The continuous weak order

Luigi Santocanale, Maria João Gouveia

THE CONTINUOUS WEAK ORDER

MARIA JOÃO GOUVEIA[1] AND LUIGI SANTOCANALE[2]

Abstract. The set of permutations on a finite set can be given the lattice structure known as the weak Bruhat order. This lattice structure is generalized to the set of words on a fixed alphabet $\Sigma = \{ x, y, z, \ldots \}$, where each letter has a fixed number of occurrences. These lattices are known as multinomial lattices and, when $\mathrm{card}(\Sigma) = 2$, as lattices of lattice paths. By interpreting the letters $x, y, z, \ldots$ as axes, these words can be interpreted as discrete increasing paths on a grid of a d-dimensional cube, with $d = \mathrm{card}(\Sigma)$.

We show how to extend this ordering to images of continuous monotone functions from the unit interval to a d-dimensional cube and prove that this ordering is a lattice, denoted by $\mathsf{L}(\mathbb{I}^d)$. This construction relies on a few algebraic properties of the quantale of join-continuous functions from the unit interval of the reals to itself: it is cyclic $\star$-autonomous and it satisfies the mix rule.

We investigate structural properties of these lattices, which are self-dual and not distributive. We characterize join-irreducible elements and show that these lattices are generated under infinite joins from their join-irreducible elements, they have no completely join-irreducible elements nor compact elements. We study then embeddings of the d-dimensional multinomial lattices into $\mathsf{L}(\mathbb{I}^d)$. We show that these embeddings arise functorially from subdivisions of the unit interval and observe that $\mathsf{L}(\mathbb{I}^d)$ is the Dedekind-MacNeille completion of the colimit of these embeddings. Yet, if we restrict to embeddings that take rational values and if $d > 2$, then every element of $\mathsf{L}(\mathbb{I}^d)$ is only a join of meets of elements from the colimit of these embeddings.

Keywords. Weak order; weak Bruhat order; permutohedron; multinomial lattice; multi-permutation; path; quantale; star-autonomous; involutive residuated lattice; join-continuous; meet-continuous.

1. Introduction

The weak Bruhat order [25, 44] on the set of permutations of an n-element set, also known as permutohedron, see [9] for an elementary exposition, is a lattice structure which has been widely studied in view if its close connections to combinatorics and geometry, see e.g. [6, 7, 34, 35]. Its algebraic structure has also been investigated and, by now, is well understood [8, 40, 42].

Multinomial lattices [4, 17, 1, 38], or lattices of multipermutations, generalize permutohedra in a natural way. Elements of a multinomial lattice are multipermutations, namely words on a totally ordered finite alphabet $\Sigma = \{ x, y, z \ldots \}$ with a fixed number of occurrences of each letter. The weak order on multipermutations is the reflexive and transitive closure of the binary relation $\prec$ defined by $wabu \prec wbau$, for $a, b \in \Sigma$ and $a < b$. If each letter of the alphabet has exactly one occurrence, then these words are permutations and the ordering is the weak Bruhat ordering. Multinomial lattices embed into permutohedra

[1] Faculdade de Ciências, Universidade de Lisboa, Portugal
[2] LIS, CNRS UMR 7020, Aix-Marseille Université, France
E-mail addresses: mjgouveia@fc.ul.pt, luigi.santocanale@lis-lab.fr.
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as principal ideals; possibly, this is a reason for the lattice theoretic literature on them not to be contained. Multipermutations have, however, a strong geometrical flavour that in our opinion justifies exploring further their lattice theoretic structure. These words can be given a geometrical interpretation as discrete increasing paths in some Euclidean cube of dimension $d = \mathrm{card}(\Sigma)$; the weak order can be thought of as a way of organizing these paths into a lattice structure. When $\mathrm{card}(\Sigma) = 2$, the connection with geometry is well-established: in this case these lattices are also known as lattices of lattice paths with North and East steps [16]; the objects these lattices are made of are among the most studied in enumerative combinatorics [29, 2] and many counting results are implicitly related to the order and lattice structures. We did not hesitate in [38] to call the multinomial lattices "lattices of paths in higher dimensions". Willing to understand the geometry of higher dimensional multinomial lattices, we started wondering whether there are full geometric relatives of these lattices. More precisely, we asked whether the weak order can be extended from discrete paths to continuous increasing paths. We present in this paper our answer to this question. Our main result sounds as follows:

Theorem. *Let $d \geq 2$. Images of increasing continuous paths from $\vec{0}$ to $\vec{1}$ in $\mathbb{R}^d$ can be given the structure of a lattice; moreover, all the permutohedra and all the multinomial lattices can be embedded into one of these lattices while respecting the dimension d.*

We call this lattice the *continuous weak order* in dimension d. While a proof of the above statement was available a few years ago, only recently we could structure and ground that proof on a solid algebraic setting, making it possible to further study these lattices. The algebra we consider is the one of the quantale $\mathsf{Q}_\vee(\mathbb{I})$ of join-continuous functions from the unit interval of the reals to itself. This is a $\star$-autonomous quantale, see [3], and moreover it satisfies the mix rule, see [11]. The construction of the continuous weak order is actually an instance of a general construction of a lattice $\mathsf{L}_d(Q)$ from a $\star$-autonomous quantale Q satisfying the mix rule. When $Q = \mathbf{2}$ (the two-element Boolean algebra) this construction yields the usual weak Bruhat order on permutations; when $Q = \mathsf{Q}_\vee(\mathbb{I})$, this construction yields the continuous weak order. Moreover, when Q is the quantale of join-continuous functions from the finite chain $\{ 0, 1, \dots, n \}$ to itself, this construction yields a multinomial lattice. The functorial properties of this construction are a key tool for analysing various embeddings. The step we took can be understood as an instance of moving to a different set of (non-commutative, in this case) truth values, as notably suggested in [31].

Let us state our algebraic results. Let $\langle Q, 1, \otimes, ^\star \rangle$ be a cyclic non-commutative $\star$-autonomous quantale satisfying the MIX rule. That is, we require that $x \otimes y \leq x \oplus y$, for each $x, y \in Q$, where $\oplus$ is the monoid structure dual to $\otimes$. Let $d \geq 2$, $[d]_2 := \{ (i, j) \mid 1 \leq i < j \leq d \}$ and consider the product $Q^{[d]_2}$. Say that a tuple $f \in Q^{[d]_2}$ is *closed* if $f_{i,j} \otimes f_{j,k} \leq f_{i,k}$ (each $i < j < k$), and that it is *open* if $f_{i,k} \leq f_{i,j} \oplus f_{j,k}$ (each $i < j < k$). Say that f is *clopen* if it is closed and open. Under these conditions, the following statement hold:

Theorem. *The set of clopen tuples of $Q^{[d]_2}$ is, with the pointwise ordering, a lattice, noted $\mathsf{L}_d(Q)$. The construction $\mathsf{L}_d(-)$ yields a limit preserving functor to the category of lattices.*

We shall make later in the text precise the domain of this functor. Paired with the following statement, relating the algebraic structure of $\mathsf{Q}_\vee(\mathbb{I})$ to the reals, we obtain a proof the main result stated above.

Theorem. *Clopen tuples of $\mathsf{Q}_\vee(\mathbb{I})^{[d]_2}$ bijectively correspond to images of monotonically increasing continuous functions $\mathsf{p} : \mathbb{I} \to \mathbb{I}^d$ such that $\mathsf{p}(0) = \vec{0}$ and $\mathsf{p}(1) = \vec{1}$.*

Let us mention that motivations for developing this work also originated from various researches undergoing in theoretical computer science, modelling the behaviour of concurrent processes via directed homotopy [22, 24] and discrete approximation of continuous paths via words [5]. The relationship between directed homotopies and congruences of two-dimensional multinomial lattices was discussed in [38]. The connection with discrete geometry appears in the conference version of this work [23]. In both cases it was distinct to us the need of developing the mathematics of a continuous weak order in dimension $d \geq 3$.

The paper is organized as follows. We recall in Section 2 some definitions and elementary results, mainly on join-continuous (or meet-continuous) functions and adjoints. In Section 3 we identify the least algebraic structure needed to perform the construction of the lattice $\mathsf{L}_d(Q)$. Therefore, we introduce and study mix ℓ-bisemigroups which, in the cases of interest to us, arise from mix $\star$-autonomous quantales. Section 4 proves that if I is what we call a perfect chain, then the quantale of join-continuous functions from I to itself is mix $\star$-autonomous. Finite chains and the unit interval of the real numbers are examples of perfect chains. Section 5 describes the construction of the lattice $\mathsf{L}_d(Q)$, for an integer $d \geq 2$ and a ℓ-bisemigroup Q. In Section 6 we focus on the particular structure of $\mathsf{Q}_\vee(\mathbb{I})$, the quantale of continuous functions from the unit interval to itself. Section 7 defines the central notion of path and discusses its equivalent characterizations. In Section 8 we show that paths in dimension 2 are in bijection with elements of the quantale $\mathsf{Q}_\vee(\mathbb{I})$. In Section 9 we argue that paths in higher dimensions bijectively correspond to clopen tuples of the product lattice $\mathsf{Q}_\vee(\mathbb{I})^{[d]_2}$, that is, to elements of $\mathsf{L}_d(\mathsf{Q}_\vee(\mathbb{I}))$. In Section 10 we discuss some structural properties of the lattices $\mathsf{Q}_\vee(\mathbb{I})$; in particular we characterize join-irreducible elements of these lattices. In Section 11 we argue that embeddings from multinomial lattices into the continuous weak order functorially arise from complete maps of perfect chains. Finally, in Section 12, we argue that when we restrict to embeddings obtained from splitting the unit interval into n intervals of the same size, the continuous weak order is not the Dedekind-MacNeille completion of the colimit of these embedding, yet every element is a join of meets (and a meet of joins) of elements from such a colimit.

2. Elementary facts on join-continuous functions

Throughout this paper, $[d]$ shall denote the set $\{1, \ldots, d\}$ while we let $[d]_2 := \{(i, j) \mid 1 \leq i < j \leq d\}$.

Let P and Q be complete posets; a function $f : P \to Q$ is *join-continuous* (resp., *meet-continuous*) if

$$f(\bigvee X) = \bigvee_{x \in X} f(x)\,, \quad (\text{resp., } f(\bigwedge X) = \bigwedge_{x \in X} f(x))\,, \tag{1}$$

for every $X \subseteq P$ such that $\bigvee X$ (resp., $\bigwedge X$) exists. We say that f is *bi-continuous* if it is both join-continuous and meet-continuous.

Recall that $\bot_P := \bigvee \emptyset$ (resp., $\top_P := \bigwedge \emptyset$) is the least (resp., greatest) element of P. Note that if f is join-continuous (resp., meet-continuous) then f is monotone and $f(\bot_P) = \bot_Q$ (resp., $f(\top_P) = \top_Q$). Let f be as above; a map $g : Q \to P$ is *left adjoint* to f if $g(q) \leq p$ holds if and only if $q \leq f(p)$ holds, for each $p \in P$ and $q \in Q$; it is *right adjoint* to f if $f(p) \leq q$ is equivalent to $p \leq g(q)$, for each $p \in P$ and $q \in Q$. Notice that there is at most one function g that is left adjoint (resp., right adjoint) to f; we write this relation by $g = f_\ell$ (resp., $g = f_\rho$). Clearly, when f has a right adjoint, then $f = (g_\rho)_\ell$, and a similar formula holds when f has a left adjoint. We shall often use the following fact:

Lemma 1. *If $f : P \to Q$ is monotone and P and Q are two complete posets, then the following are equivalent:*

(1) *f is join-continuous (resp., meet-continuous),*
(2) *f has a right adjoint (resp., left adjoint).*

If f is join-continuous (resp., meet-continuous), then we have

$$f_\rho(q) = \bigvee\{\, p \in P \mid f(p) \leq q \,\} \qquad (\text{resp., } f_\ell(q) = \bigwedge\{\, p \in P \mid q \leq f(p) \,\})\,, \quad \text{for each } q \in Q.$$

Moreover, if f is surjective, then these formulas can be strengthened so to substitute inclusions with equalities:

$$f_\rho(q) = \bigvee\{\, p \in P \mid f(p) = q \,\} \qquad (\text{resp., } f_\ell(q) = \bigwedge\{\, p \in P \mid q = f(p) \,\})\,, \quad \text{for each } q \in Q. \tag{2}$$

The set of monotone functions from P to Q can be ordered point-wise: $f \leq g$ if $f(p) \leq g(p)$, for each $p \in P$. Suppose now that f and g both have right adjoints; let us argue that $f \leq g$ implies $g_\rho \leq f_\rho$: for each $q \in Q$, the relation $g_\rho(q) \leq f_\rho(q)$ is obtained by transposing $f(g_\rho(q)) \leq g(g_\rho(q)) \leq q$, where the inclusion $g(g_\rho(q)) \leq q$ is the counit of the adjunction. Similarly, if f and g both have left adjoints, then $f \leq g$ implies $g_\ell \leq f_\ell$.

Let P be a poset, and let $\iota : P \to Q$ be an embedding of P into a complete lattice Q. Such embedding is a *Dedekind-MacNeille completion* if ι is bi-continuous and, for each $q \in Q$, there are sets $X, Y \subseteq P$ such that $q = \bigvee_{x \in X} \iota(x) = \bigwedge_{y \in Y} \iota(y)$. The Dedekind-MacNeille completion is unique up to isomorphism.

3. Lattice-ordered bi-semigroups

A (non-commutative, bounded) *lattice-ordered bi-semigroup* (ℓ-bisemigroup, for short) is a structure $\langle Q, \bot, \vee, \top, \wedge, \otimes, \oplus \rangle$ where $\langle Q, \bot, \vee, \top, \wedge \rangle$ is a bounded lattice, $\otimes$ is a binary associative operation on Q which distributes overs finite joins, $\oplus$ is a binary associative operation on Q which distributes over finite meets; moreover, the following relations

$$\beta \otimes (\gamma \oplus \delta) \leq (\beta \otimes \gamma) \oplus \delta\,, \tag{3}$$

$$(\alpha \oplus \beta) \otimes \gamma \leq \alpha \oplus (\beta \otimes \gamma)\,. \tag{4}$$

holds, for each $\alpha, \beta, \gamma, \delta \in Q$. We call these inclusions *hemidistributive* laws. We say that an ℓ-bisemigroup is *mix* if the relation

$$\alpha \otimes \beta \leq \alpha \oplus \beta\,. \tag{5}$$

holds, for each $\alpha, \beta \in Q$. We call this inclusion the *mix* rule. The inclusions (3) and (4) are non-commutative versions of the hemidistributive law of [14, §6.9] and are related to the weak distributivity of [12]. The mix rule (5) is well known in proof theory, see e.g. [11].

Remark 2. All the ℓ-bisemigroups that we shall consider have units; therefore, they are (possibly non-commutative) ℓ-bimonoids in the sense of [18]. We use 1 (resp., 0) to denote the unit of the operation $\otimes$ (resp., of $\oplus$) of an ℓ-bimonoid. The signature of ℓ-bimonoids is obtained by adding the two unit constants to the signature of ℓ-bisemigroups. Let us emphasize, however, that the morphisms between ℓ-bimonoids that we shall consider do not, in general, preserve units. This is the reason for which we emphasize the weaker structure of ℓ-bisemigroup.

We shall also use the following generalized hemidistributive laws:

$$(\alpha \oplus \beta) \otimes (\gamma \oplus \delta) \leq \alpha \oplus (\beta \otimes \gamma) \oplus \delta\,, \tag{6}$$

$$\alpha \otimes (\beta \oplus \gamma) \otimes \delta \leq (\alpha \otimes \beta) \oplus (\gamma \otimes \delta)\,, \tag{7}$$

Lemma 3. *The inclusions* (6) *and* (7) *are derivable from* (3) *and* (4)*. Moreover, in the extended language of ℓ-bimonoids (using units) these pairs of inclusions are equivalent and the mix rule* (5) *is equivalent to* $0 \leq 1$.

