_i \\
\downarrow & & \downarrow \\
X \setminus Z_i & \subseteq & X
\end{array}
\]
By lemma 2.2(1), the left hand side arrow induces a bijection on irreducible components because
\[
X \setminus Z \subseteq \bigcap_{i=1}^c \operatorname{Reg} X_i.
\]
It is therefore sufficient to prove that
\[
j^{-1}(Z) \setminus Z_{\infty} \subseteq j^{-1}(X \setminus Z), \tag{2.7}
\]
where bars denote Zariski closure in \( X_{\infty} \). For \( P \in j^{-1}(Z) \setminus Z_{\infty} \), we have \( \langle h_P(0) \rangle \subseteq Z \) and \( h_P(\eta) \notin Z \). In particular \( h \) is not constant.

We now apply proposition 2.3. There exists a *regular* irreducible \( k \)-variety \( Y \), and a birational morphism \( f : Y \rightarrow X_i \), \( X_i \) an irreducible component of \( X \), such that \( h_P \) lifts to \( Y \). Furthermore, if \( W := f^{-1}(Z) \), by proposition 2.2(2), we have
\[
P \in f_{\infty}(\overline{j_Y^{-1}(Y \setminus W)}).
\]
Therefore (2.7) follows from the self-evident inclusions
\[
\overline{f_{\infty}(\overline{j_Y^{-1}(Y \setminus W)})} = \overline{f_{\infty}(j_Y^{-1}(Y \setminus W))} \subseteq j^{-1}(X \setminus Z).
\]
3 Reminder on valuations.

Let \( k \subset K \) be a field extension. By a \( k \)-valuation \( v \) of \( K \), we mean a valuation of \( K \) which is trivial on \( k \). The corresponding valuation ring is denoted by

\[
\mathcal{O}_v := \{ f \in K : v(f) \geq 0 \} \cup \{0\},
\]

and its maximal ideal by \( M_v := \{ f \in K : v(f) > 0 \} \cup \{0\} \). The residue field \( k_v := \mathcal{O}_v/M_v \) contains \( k \) as a subfield.

Given two local subrings \((R, M)\) and \((S, N)\) of \( K \) containing \( k \), we say that \( S \) dominates \( R \) if \( R \subseteq S \) and \( M = N \cap R \).

Let \( A \subseteq K \) be a \( k \)-subalgebra. We say that a \( k \)-valuation \( v \) has a center in \( X := \text{Spec}A \) if \( A \subseteq \mathcal{O}_v \). The centre \( x \in X \) is the prime ideal \( P := M_v \cap A \), so \( \mathcal{O}_v \) dominates \( A_P \).

Suppose a \( k \)-valuation \( \overline{v} \) of the residue field \( k_v \) is given. The ring

\[
R := \{ g \in \mathcal{O}_v : g \mod M_v \in \mathcal{O}_\overline{v} \}
\]

is the valuation ring of a \( k \)-valuation \( v_0 \), called composite of \( v \) with \( \overline{v} \), and denoted by \( v_0 = v \circ \overline{v} \). It has the following property: for every \( k \)-subalgebra \( A \subseteq K \) such that \( v_0 \) has a center \( x_0 \in X = \text{Spec}A \), \( v \) has a center \( x \in X \) and \( x_0 \) is a specialization of \( x \). More precisely, the prime ideal \( P_0 := M_v \cap \mathcal{O}_{v_0} \) satisfies the properties:

\[
\mathcal{O}_{v_0}/P_0 \simeq \mathcal{O}_{\overline{v}} \text{ and } (\mathcal{O}_{v_0})_{P_0} = \mathcal{O}_v.
\]


Let \( X|k \) be a \( k \)-variety with irreducible components \( X_1, \ldots, X_c \). By a \( k \)-valuation of \( X \), we mean a \( k \)-valuation of the fraction field \( k(X_i) \) for some \( i \), with a center on \( X_i \).

**Definition 3.1.** Let \( X|k \) be an affine \( k \)-variety and \( v \) be a \( k \)-valuation of \( X \) with center \( x_v \) and associated component \( X_i \). A local uniformization of \( v \) is a birational morphism:

\[
Y_v = \text{Spec}A \to X_i \ A = \mathcal{O}_{X_i}[f_1, \ldots, f_r] \subseteq \mathcal{O}_v,
\]

such that \( A_{M_v \cap A} \) is a regular local ring. We say that Local Uniformization holds on \( X \) at some \( x \in X \) (\( \text{LU}(X, x) \) holds for short) if every \( k \)-valuation of \( X \) with center \( x \) has a local uniformization.
Remark 3.1. The definition of Local Uniformization in 3.1 is precisely the one in Zariski’s generalized theorem of Local Uniformization, in its stronger form ([13] A.III theorem U3, p. 858).

Local Uniformization results ([13] A.III th. U3 and [1] theorem on p.1839): Let $X|k$ be a $k$-variety and $x \in X$. Local Uniformization holds at $x$ if we assume that $\text{char } k = 0$ or that $(\text{char } k = p > 0$ and $\dim \mathcal{O}_{X,x} \leq 3$).

Suppose that $X$ is affine and irreducible. Recall that the Riemann-Zariski space $\mathcal{V}(X)$, consisting of all $k$-valuations of $X$, is provided with the following topology: given a subring $A$ of $k(X)$ which is a $\mathcal{O}_X$-algebra of finite type, let $\mathcal{E}(A)$ be the set of all $v \in \mathcal{V}(X)$ which are nonnegative on $A$, that is $A \subseteq \mathcal{O}_v$. Then the sets $\mathcal{E}(A)$, where $A$ runs over all $\mathcal{O}_X$-algebras of finite type contained in $k(X)$, define a basis of a topology on $\mathcal{V}(X)$. With this topology, $\mathcal{V}(X)$ is quasi-compact ([14] chap. VI, sec. 17, theorem 40) and the map $\pi_X : \mathcal{V}(X) \to X$, sending $v$ to its center on $X$, is closed and continuous ([14] chap. VI, sec. 17, lemma 4).

4 Small irreducible components of the arc space.

In this section we study the irreducible components of the arc space of a variety. Let $X|k$ be a $k$-variety, and denote by $X_1, \ldots, X_c$ its reduced irreducible components. The corresponding maximal points of $X$ are denoted by $\xi_1, \ldots, \xi_c$.