Proof. Having both (3) and (4), we derive (6) as follows:

$$(\alpha \oplus \beta) \otimes (\gamma \oplus \delta) \leq \alpha \oplus (\beta \otimes (\gamma \oplus \delta)) \leq \alpha \oplus (\beta \otimes \gamma) \oplus \delta\,.$$

Using units, we obtain (3) from (6) by instantiating α to 0; we obtain (4) from (6) by instantiating δ to 0. For the last statement, if (5) holds, then $0 \leq 1$ is derived by instantiating in (5) α with 0 and β with 1. Conversely, suppose that $0 \leq 1$ and observe then that $0 \otimes 0 \leq 0 \otimes 1 = 0$. Letting $\beta = \gamma = 0$ in (6), we derive (5) as follows:

$$\alpha \otimes \delta = (\alpha \oplus 0) \otimes (0 \oplus \delta) \leq \alpha \oplus (0 \otimes 0) \oplus \delta \leq \alpha \oplus 0 \oplus \delta = \alpha \oplus \delta\,. \qquad \square$$

All the ℓ-bisemigroups that we shall consider arise from non-commutative bounded involutive residuated lattice.

A (non-commutative, bounded) *residuated lattice* is a structure $\langle Q, \bot, \vee, \top, \wedge, 1, \otimes, \multimap, \circ\!-\rangle$ such that $\langle Q, \bot, \vee, \top, \wedge\rangle$ is a bounded lattice, $\langle Q, 1, \otimes\rangle$ is a monoid structure compatible with the lattice ordering (noted $\leq$) which moreover is related to the binary operations $\multimap, \circ\!-$ as follows:

$$\alpha \otimes \beta \leq \gamma \quad \text{iff} \quad \alpha \leq \gamma \circ\!- \beta \quad \text{iff} \quad \beta \leq \alpha \multimap \gamma\,, \quad \text{for each } \alpha, \beta, \gamma \in Q. \tag{8}$$

The operations $\multimap, \circ\!-$ are called the residuals (or adjoints) of $\otimes$. Let us recall that the following inclusions are valid:

$$\alpha \otimes (\alpha \multimap \beta) \leq \beta\,, \qquad\qquad (\beta \circ\!- \alpha) \otimes \alpha \leq \beta\,. \tag{9}$$

A (*unital*) *quantale* [36] is a complete lattice Q coming with a monoid structure $1, \otimes$ such that $\otimes$ distributes over arbitrary joins in both variables. A quantale is a residuated lattice in a canonical way, as distribution over arbitrary joins ensures the existence of the residuals.

A residuated lattice is said to be *involutive* if it comes with an element $0 \in Q$ such that

$$x \multimap 0 = 0 \circ\!- x\,,$$
$$0 \circ\!- (x \multimap 0) = x\,,$$

for each $x \in Q$. Such an element 0 is called *cyclic* and *dualizing*. In [23] we called a complete involutive residuated lattice a $\star$-*autonomous quantale*, as these structures are posetal version of $\star$-autonomous categories [3]. Similar namings, such as *(pseudo)* $\star$-*autonomous lattice*, have also been used in the literature, see e.g. [32, 15]. We shall stick to this naming in the future sections as all the involutive residuated lattices that we consider are complete. Given an involutive residuated lattice $\langle Q, \bot, \vee, \top, \wedge, 1, \otimes, \multimap, \circ\!-, 0\rangle$, we obtain an ℓ-bimonoid by defining

$$x^\star := x \multimap 0\,, \qquad\qquad f \oplus g := (g^\star \otimes f^\star)^\star\,. \tag{10}$$

From these definitions it follows that $(-)^\star$ is an antitone involution of Q and that $0 = 1^\star$. Moreover, considering that

$$
\begin{aligned}
(x \otimes y)^\star &= y \multimap x^\star = y^\star \multimapinv x\,, \\
x^\star \oplus y &= (y^\star \otimes x)^\star = x \multimap y^{\star\star} = x \multimap y\,, \\
x \oplus y^\star &= x \multimapinv y\,,
\end{aligned}
$$

the relations in (8) can be expressed as follows:

$$
\alpha \otimes \beta \leq \gamma \quad \text{iff} \quad \alpha \leq \gamma \oplus \beta^\star \quad \text{iff} \quad \beta \leq \alpha^\star \oplus \gamma\,, \quad \text{for each } \alpha, \beta, \gamma \in Q. \tag{11}
$$

Lemma 4. *With the definitions given in equation* (10)*, each involutive residuated lattice is a ℓ-bimonoid, and therefore an ℓ-bisemigroup.*

Proof. Since $0, \oplus$ are dual to $1, \otimes$, $\oplus$ is a monoid operation on Q with unit 0 and which distributes over meets.

We therefore verify that the hemidistributive laws holds in Q. Considering that $\alpha \oplus \beta = \alpha^{\star\star} \oplus \beta = \alpha^\star \multimap \beta$ and, similarly, $\gamma \oplus \delta = \gamma \multimapinv \delta^\star$, we derive

$$
\alpha^\star \otimes (\alpha \oplus \beta) \otimes (\gamma \oplus \delta) \otimes \delta^\star = \alpha^\star \otimes (\alpha^\star \multimap \beta) \otimes (\gamma \multimapinv \delta^\star) \otimes \delta^\star \leq \beta \otimes \gamma\,,
$$

using (9). Yet, the inequality so deduced is equivalent to (6) by adjointness (11). □

According to our previous observations, we could have defined a involutive residuated lattice as a structure $\langle Q, \bot, \vee, \top, \wedge, 1, \otimes, 0, (-)^\star \rangle$ where $\langle Q, \bot, \vee, \top, \wedge \rangle$ is a bounded lattice, $\otimes$ is a monoid operation (with unit 1) on Q that distributes over joins, $(-)^\star : Q \to Q$ is an antitone involution of Q, subject to the residuation laws as in (11), where the structure $(0, \oplus)$ on Q is defined by duality:

$$
0 := 1^\star \quad \text{and} \quad f \oplus g := (g^\star \otimes f^\star)^\star\,. \tag{12}
$$

This shall be our preferred way to verify that a residuated lattice with a distinct element 0 is an involutive residuated lattice. For the sake of verifying that a structure is an involutive residuated lattice, let us remark that we can simplify our work according to the following statement.

Lemma 5. *Consider a structure $\langle Q, \bot, \vee, \top, \wedge, 1, \otimes, 0, (-)^\star \rangle$ as above, where we only require that $\alpha \otimes \beta \leq \gamma$ is equivalent to $\alpha \leq \gamma \oplus \beta^\star$, for each $\alpha, \beta, \gamma \in Q$. Then $\alpha \otimes \beta \leq \gamma$ is also equivalent to $\beta \leq \alpha^\star \oplus \gamma$, for each $\alpha, \beta, \gamma \in Q$.*

Proof. Suppose that $\alpha \otimes \beta \leq \gamma$, so $\alpha \leq \gamma \oplus \beta^\star$. Apply $(-)^\star$ to this relation and derive $\beta \otimes \gamma^\star = (\gamma \oplus \beta^\star)^\star \leq \alpha^\star$; derive then $\beta \leq \alpha^\star \oplus \gamma^{\star\star} = \alpha^\star \oplus \gamma$. For the converse direction, observe that all these transformations are reversible. □

Example 6. Boolean algebras are the involutive residuated lattices such that $\wedge = \otimes$ and $\vee = \oplus$. Similarly, distributive lattices are the ℓ-bisemigroups such that $\wedge = \otimes$ and $\vee = \oplus$.

Example 7. Consider the following structure on the ordered set $\{\,-1 < 0 < 1\,\}$:

$\otimes$	-1	0	1
-1	-1	-1	-1
0	-1	0	1
1	-1	1	1

$\oplus$	-1	0	1
-1	-1	-1	1
0	-1	0	1
1	1	1	1

	$\star$
-1	1
0	0
1	-1

Together with the lattice structure on the chain, this structure yields a mix involutive residuated lattice, known in the literature as the Sugihara monoid on the three-element chain, see e.g. [19].

Example 8. As the category of complete lattices and join-continuous functions is a symmetric monoidal closed category, for every complete lattice X the set of join-continuous functions from X to itself is a monoid object in that category, that is, a quantale, see [26, 36], and therefore a residuated lattice. We review this next. For a complete lattice X, let $\mathsf{Q}_{\vee}(X)$ denote the set of join-continuous functions from X to itself. For $f, g \in \mathsf{Q}_{\vee}(X)$ define $f \otimes g := g \circ f$. Considering that the ordering in $\mathsf{Q}_{\vee}(X)$ is pointwise, let us verify that $\otimes$ distributes over arbitrary joins:

$$\begin{aligned}
((\bigvee_{i\in I} f_i) \otimes g)(x) &= (g \circ \bigvee_{i\in I} f_i)(x) = g((\bigvee_{i\in I} f_i)(x)) = g(\bigvee_{i\in I} f_i(x)) \\
&= \bigvee_{i\in I} g(f_i(x)) = \bigvee_{i\in I}((g \circ f_i)(x)) = (\bigvee_{i\in I} g \circ f_i)(x) = (\bigvee_{i\in I}(f_i \otimes g))(x)\,, \\
(f \otimes (\bigvee_{i\in I} g_i))(x) &= ((\bigvee_{i\in I} g_i) \circ f)(x) = (\bigvee_{i\in I} g_i)(f(x)) \\
&= \bigvee_{i\in I} g_i(f(x)) = (\bigvee_{i\in I}(g_i \circ f))(x) = (\bigvee_{i\in I}(f \otimes g_i))(x)\,.
\end{aligned}$$

Obviously, the identity is the unit for $\otimes$. We argue in the next Section that if I is a finite chain or the interval $[0, 1]$, then $\mathsf{Q}_{\vee}(I)$ has a cyclic dualizing element, thus a involutive residuated lattice extending the residuated lattice structure.

4. Mix $\star$-autonomous quantales from perfect chains

We consider complete chains I such that the two transformations

$$f^{\wedge}(x) = \bigwedge_{x<x'} f(x')\,, \qquad\qquad f^{\vee}(x) = \bigvee_{x'<x} f(x')\,. \tag{13}$$

yield an order isomorphism from $\mathsf{Q}_{\vee}(I)$ to $\mathsf{Q}_{\wedge}(I)$. We shall say that such a chain is *perfect*.

Example 9. Let $n \geq 0$ and let $\mathbb{I}_n$ be the chain $\{0, \dots, n\}$. A join-continuous function from $\mathbb{I}_n$ to $\mathbb{I}_n$ is uniquely determined by the value on the set $\{1, \dots, n\}$ of its join-prime elements. Similarly, a meet-continuous function from $\mathbb{I}_n$ to $\mathbb{I}_n$ is uniquely determined by its restriction to the set $\{0, \dots, n-1\}$ of its meet-prime elements. We immediately deduce that $\mathsf{Q}_{\vee}(\mathbb{I}_n)$ and $\mathsf{Q}_{\wedge}(\mathbb{I}_n)$ are order isomorphic. The functions defined in (13) realize this isomorphism. Observe that, for $I = \mathbb{I}_n$, we have

$$f^{\wedge}(x) = \begin{cases} n\,, & x = n\,, \\ f(x+1)\,, & \text{otherwise}\,, \end{cases} \qquad f^{\vee}(x) = \begin{cases} 0\,, & x = 0\,, \\ f(x-1)\,, & \text{otherwise}\,. \end{cases}$$

Example 10. We shall see with Proposition 33 that the interval $[0, 1]$ of the reals, later on denoted by $\mathbb{I}$, is perfect. The quantale $\mathsf{Q}_{\vee}(\mathbb{I})$ shall be investigated further in Section 6.

Recalling that the correspondences sending $f \in \mathsf{Q}_{\vee}(I)$ to $f_{\rho} \in \mathsf{Q}_{\wedge}(I)$ and $g \in \mathsf{Q}_{\wedge}(I)$ to $g_{\ell} \in \mathsf{Q}_{\vee}(I)$ are inverse is antitone, let us observe the following:

Proposition 11. *For each $f \in \mathsf{Q}_{\vee}(I)$, the relation $(f_{\rho})^{\vee} = (f^{\wedge})_{\ell}$ holds. Therefore, the function $(-)^{\star}$ defined by*

$$f^{\star} := (f_{\rho})^{\vee} = (f^{\wedge})_{\ell}\,,$$

is an involution of $\mathsf{Q}_{\vee}(I)$.

Proof. Let $f \in \mathsf{Q}_{\vee}(I)$; we shall argue that $(f_{\rho})^{\vee}$ is left adjoint to $f^{\wedge}$, namely that $x \leq f^{\wedge}(y)$ if and only if $(f_{\rho})^{\vee}(x) \leq y$, for each $x, y \in \mathbb{I}$.

We begin by proving that $x \leq f^{\wedge}(y)$ implies $(f_\rho)^{\vee}(x) \leq y$. Suppose $x \leq f^{\wedge}(y)$ so, for each z with $y < z$, we have $x \leq f(z)$. Suppose that $(f_\rho)^{\vee}(x) \not\leq y$, thus there exists $w < x$ such that $f_\rho(w) \not\leq y$. Then $y < f_\rho(w)$ so, from $x \leq f^{\wedge}(y) = \bigwedge_{y<y'} f(y')$, we deduce $x \leq f(f_\rho(w))$. Considering that $f(f_\rho(w)) \leq w$, we deduce $x \leq w$, contradicting $w < x$. Therefore, $(f_\rho)^{\vee}(x) \leq y$.

Dually, we can argue that, for $g \in \mathsf{Q}_\wedge(I)$, $g^{\vee}(x) \leq y$ implies $x \leq (g_\ell)^{\wedge}(y)$, for each $g \in \mathsf{Q}_\wedge(\mathbb{I})$. Letting in this statement $g := f_\rho$, we obtain the converse implication: $(f_\rho)^{\vee}(x) \leq y$ implies $x \leq ((f_\rho)_\ell)^{\wedge}(y) = f^{\wedge}(y)$.

For the last statement, observe that the correspondence $(-)^\star$ is order reversing since it is the composition of an order reversing function with a monotone one; it is an involution since $f^{\star\star} = (((f_\rho)^{\wedge})^{\vee})_\ell = (f_\rho)_\ell = f$. □

Lemma 12. *We have*

$$f^\star(x) = \bigvee \{\, y \in I \mid f(y) < x \,\} \,. \tag{14}$$

Proof. Recall that $f^\star$ has been defined as $(f^{\wedge})_\ell$. Let us show that the expression on the right of equation (14) yields a left adjoint for $f^{\wedge}$. For each $x, z \in I$, we have

$$\begin{aligned} \bigvee \{\, y \in I \mid f(y) < x \,\} \leq z \ &\text{iff } \forall y (\, f(y) < x \text{ implies } y \leq z \,) \\ &\text{iff } \forall y (\, z < y \text{ implies } x \leq f(y) \,) \\ &\text{iff } x \leq \bigwedge_{z<y} f(y) = f^{\wedge}(z) \,. \end{aligned}$$ □

For $f, g \in \mathsf{Q}_\vee(I)$, let us define

$$f \otimes g := g \circ f \,, \qquad\qquad 1 := id_I$$

and, using duality as in (12),

$$f \oplus g := (g^\star \otimes f^\star)^\star \qquad\qquad 0 := 1^\star \,.$$

Let us remark that the operation $\oplus$ is obtained by transporting composition in $\mathsf{Q}_\wedge(I)$ to $\mathsf{Q}_\vee(I)$ via the isomorphism:

$$f \oplus g = (g^\star \otimes f^\star)^\star = (f_\ell^{\wedge} \circ g_\ell^{\wedge})_\rho^{\vee} = ((g^{\wedge} \circ f^{\wedge})_\ell)_\rho^{\vee} = (g^{\wedge} \circ f^{\wedge})^{\vee} \,.$$

In a similar way, 0 is the image via the isomorphism of the identity of the chain I, as an element of $\mathsf{Q}_\wedge(I)$. Using Lemma 12, a useful expression for 0 is the following:

$$0(x) := \bigvee_{x'<x} x' \,. \tag{15}$$

Proposition 13. *For each $f, g, h \in \mathsf{Q}_\vee(I)$, $f \otimes g \leq h$ if and only if $f \leq h \oplus g^\star$.*

Proof. Suppose $f \otimes g \leq h$, that is, $g \circ f \leq h$. We aim at showing that $f^{\wedge} \leq g_\rho \circ h^{\wedge}$, since then, by applying $(-)^{\vee}$ to this relation, we shall obtain $f \leq (g_\rho \circ h^{\wedge})^{\vee} = (g_\rho^{\vee\wedge} \circ h^{\wedge})^{\vee} = (g^{\star\wedge} \circ h^{\wedge})^{\vee} = h \oplus g^\star$.