Let $\zeta \in X$ and $Z := \overline{\{\zeta\}} \subseteq X$. Recall from proposition 2.1 that $Z_\infty \setminus (\text{Sing}Z)_\infty$ is irreducible. The Zariski closure

$$Z_\infty \setminus (\text{Sing}Z)_\infty \subseteq Z_\infty \subseteq X_\infty$$

of $Z_\infty \setminus (\text{Sing}Z)_\infty$ is denoted by $Z_\infty^0$, its generic point by $\zeta_\infty$. Note that $\zeta_\infty \in (Z \setminus \text{Sing}Z)_\infty$, hence

$$Z_\infty^0 = \overline{(Z \setminus \text{Sing}Z)_\infty}$$

(4.1)

This applies in particular to the maximal points $\xi_1, \ldots, \xi_c$ of $X$ and the corresponding irreducible components of $X_\infty$ are denoted by $X_1^\circ, \ldots, X_c^\circ$. These may not be all irreducible components of $X_\infty$ and motivates the following definition:
Definition 4.1. Let $X | k$ be a $k$-variety, $\zeta \in X$ and $Z := \{ \zeta \} \subseteq X$. We say that $X$ is arc-sharp at $\zeta$ if $Z^\circ_\infty = X^\circ_\infty$ for some $i$ (i.e. $\zeta = \zeta_i$), or if

$$Z^\circ_\infty \not\subseteq \bigcup_{i=1}^{c} X^\circ_\infty.$$ 

If $X$ is not arc-sharp at $\zeta$, we say that $X$ is arc-blunt at $\zeta$.

Remark 4.1. The variety $X$ is arc-sharp at $\zeta$ if and only if $X_i$ is arc-sharp at $\zeta$ for each $i$ such that $\zeta \in X_i$.

Furthermore $Z^\circ_\infty$ is an irreducible component of $X^\circ_\infty$ if and only if for every irreducible subvariety $F$, $Z \subseteq F \subseteq X$, $F$ is arc-sharp at $\zeta$.

In order to characterize irreducible components of $X^\circ_\infty$, we need to introduce prime divisors ([14] sec. 14). It is well known that the residue field $k(v)$ of a prime divisor $v$ of $X$ over $\zeta$ is a finitely generated field extension of $k(\zeta)$.

Definition 4.2. A prime divisor of $X$ over $\zeta$ is a discrete valuation $v$ of some $k(X_i)$, $\zeta \in X_i$, with ring $(O_v, M_v, k_v)$ such that

$$O_{X_i, \zeta} \subseteq O_v, \ M_v \cap O_{X_i, \zeta} = \zeta, \text{ and } \text{tr.deg}_{k(\zeta)} k_v = \dim O_{X_i, \zeta} - 1.$$ 

We say that $O_v$ is generically smooth over $\zeta$ if

$$\dim_{k_v} \Omega_{k_v} = \text{tr.deg}_{k(\zeta)} k_v.$$ 

Lemma 4.2. Let $X | k$ be an irreducible $k$-variety, $\zeta \in X$ and $Z := \{ \zeta \} \subseteq X$. Let $v$ be a prime divisor of $X$ over $\zeta$. There exists a birational morphism of varieties $\pi : Y \to X$ such that

(1) $Y$ is regular, and

(2) $E := \pi^{-1}(Z)_{\text{red}}$ is irreducible and $O_{Y, E} = O_v$.

If $v$ is generically smooth over $\zeta$, we may furthermore take the induced map $E \to Z$ smooth.

Proof. Let $\pi_1 : Y_1 \to X$ be a projective birational morphism such that the center $E_1$ of $v$ in $Y_1$ is a divisor, $Y_1$ normal. In particular, the generic point of $E_1$ is regular. We have $\pi_1^{-1}(Z)_{\text{red}} = E_1 \cup E_2$, where $E_2$ does not contain $E_1$. Take $Y := \text{Reg}Y_1 \setminus E_2$.

Note that we may replace $Y$ with $Y \cap \pi^{-1}(U)$, where $U \subseteq X$ is any Zariski neighborhood of $\zeta$. If $v$ is generically smooth over $\zeta$, then $E \cap \pi^{-1}(U) \to Z \cap U$ is smooth for suitable $U$ and this concludes the proof. \hfill \Box
Proposition 4.3. Let $X/k$ be an irreducible $k$-variety, $\zeta \in X$ and $Z := \{\zeta\} \subseteq X$. We consider the following properties:

(1) there exists a generically smooth prime divisor over $\zeta$;

(2) there exists $P \in X_{\infty}$ such that $h_P(0) = \zeta$, $h_P(\eta) \in \text{Reg}(X)$ and $\kappa(P)|k(\zeta)$ is a finite and separable field extension;

(3) there exists a $k$-valuation $v$ of $X$ with center $\zeta$ in $X$ and such that $k_v|k(\zeta)$ is a separable field extension;

(4) for every proper and birational morphism $\pi: Y \to X$, $\pi^{-1}(\zeta)$ has a point over the separable closure $k(\zeta)^{\text{sep}}$ of $k(\zeta)$;

We have equivalences (1) $\iff$ (2) and (3) $\iff$ (4). Furthermore, we have an implication (2) $\implies$ (3); the converse holds if $LU(X, \zeta)$ holds.

Proof. We get (1) $\implies$ (2) by applying lemma 4.2: when $E \to Z$ is smooth, $\pi^{-1}(\zeta)$ has points over $k(\zeta)^{\text{sep}}$. Let $\zeta' \in E$ be such that $k(\zeta')|k(\zeta)$ is separable. There exists an arc $\tilde{h}: \text{Spec} \ k(\zeta')[[[t]]] \to Y$ centered at $\zeta'$ and such that $\tilde{h}(\eta) \notin E$. Then the arc $h = \pi \circ \tilde{h}$ satisfies (2).

Conversely, we apply proposition 2.3 to obtain a birational morphism $\pi: X' \to X$ and a point

$$\zeta' := h_{P'}(0) \in \text{Reg}(X') \cap \pi^{-1}(\zeta),$$

where $h_P = \pi \circ h_{P'}$. Then $k(\zeta')|k(\zeta)$ is a finite and separable field extension since $k(\zeta') \subseteq \kappa(P)$. Blowing up at $\zeta'$ produces the required generically smooth prime divisor over $\zeta$.

The previous argument also shows that (2) $\implies$ (3): the regular local ring $\mathcal{O}_{X', \zeta'}$ has a valuation $v$ centered at $\zeta'$ and residue field $k(\zeta')$. Conversely, suppose that $LU(X, \zeta)$ holds. There exists a birational morphism of $k$-varieties $Y_v \to X$ such that the center $y_v \in Y_v$ of $v$ sits inside $\text{Reg}(Y_v)$. Let $\zeta' \in \text{Reg}(Y_v) \cap \{y_v\}$ be such that $k(\zeta')|k(\zeta)$ is a finite and separable field extension. Blowing up at $\zeta'$ produces the required generically smooth prime divisor over $\zeta$, hence (3) $\implies$ (1) holds.