This is achieved as follows. From $g(f(x)) \leq h(x)$, for all $x \in I$, deduce $f(x) \leq g_\rho(h(x))$, for all $x \in I$, and therefore

$$f^{\wedge}(x) = \bigwedge_{x<y} f(y) \leq \bigwedge_{x<y} g_\rho(h(y)) = g_\rho(\bigwedge_{x<y} h(y)) = g_\rho(h^{\wedge}(x)) \,,$$

for each $x \in I$, using the fact that g_ρ is meet-continuous.

A similar argument, shows that if $f, g, h \in \mathsf{Q}_\wedge(I)$ and $f \leq g \circ h$, then $g_\ell \circ f^\vee \leq h^\vee$. For $f, g, h \in \mathsf{Q}_\vee(I)$, this yields that $f \leq h \oplus g$ implies $f \otimes g^\star \leq h$. Therefore, if $f \leq h \oplus g^\star$, then $f \otimes g = f \otimes g^{\star\star} \leq h$. □

Corollary 14. *For each perfect chain I the residuated lattice $\mathsf{Q}_\vee(I)$ of join-continuous functions from I to itself is a mix $\star$-autonomous quantale.*

Proof. By the previous Lemma and by Lemma 5, the antitone involution $(-)^\star$ yields the dual operation $\oplus$ satisfying the residuation relations (11). By equation (15), it is also clear that the relation $0 \leq 1$, so the mix rule holds in $\mathsf{Q}_\vee(I)$. □

Remark 15. The $\star$-autonomous quantale structure on $\mathsf{Q}_\vee(\mathbb{I}_n)$ is the unique possible one. It was shown in [39, §4.1] using duality theory that dualizing objects of in $\mathsf{Q}_\vee(\mathbb{I}_n)$ are in bijection with isomorphisms of the ordered set $\{1, \ldots, n\}$. Obviously, there is just one such isomorphism. On the other hand, the dualizing elements of an involutive residuated lattice such that $1 = 0$ are exactly the elements f that are invertible (in particular, this is the case for the quantale $\mathsf{Q}_\vee(\mathbb{I})$). We sketch a proof of this. If f is dualizing, then $1 = f \multimapinv (1 \multimap f) = f \multimapinv f = f \oplus f^\star$. Similarly, $1 = f^\star \oplus f$ and, dually, $1 = f \otimes f^\star = f^\star \otimes f$. Vice versa, if f has an inverse f^{-1}, then $f \oplus g = f \otimes f^{-1} \otimes (f \oplus g) \leq f \otimes (((f^{-1} \otimes f) \oplus g)) = f \otimes g$, so $f \otimes g = f \oplus g$, for any g. Then $f^{-1} = f^\star$, since $1 \leq f \oplus f^\star = f \otimes f^\star \leq 0 = 1$, and $f \multimapinv (g \multimap f) = f \oplus (f^\star \otimes g) = f \otimes f^\star \otimes g = g$.

5. Lattices from mix lattice-ordered bi-semigroups

In this section d shall be a fixed integer greater than or equal to 2 (the case $d = 2$ being trivial). Given an ℓ-bisemigroup Q, consider the product $Q^{[d]_2} := \prod_{1 \leq i < j \leq d} Q$. We say that a tuple $f = \langle f_{i,j} \mid 1 \leq i < j \leq d \rangle$ of this product is *closed* (resp., *open*) if

$$f_{i,j} \otimes f_{j,k} \leq f_{i,k} \qquad (resp.,\ f_{i,k} \leq f_{i,j} \oplus f_{j,k})\,.$$

Recall that $Q^{[d]_2}$ has a lattice structure induced by the coordinate-wise meets and joins. It is then easily verified that closed tuples are closed under arbitrary meets and open tuples are closed under arbitrary joins.

Remark 16. If Q is an involutive residuated lattice, then f is closed if and only if $f^\star := \langle (f_{\sigma(j),\sigma(i)})^\star \mid 1 \leq i < j \leq d \rangle$ is open, where $\sigma(i) := d - i + 1$, for each $i \in [d]$. In this case the correspondence sending f to $f^\star$ is an antitone involution of $Q^{[d]_2}$, sending closed tuples to open ones, and vice versa.

For $(i, j) \in [d]_2$, a subdivision of the interval $[i, j]$ is a subset of this interval containing the endpoints i and j. We write such a subdivision as sequence of the form $i = \ell_0 < \ell_1 < \ldots \ell_{k-1} < \ell_k = j$ with $i < \ell_i < j$, for $i = 1, \ldots, k-1$. We shall use then $S_{i,j}$ to denote the set of subdivisions of the interval $[i, j]$.

Lemma 17. *For each $f \in Q^{[d]_2}$, the tuple $\overline{f}$ defined by*

$$\overline{f}_{i,j} := \bigvee_{i<\ell_1<\ldots\ell_{k-1}<j \in S_{i,j}} f_{i,\ell_1} \otimes f_{\ell_1,\ell_2} \otimes \ldots \otimes f_{\ell_{k-1},j}\,.$$

is the least closed tuple g such that $f \leq g$. Dually, if we set

$$f^\circ_{i,j} := \bigwedge_{i<\ell_1<\ldots\ell_{k-1}<j \in S_{i,j}} f_{i,\ell_1} \oplus f_{\ell_1,\ell_2} \oplus \ldots \oplus f_{\ell_{k-1},j}\,.$$

then f° is the greatest open tuple below f.

Proof. It suffices to prove the first statement. Since $\{ i < j \} \in S_{i,j}$, then $f_{i,j} \leq \overline{f}_{i,j}$, for each $(i,j) \in [d]_2$, thus $f \leq \overline{f}$. Now, if g is closed and $f \leq g$, then, for each subdivision $i < \ell_1 < \ldots \ell_{k-1} < j$, we have

$$f_{i,\ell_1} \otimes \ldots \otimes f_{\ell_{k-1},j} \leq g_{i,\ell_1} \otimes \ldots \otimes g_{\ell_{k-1},j} \leq g_{i,j} .$$

We are left to prove that $\overline{f}$ is closed. To the sake of being concise, if $\varsigma \in S_{i,j}$ is $i = \ell_0 < \ell_1 < \ldots \ell_{k-1} < \ell_k = j$, then we let $\pi(f,\varsigma)$ be $f_{i,\ell_1} \otimes f_{\ell_1,\ell_2} \otimes \ldots \otimes f_{\ell_{k-1},j}$. Observe next that if $\varsigma \in S_{i,j}$ and $\varsigma' \in S_{j,k}$, then the set theoretic union $\varsigma \cup \varsigma'$ belongs to $S_{i,k}$ and, moreover, $\pi(\varsigma, f) \otimes \pi(\varsigma', f) = \pi(\varsigma \cup \varsigma', f)$. We have therefore

$$\begin{aligned}
\overline{f}_{i,j} \otimes \overline{f}_{j,k} &= \bigvee_{\varsigma \in S_{i,j}} \pi(\varsigma, f) \otimes \bigvee_{\varsigma' \in S_{j,k}} \pi(\varsigma', f) = \bigvee_{\varsigma \in S_{i,j}, \varsigma' \in S_{j,k}} \pi(\varsigma, f) \otimes \pi(\varsigma', f) \\
&= \bigvee_{\varsigma \in S_{i,j}, \varsigma' \in S_{j,k}} \pi(\varsigma \cup \varsigma', f) \leq \bigvee_{\varsigma'' \in S_{i,k}} \pi(\varsigma'', f) = \overline{f}_{i,k} .
\end{aligned}$$
□

We call the map $f \mapsto \overline{f}$ the *closure*, and the map $f \mapsto f^{\circ}$ the *interior*. Then a tuple is closed if and only of it is equal to its closure, and a tuple is open if and only of it is equal to its interior. We shall be interested in tuples $f \in Q^{[d]_2}$ that are *clopen*, that is, they are at the same time closed and open.

Proposition 18. *Let Q be a mix ℓ-bisemigroup and let $f \in Q^{[d]_2}$. If f is closed, then so is f°.*

Proof. Let $i, j, k \in [d]$ with $i < j < k$. We need to show that

$$f^{\circ}_{i,j} \otimes f^{\circ}_{j,k} \leq f_{i,\ell_1} \oplus \ldots \oplus f_{\ell_{n-1},k}$$

whenever $i < \ell_1 < \ldots \ell_{n-1} < k \in S_{i,k}$. This is achieved as follows. Let $u \in \{ 0, 1, \ldots, n-1 \}$ be such that $j \in [\ell_u, \ell_{u+1})$. Firstly suppose that $\ell_u < j$; put then

$$\begin{aligned}
\alpha &:= f_{i,\ell_1} \oplus \ldots \oplus f_{\ell_{u-1},\ell_u} , & \delta &:= f_{\ell_{u+1}} \oplus \ldots \oplus f_{\ell_{n-1},k} \\
\beta &:= f_{\ell_u,j} & \gamma &:= f_{j,\ell_{u+1}} .
\end{aligned}$$

Then

$$\begin{aligned}
f^{\circ}_{i,j} \otimes f^{\circ}_{j,k} &\leq (\alpha \oplus \beta) \otimes (\gamma \oplus \delta) , && \text{by definition of } f^{\circ}_{i,j} \text{ and } f^{\circ}_{j,k}, \\
&\leq \alpha \oplus (\beta \otimes \gamma) \oplus \delta , && \text{by the inequation (6)}, \\
&\leq \alpha \oplus f_{\ell_u,\ell_{u+1}} \oplus \delta = f_{i,\ell_1} \oplus \ldots \oplus f_{\ell_{n-1},k} , && \text{since } f \text{ is closed.}
\end{aligned}$$

Notice that we might have that α defined above is an empty (co)product (e.g. when $u = 0$), in which case we can use the inclusion (3) in place of (6). A similar remark has to be raised when δ defined above is an empty (co)product (when $u = n-1$), in which case we use inclusion (4). Finally, if $j = \ell_u$, then let α, γ, δ as above, we derive

$$f^{\circ}_{i,j} \otimes f^{\circ}_{j,k} \leq \alpha \otimes (\gamma \oplus \delta) \leq \alpha \oplus (\gamma \oplus \delta) = f_{i,\ell_1} \oplus \ldots \oplus f_{\ell_{n-1},k} ,$$

using the mix rule (5). □

Since the definition of ℓ-bisemigroup is auto-dual, we also have the following statement:

Proposition 19. *Let Q be a mix ℓ-bisemigroup and let $f \in Q^{[d]_2}$. If f is open, then so is $\overline{f}$.*

Definition 20. For Q a mix ℓ-bisemigroup, $\mathsf{L}_d(Q)$ shall denote the set of clopen tuples of $Q^{[d]_2}$.

Theorem 21. *The set $\mathsf{L}_d(Q)$ is, with the ordering inherited from $Q^{[d]_2}$, a lattice.*

Proof. For a family $\{ f_i \mid i \in I \}$, with each f_i clopen, define

$$\bigvee_{\mathsf{L}_d(Q)} \{ f_i \mid i \in I \} := \overline{\bigvee \{ f_i \mid i \in I \}}, \qquad \bigwedge_{\mathsf{L}_d(Q)} \{ f_i \mid i \in I \} := (\bigwedge \{ f_i \mid i \in I \})^\circ , \tag{16}$$

whenever the supremum $\bigvee\{ f_i \mid i \in I \}$ (resp., infimum $\bigwedge\{ f_i \mid i \in I \}$) exists in $Q^{[d]_2}$. Since this join (resp., meet) is open, its closure is clopen by Proposition 19 (resp., Proposition 18) and therefore it belongs to $\mathsf{L}_d(Q)$. Then It is easily seen that this is the supremum (resp., infimum) of the family $\{ f_i \mid i \in I \}$ in the poset $\mathsf{L}_d(Q)$. □

Example 22. Let $Q = \mathbf{2}$ be the two element Boolean algebra $\mathbf{2}$. We identify a tuple $\chi \in \mathbf{2}^{[d]_2}$ with the characteristic map of a subset S_χ of $\{ (i, j) \mid 1 \leq i < j \leq d \}$. Think of this subset as a relation. Then χ is clopen if both S_χ and its complement in $\{ (i, j) \mid 1 \leq i < j \leq d \}$ are transitive relations. These subsets are in bijection with permutations of the set $[d]$, see [9]; the lattice $\mathsf{L}_d(\mathbf{2})$ is therefore isomorphic to the well-known permutohedron, aka the weak Bruhat order.

Example 23. On the other hand, if Q is the Sugihara monoid on the three-element chain described in Example 7, then the lattice of clopen tuples is isomorphic to the lattice of pseudo-permutations, see [30, 41, 13].

Example 24. Let us consider a finite chain $\mathbb{I}_n = \{ 0, \ldots, n \}$ and the quantale $\mathsf{Q}_\vee(\mathbb{I}_n)$. Let $d \cdot n$ be the integer vector of length d whose all entries are equal to n. We claim that the lattice $\mathsf{L}_d(\mathsf{Q}_\vee(\mathbb{I}_n))$ is isomorphic to the multinomial lattice $\mathsf{L}(d \cdot n)$ of [4], see also [41, §8-10]. It is argued in [41] that elements of these multinomial lattices are in bijection with some clopen tuples of the product $\mathsf{L}(n, n)^{[d]_2}$. Considering that a binomial lattice $\mathsf{L}(n, n)$ is isomorphic (as a lattice) to the quantale $\mathsf{Q}_\vee(\mathbb{I}_n)$, we are left to verify that the two notions of closed/open tuple coincide via the bijection.

For $x, y \in [n]$, let $\langle x, y \rangle$ denote the least join-continuous function $f \in \mathsf{Q}_\vee(\mathbb{I}_n)$ such that $y \leq f(x)$. Elements of the form $\langle x, y \rangle$ are the join-prime elements of $\mathsf{Q}_\vee(\mathbb{I}_n)$. A tuple $f \in \mathsf{Q}_\vee(\mathbb{I}_n)^{[d]_2}$ is called closed in [41] if, for each i, j, k with $1 \leq i < j < k \leq d$ and each triple $x, y, z \in [n]$, $\langle x, y \rangle \leq f_{i,j}$ and $\langle y, z \rangle \leq f_{j,k}$ imply $\langle x, z \rangle \leq f_{i,k}$. Considering that, for $f \in \mathsf{Q}_\vee(\mathbb{I}_n)$, $\langle x, y \rangle \leq f$ if and only if $y \leq f(x)$, closedness is easily seen to be equivalent to the condition $f_{j,k} \circ f_{i,j} \leq f_{i,k}$, that is, to the notion of closedness introduced in this section. Let us argue that a tuple is open as defined in [41] if and only if it is open as defined in this section. To this goal, for $x, y \in [n]$, let $[x, y]$ be the greatest join-continuous function $f \in \mathsf{Q}_\vee(\mathbb{I}_n)$ such that $f(x) \leq y - 1$. Elements of the form $[x, y]$ are the meet-irreducible elements of $\mathsf{Q}_\vee(\mathbb{I}_n)$ and, moreover, $[x, y]^\star = \langle y, x \rangle$. In [41] a tuple $f \in \mathsf{Q}_\vee(\mathbb{I}_n)^{[d]_2}$ is said to be open if $f_{i,j} \leq [x, y]$ and $f_{j,k} \leq [y, z]$ imply $f_{i,k} \leq [x, z]$, for each $x, y, z \in [n]$ and whenever $1 \leq i < j < k \leq d$. This condition is equivalent to $\langle y, x \rangle = [x, y]^\star \leq f_{i,j}{}^\star$ and $\langle z, y \rangle = [y, z]^\star \leq f_{j,k}{}^\star$ imply $\langle z, x \rangle = [x, z]^\star \leq f_{i,k}{}^\star$, for each $z, y, x \in [n]$ and $1 \leq i < j < k \leq d$. As before, this is equivalent to $f_{j,k}{}^\star \otimes f_{i,j}{}^\star \leq f_{i,k}{}^\star$ and then to $f_{i,k} \leq (f_{j,k}{}^\star \otimes f_{i,j}{}^\star)^\star = f_{i,j} \oplus f_{j,k}$, yielding the notion of openness as defined here.

Proposition 25. $\mathsf{L}_d(-)$ *is a limit-preserving functor from the category of mix ℓ-bisemigroups to the category of lattices.*

Proof. Let $\psi : Q_0 \to Q_1$ be an ℓ-bisemigroup morphism (that is, a lattice morphism which, moreover, preserves $\otimes$ and $\oplus$). The map $\psi^{[d]_2} : Q_0{}^{[d]_2} \to Q_1{}^{[d]_2}$ defined by $[\psi^{[d]_2}(f)]_{i,j} := \psi(f_{i,j})$ commutes both with the closure map and with the interior map, since these maps are defined by means of the operations preserved by ψ. Consequently, the image by $\psi^{[d]_2}$ of a clopen is clopen. Similarly, the lattice operations on clopens, defined in equation (16), are preserved by $\psi^{[d]_2}$ since (for example for the joins) this function preserves the joins of ${Q_0}^{[d]_2}$ and the closure.

Since the forgetful functor from the category of lattices to the category of sets creates limits, in order to argue that the functor $\mathsf{L}_d(-)$ preserves limits, we can consider it as a functor form the category of mix ℓ-bisemigroups to the category of sets and functions and show that it preserves limits.

Let C be the category of ℓ-bisemigroups and their morphisms and consider the category of limit preserving functors from C to the category $\mathcal{S}$ of sets and functions. This category contains the forgetful functor (that we note here X) and is closed under limits. This holds since limits in the category of functors from C to $\mathcal{S}$ are computed pointwise. It is then enough to observe that $\mathsf{L}_d(-) : C \to \mathcal{S}$ is the following equalizer:

$$\begin{tikzcd} \mathsf{L}_d(X) \arrow[r, hook] & X^{[d]_2} \arrow[r, shift left=3, "(-)^{\circ}"] \arrow[r, "id" description] \arrow[r, shift right=3, "\overline{(-)}"'] & X^{[d]_2} \end{tikzcd}$$

□

In particular, from the previous proposition we obtain the following statement, that we shall use in Section 11.

Proposition 26. *If $i : Q_0 \to Q_1$ is an injective homomorphism of mix ℓ-bisemigroups, then $\mathsf{L}_d(i) : \mathsf{L}_d(Q_0) \to \mathsf{L}_d(Q_1)$ is an embedding.*

The goal of the rest of this section is to argue that clopen tuples naturally arise as some sort of enrichment (in the sense of [31, 28, 43]) or metric of a set X. For the sake of this discussion, we shall fix an involutive residuated lattice Q with the property that $0 = 1$. This equality holds in the quantale $\mathsf{Q}_\vee(\mathbb{I})$ studied in Section 6, but fails in other mix involutive residuated lattices, e.g. in the quantales $\mathsf{Q}_\vee(\mathbb{I}_n)$.

A *skew metric* of X over Q is a map $\delta : X \times X \to Q$ such that, for all $x, y, z \in X$,

$$\begin{aligned} &\delta(x, x) \leq 0\,, \\ &\delta(x, z) \leq \delta(x, y) \oplus \delta(y, z)\,, \\ &\delta(x, y) = \delta(y, x)^{\star}\,. \end{aligned}$$

That is, a skew metric is a semi-metric (see e.g. [33]) with values in Q, where the symmetry condition has been replaced by the last requirement, skewness. Similar kind of metrics have been considered in the literature, for example in [27]. Observe that (when $X \neq \emptyset$) $1 = 0^{\star} \leq \delta(x, x)^{\star} = \delta(x, x) \leq 0$, so if Q is mix, then necessarily $1 = 0$.

Lemma 27. *Suppose in Q the equality $1 = 0$ holds. By defining*

$$\delta_f(i, j) := \begin{cases} f_{i,j}\,, & i < j\,, \\ 0\,, & i = j\,, \\ f^{\star}_{j,i}\,, & j < i\,, \end{cases}$$

every clopen tuple f of $Q^{[d]_2}$ yields a unique skew metric on the set $[d]$. Every skew metric on the set $[d]$ with values in Q arises in this way.

The Lemma is an immediate consequence of the following statement:

Lemma 28. *A tuple f is clopen if and only if δ_f is a skew metric.*

Proof. Suppose δ_f is a skew metric. For $1 \leq i < j < k \leq d$, we have $f_{i,k} \leq f_{i,j} \oplus f_{j,k}$ (openness) and $f_{k,i} \leq f_{k,j} \oplus f_{j,i}$ which in turn is equivalent to $f_{i,j} \otimes f_{j,k} \leq f_{i,k}$ (closedness).

Conversely, suppose that f is clopen. Say that the pattern (ijk) is satisfied by f if $f_{i,k} \leq f_{i,j} \oplus f_{i,j}$. If $\mathrm{card}(\{ i, j, k \}) \leq 2$, then f satisfies the pattern (ijk) if $i = j$ or $j = k$, since then $f_{i,j} = 0$ or $f_{j,k} = 0$. If $i = k$, then $0 \leq f_{i,j} \oplus f_{j,i}$ is equivalent to $f_{i,j} \leq f_{i,j}$.

Suppose therefore that $\mathrm{card}(\{ i, j, k \}) = 3$. By assumption, f satisfies (ijk) and (kji) whenever $i < j < k$. Then it is possible to argue that all the patterns on the set $\{ i, j, k \}$ are satisfied by observing that if (ijk) is satisfied, then (jki) is satisfied as well: from $f_{i,k} \leq f_{i,j} \oplus f_{j,k}$, derive $f_{i,k} \otimes f_{k,j} = f_{i,k} \otimes f^{\star}_{j,k} \leq f_{i,j}$ and then $f_{j,i} = f^{\star}_{i,j} \leq (f_{i,k} \otimes f_{k,j})^{\star} = f_{j,k} \oplus f_{k,i}$. □

Remark 29. In the next sections we shall often need to verify that some tuple $f \in Q^{[d]_2}$ is clopen. A simple sufficient condition is that, for each $i, j, k \in [d]$ with $i < j < k$, either $f_{i,k} = f_{i,j} \otimes f_{j,k}$, or $f_{i,k} = f_{i,j} \oplus f_{j,k}$. Indeed, from $f_{i,k} = f_{i,j} \otimes f_{j,k}$ we derive $f_{i,j} \otimes f_{j,k} \leq f_{i,k} = f_{i,j} \otimes f_{j,k} \leq f_{i,j} \oplus f_{j,k}$, using the mix rule. Similarly, $f_{i,k} = f_{i,j} \oplus f_{j,k}$ implies $f_{i,j} \otimes f_{j,k} \leq f_{i,k} \leq f_{i,j} \oplus f_{j,k}$.

6. The mix $\star$-autonomous quantale $\mathsf{Q}_{\vee}(\mathbb{I})$

From this section onward $\mathbb{I}$ denotes the unit interval of the reals, $\mathbb{I} := [0, 1]$. Recall that we use $\mathsf{Q}_{\vee}(\mathbb{I})$ for the set of join-continuous functions from $\mathbb{I}$ to itself. Notice that a monotone function $f : \mathbb{I} \to \mathbb{I}$ is join-continuous if and only if

$$f(x) = \bigvee_{y<x} f(y)\,, \quad \text{or even} \quad f(x) = \bigvee_{y<x,\, y \in \mathbb{I} \cap \mathsf{Q}} f(y)\,, \tag{17}$$

see Proposition 2.1, Chapter II of [21]. According to Example 8, we have:

Lemma 30. *Composition induces a quantale structure on* $\mathsf{Q}_{\vee}(\mathbb{I})$.

Let now $\mathsf{Q}_{\wedge}(\mathbb{I})$ denote the collection of meet-continuous functions from $\mathbb{I}$ to itself. By duality, we obtain:

Lemma 31. *Composition induces a dual quantale structure on* $\mathsf{Q}_{\wedge}(\mathbb{I})$.

With the next set of observations we shall see $\mathsf{Q}_{\vee}(\mathbb{I})$ and $\mathsf{Q}_{\wedge}(\mathbb{I})$ are order isomorphic. For a monotone function $f : \mathbb{I} \to \mathbb{I}$, define

$$f^{\wedge}(x) = \bigwedge_{x<x'} f(x')\,, \qquad\qquad f^{\vee}(x) = \bigvee_{x'<x} f(x')\,.$$

Lemma 32. *Let $f : \mathbb{I} \to \mathbb{I}$ be monotone. If $x < y$, then $f^{\wedge}(x) \leq f^{\vee}(y)$.*

Proof. Pick $z \in \mathbb{I}$ such that $x < z < y$ and observe then that $f^{\wedge}(x) \leq f(z) \leq f^{\vee}(y)$. □

Proposition 33. *For a monotone $f : \mathbb{I} \to \mathbb{I}$, the following statements hold:*

1. *$f^{\wedge}$ is the least meet-continuous function above f and $f^{\vee}$ is the greatest join-continuous function below f,*
2. *the relations $f^{\vee\wedge} = f^{\wedge}$ and $f^{\wedge\vee} = f^{\vee}$ hold,*
3. *the operations $(\,\cdot\,)^{\vee} : \mathsf{Q}_{\wedge}(\mathbb{I}) \to \mathsf{Q}_{\vee}(\mathbb{I})$ and $(\,\cdot\,)^{\wedge} : \mathsf{Q}_{\vee}(\mathbb{I}) \to \mathsf{Q}_{\wedge}(\mathbb{I})$ are inverse order preserving bijections.*

Proof. (1) We only prove the first statement. Let us show that $f^{\wedge}$ is meet-continuous; to this goal, we use equation (17):

$$\bigwedge_{x<t} f^{\wedge}(t) = \bigwedge_{x<t} \bigwedge_{t<t'} f(t') = \bigwedge_{x<t} f(t') = f^{\wedge}(x)\,.$$

We observe next that $f \leq f^{\wedge}$, as if $x < t$, then $f(x) \leq f(t)$. This implies that if $g \in \mathsf{Q}_{\wedge}(\mathbb{I})$ and $f^{\wedge} \leq g$, then $f \leq f^{\wedge} \leq g$. Conversely, if $g \in \mathsf{Q}_{\wedge}(\mathbb{I})$ and $f \leq g$, then

$$f^{\wedge}(x) = \bigwedge_{x<t} f(t) \leq \bigwedge_{x<t} g(t) = g(x)\,.$$

Let us prove (2) and (3). Clearly, both maps are order preserving. Let us show that $f^{\vee\wedge} = f^{\wedge}$ whenever f is order preserving. We have $f^{\vee\wedge} \leq f^{\wedge}$, since $f^{\vee} \leq f$ and $(-)^{\wedge}$ is order preserves the pointwise ordering. For the converse inclusion, recall from the previous lemma that if $x < y$, then $f^{\wedge}(x) \leq f^{\vee}(y)$, so

$$f^{\wedge}(x) \leq \bigwedge_{x<y} f^{\vee}(y) = f^{\vee\wedge}(x)\,,$$

for each $x \in \mathbb{I}$. Finally, to see that $(-)^{\wedge}$ and $(-)^{\vee}$ are inverse to each other, observe that of $f \in \mathsf{Q}_{\wedge}(\mathbb{I})$, then $f^{\vee\wedge} = f^{\wedge} = f$. The equality $f^{\wedge\vee} = f$ for $f \in \mathsf{Q}_{\vee}(\mathbb{I})$ is derived similarly. □

Corollary 34. $\mathsf{Q}_{\vee}(\mathbb{I})$ *is a complete distributive lattice.*

Proof. The interval $\mathbb{I}$ is a complete distributive lattice, whence the set $\mathbb{I}^{\mathbb{I}}$ of all functions from $\mathbb{I}$ to $\mathbb{I}$, is also a complete distributive lattice, under the pointwise ordering and the pointwise operations. The subset of monotone functions from $\mathbb{I}$ to $\mathbb{I}$ is closed under infs and sups from $\mathbb{I}^{\mathbb{I}}$. In view of Proposition 33, join-continuous functions are the monotone functions that are fixed points of the interior operator $f \mapsto f^{\vee}$. As from standard theory, it follows that $\mathsf{Q}_{\vee}(\mathbb{I})$ is a complete lattice, that join-continuous functions are closed under pointwise suprema, and that infima in $\mathsf{Q}_{\vee}(\mathbb{I})$ are computed as follows:

$$(\bigwedge_{i\in I} f_i)(x) = \bigvee_{y<x} \inf\{\, f_i(y) \mid i \in I\,\}\,.$$

Finally notice that, in case $I = \{\,1, 2\,\}$, then

$$\begin{aligned}
(f_1 \wedge f_2)(x) &= \bigvee_{y<x} \min(f_1(y), f_2(y)) \\
&= \bigvee_{y_1<x} \bigvee_{y_2<x} \min(f_1(y_1), f_2(y_2))\,, \\
&\qquad\qquad \text{since the set } \{\, y \in \mathbb{I} \mid y < x\,\} \text{ is upward directed,} \\
&= \min(\bigvee_{y_1<x} f_1(y_1), \bigvee_{y_2<x} f_2(y_2)) = \min(f_1(x), f_2(x))\,,
\end{aligned}$$

where the last step follows from $f_i^{\vee} = f_i$, $i \in \{\,1, 2\,\}$. Therefore finite (non-empty) meets are computed pointwise, and this implies that $\mathsf{Q}_{\vee}(\mathbb{I})$ is a distributive lattice. □

Considering that $\mathbb{I}$ is a complete lattice, Proposition 33 shows that it is also a perfect chain and therefore. According to Corollary 14, we deduce the following statement.

Corollary 35. $\mathsf{Q}_{\vee}(\mathbb{I})$ *is a mix $\star$-autonomous quantale.*

7. Paths

Let in the following $d \geq 2$ be a fixed integer; we shall use $\mathbb{I}^d$ to denote the d-fold product of $\mathbb{I}$ with itself. That is, $\mathbb{I}^d$ is the usual geometric cube in dimension d. Let us recall that $\mathbb{I}^d$, as a product of the poset $\mathbb{I}$, has itself the structure of a poset (the order being coordinate-wise) and, moreover, of a complete lattice.

Definition 36. A *path in* $\mathbb{I}^d$ is a chain $C \subseteq \mathbb{I}^d$ with the following properties:

(1) if $X \subseteq C$, then $\bigwedge X \in C$ and $\bigvee X \in C$,
(2) C is dense as an ordered set: if $x, y \in C$ and $x < y$, then $x < z < y$ for some $z \in C$.

That is, we have defined a path in $\mathbb{I}^d$ as a totally ordered dense sub-complete-lattice of $\mathbb{I}^d$. We are going to see that paths in $\mathbb{I}^d$ can be characterized in many ways.