Statement (3) $\implies$ (4) is trivial: let $y \in Y$ be the center of the valuation $v$ provided by (3). Since $k(y)|k(\zeta)$ is separable, there exists $\zeta' \in \text{Reg}(\{y\})$ with $k(\zeta')|k(\zeta)$ a finite and separable field extension.
Finally let us prove \((4) \implies (3)\). Let \(\pi_\alpha : X_\alpha \to X\) an arbitrary proper and birational morphism. The birational correspondence map is further denoted by
\[
\pi_{\alpha\beta} : X_\beta \cdots \to X_\alpha.
\]
Let

\[
S_\alpha := \{\zeta' \in \pi_\alpha^{-1}(\zeta) : k(\zeta')|k(\zeta) \text{ is a finite and separable extension}\}.
\]

We remark that every maximal point \(c_\alpha\) of \(S_\alpha\) corresponds to a separable field extension \(k(c_\alpha)|k(\zeta)\). It follows from these definitions that, whenever \(\pi_{\alpha\beta}\) is a morphism, we have
\[
\pi_{\alpha\beta}(S_\beta) \subseteq S_\alpha.
\tag{4.2}
\]

We define \(F_\alpha\) to be the intersection of \(\pi_{\alpha\beta}(S_\beta)\) whenever \(\pi_{\alpha\beta}\) is a morphism. Since \(X_\alpha\) is Noetherian, \((4.2)\) implies that \(F_\alpha \neq \emptyset\). Furthermore, we have
\[
\pi_{\alpha\beta}(F_\beta) = F_\alpha
\tag{4.3}
\]

Let
\[
\mathcal{F} := \varprojlim F_\alpha \subseteq \pi_X^{-1}(\zeta) \subseteq \varprojlim X_\alpha = \mathcal{V}(X).
\]

We have that \(\mathcal{F}\) is a closed set and \(F_\alpha = \pi_{X_\alpha}(\mathcal{F})\) where we denote by \(\pi_{X_\alpha} : \mathcal{V}(X) \to X_\alpha\) the projection maps (recall that they are closed maps, [14] VI sec. 17, lemma 4). Besides, \(F_\alpha \neq \emptyset\) for all \(\alpha\), \((4.3)\) and the quasicompacity of the Zariski-Riemann surface imply that \(\mathcal{F}\) is nonempty. We may therefore pick \(v \in \mathcal{F}\) such that \(y_\alpha := \pi_{X_\alpha}(v) \in X_\alpha\) is the generic point of some irreducible component of \(I_\alpha\) for every \(\alpha\). Note that \(k(y_\alpha)|k(\zeta)\) is a separable field extension: since \(\pi_{\alpha\beta}(F_\beta) = I_\alpha\) whenever \(\pi_{\alpha\beta}\) is defined at \(x_\beta\), we have \(k(y_\alpha) \subseteq k(c_\beta)\), where \(c_\beta\) is any maximal point of \(F_\beta\) specializing to \(y_\beta\). In particular, the residue field \(k_v = \varprojlim k(y_\alpha)\) of \(v\) is a separable field extension of \(k(\zeta)\). This proves the proposition.

The main result of this section is the following theorem. We list below some corollaries.

**Theorem 4.4.** Let \(X|k\) be a \(k\)-variety, \(\zeta \in X\) and \(Z := \{\zeta\} \subseteq X\). Assume that there exists a generically smooth prime divisor over \(\zeta\). Then \(X\) is arc-blunt at \(\zeta\).
Proof. The argument is the same as in the proof of proposition 2.1. To begin with, it can be assumed that \( X \) is irreducible. We use property (2) of proposition 4.3 which provides some \( h_P \).

By proposition 2.3, there exists a regular irreducible \( k \)-variety \( X' \), and a birational morphism \( f : X' \to X \), such that \( h_P \) lifts to \( X' \). Furthermore, the exceptional locus \( E \) of \( X' \to X \) maps to \( Z \). Let \( h_P \) lift with center \( \zeta' \in X' \).

Since \( \kappa(P)|k(\zeta) \) is finite and separable, so is \( k(\zeta')|k(\zeta) \). Let \( Z' := \{ \zeta' \} \subseteq X' \). Since \( k(\zeta')|k(\zeta) \) is finite and separable, we have \( f_\infty(\zeta'_\infty) = \zeta_\infty \). Here \( \zeta_\infty \) (resp. \( \zeta'_\infty \)) is the generic point of \( Z'_\infty \) (resp. \( Z'_0 \)).

By lemma 2.2(2), we have

\[
\zeta_\infty \in f_\infty(j_{X'}^{-1}(X'\setminus E))
\]

and the conclusion follows from the self-evident inclusions

\[
f_\infty(j_{X'}^{-1}(X'\setminus E)) \supseteq j^{-1}(X_\infty) = X_\infty.
\]

\[\square\]

**Corollary 4.5.** Let \( X|k \) be an irreducible \( k \)-variety, \( \zeta \in X \) and \( Z := \{ \zeta \} \subseteq X \). Let \( C_{Z|X} \to Z \) be the corresponding normal cone and denote by \( C_1, \ldots, C_r \) those irreducible components of \( C_{Z|X} \) which map dominantly to \( Z \). Let

\[
[C_{Z|X}] = m_1[C_1] + \cdots + m_r[C_r]
\]

be the corresponding fundamental cycle. If there exists \( i \), \( 1 \leq i \leq r \), such that \( (p \nmid m_i \text{ and } C_i \text{ is generically smooth over } k(\zeta)) \), then \( X \) is arc-blunt at \( \zeta \).

In particular, \( X \) is arc-blunt at \( \zeta \) whenever the multiplicity \( m(\zeta) \) of \( X \) at \( \zeta \) is prime to \( p \).

**Proof.** It can be assumed that \( \zeta \in X \) is a point of codimension one by cutting locally at \( \zeta \). Let \( X' \to X \) be an étale covering over a neighborhood of \( \zeta \). Changing \( X \) by some irreducible component of \( X' \), it can be assumed that \( r = 1 \), \( [C_1] \) is an affine line over \( k(\zeta) \), i.e. the inverse image of \( \zeta \) by the normalization map \( n := \overline{X} \to X \) is a unique point \( \overline{\zeta} \in \overline{X} \), rational over \( k(\zeta) \). Then,

\[
m_1 = \dim_{k(\zeta)} \left( \mathcal{O}_{X, \overline{\zeta}} / \mathcal{M}_{X, \overline{\zeta}} \mathcal{O}_{X, \overline{\zeta}} \right)
\]
where $\mathcal{M}_{X, \xi}$ is the maximal ideal of $\mathcal{O}_{X, \xi}$. Hence $[k(\xi) : k(\xi)]$ divides $m_1$ and, since $p \nmid m_1$ by hypothesis, it follows that $p \nmid [k(\xi) : k(\xi)]$. Thus the extension $k(\xi)|k(\xi)$ is separable. The conclusion follows from theorem 4.4. The last statement follows from the fact

$$m(\xi) = \sum_{i=1}^{r} m_i \deg P(C_i),$$

so $X$ satisfies the assumptions of the theorem if $p \nmid m(\xi)$.

\section{Existence of small irreducible components.}

The main result characterizing small irreducible components of $X_\infty$ is theorem 5.5 below. We first recall some general facts about radicial morphisms ([3] chap. 1, sec. 3.5).

Let $f : \mathcal{Y} \to \mathcal{X}$ be a morphism of schemes having finitely many irreducible components. We say that $f$ is \textit{dominant} if $f(\mathcal{Y}) = \mathcal{X}$. Equivalently, the induced map $\mathcal{O}_{\mathcal{X}_\text{red}} \to \mathcal{O}_{\mathcal{Y}_\text{red}}$ is injective.

We say that $f$ is \textit{strongly dominant} if $f$ induces a surjective application from the set of maximal points $\{y_1, \ldots, y_n\}$ of $\mathcal{Y}$ into the set of maximal points $\{x_1, \ldots, x_m\}$ of $\mathcal{X}$. If $f$ is strongly dominant, there is an induced inclusion

$$\text{Tot}(\mathcal{X}_\text{red}) = \prod_{i=1}^{m} k(x_i) \subseteq \text{Tot}(\mathcal{Y}_\text{red}) = \prod_{j=1}^{n} k(y_j)$$

between total rings of fractions. We say that $f$ is \textit{birational} if $f$ is strongly dominant, and if the above inclusion is an equality.

A morphism $f : \mathcal{X}' \to \mathcal{X}$ of schemes is said to be \textit{radicial} if every nonempty fiber $f^{-1}(x)$ of $f$ has only one element $\{x'\}$ and is such that the induced field extension $k(x) \subseteq k(x')$ is radicial, i.e. algebraic and separably closed ([3] Def. 3.5.4 and Prop. 3.5.8). In other terms, if $\text{char} k(x) = 0$, we have $k(x) = k(x')$; if $\text{char} k(x) = p > 0$, every $\lambda \in k(x')$ satisfies:

$$\lambda^{p^\alpha} \in k(x) \text{ for some } \alpha \geq 0. \quad (5.1)$$

We remark at this point that an integral, radicial and dominant morphism $f : \mathcal{Y} \to \mathcal{X}$ is actually strongly dominant. Namely, let $z \in \mathcal{Y}$ be a maximal point mapping to a non maximal point of $\mathcal{X}$ and let $\mathcal{Y}_0 := \mathcal{Y} \setminus \{z\}$, where
Let $f : X' \to X$ be a finite, radicial and dominant morphism of $k$-varieties, with $\text{char} k = p > 0$. There exists $\alpha \geq 0$ and inclusions

$$O_{X'}^\alpha \subseteq O_X \subseteq O_{X'}.$$  

**Proof.** It can be assumed that $X = \text{Spec} R$, hence $X' = \text{Spec} R'$ is affine, and that $R' = R[x]$ is generated by one element as a $R$-algebra. The right hand side map is an inclusion $R \subseteq R[x]$ because $f$ is dominant. Let $P_1, \ldots, P_c$ be the minimal primes of $R$. Since $f$ is radicial and surjective (finite and dominant), we may label $P'_1, \ldots, P'_c$ the minimal primes of $R'$, with

$$P'_i \cap R = P_i, \ K_i := QF(R/P_i) \subseteq K'_i := QF(R'/P'_i) \text{ radicial, } 1 \leq i \leq c.$$  

By (5.1), we may pick $\alpha \geq 0$ such that

$$\text{Tot}(R')^\alpha \subseteq \text{Tot}(R) = \prod_{i=1}^c K_i \subseteq \text{Tot}(R') = \prod_{i=1}^c K'_i. \quad (5.2)$$  

In particular, we may assume that $x \in \text{Tot}(R)$ in order to prove the lemma, i.e. $f$ birational.

We now argue by induction on the pair $(d = \dim R, c)$. For $d = 0$, $f$ birational implies $R = R'$ so there is nothing more to prove. Assume that $d \geq 1$. Using induction on $c$, we may furthermore assume that $x$ is not a zero divisor in $R'$. Then we may pick $h \in R$ such that

$$hR' \subseteq R, \ h \text{ not a zero-divisor.} \quad (5.3)$$  

To see this, write $x = f/g$, $f, g \in R$, $g$ not a zero-divisor, a relation

$$x^m + f_1 x^{m-1} + \cdots + f_m = 0, \ f_1, \ldots, f_m \in R$$  

and take $h := g^{m-1}$. By (5.3), we are reduced to proving that $x^{\alpha^\alpha} \in hR'$ for some $\alpha \geq 0$.

To complete the proof, let

$$\overline{R} := \frac{R}{\sqrt{(h)}}, \ \overline{R'} := \frac{R'}{\sqrt{(hR')}},$$

$$Z := \{z\}. \text{ The map } f_0 : \mathcal{Y}_0 \to \mathcal{X} \text{ is again integral, radicial and dominant. In particular } f_0 \text{ is surjective by the going up theorem. Since } f \text{ is radicial, } f^{-1}(f(z)) \text{ has only one element: a contradiction.}$$

We now state a couple of lemmas.

**Lemma 5.1.** Let $f : X' \to X$ be a finite, radicial and dominant morphism of $k$-varieties, with $\text{char} k = p > 0$. There exists $\alpha \geq 0$ and inclusions

$$O_X^\alpha \subseteq O_X \subseteq O_{X'}.$$  

**Proof.** It can be assumed that $X = \text{Spec} R$, hence $X' = \text{Spec} R'$ is affine, and that $R' = R[x]$ is generated by one element as a $R$-algebra. The right hand side map is an inclusion $R \subseteq R[x]$ because $f$ is dominant. Let $P_1, \ldots, P_c$ be the minimal primes of $R$. Since $f$ is radicial and surjective (finite and dominant), we may label $P'_1, \ldots, P'_c$ the minimal primes of $R'$, with

$$P'_i \cap R = P_i, \ K_i := QF(R/P_i) \subseteq K'_i := QF(R'/P'_i) \text{ radicial, } 1 \leq i \leq c.$$  

By (5.1), we may pick $\alpha \geq 0$ such that

$$\text{Tot}(R')^\alpha \subseteq \text{Tot}(R) = \prod_{i=1}^c K_i \subseteq \text{Tot}(R') = \prod_{i=1}^c K'_i. \quad (5.2)$$  

In particular, we may assume that $x \in \text{Tot}(R)$ in order to prove the lemma, i.e. $f$ birational.

We now argue by induction on the pair $(d = \dim R, c)$. For $d = 0$, $f$ birational implies $R = R'$ so there is nothing more to prove. Assume that $d \geq 1$. Using induction on $c$, we may furthermore assume that $x$ is not a zero divisor in $R'$. Then we may pick $h \in R$ such that

$$hR' \subseteq R, \ h \text{ not a zero-divisor.} \quad (5.3)$$  

To see this, write $x = f/g$, $f, g \in R$, $g$ not a zero-divisor, a relation

$$x^m + f_1 x^{m-1} + \cdots + f_m = 0, \ f_1, \ldots, f_m \in R$$  

and take $h := g^{m-1}$. By (5.3), we are reduced to proving that $x^{\alpha^\alpha} \in hR'$ for some $\alpha \geq 0$.