Lemma 37. *Paths in $\mathbb{I}^d$ are exactly the maximal chains of the poset $\mathbb{I}^d$.*

Proof. We firstly argue that every path in $\mathbb{I}^d$ is a maximal chain of $\mathbb{I}^d$.

Let $C \subseteq \mathbb{I}^d$ be a path and suppose that there exists $z \in \mathbb{I}^d \setminus C$ such that $C \cup \{ z \}$ is a chain. Let $z^- = \{ c \in C \mid c < z \}$ and $z^+ = \{ c \in C \mid z < c \}$. Since $z \notin C$ and C is closed under meets and joins, we have $\bigvee z^- < z < \bigwedge z^+$, with $\bigvee z^-, \bigwedge z^+ \in C$. By density, let $w \in C$ be such that $\bigvee z^- < w < \bigwedge z^+$. Since $w \in C \subseteq C \cup \{ z \}$ and the latter is a chain, then $w < z$ or $z < w$. In the first case we obtain $w \leq \bigvee z^-$ and in the second case $\bigwedge z^+ \leq w$ and, in both cases, we have a contradiction.

Next, we argue that every maximal chain of $\mathbb{I}^d$ is a path in $\mathbb{I}^d$. Let C be a maximal chain of $\mathbb{I}^d$. Take $X \subseteq C$ and let $a := \bigwedge X \in \mathbb{I}^d$. The maximality of C implies that $0, 1 \in C$ and so $a \in C$ whenever $X = \emptyset$ or $X = C$. Suppose that $X \neq \emptyset$. We claim that $C \cup \{ a \}$ is a chain and consequently $a \in C$ by the maximality of C. Let $c \in C$; if $c \not\leq a$ then $c \not\leq x$, for some $x \in X$, which implies $x < c$ and so $a < c$; if $a \not\leq c$, then $x \not\leq c$ for every $x \in X$, which implies $c < x$ for every $x \in X$, and so $c \leq a$. Thus $C \cup \{ a \}$ is a chain as aimed. Let us now prove that C is dense. Let $x < y$ in C. Suppose that for every $c \in C$ we have $y \leq c$ or $c \leq x$. Since $x < y$, there exists $j \in [d]$ such that $x_j < y_j$. The density of $\mathbb{I}$ implies the existence of $z_j \in I$ such that $x_j < z_j < y_j$. Take $w \in \mathbb{I}^d$ to be defined by $w_j = z_j$ and $w_i = x_i$ for $i \neq j$. Clearly $x < w < y$. If $w \notin C$, then $C \cup \{ w \}$ is not a chain and there exists $c \in C$ such that $w \not\leq c$ and $c \not\leq w$; consequently, $y \not\leq c$ and $c \not\leq x$, which contradicts the assumption that $y \leq c$ or $c \leq x$, for each $c \in C$. Thus there must be $c \in C$ such that $x < c < y$. □

We carry over with a characterization of maximal chains of $\mathbb{I}^d$ which justifies naming them paths.

Lemma 38. *A monotone function $\mathsf{p} : \mathbb{I} \to \mathbb{I}^d$ such that $\mathsf{p}(0) = \vec{0}$ and $\mathsf{p}(1) = \vec{1}$ is topologically continuous if and only if it is bi-continuous. Consequently, its image in $\mathbb{I}^d$ is a path.*

Proof. Let p be as in the statement of the Lemma. For each $i \in \{ 1, \ldots, d \}$, let $\pi_i : \mathbb{I}^d \to \mathbb{I}$ be the projection on the i-th coordinate, and set $f_i := \pi_i \circ f$, so each f_i is monotone. Recall the standard theorem on existence/characterization of left limits of monotone functions: $\lim_{y \to x^-} f_i(y) = \bigvee f_i([0, x))$.

If p is topologically continuous, then each f_i is topologically continuous. Let $X \subseteq \mathbb{I}$ and observe that X is cofinal in $[0, \bigvee X)$ (that is, for each $y \in [0, \bigvee X)$ here exists $x \in X$ such that $y \leq x$. This implies that $\bigvee g([0, \bigvee X)) \leq \bigvee g(X)$, for each monotone function g. It follows that

$$f_i(\bigvee X) = \lim_{y \to (\bigvee x)^-} f_i(y)\,, \qquad \text{since } f_i \text{ is topologically continuous,}$$
$$= \bigvee f_i([0, \bigvee X)) \leq \bigvee f_i(X)\,.$$

Since the opposite inclusion holds by monotonicity, this shows that each f_i is join-continuous, so f is join-continuous. In a similar way, f is meet-continuous.

Conversely, let us suppose that f is bi-continuous. Thus, for each $x \in \mathbb{I}$, we have

$$\lim_{y \to x^-} f_i(y) = \bigvee f_i([0, x)) = f(x) = \bigwedge f_i((x, 1]) = \lim_{z \to x^+} f_i(z)\,,$$

showing that each f_i (and therefore f) is topologically continuous.

For the last statement, let $C = \mathsf{p}(\mathbb{I})$. Let $X \subseteq C$ and $Y \subseteq \mathbb{I}$ be such that $\mathsf{p}(Y) = X$. Then $\bigvee X = \bigvee \mathsf{p}(Y) = \mathsf{p}(\bigvee Y) \in C$; in a similar way, $\bigwedge Y \in C$. Let us show that C is dense. Let $x, y \in \mathbb{I}$ be such that $\mathsf{p}(x) < \mathsf{p}(y)$. Since p is monotone, we also have $x < y$ (use Lemma 39). Consider then the image of the connected interval $[x, y]$. Since p is topologically continuous, its image cannot be the disconnected two points set $\{ \mathsf{p}(x), \mathsf{p}(y) \}$. Therefore there exists $z \in (x, y)$ such that $\mathsf{p}(z) \notin \{ \mathsf{p}(x), \mathsf{p}(y) \}$; then, by monotonicity, we get $\mathsf{p}(x) < \mathsf{p}(z) < \mathsf{p}(y)$. □

Thus, if $\mathsf{p} : \mathbb{I} \to \mathbb{I}^d$ is a monotone topologically continuous function with $\mathsf{p}(0) = \vec{0}$ and $\mathsf{p}(1) = \vec{1}$, then $\mathsf{p}(\mathbb{I}) \subseteq \mathbb{I}^d$ is a path. We are going to show that every path arises in this way.

Lemma 39. *Consider a monotone function $f : C \to P$ where C is a chain and P is any poset. Then f reflects the strict order: $f(x) < f(y)$ implies $x < y$.*

Proof. Suppose $f(x) < f(y)$. We have $y \leq x$ or $x < y$. However, if $y \leq x$, then $f(y) \leq f(x)$ as well, contradicting $f(x) < f(y)$. Whence $x < y$. □

Lemma 40. *Any bi-continuous function $f : C \to \mathbb{I}$, where C is a path, is surjective.*

Proof. Since f is bi-continuous, it has left and right adjoints, say $\ell \dashv f \dashv \rho$. We shall show that $\ell \leq \rho$; from this and the unit/counit relations $h(\rho(t)) \leq t \leq h(\ell(t))$ it follows that both $\ell(t)$ and $\rho(t)$ are preimages of $t \in \mathbb{I}$.

Let $t \in \mathbb{I}$ be arbitrary; since C is a chain, either $\ell(t) \leq \rho(t)$ holds, or $\rho(t) < \ell(t)$ holds. In the latter case, let $c \in C$ be such that $\rho(t) < c < \ell(t)$. As $\mathbb{I}$ is a chain, either $f(c) \leq t$, or $t \leq f(c)$. If $f(c) \leq t$, then we have $c \leq \rho(t)$, contradicting $\rho(t) < c$; if $t \leq f(c)$, then $\ell(t) \leq c$, contradicting $c < \ell(t)$. Therefore the relation $\ell(t) \leq \rho(t)$ holds, for each $t \in C$. □

For a path $C \subseteq \mathbb{I}^d$ and $i = 1, \ldots, d$, let us define $\pi_i : C \to \mathbb{I}$ as the inclusion of C into $\mathbb{I}^d$ followed by the projection to the i-component. Observe that π_i is bi-continuous (since it is the composition of two bi-continuous functions), thus it is surjective by the previous Lemma.

Proposition 41. *Every path C is order isomorphic to $\mathbb{I}$. In particular, there exists a monotone continuous function $\mathsf{p} : C \to \mathbb{I}$ such that $\mathsf{p}(0) = \vec{0}$, $\mathsf{p}(1) = \vec{1}$, and $\mathsf{p}(\mathbb{I}) = C$.*

Proof. We shall show that C has a dense countable subset C_{Q} without endpoints which generates C both under infinite joins and under infinite meets. By a well known theorem by Cantor, see e.g. [10, Proposition 1.4.2], C_{Q} is order isomorphic to $\mathbb{I} \cap \mathbb{Q} \setminus \{ 0, 1 \}$. Then C is order isomorphic to the Dedekind-MacNeille completion of $\mathbb{I} \cap \mathbb{Q} \setminus \{ 0, 1 \}$, namely to $\mathbb{I}$. For each $i \in \{ 1, \ldots, d \}$ and $q \in \mathbb{I} \cap \mathbb{Q} \setminus \{ 0, 1 \}$, pick $c_{i,q} \in C$ such that $\pi_i(c_{i,q}) = q$. Let

$$C_{\mathsf{Q}} := \{ c_{i,q} \mid i \in \{ 1, \ldots, d \},\ q \in \mathbb{I} \cap \mathbb{Q} \setminus \{ 0, 1 \} \},$$

and observe that C_{Q} is countable. We firstly argue that C_{Q} is dense in C. Let $c, c' \in C$ such that $c < c'$. By definition of the order on $\mathbb{I}^d$, $\pi_i(c) < \pi_i(c')$ for some $i \in \{ 1, \ldots, d \}$. Let $q \in \mathbb{I} \cap \mathbb{Q}$ be such that $\pi_i(c) < q < \pi_i(c')$. Then, by Lemma 39, we deduce $c < c_{i,q} < c'$, with $c_{i,q} \in C_{\mathsf{Q}}$.

Also C_{Q} has no endpoints. For example, if $c = c_{i,q} \in C_{\mathsf{Q}}$ and $q' \in \mathbb{I} \cap \mathbb{Q}$ is such that $q' < q$, then necessarily $c_{i,q'} < c_{i,q}$, so C_{Q} has no least element.

Finally, we prove that C_{Q} generates C under infinite joins. Let $c \in C$ and consider the set $D := \{ x \in C_{\mathsf{Q}} \mid x < c \}$; suppose that $\bigvee D < c$. There exists $i \in \{ 1, \ldots, d \}$ such that $\pi_i(\bigvee D) < \pi_i(c)$, and we can pick $q \in \mathsf{Q}$ such that $\pi_i(\bigvee D) < q < \pi_i(c)$. Let $c_{i,q}$ be such that $\pi_i(c_{i,q}) = q$, then, by Lemma 39, we have $\bigvee D < c_{i,q} < c$. Yet, this is a contradiction,

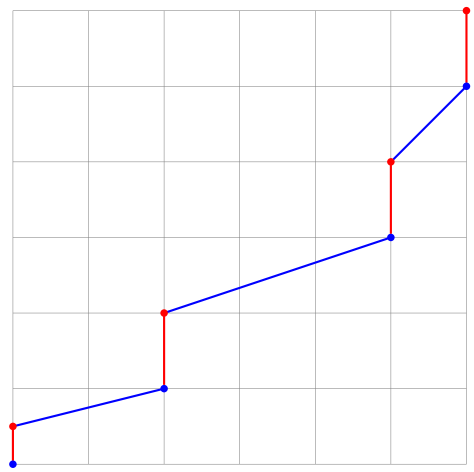

FIGURE 1. The path C_f of $f \in \mathsf{Q}_\vee(\mathbb{I})$

as $c_{i,q} \in C_{\mathsf{Q}}$ and $c_{i,q} < c$ imply $c_{i,q} \in D$, whence $c_{i,q} \leq \bigvee D$. In a similar way, we can show that every element of C is a meet of elements of C_{Q}. □

8. Paths in dimension 2

We give next a further characterization of the notion of path, valid in dimension 2. The principal result of this Section, Theorem 45, states that paths in dimension 2 bijectively correspond to elements of the quantale $\mathsf{Q}_\vee(\mathbb{I})$.

For a monotone function $f : \mathbb{I} \to \mathbb{I}$ define $C_f \subseteq \mathbb{I}^2$ by the formula

$$C_f := \bigcup_{x \in \mathbb{I}} \{ x \} \times [f^\vee(x), f^\wedge(x)] \,. \tag{18}$$

Notice that, by Proposition 33, $C_f = C_{f^\vee} = C_{f^\wedge}$. As suggested in figure 8, when $f \in \mathsf{Q}_\vee(\mathbb{I})$, then C_f is the graph of f (in blue in the figure) with the addition of the intervals $(f^\vee(x), f^\wedge(x)]$ (in red in the figure) when x is a discontinuity point of f.

Proposition 42. *C_f is a path in $\mathbb{I}^2$.*

Proof. We prove first that C_f, with the product ordering induced from $\mathbb{I}^2$, is a linear order. To this goal, we shall argue that, for $(x, y), (x', y') \in C_f$, we have $(x, y) < (x', y')$ iff either $x < x'$ or $x = x'$ and $y < y'$. That is, C_f is a lexicographic product of linear orders, whence a linear order. Let us suppose that one of these two conditions holds: a) $x < x'$, b) $x = x'$ and $y < y'$. If a), then $f^\wedge(x) \leq f^\vee(x')$. Considering that $y \in [f^\vee(x), f^\wedge(x)]$ and $y' \in [f^\vee(x'), f^\wedge(x')]$ we deduce $y \leq y'$. This proves that $(x, y) < (x', y')$ in the product ordering. If b) then we also have $(x, y) < (x', y')$ in the product ordering. The converse implication, $(x, y) < (x', y')$ implies $x < x'$ or $x = x'$ and $y < y'$, trivially holds.

We argue next that C_f is closed under joins from $\mathbb{I}^2$. Let (x_i, y_i) be a collection of elements in C_f, we aim to show that $(\bigvee x_i, \bigvee y_i) \in C_f$, i.e. $\bigvee y_i \in [f^\vee(\bigvee x_i), f^\wedge(\bigvee x_i)]$. Clearly, as $y_i \leq f^\wedge(x_i)$, then $\bigvee y_i \leq \bigvee f^\wedge(x_i) \leq f^\wedge(\bigvee x_i)$. Next, $f^\vee(x_i) \leq y_i$, whence $f^\vee(\bigvee x_i) = \bigvee f^\vee(x_i) \leq \bigvee y_i$. By a dual argument, we have that $(\bigwedge x_i, \bigwedge y_i) \in C_f$.

Finally, we show that C_f is dense; to this goal let $(x, y), (x', y') \in C_f$ be such that $(x, y) < (x', y')$. If $x < x'$ then we can find a z with $x < z < x'$; of course, $(z, f(z)) \in C_f$ and, by the previous characterisation of the order, $(x, y) < (z, f(z)) < (x', y')$ holds. If $x = x'$ then $y < y'$ and we can find a w with $y < w < y'$; as $w \in [y, y'] \subseteq [f^\vee(x), f^\wedge(x)]$, then $(x, w) \in C_f$; clearly, we have then $(x, y) < (x, w) < (x, y') = (x', y')$. $\square$

For C a path in $\mathbb{I}^2$, define

$$f_C^-(x) := \bigwedge \{\, y \mid (x, y) \in C \,\}, \qquad\qquad f_C^+(x) := \bigvee \{\, y \mid (x, y) \in C \,\}. \tag{19}$$

Recall that a path $C \subseteq \mathbb{I}^2$ comes with bi-continuous surjective projections $\pi_1, \pi_2 : C \to \mathbb{I}$. Observe that the following relations hold:

$$f_C^- = \pi_2 \circ (\pi_1)_\ell\,, \qquad\qquad f_C^+ = \pi_2 \circ (\pi_1)_\rho\,. \tag{20}$$

Indeed, we have

$$\begin{aligned} \pi_2((\pi_1)_\ell(x)) &= \pi_2(\bigwedge \{\, (x', y) \in C \mid x = x' \,\})\,, && \text{using equation (2)} \\ &= \bigwedge \pi_2(\{\, (x', y) \in C \mid x = x' \,\}) = \bigwedge \{\, y \mid (x, y) \in C \,\}\,. \end{aligned}$$

The other expression for f_C^+ is derived similarly. In particular, the expressions in (20) show that $f_C^- \in \mathsf{Q}_\vee(\mathbb{I})$ and $f_C^+ \in \mathsf{Q}_\wedge(\mathbb{I})$.