To complete the proof, let

$$\overline{R} := \frac{R}{\sqrt{(h)}}, \ \overline{R'} := \frac{R'}{\sqrt{(hR')}},$$

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so the map Spec $\overline{R'} \rightarrow$ Spec $\overline{R}$ is again finite, dominant and radicial. Let $\overline{x} \in \overline{R'}$ be the image of $x$. Applying now induction on $d$, there exists $r \in R$ and $\overline{\alpha} \geq 0$ such that

$$x^{\overline{\alpha}} - r \in \sqrt{(hR')}.$$ 

Since there exists $n_0 \geq 1$ such that $(\sqrt{(hR')})^n \subseteq hR'$ for every $n \geq n_0$, we have

$$x^{\overline{\alpha}} \in hR'$$

for every $\alpha \geq \overline{\alpha} + \log_p n_0$.

This concludes the proof.

**Lemma 5.2.** Let $f : X' \rightarrow X$ be a finite and radicial morphism of $k$-varieties, with char $k = p > 0$. Then $f_\infty : X'_\infty \rightarrow X_\infty$ is integral and radicial. More precisely, there exists $\alpha \geq 0$ such that

$$\mathcal{O}_{X'\infty}^{\rho^\alpha} \subseteq \text{Im}(\mathcal{O}_{X_\infty} \rightarrow \mathcal{O}_{X'\infty}).$$

**Proof.** It can be assumed that $X$ is affine and $f$ is dominant. By lemma 5.1, there exists $\alpha \geq 0$ and inclusions

$$\mathcal{O}_{X'}^{\rho^\alpha} \subseteq \mathcal{O}_X \subseteq \mathcal{O}_{X'}.$$

Let $u_1, \ldots, u_r$ be generators of the $\mathcal{O}_X$-algebra $\mathcal{O}_{X'}$ and,

$$u_j^{\rho^\alpha} - f_j = 0, \quad f_j \in \mathcal{O}_X, \quad 1 \leq j \leq r,$$

be relations satisfied by the generators. Then $\mathcal{O}_{X_\infty}$ is generated as an $\mathcal{O}_{X_\infty}$-algebra by elements $U_{j,n}$, $1 \leq j \leq r$ and $n \geq 0$; there are relations

$$U_j^{\rho^\alpha} - F_{j,\rho^\alpha} = 0, \quad F_{j,\rho^\alpha} \in \mathcal{O}_{X_\infty}, \quad 1 \leq j \leq r, \quad n \geq 0$$

in $\mathcal{O}_{X_\infty}$ as required.

**Remark 5.3.** We do not know if $f_\infty$ is an integral morphism when removing the assumption char $k = p > 0$.

**Lemma 5.4.** Let $f : X' \rightarrow X$ be a dominant morphism of irreducible $k$-varieties. Let $\zeta' \in X'$ and $\zeta := f(\zeta')$. Assume that both field extensions $k(X')|k(X)$ and $k(\zeta')|k(\zeta)$ are separable.

If $X'$ is arc-blunt at $\zeta'$, so is $X$ arc-blunt at $\zeta$. 18
Proof. Since \( k(X')|k(X) \) is separable and \( f \) is dominant, we have:

\[
\overline{f_\infty(j'^{-1}(\mathrm{Reg}X'))} = X^o_\infty. \tag{5.5}
\]

On the other hand, \( f_\infty(\zeta'_\infty) = \zeta_\infty \) because \( k(\zeta')|k(\zeta) \) is separable.

If \( X' \) is arc-blunt at \( \zeta' \), then \( \zeta'_\infty \in j'^{-1}(\overline{\mathrm{Reg}X'}) \) (see (4.1)) and the conclusion follows from (5.5), since

\[
\overline{f_\infty(j'^{-1}(\mathrm{Reg}X'))} = \overline{f_\infty(j'^{-1}(\mathrm{Reg}X'))}.
\]

\( \square \)

**Theorem 5.5.** Let \( f : X' \to X \) be a finite and birational morphism of \( k \)-varieties, with \( \text{char} k = p > 0 \). Let \( \zeta \in X \) and \( Z := \{\zeta\} \subseteq X \). We assume that the following additional property holds:

the map \( X' \times_X S \to X \times_X S \) is radicial, where \( S := \text{Spec} \mathcal{O}_{X,\zeta} \setminus \{\zeta\} \). \( \text{(5.6)} \)

The following properties are equivalent:

(1) \( X \) is arc-sharp at \( \zeta \);

(2) \( X' \) is arc-sharp at each \( \zeta' \in f^{-1}(\zeta) \) such that the field extension \( k(\zeta')|k(\zeta) \) is separable.

**Remark 5.6.** We do not know if assumption (5.6) can be avoided. It is satisfied in the following two situations:

(1) \( \dim \mathcal{O}_{X,\zeta} = 1 \);

(2) there exists a finite and strongly dominant morphism \( X \to X_0 \) which is generically purely inseparable with \( X_0 \) normal.

Examples of (2) include hypersurfaces with equation of the form

\[
X := \text{Spec} \frac{k[x_1, \ldots, x_n, y]}{(y^{\alpha^2} + f(x_1, \ldots, x_n))}, \quad \alpha \geq 1.
\]

Such singularities have played an important role in Resolution of Singularities and have been pointed out by Zariski as a test case for Resolution, see [15] p.88.
Proof. Part (1) \implies (2) is an immediate application of lemma 5.4 and holds more generally for separable and strongly dominant morphisms of \( k \)-varieties with no extra assumption.

Assume now that (2) holds. We first reduce to the case where \( f \) is a radicial morphism. To begin with, we may replace \( X \) by any Zariski neighborhood of \( \zeta \), thus assume that \( X \) is affine and \( Z \) regular. By remark 4.1, we may furthermore assume that \( X \) is irreducible. Let \( e : Y \to X \) be a finite and étale neighborhood of \( \zeta \), and consider the diagram:

\[
\begin{array}{ccc}
Y & \xleftarrow{g} & Y' := Y \times_X X' \\
\downarrow{e} & & \downarrow{e'} \\
X & \xleftarrow{f} & X'
\end{array}
\]

By [4] lemma 6.15.3.1 (ii), \( g \) satisfies the extra assumption (5.6) at each \( \tau \in e^{-1}(\zeta) \). Also some \( \tau' \in e'^{-1}(\tau) \) induces a separable residue extension \( k(\tau')|k(\tau) \) if and only if \( k(\zeta')|k(\zeta) \) is separable, with \( \zeta' := e'(\tau') \). Finally, we have

\[
Y_\infty \simeq Y \times_X X_\infty, 
Y'_\infty \simeq Y' \times_{X'} X'_\infty
\]

because \( e, e' \) are étale morphisms. Therefore the theorem holds if and only if it holds for every \( \tau \in e^{-1}(\zeta) \), and for each morphism \( Y_i \times_X X' \to Y_i \), \( Y_i \) an irreducible component of \( Y \) with \( \tau \in Y_i \). Choosing a suitable étale neighborhood of \( \zeta \), it can be assumed the following for each \( i \):

- the integral closure of the local ring \( \mathcal{O}_{Y_i, \tau} \) is local, and
- the residue extension \( k(\tau')|k(\tau) \) is radicial for each \( \tau' \in e'^{-1}(\tau) \).