Lemma 43. *We have*

$$f_C^- = (f_C^+)^\vee\,, \quad f_C^+ = (f_C^-)^\wedge\,, \quad \textit{and} \quad C = C_{f_C^+} = C_{f_C^-}\,.$$

Proof. Firstly, let us argue that $f_C^+ = (f_C^-)^\wedge$; we do this by showing that f_C^+ is the least meet-continuous function above f_C^-. We have $f_C^-(x) \leq f_C^+(x)$ for each $x \in \mathbb{I}$, since π_1 is surjective so the fibers $\pi_1^{-1}(x) = \{\, (x', y) \in C \mid x' = x \,\}$ are non empty. Suppose now that $f_C^- \leq g \in \mathsf{Q}_\wedge(\mathbb{I})$. In order to prove that $f_C^+ \leq g$ it will be enough to prove that $f_C^+(x) \leq g(x')$ whenever $x < x'$. Observe that if $x < x'$ then $f_C^+(x) \leq f_C^-(x')$: this is because if $(x, y), (x', y') \in C$, then $x < x'$ and C a chain imply $y \leq y'$. We deduce therefore $f_C^+(x) \leq f_C^-(x') \leq g(x')$. The relation $f_C^- = (f_C^+)^\vee$ is proved similarly.

Next we argue that $(x, y) \in C$ if and only if $f_C^-(x) \leq y \leq f_C^+(x)$. The direction from left to right is obvious. Conversely, we claim that if $f_C^-(x) \leq y \leq f_C^+(y)$, then the pair (x, y) is comparable with all the elements of C. It follows then that $(x, y) \in C$, since C is a maximal chain. Let us verify the claim. Let $(x', y') \in C$, if $x = x'$ then our claim is obvious, and if $x' < x$, then $y' \leq f_C^+(x') \leq f_C^-(x) \leq y$, so $(x', y') \leq (x, y)$; the case $x < x'$ is similar. $\square$

Lemma 44. *Let $f : \mathbb{I} \to \mathbb{I}$ be monotone and consider the path C_f. Then $f^\vee = f_{C_f}^-$ and $f^\wedge = f_{C_f}^+$.*

Proof. For a monotone $f : \mathbb{I} \to \mathbb{I}$, let $f' : \mathbb{I} \to C_f$ by $f' := \langle id_{\mathbb{I}}, f^\vee \rangle$, so $f^\vee = \pi_2 \circ f'$, as in the diagram below:

$$\begin{tikzcd} \mathbb{I} \arrow[r, shift left=2, "(\pi_1)_\ell"] \arrow[r, shift right=2, "\langle id, f^\vee \rangle"'] \arrow[rr, bend left, "f_{C_f}^-"] \arrow[rr, bend right, "f^\vee"'] & C_f \arrow[l, "\pi_1"] \arrow[r, "\pi_2"] & \mathbb{I} \end{tikzcd}$$

Recall that $f_{C_f}^- = \pi_2 \circ (\pi_1)_\ell$. Therefore, in order to prove the relation $f^\vee = f_{C_f}^- = \pi_2 \circ (\pi_1)_\ell$ it shall be enough to prove that $\langle id, f^\vee \rangle$ is left adjoint to the first projection (that is, we prove that $\langle id, f^\vee \rangle = (\pi_1)_\ell$, from which it follows that $f^\vee = \pi_1 \circ \langle id, f^\vee \rangle = \pi_2 \circ (\pi_1)_\ell$). This amounts to verify that, for $x \in \mathbb{I}$ and $(x', y) \in C_f$ we have $x \leq \pi_1(x', y)$ if and only if

$(x, f^{\vee}(x)) \leq (x', y)$. To achieve this goal, the only non trivial observation is that if $x \leq x'$, then $f^{\vee}(x) \leq f^{\vee}(x') \leq y$. The relation $f^{\wedge} = \pi_2 \circ (\pi_1)_{\rho}$ is proved similarly. □

Theorem 45. *There is a bijective correspondence between the following data:*

(1) *paths in* $\mathbb{I}^2$,
(2) *join-continuous functions in* $\mathsf{Q}_{\vee}(\mathbb{I})$,
(3) *meet-continuous functions in* $\mathsf{Q}_{\wedge}(\mathbb{I})$.

Proof. According to Lemmas 43 and 44, the correspondence sending a path C to $f^{-}_{C} \in \mathsf{Q}_{\vee}(\mathbb{I})$ has the mapping sending f to C_f as an inverse. Similarly, the correspondence $C \mapsto f^{+}_{C} \in \mathsf{Q}_{\wedge}(\mathbb{I})$ has $f \mapsto C_f$ as inverse. □

9. Paths in higher dimensions

We show in this section that paths in dimension d, as defined in Section 7, are in bijective correspondence with clopen tuples of $\mathsf{Q}_{\vee}(\mathbb{I})^{[d]_2}$, as defined in Section 5; therefore, as established in that Section, there is a lattice $\mathsf{L}_d(\mathsf{Q}_{\vee}(\mathbb{I}))$ whose underlying set can be identified with the set of paths in dimension d.

Let $f \in \mathsf{Q}_{\vee}(\mathbb{I})^{[d]_2}$, so $f = \{ f_{i,j} \mid 1 \leq i < j \leq d \}$. We define then, for $1 \leq i < j \leq d$,

$$f_{j,i} := (\, f_{i,j}\,)^{\star} = ((f_{i,j})_{\rho})^{\vee}\,.$$

Moreover, for $i \in [d]$, we let $f_{i,i} := id$. We say shall say that a tuple $f \in \mathsf{Q}_{\vee}(\mathbb{I})^{[d]_2}$ is *compatible* if $f_{j,k} \circ f_{i,j} \leq f_{i,k}$, for each triple of elements $i, j, k \in [d]$. It is readily seen that a tuple is compatible if and only if δ_f, defined in Lemma 27, is a skew metric on $[d]$. Therefore, according to Lemma 28, *a tuple is compatible if and only if it is clopen.*

If $C \subseteq \mathbb{I}^d$ is a path, then we shall use $\pi_i : C \to \mathbb{I}$ to denote the projection onto the i-th coordinate. Then $\pi_{i,j} := \langle \pi_i, \pi_j \rangle : C \to \mathbb{I} \times \mathbb{I}$.

Definition 46. For a path C in $\mathbb{I}^d$, let us define $\nu(C) \in \mathsf{Q}_{\vee}(\mathbb{I})^{[d]_2}$ by the formula:

$$\nu(C)_{i,j} := \pi_j \circ (\pi_i)_{\ell}\,, \quad (i,j) \in [d]_2. \tag{21}$$

Remark 47. An explicit formula for $\nu(C)_{i,j}(x)$ is as follows:

$$\nu(C)_{i,j}(x) = \bigwedge \{\, \pi_j(y) \in C \mid \pi_i(y) = x \,\}\,. \tag{22}$$

Let $C_{i,j}$ be the image of C via the projection $\pi_{i,j}$. Then $C_{i,j}$ is a path, since it is the image of a bi-continuous function from $\mathbb{I}$ to $\mathbb{I} \times \mathbb{I}$. Some simple diagram chasing (or the formula in (22)) shows that $\nu(C)_{i,j} = f^{-}_{C_{i,j}}$ as defined in (19).

Definition 48. For a compatible $f \in \mathsf{Q}_{\vee}(\mathbb{I})^{[d]_2}$, define

$$C_f := \{\, (x_1, \ldots, x_d) \mid f_{i,j}(x_i) \leq x_j, \text{ for all } i, j \in [d] \,\}\,.$$

Remark 49. Notice that the condition $f_{i,j}(x) \leq y$ is equivalent (by definition of $f_{i,j}$ or $f_{j,i}$) to the condition $x \leq f^{\wedge}_{j,i}(y)$. Thus, there are in principle many different ways to define C_f; in particular, when $d = 2$ (so any tuple $\mathsf{Q}_{\vee}(\mathbb{I})^{[d]_2}$ is compatible), the definition given above is equivalent to the one given in (18).

Proposition 50. *C_f is a path.*

The proposition is an immediate consequence of the following Lemmas 51, 52 and 54.

Lemma 51. *C_f is a total order.*

Proof. Let $x, y \in C_f$ and suppose that $x \not\leq y$, so there exists $i \in [d]$ such that $x_i \not\leq y_i$. W.l.o.g. we can suppose that $i = 1$, so $y_1 < x_1$ and then, for $i > 1$, we have $f^{\wedge}_{1,i}(y_1) \leq f_{1,i}(x_1)$, whence $y_i \leq f^{\wedge}_{1,i}(y_1) \leq f_{1,i}(x_1) \leq x_1$. This shows that $y < x$. ◻

Lemma 52. *C_f is closed under arbitrary meets and joins.*

Proof. Let $\{ x^\ell \mid \ell \in I \}$ be a family of tuples in C_f. For all $i, j \in [d]$ and $\ell \in I$, we have $f_{i,j}(\bigwedge_{\ell \in I} x^\ell_i) \leq f_{i,j}(x^\ell_i) \leq x^\ell_j$, and therefore $f_{i,j}(\bigwedge_{\ell \in I} x^\ell_i) \leq \bigwedge_{\ell \in I} x^\ell_j$. Since meets in $\mathbb{I}^d$ are computed coordinate-wise, this shows that C_f is closed under arbitrary meets. Similarly, $f_{i,j}(x^\ell_i) \leq \bigvee_{\ell \in I} x^\ell_j$ and

$$f_{i,j}(\bigvee_{\ell \in I} x^\ell_i) = \bigvee_{\ell \in I} f_{i,j}(x^\ell_i) \leq \bigvee_{\ell \in I} x^\ell_j \,,$$

so C_f is also closed under arbitrary joins. ◻

Lemma 53. *Let $f \in \mathsf{Q}_\vee(\mathbb{I})^{[d]_2}$ be compatible. Let $i_0 \in [d]$ and $x_0 \in \mathbb{I}$; define $x \in \mathbb{I}^d$ by setting $x_i := f_{i_0,i}(x_0)$ for each $i \in [d]$. Then $x \in C_f$ and $x = \bigwedge\{ y \in C_f \mid \pi_{i_0}(y) = x_0 \}$.*

Proof. Since f is compatible, $f_{i,j} \circ f_{i_0,i} \leq f_{i_0,j}$, for each $i, j \in [d]$, so

$$f_{i,j}(x_i) = f_{i,j}(f_{i_0,i}(x_0)) \leq f_{i_0,j}(x_0) = x_j \,.$$

Therefore, $x \in C_f$. Observe that since $f_{i_0,i_0} = id$, we have $x_{i_0} = x_0$ and x so defined is such that $\pi_{i_0}(x) = x_0$. On the other hand, if $y \in C_f$ and $x_0 \leq \pi_{i_0}(y) = y_{i_0}$, then $x_i = f_{i,i_0}(x_0) \leq f_{i,i_0}(y_{i_0}) \leq y_i$, for all $i \in [d]$. Thus $x = \bigwedge\{ y \in C_f \mid \pi_{i_0}(y) = x_0 \}$. ◻

Lemma 54. *C_f is dense.*

Proof. Let $x, y \in C_f$ and suppose that $x < y$, so there exists $i_0 \in [d]$ such that $x_{i_0} < y_{i_0}$. Pick $z_0 \in \mathbb{I}$ such that $x_{i_0} < z_0 < y_{i_0}$ and define $z \in C_f$ as in Lemma 53, $z_i := f_{i_0,i}(z_0)$, for all $i \in [d]$. We claim that $x_i \leq z_i \leq y_i$, for each $i \in [d]$. From this and $x_{i_0} < z_{i_0} < y_0$ it follows that $x < z < y$. Indeed, we have $z_i = f_{i_0,i}(z_0) \leq f_{i_0,i}(y_{i_0}) \leq y_i$. Moreover, $x_{i_0} < z_0$ implies $f^{\wedge}_{i_0,i}(x_{i_0}) \leq f_{i_0,i}(z_0)$; by Remark 49, we have $x_i \leq f^{\wedge}_{i_0,i}(x_{i_0})$. Therefore, we also have $x_i \leq f^{\wedge}_{i_0,i}(x_{i_0}) \leq f_{i_0,i}(z_0) = z_i$. ◻

Lemma 55. *If $f \in \mathsf{Q}_\vee(\mathbb{I})^{[d]_2}$ is compatible, then $\nu(C_f) = f$.*

Proof. By Lemma 53, the correspondence sending x to $(f_{i,1}(x), \ldots, f_{d,1}(x))$ is left adjoint to the projection $\pi_i : C_f \to \mathbb{I}$. In turn, this gives that $\nu(C_f)_{i,j}(x) = \pi_j((\pi_i)_\ell(x)) = f_{i,j}(x)$, for any $i, j \in [d]$. It follows that $\nu(C_f) = f$. ◻

Lemma 56. *For C a path in $\mathbb{I}^d$, we have $C_{\nu(C)} = C$.*

Proof. Let us show that $C \subseteq C_{\nu(C)}$. Let $c \in C$; notice that for each $i, j \in [d]$, we have

$$\nu(C)_{i,j}(c_i) = \pi_j((\pi_i)_\ell(c_i)) = \pi_j((\pi_i)_\ell(\pi_i(c)) \leq \pi_j(c) = c_j \,,$$

so $c \in C_{\nu(C)}$. For the converse inclusion, notice that $C \subseteq C_{\nu(C)}$ implies $C = C_{\nu(C)}$, since every path is a maximal chain. ◻

Putting together Lemmas 55 and 56 we obtain:

Theorem 57. *The correspondences, sending a path C in $\mathbb{I}^d$ to the tuple $\nu(C)$, and a compatible tuple f to the path C_f, are inverse bijections.*

10. Structure of the continuous weak orders

As established in Section 5, there is a lattice structure $\mathsf{L}_d(\mathsf{Q}_\vee(\mathbb{I}))$ whose underlying set is the set of clopen tuples of the product $\mathsf{Q}_\vee(\mathbb{I})^{[d]_2}$. By the results in the previous section, these tuples can be identified with paths in dimension d. We give in this section a minimum of structural theory of these lattices by characterizing their join-irreducible elements.

10.1. **Join-prime elements of $\mathsf{Q}_\vee(\mathbb{I})$.** Recall from Corollary 34 that $\mathsf{Q}_\vee(\mathbb{I})$ is a complete distributive lattice and that, in distributive lattices, join-prime and join-irreducible elements coincide. We determine therefore the join-prime elements of $\mathsf{Q}_\vee(\mathbb{I})$. For $x, y \in \mathbb{I}$, let us put

$$e_{x,y}(t) := \begin{cases} 0\,, & 0 \leq t \leq x\,,\\ y\,, & x < t\,, \end{cases} \qquad\qquad E_{x,y}(t) := \begin{cases} 0\,, & 0 \leq t < x\,,\\ y\,, & x \leq t < 1\,,\\ 1\,, & t = 1\,, \end{cases} \tag{23}$$

so $e_{x,y} \in \mathsf{Q}_\vee(\mathbb{I})$, $E_{x,y} \in \mathsf{Q}_\wedge(\mathbb{I})$ and $E_{x,y} = e^{\wedge}_{x,y}$.

Definition 58. A *one step function* is a function of the form $e_{x,y}$ where $x, y \in \mathbb{I}$. We say that $e_{x,y}$ is *prime* if $e_{x,y} \neq \bot$. We say that $e_{x,y}$ is *rational* if $x, y \in \mathbb{I} \cap \mathbb{Q}$.

Lemma 59. *For each $x, y \in \mathbb{I}$, $e_{x,y} = \bot$ if and only of $x = 1$ or $y = 0$.*

Proof. If $x = 1$ or $y = 0$, then $e_{x,y}$ is the constant function that takes 0 as its unique value, i.e. $e_{x,y} = \bot$. Conversely, if $x < 1$ and $0 < y$, then, $e_{x,y}(1) = y \neq 0$, so $e_{x,y} \neq \bot$. □

From the lemma it also follows that $e_{x,y} \neq \bot$ if and only if $x < 1$ and $0 < y$. Notice therefore that $e_{x,y} \neq \bot$ if and only if the point $(x, y) \in \mathbb{I}^2$ does not lie on the path $\{\,(x, 0) \mid x \in \mathbb{I}\,\} \cup \{\,(1, y) \mid y \in \mathbb{I}\,\}$.

Lemma 60. *For $f \in \mathsf{Q}_\vee(\mathbb{I})$ and $x, y \in \mathbb{I}$, $e_{x,y} \leq f$ if and only if $y \leq f^\wedge(x)$.*

Proof. If $e_{x,y} \leq f$ then $y = e^{\wedge}_{x,y}(x) \leq f^\wedge(x)$. Conversely, suppose that $y \leq f^\wedge(x)$. If $t \leq x$, then $e_{x,y}(t) = 0 \leq f(t)$. If $x < t \leq 1$, then $e_{x,y}(t) = y \leq f^\wedge(x) \leq f(t)$, where the last inequality follows from Lemma 32. □

Corollary 61. *Let $x, y, z, w \in \mathbb{I}$ and suppose that $e_{x,y}, e_{z,w} \neq \bot$. Then $e_{x,y} \leq e_{z,w}$ if and only if $z \leq x$ and $y \leq w$.*

Proof. If $e_{x,y} \leq e_{z,w}$, then $y \leq e^{\wedge}_{z,w}(x)$. Since $0 < y$, we derive then $z \leq x$. Since $x < 1$ we also have $e^{\wedge}_{z,w}(x) = w$, so $y \leq e^{\wedge}_{z,w}(x) = w$. Conversely, suppose $z \leq x$ and $y \leq w$. From $z \leq x < 1$ we deduce $e^{\wedge}_{z,w}(x) = w$, so $y \leq w = e^{\wedge}_{z,w}(x)$ yields, according to the previous lemma, $e_{x,y} \leq e_{z,w}$. □

For $f \in \mathsf{Q}_\vee(\mathbb{I})$ and $x_0, x_1 \in \mathbb{I}$ with $x_0 \leq x_1$, we define $f_{(x_0,x_1]} \in \mathsf{Q}_\vee(\mathbb{I})$ as follows:

$$f_{(x_0,x_1]}(t) := \begin{cases} 0\,, & 0 \leq t \leq x_0\,,\\ f(t)\,, & x_0 < t \leq x_1\,,\\ f^\wedge(x_1)\,, & x_1 < t\,. \end{cases}$$

In particular, for any $x \in \mathbb{I}$, we have

$$f_{(0,x]}(t) = \begin{cases} f(t)\,, & 0 \leq t \leq x\,,\\ f^\wedge(x)\,, & x < t\,, \end{cases} \qquad\qquad f_{(x,1]}(t) = \begin{cases} 0\,, & 0 \leq t \leq x\,,\\ f(t)\,, & x < t \leq 1\,, \end{cases}$$

so

$$f = f_{(0,x]} \vee f_{(x,1]}\,.$$

Proposition 62. *Prime one step functions are exactly the join-prime elements of* $\mathsf{Q}_\vee(\mathbb{I})$.

Proof. Consider $e_{x,y}$ and suppose that $e_{x,y} \leq f \vee g$. This relation holds if and only if $y \leq \max(f^\wedge(x), g^\wedge(x))$, if and only if $y \leq f^\wedge(x)$ or $y \leq g^\wedge(x)$, that is $e_{x,y} \leq f$ or $e_{x,y} \leq g$. Thus every function of the form $e_{x,y}$ which is different from $\perp$ is join-prime.

Conversely, let $f \in \mathsf{Q}_\vee(\mathbb{I})$ be join-prime (so f is join-irreducible) and recall that, for any $x \in \mathbb{I}$, $f = f_{(0,x]} \vee f_{(x,1]}$. Therefore, for each $x \in \mathbb{I}$, $f = f_{(0,x]}$ or $f = f_{(x,1]}$. Observe also that if $f = f_{(0,x]}$ and $f = f_{(x,1]}$, then $f = e_{x,f^\wedge(x)}$.

Let now $I_f := \{\, x \in \mathbb{I} \mid f = f_{(x,1]} \,\}$ and $F_f := \{\, x \in \mathbb{I} \mid f = f_{(0,x]} \,\}$, so $I_f \cup F_f = \mathbb{I}$. Notice that $x \in I_f$ if and only if $f(x) = 0$ and $x \in F_f$ if and only if the restriction of f to the interval $(x, 1]$ is constant. From these considerations it immediately follows that I_f is a downset and F_f is an upset; moreover, I_f is closed under joins (since f is join-continuous) and F_f is closed under meets. If $x \in I_f$, $y \in F_f$, and $y < x$, then f is constant with value 0, which contradicts f being join-irreducible (thus distinct from $\perp$). Therefore, if $x \in I_f$ and $y \in F_f$, then $x \leq y$. Then $x_0 = \bigvee I_f = \bigwedge F_f \in I_f \cap F_f$ and $f = e_{x_0,f^\wedge(x_0)}$. ◻

Proposition 63. *Every* $f \in \mathsf{Q}_\vee(\mathbb{I})$ *is a (possibly infinite) join of prime one step functions.*

Proof. Clearly we have $\bigvee\{\, e_{x,y} \mid e_{x,y} \leq f \,\} \leq f$, so let us argue that this inclusion is an equality. Let g be such that $e_{x,y} \leq g$ whenever $e_{x,y} \leq f$. In particular, for x arbitrary and $y = f^\wedge(x)$, we have $e_{x,y} \leq g$, that is $f^\wedge(x) \leq g^\wedge(x)$. We argued therefore that, within $\mathsf{Q}_\wedge(\mathbb{I})$, $f^\wedge \leq g^\wedge$. We have, therefore, $f = f^{\wedge\vee} \leq g^{\wedge\vee} = g$. ◻

Remark 64. Proposition 63 implies that $\mathsf{Q}_\vee(\mathbb{I})$ is the Dedekind-MacNeille completion of the sublattice generated by the prime one step functions. The statement of the Proposition can be further strengthened as follows: every $f \in \mathsf{Q}_\vee(\mathbb{I})$ is a (possibly infinite) join of prime *rational* one step functions, implying that $\mathsf{Q}_\vee(\mathbb{I})$ is the Dedekind-MacNeille completion of the sublattice generated by the rational one step functions. To see why this is the case, observe that every one step function is the the join of the rational one step functions below it.

Finally, we verify the following relations, that we shall need to understand the structure of join-irreducible elements in higher dimensions.

Lemma 65. *For each* $x, y, y', z \in \mathbb{I}$,

$$e_{y',z} \circ e_{x,y} = \begin{cases} \perp\,, & y \leq y'\,, \\ e_{x,z}\,, & \textit{otherwise}\,. \end{cases}$$

In particular, $e_{y,z} \circ e_{x,y} = \perp$.

Proof. Let us study the formula for the composition:

$$e_{y',z}(e_{x,y}(t)) = \begin{cases} 0\,, & e_{x,y}(t) \leq y'\,, \\ z\,, & y' < e_{x,y}(t)\,. \end{cases}$$

Now, if $y \leq y'$, then $e_{x,y}(t) \leq y'$, for each $t \in \mathbb{I}$, so $e_{y',z} \circ e_{x,y} = \perp$. If $y' < y$, then $y' < e_{x,y}(t)$ if and only if $e_{x,y}(t) = y$, i.e. iff $x < t$. This yields $e_{y',z} \circ e_{x,y} = e_{x,z}$. ◻

The following Lemma is verified in a similar way.

Lemma 66. *For each $x, y, z \in \mathbb{I}$,*

$$e^{\wedge}_{y,z} \circ e^{\wedge}_{x,y} = \begin{cases} e^{\wedge}_{y,z}\,, & y = 0\,, \\ e^{\wedge}_{x,z}\,, & 0 < y < 1\,, \\ e^{\wedge}_{x,y}\,, & y = 1\,. \end{cases}$$

10.2. **Join-irreducible elements of $\mathsf{L}(\mathbb{I}^d)$.** We study next join-irreducible elements of the lattice $\mathsf{L}_d(\mathsf{Q}_\vee(\mathbb{I}))$, for $d \geq 3$. To ease reading, we shall use the notation $\mathsf{L}(\mathbb{I}^d)$ for $\mathsf{L}_d(\mathsf{Q}_\vee(\mathbb{I}))$.

For $p \in \mathbb{I}^d$, let $e_p \in \mathsf{Q}_\vee(\mathbb{I})^{[d]_2}$ be the tuple defined as follows:

$$e_p := \langle\, e_{p_i,p_j} \mid (i,j) \in [d]_2 \,\rangle\,.$$

Let also define

$$\mu^\vee_p := \min\{\, i \in [d] \mid p_i < 1 \,\}\,, \qquad M^\vee_p := \max\{\, j \in [d] \mid 0 < p_j \,\}\,,$$

(where we let in these formulas $\min\emptyset = d+1$ and $\max\emptyset = 0$) and

$$\dim^\vee(p) := M^\vee_p - \mu^\vee_p\,.$$

Therefore, for each $p \in \mathbb{I}^d$, $p_i = 1$ if $i < \mu^\vee_p$ and $p_j = 0$ if $j > M^\vee_p$. In particular, we cannot have $M^\vee_p < \mu^\vee_p - 1$, so $\dim^\vee(p) \geq -1$.

Lemma 67. *For each $p \in \mathbb{I}^d$, the relation $e_p \neq \perp$ holds if and only if $\dim^\vee(p) > 0$.*

Proof. Recall that $e_{p_i,p_j} = \perp$, if $p_i = 1$ or $p_j = 0$. Suppose that $\dim^\vee(p) \leq 0$, so $M^\vee_p \leq \mu^\vee_p$ and consider $(i,j) \in [d]_2$: we have then $p_i = 1$ or $p_j = 0$. Therefore, $e_{p_i,p_j} = \perp$ for each $(i,j) \in [d]_2$, and $e_p = \perp$.

Suppose next that $\dim^\vee(p) > 0$, so $\mu^\vee_p < M^\vee_p$. To ease reading, let $\mu = \mu^\vee_p$ and $M = M^\vee_p$. Since $1 \leq \mu$ and $M \leq d$, we have $(\mu, M) \in [d]_2$ and since $p_\mu \neq 1$ and $p_M \neq 0$, we have $e_{p_\mu,p_M} \neq \perp$ and therefore $e_p \neq \perp$. □

Proposition 68. *For each $p \in \mathbb{I}^d$, e_p is a clopen tuple of $\mathsf{Q}_\vee(\mathbb{I})^{[d]_2}$. That is, $e_p \in \mathsf{L}(\mathbb{I}^d)$.*

Proof. We use Remark 29 to establish that e_p is compatible and, to this goal, we use the relations established with Lemmas 65 and 66. The relation $e^{\wedge}_{p_i,p_k} = e^{\wedge}_{p_j,p_k} \circ e^{\wedge}_{p_i,p_j}$ holds unless $p_j \in \{0,1\}$. If $p_j = 0$, then

$$e_{p_j,p_k} \circ e_{p_i,p_j} = \perp \leq e_{p_i,p_k} \leq e^{\wedge}_{p_i,p_k} \leq e^{\wedge}_{0,p_k} = e^{\wedge}_{p_j,p_k} \circ e^{\wedge}_{p_i,p_j}\,.$$

If $p_j = 1$, then

$$e_{p_j,p_k} \circ e_{p_i,p_j} = \perp \leq e_{p_i,p_k} \leq e^{\wedge}_{p_i,p_k} \leq e^{\wedge}_{p_i,1} = e^{\wedge}_{p_j,p_k} \circ e^{\wedge}_{p_i,p_j}\,.$$ □

Notice that $p \in \mathbb{I}^d$ has $\dim^\vee(p) \leq 0$ if and only if it lies on the path

$$\bigcup_{i\in[d]} \{\, (\underbrace{1,\ldots,1}_{i-1}, x, \underbrace{0,\ldots,0}_{d-i}) \mid x \in \mathbb{I} \,\}\,.$$

It is readily seen that this path corresponds to the tuple that is the bottom of the lattice $\mathsf{L}(\mathbb{I}^d)$ (as well as of the lattice $\mathsf{Q}_\vee(\mathbb{I})^{[d]_2}$).

Lemma 69. *For each $f \in \mathsf{L}(\mathbb{I}^d)$, $x \in \mathbb{I}$, and $(m, M) \in [d]_2$, there exists $p(f,x,m,M) \in \mathbb{I}^d$ such that $e_{p(f,x,m,M)} \leq f$, $p(f,x,m,M)_m = x$ and $p(f,x,m,M)_M = f^{\wedge}_{m,M}(x)$.*

Proof. We construct $p = p(f, x, m, M)$ as follows. We let $p_m = x$ and, for i with $m < i \leq M$, we let $p_i = f^{\wedge}_{m,i}(x)$. If $i < m$ then we let $p_i = 1$, and if $M < i$, then we let $p_i = 0$.

Let us verify that $e_p \leq f$, that is $e_{p_i,p_j} \leq f_{i,j}$ for each $(i, j) \in [d]_2$. If $i < m$, then $e_{p_i,p_j} = \perp \leq f_{i.j}$. Similarly, if $M < j$, then $e_{p_i,p_j} = \perp \leq f_{i,j}$. Therefore we can assume that $m \leq i < j \leq M$. We verify that $e_{p_i,p_j} \leq f_{i,j}$ using Lemma 60. If $i = m$, $p_j = f^{\wedge}_{m,j}(x) \leq f^{\wedge}_{m,j}(p_m)$, simply because $p_m = x$; if $m < i$, then $p_j = f^{\wedge}_{m,j}(x) \leq f^{\wedge}_{i,j}(f^{\wedge}_{m,i}(x)) = f^{\wedge}_{i,j}(p_i)$, recalling that $p_i = f^{\wedge}_{m,i}(x)$ and using openedness of f. □

Corollary 70. *For each $f \in \mathsf{L}(\mathbb{I}^d)$ the relation*

$$f = \bigvee\{\, e_p \mid p \in \mathbb{I}^d \quad \textit{and} \quad e_p \leq f \,\}\,. \tag{24}$$

holds in $\mathsf{L}(\mathbb{I}^d)$.

Proof. Using Lemma 69, we see that relation (24) holds in $\mathsf{Q}_\vee(\mathbb{I})^{[d]_2}$, for any $f \in \mathsf{L}(\mathbb{I}^d)$. A fortiori, the same relation holds in $\mathsf{L}(\mathbb{I}^d)$. □

Proposition 71. *For each $p \in \mathbb{I}^d$, if $e_p \neq \perp$, then e_p is join-irreducible within $\mathsf{L}(\mathbb{I}^d)$.*

Proof. Assume that the relation $e_p = \alpha \vee \beta$ holds in $\mathsf{L}(\mathbb{I}^d)$. Let $m := \mu^{\vee}_p$ and $M := M^{\vee}_p$. Observe that, for $(i, j) \in [d]_2$, if $i < m$ or $M < j$, then $e_{i,j} = \alpha_{i,j} = \beta_{i,j} = \perp$; so we only need to show that either $e_{p_{i,j}} = \alpha_{i,j}$ whenever $m \leq i < j \leq M$, or $e_{p_{i,j}} = \beta_{i,j}$ whenever $m \leq i < j \leq M$. Said otherwise, we can assume that $m = 1$ and $M = d$ (so $p_1 \neq 1$ and $p_d \neq 0$).