In view of assumption (5.6), we have now achieved the following reduction:

the map \( X' \times_X \text{Spec}\mathcal{O}_{X, \zeta} \to X \times_X \text{Spec}\mathcal{O}_{X, \zeta} \) is radicial. \hspace{1cm} (5.7)

Let now \( g : Y' \to Y \) be a finite and strongly dominant morphism of \( k \)-varieties, \( \tau \in Y \). Suppose that the map

\[
Y' \times_Y \text{Spec}\mathcal{O}_{Y, \tau} \to Y \times_Y \text{Spec}\mathcal{O}_{Y, \tau}
\]

is radicial. After possibly replacing \( Y \) by some Zariski neighborhood \( U \) of \( \zeta \), \( Y' \) by \( g^{-1}(U) \), there exists a nowhere dense closed subset \( W \subset Y \) such that
belongs to each irreducible component of \( W \), and \( g^{-1}(Y \setminus W) \to Y \setminus W \) is a radicial morphism. Note that the map

\[
W' \times_W \text{Spec} \mathcal{O}_{W, \tau} \to W \times_W \text{Spec} \mathcal{O}_{W, \tau}
\]

is also radicial. Arguing by induction on the dimension, we may eventually assume that the map \( f : X' \to X \) is radicial. Let \( \{ \zeta' \} := f^{-1}(\zeta)_{\text{red}}, Z' := \{ \zeta' \} \subseteq X' \). There are two cases to consider:

**Case 1:** \( k(\zeta') = k(\zeta) \). By assumption, \( X' \) is arc-sharp at \( \zeta' \). Therefore there exists \( F \in \mathcal{O}_{X_{\infty}} \) vanishing on \( X_{\infty}' \) but not on \( Z_{\infty}' \). By lemma 5.2, we may assume that \( F \in \mathcal{O}_{X_{\infty}} \) by replacing the original \( F \) with \( F^{p^n} \). Since the morphisms \( X_{\infty}' \to X_{\infty}' \) and \( Z_{\infty}' \to Z_{\infty}' \) are birational, \( F \in \mathcal{O}_{X_{\infty}} \) vanishes on \( X_{\infty}' \) but not on \( Z_{\infty}' \). Therefore \( X \) is arc-sharp at \( \zeta \).

**Case 2:** \( k(\zeta') \mid k(\zeta) \) is purely inseparable. Let \( r' \in \mathcal{O}_{X'} \) satisfy \( r'^p - r \in I_{Z'} \), \( r \in \mathcal{O}_{X} \) and the residue \( \tau \) of \( r \) in \( k(\zeta) \) is not in \( k(\zeta)^p \). Let \( x, y \in I_{Z} \) satisfy \( r' = y/x \). By lemma 5.1, there exists \( \alpha \geq 0 \) such that

\[
s := (r'^p - r)^p \in \mathcal{O}_{X}.
\]

We deduce that there exists a relation

\[
y^{p^{\alpha+1}} - x^{p^{\alpha+1}}(r^{p^\alpha} + s) = 0
\]

in \( \mathcal{O}_{X} \). In particular, there is a relation

\[
X_0^{p^{\alpha+1}}(R_1^{p^\alpha} + S_{p^\alpha}) = 0
\]

in \( \mathcal{O}_{X_{\infty}} \). The rational function \( X_0 \) is not a zero-divisor on \( X_{\infty}' \), \( R_1^{p^\alpha} + S_{p^\alpha} = 0 \) in \( \mathcal{O}_{X_{\infty}} \). Since \( \tau \notin k(\zeta)^p \), \( r \) is a differential parameter on some nonempty open set of \( Z \), whence none of functions \( R_l, l \geq 0 \), vanishes on \( Z \). On the other hand, we have \( S_l \in I_{Z_{\infty}} \) for every \( l \geq 0 \) because \( s \in I_{Z'} \cap \mathcal{O}_X = I_Z \). This proves that the function \( R_1^{p^\alpha} + S_{p^\alpha} \in \mathcal{O}_{X_{\infty}} \) vanishes identically on \( X_{\infty}' \), but does not on \( Z_{\infty}' \). Therefore \( X \) is arc-sharp at \( \zeta \) as stated.

**Example 5.7.** Let \( k \) be a field of characteristic \( p > 0 \). For \( n \geq 1 \), let \( X_n \) be the \( 2n \)-dimensional variety given by

\[
(\cdots((y^p + z_1x_1^p)^p + z_2x_2^p)^p + \cdots)^p + z_nx_n^p = 0
\]

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in \( k_{n+1}^{2n+1} \). Applying theorem 5.4 to the normalization \( f : \overline{X}_n \to X_n \) of \( X_n \) it can be proved that \( (X_n)_\infty \) has exactly \( n + 1 \) irreducible components. More precisely, fix \( n \) and set \( X := X_n \), we have \( \mathcal{O}_X = \mathcal{O}_X[u] \) where

\[
u = \left( \cdots ((y^p + z_1x_1^p)^p + z_2x_2^p)^p + \cdots )^p + z_{n-1}x_{n-1}^p \right).
\]

Since \( u^p = z_n \), the hypothesis (5.6) in th. 5.4 holds. For \( 0 \leq r \leq n \) let \( Y_r \) be the subvariety of \( X \) given by

\[
\left( \cdots ((y^p + z_1x_1^p)^p + z_2x_2^p)^p + \cdots )^p + z_rx_r^p \right) = x_{r+1} = \cdots = x_n = 0
\]

so that \( Y_n = X \) and \( Y_{r-1} = \text{Sing} Y_r, 1 \leq r \leq n \). Let \( \zeta_r \) be the generic point of \( Y_r \). Then \( f^{-1}(\zeta_r) \) contains only one point \( \zeta'_r \) and \( k(\zeta'_r) = k(\zeta_r)(u) \), hence the extension \( k(\zeta'_r)|k(\zeta_r) \) is purely inseparable. Thus \( X = X_n \) is arc-sharp at \( \zeta_r \). Arguing by induction on \( n \), we conclude that \( \{Y_r^0\}_{r=0}^n \) are the irreducible components of \( (X_n)_\infty \).

We state two immediate corollaries of theorem 5.4:

**Corollary 5.8.** Let \( X|k \) be a \( k \)-variety, \( n : \overline{X} \to X \) be the normalization morphism. Let \( \zeta \in X, Z := \{\overline{\zeta}\} \subseteq X \) and assume that \( \dim \mathcal{O}_{X,\zeta} = 1 \).

Then \( X \) is arc-sharp at \( \zeta \) if and only if for each \( \overline{\zeta} \in n^{-1}(\zeta) \), the field extension \( k(\overline{\zeta})|k(\zeta) \) is inseparable.

**Proof.** It follows from theorems 4.3 and 5.4.

**Corollary 5.9.** Let \( X|k \) be a \( k \)-variety of dimension two and \( n : \overline{X} \to X \) be the normalization morphism. The decomposition into irreducible components of \( X_\infty \) is

\[
X_\infty = \left( \bigcup_{i=1}^c X_{i,\infty}^0 \right) \cup \left( \bigcup_{j=1}^d Z_{j,\infty}^0 \right),
\]

where: \( X = X_1 \cup \cdots \cup X_c \) is the decomposition of \( X \) into irreducible components, and the generic points \( \zeta_1, \ldots, \zeta_d \) of \( Z_1, \ldots, Z_d \) are precisely those \( \zeta \in X \) such that

1. \( \dim \mathcal{O}_{X,\zeta} = 1 \), and

2. for each \( \overline{\zeta} \in n^{-1}(\zeta) \), the field extension \( k(\overline{\zeta})|k(\zeta) \) is inseparable.

**Proof.** It follows from corol. 5.5. Note that \( X \) is arc-blunt at every closed point since \( k \) is a perfect field.
6 Some open problems and an example.

This section contains the main open questions about small components. We test and illustrate this material with a family of hypersurfaces with equation of the type (1.2).

The definition of arc-blunt/sharp is for the whole arc space $X_\infty$. A variant consists in replacing $X_\infty$ by $j^{-1}(\text{Sing} X) = X^{\text{Sing}X}_\infty$.

**Question 6.1.** Let $X/k$ be an irreducible $k$-variety, $\zeta \in \text{Sing} X$ and $Z := \{\zeta\}$. Assume that $X$ is arc-blunt at $\zeta$, i.e.

$$Z_\infty \subseteq X_\infty \setminus (\text{Sing} X)_\infty.$$

Is it true that

$$j^{-1}(\text{Sing} X) \setminus (\text{Sing} X)_\infty ?$$

In other terms, we ask whether each small irreducible component of the closed set $j^{-1}(\text{Sing} X)$ is an irreducible component of $X_\infty$. The answer is affirmative when $\dim \mathcal{O}_{X,\zeta} = 1$: this follows from the criterion for arc-sharpness in corollary 5.8 and theorem 4.4.

**Question 6.2.** Let $f : X' \to X$ be a proper and birational morphism of $k$-varieties, with $\text{char} k = p > 0$, and let $\zeta \in X$. Under which conditions are the following properties equivalent:

1. $X$ is arc-sharp at $\zeta$;
2. $X'$ is arc-sharp at each $\zeta' \in f^{-1}(\zeta)$ such that the field extension $k(\zeta')|k(\zeta)$ is separable?

Implication (1) $\implies$ (2) holds in a more general context for proper and surjective morphisms which are generically smooth (lemma 5.3). Theorem 5.5 gives a special case of the converse implication. Example 6.3 below points out a necessary restriction to big enough characteristics when compared to the dimension (tentatively, $p > \dim \mathcal{O}_{X,\zeta}$). It is possible that (1) $\iff$ (2) holds without further restriction on $f$.

We restate this question in the case of an irreducible variety $X$ of dimension 3 in characteristic $p \geq 3$: is it true that $X_\infty$ is irreducible if and only if for every resolution of singularities $\pi : Y \to X$ and every $\zeta \in X$, $\pi^{-1}(\zeta)$ has a point over the separable closure of $k(\zeta)$?
Example 6.3. The condition \( p > \dim \mathcal{O}_{X, \xi} \) is necessary. For example, let \( n \geq p > 0 \) and let \( X = \{ f = 0 \} \subset \mathbb{A}^{2n+1}_k \) with
\[
f = y^p + z_1x_1^p + z_2x_2^p \cdots + z_nx_n^p.
\]
Then \( Z := \text{Sing } X = \{ y = x_1 = \ldots = x_n = 0 \} \) is regular. If \( \xi \) is the generic point of \( Z \) and \( \pi : X' \to X \) the blowing up along \( Z \), then \( \pi^{-1}(\xi) \) has no separable point. But \( p \leq \dim \mathcal{O}_{X, \xi} = n \) and we will next show that \( X_\infty \) is irreducible. In fact, note that
\[
F_0 = Y_0^p + Z_{1,0}X_{1,0}^p + \cdots + Z_{n,0}X_{n,0}^p,
\]
\[
F_i = Z_{i,1}X_{i,1}^p + \cdots + Z_{n,1}X_{n,1}^p \quad \text{for } 1 \leq i \leq p - 1.
\]
Hence there exist polynomials
\[
g_{i,0}, h_{0} \in \left( k \left[ \{ Z_{i,r} \}_{0 \leq r \leq p-1} \right] \right)_L [W_{p-1,0}, \ldots, W_{n,0}],
\]
where \( L := Z_{1,1}Z_{2,2} \cdots Z_{p-1,p-1} \) and \( W_{p-1,0}, \ldots, W_{n,0} \) are indeterminacies, such that \( \{ F_i = 0 \}_{i=0}^{p-1} \) if and only if \( X_{i,0}^p = g_{i,0}(X_{p-1,0}^p, \ldots, X_{n,0}^p), 1 \leq i \leq p - 1, \) and \( Y_0^p = h_{0}(X_{p-1,0}^p, \ldots, X_{n,0}^p) \). We extend this argument by induction and we obtain that there exist
\[
g_{i,r}, h_{r} \in \left( k \left[ \{ Z_{i,r} \}_{0 \leq r \leq n} \right] \right)_L [W_{p-1,0}, \ldots, W_{n,0}, W_{p-1,r}, \ldots, W_{n,r}]
\]
such that the ring \( (\mathcal{O}_{X_\infty})_L \) is isomorphic to
\[
\left( k \left[ \{ Y_r, Z_{i,r}, X_{i,r} \}_{0 \leq i \leq n, r \geq 0} \right] \right)_L / \left( k \left[ \{ Y_r - h_r(X_{p-1,0}^p, \ldots, X_{n,r}^p), X_{p-1,0}^p - g_{i,r}(X_{p-1,0}^p, \ldots, X_{n,r}^p) \}_{1 \leq i \leq p-1, r \geq 0} \right] \right)_L.
\]
Now, let us consider the field \( K := k(\{ \overline{Z}_{i,r} \}_{0 \leq i \leq n, r \geq 0}) \) where \( \overline{Z}_{i,r} = Z_{i,r} \).
From the previous study it follows that we may define an arc in \( X_\infty \)
\[
\phi : \text{Spec } K[[\xi]] \to X_\infty
\]
by \( Z_{i,r} = \overline{Z}_{i,r}, 1 \leq i \leq n, r \geq 0, X_{p,0} = \xi, X_{i,r} = 0 \) for \( i = p \) and \( r \geq 1 \) or \( i \geq p + 1, r \geq 0 \) and
\[
X_{i,r} = \overline{f}_{i,r} (\overline{Z}_{i,r}, X_{p-1,0}, \ldots, X_{n,r}), \quad Y_r = \overline{h}_r (\overline{Z}_{i,r}, X_{p-1,0}, \ldots, X_{n,r})
\]
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Fix irreducible components since $X_z$, which contradicts the previous statement. This will imply that

$$K = \sum_{t=1}^n a_i(x_1, \ldots, x_n),$$

where $a_i(x_1, \ldots, x_n)$ are regular functions on some regular variety $U$ with function field $K$. $\lambda_1, \ldots, \lambda_s$ defines a $p$-basis of $K$ over the prime field and $s \geq n$. For simplicity, write $a_i(\lambda) = a_i(\lambda_1, \ldots, \lambda_s)$. Assume that

$$\text{rank}(d_{ij}(\lambda)), i \neq j = n.\quad \text{This condition implies that}$$

$$Z := \text{Sing} X = \{y = x_1 = \ldots = x_n = 0\}.\quad \text{In particular}$$

where $\sum_{i=1}^n A_{j,n} t^n$ is the Taylor development of $a_j(\sum_{i=1}^n \Delta_i t^n)$ in the same way of (2.1). Note that $A_{j,1} = \sum_{i=1}^n d_{ij}(\lambda_i) \Lambda_{i,1}$ for $1 \leq j \leq n$.

The condition $\text{rank}(d_{ij}(\lambda)), i \neq j = n$ together with the expression of the $A_{j,1}$ implies that no linear combination of the $A_{j,1}$ is equal to 0.

Two different cases occur, first consider that $n \leq p - 1$. There exists a $n \times n$ minor, say,

$$\Delta = \begin{pmatrix} A_{1,1} & \cdots & A_{n,1} \\ \vdots & \ddots & \vdots \\ A_{1,n} & \cdots & A_{n,n} \end{pmatrix},$$

such that $L = \det(\Delta) \neq 0$. In fact, if all minors have determinant equal zero, then there is a linear combination of the columns of the matrix that is equal to zero. Hence, there exists $c_1, \ldots, c_n \in K$ such that $c_1 A_{1,1} + \cdots + c_n A_{n,1} = 0$ which contradicts the previous statement. This will imply that $X_\infty$ has two irreducible components since $Z_\infty = \{X_{i,m} = 0\} \not\subset \{L = 0\}$ and $Z$ is regular.
We have to prove that \( X_0^0 \subset \{ L = 0 \} \). Take \( h \in X_\infty \setminus Z_\infty \). There exists indices \((i, m)\) such that \( X_{i,m}(h) \neq 0 \) with \( 1 \leq i \leq n \) and \( m \geq 0 \). Take \( m \) minimum with this property and consider \( F_{mp}(h), F_{mp+1}(h), \ldots, F_{mp+n}(h) \):

\[
F_{mp+j}(h) = (A_{1,j}(h)X_{1,m}(h)^p + \cdots + A_{n,j}(h)X_{n,m}(h)^p) = 0, \quad 1 \leq j \leq n. \tag{6.1}
\]

Here, we make use of the following notation: given \( G \in \mathcal{O}_{X_\infty} \), by \( G(h) \) we mean the image of \( G \) by the morphism \( \mathcal{O}_{X_\infty} \to K \) induced by \( h : \text{Spec} K[[t]] \to X \).

Recall now that \( n \leq p-1 \). The statement in (6.1) implies that \( L(h) = 0 \), and hence that \( X_0^0 \subset \{ L = 0 \} \).

Consider now the case \( n > p-1 \). The same argument used in the previous case, implies that there exists a \((p-1) \times (p-1)\) minor, say

\[
\Delta = \begin{pmatrix}
A_{1,1} & \cdots & A_{1,p-1} \\
\vdots & & \vdots \\
A_{p-1,1} & \cdots & A_{p-1,p-1}
\end{pmatrix},
\]

such that \( L = \det(\Delta) \neq 0 \). Following the ideas of Example 6.3, we can find functions \( g_{i,0}, h_0 \in \left( k \left[ \left\{ \begin{array}{c} A_{i,r} \\ 0 \leq r \leq p-1 \end{array} \right\} \right] \right)_L \left[ W_{p-1,0}, \ldots, W_{0,0} \right] \), where \( W_{p-1,0}, \ldots, W_{0,0} \) are indeterminacies, such that \( \{ F_i = 0 \}_{i=0}^{p-1} \) if and only if \( X_{i,0}^p = g_{i,0}(X_{p-1,0}^p, \ldots, X_{n,0}^p) \), \( 1 \leq i \leq p-1 \), and \( Y_0^p = h_0(X_{p-1,0}^p, \ldots, X_{n,0}^p) \). The same argument used in Example 6.3 implies that \( Z_\infty \subset X_0^0 \), and hence \( X_\infty \) is irreducible.

The next question is about existence of a curve selection lemma. Computing a Zariski closure is a non trivial problem in arc spaces and a curve selection lemma is useful.

**Question 6.5. (Curve Selection)** Let \( X|k \) be an irreducible \( k \)-variety, \( \zeta \in X \) and \( f \in \mathcal{O}_X \), \( f \neq 0 \). Assume that \( X \) is arc-blunt at \( \zeta \). Does there exist a field extension \( K \supseteq k(\zeta_\infty) \) and a wedge \( \phi : \text{Spec} K[[u]] \to X_\infty \) such that \( \phi(0) = \zeta_\infty \) and \( \phi(u) \notin (V(f))_\infty \)?

**Question 6.6.** Let \( X|k \) be an irreducible \( k \)-variety and \( v \) be a valuation of \( k(X)|k \). Denote by \( z \in X \) the center of \( v \). Does there exist a birational morphism \( X_v \to X \) such that \( v \) has a center \( z_v \in X_v \) and \( (X_v)_\infty \) is irreducible?
Using a classical theorem of Albanese, we can prove the following:

**Proposition 6.7.** Let \( k \) be an algebraically closed field of characteristic \( p > 0 \) and \( K|k \) be a function field of dimension \( d \geq 1 \). If \( p > d! \), there exists a projective variety \( X|k \) such that \( k(X) = K \) and \( X_\infty \) is irreducible.

**Proof.** It follows from [9], last sentence on p. 202, and corol. 4.4 from this article.

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**References**


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