Firstly, we claim that $e_{p_1,p_d} = \alpha_{1,d} \vee \beta_{1,d}$. If not, then we have $\alpha_{1,d} \vee \beta_{1,d} < e_{p_1,p_d}$; consider then the tuple $f \in \mathsf{Q}_\vee(\mathbb{I})^{[d]_2}$ such that $f_{i,j} = e_{p_i,p_j}$ if $i \neq 1$ or $j \neq d$, and $f_{1,d} = \alpha_{1,d} \vee \beta_{1,d}$; trivially, f is closed, since for $i < j < k$, $e_{p_j,p_k} \circ e_{p_i,p_j} = \perp \leq e_{p_i,p_k}$. We obtain then the following contradiction:

$$e_p = \alpha \vee_{\mathsf{L}(\mathbb{I}^d)} \beta = \overline{\alpha \vee_{\mathsf{Q}_\vee(\mathbb{I})^{[d]_2}} \beta} \leq \overline{f} = f < e_p\,.$$

Thus we have $e_{p_1,p_d} = \alpha_{1,d} \vee \beta_{1,d}$ in $\mathsf{Q}_\vee(\mathbb{I})$, and therefore $e_{p_1,p_d} = \alpha_{1,d}$ or $e_{p_1,p_d} = \beta_{1,d}$. Let us suppose that $e_{p_1,p_d} = \alpha_{1,d}$, we shall prove that $e_p = \alpha$. (A similar argument proves that $e_p = \beta$ if $e_{p_1,p_d} = \beta_{1,d}$).

Notice first that $e_p = \alpha \vee \beta$ implies $\alpha_{i,j} \leq e_{p_i,p_j}$ for each $(i, j) \in [d]_2$. On the other hand, if $1 < i < d$, then

$$p_d = e^{\wedge}_{p_1,p_d}(p_1) = \alpha^{\wedge}_{1,d}(p_1) \leq \alpha^{\wedge}_{i,d}(\alpha^{\wedge}_{1,i}(p_1)) \leq \alpha^{\wedge}_{i,d}(e^{\wedge}_{p_1,p_i}(p_1)) = \alpha^{\wedge}_{i,d}(p_i)\,,$$

showing that $e_{p_i,p_d} \leq \alpha_{i,d}$ and, consequently, $e_{p_i,p_d} = \alpha_{i,d}$, for $i = 1, \ldots, d-1$.

Suppose now that $e_{p_i,p_j} \nleq \alpha_{i,j}$ for some $(i, j) \in [d]_2$ with $j < d$. This relation amounts to $p_j \nleq \alpha^{\wedge}_{i,j}(p_i)$, thus to $\alpha^{\wedge}_{i,j}(p_i) < p_j$. Then

$$p_d = e^{\wedge}_{p_i,p_d}(p_i) = \alpha^{\wedge}_{i,d}(p_i) \leq \alpha^{\wedge}_{j,d}(\alpha^{\wedge}_{i,j}(p_i)) = e^{\wedge}_{p_j,p_d}(\alpha^{\wedge}_{i,j}(p_i)) = 0\,,$$

against the hypothesis. Thus we have $\alpha_{i,j} = e_{p_i,p_j}$ for each $(i, j) \in [d]_2$, that is $e_p = \alpha$. □

In the rest of this section we aim shall prove the converse of Proposition 71: if α is join-irreducible, then $\alpha = e_p$ for some $p \in \mathbb{I}^d$. Let $p, q \in \mathbb{I}^d$ with $p \leq q$; as usual, $[p, q]$ denotes the set $\{\, r \in \mathbb{I}^d \mid p \leq r \leq q \,\}$; define then $e_{[p,q]}$ by

$$e_{[p,q]} := \bigvee\{\, e_r \mid r \in [p, q] \,\}$$

where this infinite join is computed in $\mathsf{Q}_{\vee}(\mathbb{I})^{[d]_2}$ (we shall argue few lines below that $e_{[p,q]}$ is clopen). We notice in the meantime the following expression of $e_{[p,q]}$. For each $(i,j) \in [d]_2$,

$$
\begin{aligned}
(\, e_{[p,q]} \,)_{i,j} &= (\, \bigvee \{\, e_r \mid r \in [p,q] \,\} \,)_{i,j} \\
&= \bigvee \{\, e_{r_i,r_j} \mid p_i \leq r_i \leq q_i \,,\ p_j \leq r_j \leq q_j \,\} = e_{p_i,q_j} \,.
\end{aligned}
$$

Lemma 72. *For each $p,q \in \mathbb{I}^d$ with $p \leq q$, $e_{[p,q]}$ is clopen. Moreover, for $r \in \mathbb{I}^d$ such that $\perp < e_r$, the relation $e_r \leq e_{[p,q]}$ holds if and only*

1. $p_{\mu_r^\vee} \leq r_{\mu_r^\vee}$ *and* $r_{M_r^\vee} \leq q_{M_r^\vee}$,
2. $r_i \in [p_i,q_i]$, *for each* i *such that* $\mu_r^\vee < i < M_r^\vee$.

Proof. Clearly $e_{[p,q]}$ is open since it is a join of open tuples; it is also closed since the relations $e_{p_j,q_k} \circ e_{p_i,q_j} \leq e_{p_i,q_k}$ holds by Lemma 65.

Let now $r \in \mathbb{I}^d$ be such that $\perp < e_r$; to ease the reading, let also $m := \mu_r^\vee$ and $M := M_r^\vee$. Suppose that $e_r \leq e_{[p,q]}$, that is $e_{r_i,r_j} \leq e_{p_i,q_j}$, for each $(i,j) \in [d]_2$. Since $\perp < e_{r_m,r_M} \leq e_{p_m,q_M}$, we have $p_m \leq r_m$ and $r_M \leq q_M$. Also, for $m < i < M$, $e_{r_m,r_i} \leq e_{p_m,q_i}$ with $r_m < 1$ yields $r_i \leq q_i$, and $e_{r_i,r_M} \leq e_{p_i,q_M}$ with $0 < r_M$ yields $p_i \leq r_i$. Thus, for such an i, $p_i \leq r_i \leq q_i$.

Let us verify that (1) and (2) imply $e_r \leq e_{[p,q]}$. Consider $(i,j) \in [d]_2$: if $i < m$ or $M < j$, then $e_{r_i,r_j} = \perp$, so $e_{r_i,r_j} \leq e_{p_i,q_j}$ trivially holds; otherwise $m \leq i < j \leq M$, and conditions (1) and (2) imply that $p_i \leq r_i$ and $r_j \leq q_j$. $\square$

As a particular instance of the previous Lemma (i.e. when $p = q$ in the statement of the Lemma) we deduce the following statement:

Proposition 73. *Let $r,p \in \mathbb{I}^d$ be such that $\perp < e_r$. Then $e_r \leq e_p$ if, and only if,*

1. $p_{\mu_r^\vee} \leq r_{\mu_r^\vee}$, $r_{M_r^\vee} \leq p_{M_r^\vee}$,
2. $r_i = p_i$ *for all* $i \in [d]$ *with* $\mu_r^\vee < i < M_r^\vee$.

Notice that the relation $\perp < e_r \leq e_p$ also implies that

$$
\mu_p^\vee \leq \mu_r^\vee < M_r^\vee \leq M_p^\vee \,.
$$

Suppose for example that $\mu_r^\vee < \mu_p^\vee$, so p has in position $m := \mu_r^\vee$ a 1. This implies that $e_{p_m,p_j} = \perp$ for each $j > m$. Therefore, $e_r \leq e_p$ implies that $e_{r_m,r_j} = \perp$ for each $j > m$. Since by definition $r_m < 1$, the relations $e_{r_m,r_j} = \perp$ imply that $r_j = 0$ for each $r_j > m$. Yet this means that $\dim^\vee(r) \leq 0$, so $e_r = \perp$, against the assumption.

Proposition 74. *If $\alpha \in \mathsf{L}(\mathbb{I}^d)$ is join-irreducible, then $\alpha = e_p$ for some $p \in \mathbb{I}^d$.*

Proof. We claim first that there exists $p \in \mathbb{I}^d$ such that $\alpha \leq e_p$. To prove the claim, we define an infinite sequence of intervals $I_n := [p^n, q^n]$, $n \geq 0$, with the following properties:

1. $I_{n+1} \subseteq I_n$, for each $n \geq 0$,
2. $q_i^n - p_i^n = \frac{1}{2^n}$, for each $i \in [d]$,
3. $\alpha = \bigvee \{\, e_r \mid e_r \leq \alpha,\ r \in I_n \,\}$.

Notice that the last condition implies that $\alpha \leq \bigvee I_n$.

We let $I_0 := \mathbb{I}$, so, for example, (3) holds by Corollary 70.

Given I_n, we define I_{n+1} as follows. For each $i \in [d]$, let $I_{i,0} := [p_i^n, \frac{q_i^n+p_i^n}{2}]$ and $I_{i,1} := [\frac{q_i^n+p_i^n}{2}, q_i^n]$; given a function $f : [d] \to \{\, 0,1 \,\}$, we let

$$
I_f := I_{1,f(1)} \times \ldots \times I_{d,f(d)} \,.
$$

Since

$$I_n = \bigcup_{f:[d] \longrightarrow \{0,1\}} I_f$$

then

$$\alpha = \bigvee \{ e_r \mid e_r \leq \alpha, r \in I_n \} = \bigvee_{f:[d] \longrightarrow \{l,r\}} \bigvee \{ e_r \mid e_r \leq \alpha, r \in I_f \},$$

so, since α is join-irreducible, there exists f such that

$$\alpha = \bigvee \{ e_r \mid e_r \leq \alpha, r \in I_f \},$$

We let then $I_{n+1} := I_f$.

Let $\beta_n = \bigvee_n I_n$ and let p^ω be the unique element of $\bigcap_{n \geq 0} I_n$. Observe that, since the sequences $\{ p_i^n \}_{n \geq 0}$, are increasing while the sequences $\{ q_i^n \}_{n \geq 0}$, are decreasing, $p_i^\omega = \bigvee_{n \geq 0} p_i^\omega = \bigwedge_{n \geq 0} q_i^n$. We verify next that $\bigwedge_{n \geq 0} \beta_n = e_{p^\omega}$. Let $r \in \mathbb{I}^d$ be such that $r \leq \beta_n$ for each $n \geq 0$, and put $m := \mu_r^\vee$ and $M := M_r^\vee$. Then, for each $n \geq 0$, $p_m^n \leq r_m$, $r_i \in [p_i^n, q_i^n]$, $r_M \leq q_M^n$. It follows that $p_m^\omega \leq r_m$, $r_i = p_i^\omega$ for i such that $m < i < M$, that is $e_r \leq e_{p^\omega}$. Since $\alpha \leq \beta_n$ for each $n \geq 0$ and $\alpha = \bigvee \{ e_r \mid e_r \leq \alpha \}$, then $\alpha \leq e_{p^\omega}$. This proves our claim.

Observe now that

$$\alpha = \bigvee_{1 \leq m < M \leq d} J(\alpha, m, M)$$

where

$$J(\alpha, m, M) = \{ e_r \mid \bot < e_r, \mu_r^\vee = m, M_r^\vee = M \},$$

so, since α is join-irreducible, then for some $m, M \in \{ d \}$ with $m < M$,

$$\alpha = \bigvee J(\alpha, m, M).$$

Observe now that if $r \in \mathbb{I}^d$ is such that $\bot < e_r$ and $e_r \in J(\alpha, m, M)$, then $e_r \leq \alpha \leq e_p$, whence by Lemma 73 r is of the form

$$r = (1, \ldots, 1, r_m, p_{m+1}, \ldots, p_{M-1}, r_M, 0, \ldots, 0).$$

The join $\alpha = \bigvee J(\alpha, m, M)$ is then e_q with

$$q = (1, \ldots, 1, \bigwedge_{e_r \in J(\alpha,m,M)} r_m, p_{m+1}, \ldots, p_{M-1}, \bigvee_{e_r \in J(\alpha,m,M)} r_M, 0, \ldots, 0). \qquad \square$$

10.3. **Lack of compact elements.** Let L be a complete lattice. An element j of L is *completely join-irreducible* if, for any $X \subseteq L$, $j = \bigvee X$ implies $j \in X$; it is *completely join-prime* if, for any $X \subseteq L$, $j \leq \bigvee X$ implies $j \in x$, for some $x \in X$. Every completely join-prime element is also completely join-irreducible. If L is a frame, that is, if $x \wedge \bigvee Y = \bigvee_{y \in Y} x \wedge y$ for each $x \in L$ and $Y \subseteq L$, then the converse holds as well.

A family $\mathcal{F} \subseteq L$ is *directed* if every finite (possibly empty) subset of $\mathcal{F}$ has an upper bound in $\mathcal{F}$. An element $c \in L$ is *compact* if, for every directed family $\mathcal{F} \subseteq L$, $c \leq \bigvee \mathcal{F}$ implies $c \leq f$ for some $f \in \mathcal{F}$.

Let us remark that there are no completely join-prime (equivalently, completely join-irreducible) elements in $\mathsf{Q}_\vee(\mathbb{I})$. Indeed, for every prime one step function $e_{x,y}$, we can write $e_{x,y} = \bigvee_{\ell \in L} e_{x_\ell, y_\ell}$ where the set $\{ e_{x_\ell, y_\ell} \mid \ell \in L \}$ is a chain and $e_{x_\ell, y_\ell} < e_{x,y}$, for each $\ell \in L$. Similarly, there are no compact elements in $\mathsf{Q}_\vee(\mathbb{I})$. Indeed, if f is compact, then

Proposition 63 implies that f is a finite join of join-irreducible elements below it, say $f = \bigvee_{i=1,\dots,n} e_{x_i,y_i}$. We can assume that $\{ e_{x_i,y_i} \mid i = 1,\dots,n \}$ is an antichain. Now, if $\{ e_{x_{1,\ell},y_{1,\ell}} \mid \ell \in L \}$ is a chain approximating strictly from below e_{x_1,y_1}, then $f = \bigvee_{\ell \in L} e_{x_{1,\ell}y_{1,\ell}} \vee \bigvee_{i=2,\dots,n} e_{x_i,y_i}$, so $f = e_{x_{1,\ell}y_{1,\ell}} \vee \bigvee_{i=2,\dots,n} e_{x_i,y_i}$ for some $\ell \in L$. It follows that $e_{x_1,y_1} \leq f = e_{x_{1,\ell}y_{1,\ell}} \vee \bigvee_{i=2,\dots,n} e_{x_i,y_i}$, so either $e_{x_1,y_1} \leq e_{x_{1,\ell}y_{1,\ell}}$, or $e_{x_1,y_1} \leq e_{x_i,y_i}$ for some $i = 2,\dots,n$. In all the cases we obtain a contradiction.

For a similar reason, the lattices $\mathsf{L}(\mathbb{I}^d)$ have no completely join-irreducible elements. Indeed, given $p \in \mathbb{I}^d$ such that $\perp < e_p$, it is easy to construct (using Proposition 73) a chain of join-irreducible elements strictly below e_p whose join is e_p.

In the rest of this section we argue that the lattices $\mathsf{L}(\mathbb{I}^d)$ do not have any compact element.

Lemma 75. *Let $\mathcal{F} \subseteq \mathsf{Q}_\vee(\mathbb{I})^{[d]_2}$ be a directed family of closed elements. Then then join $\bigvee_{\mathsf{Q}_\vee(\mathbb{I})^{[d]_2}} \mathcal{F} \in \mathsf{Q}_\vee(\mathbb{I})^{[d]_2}$ is closed.*

Proof. A straightforward verification:

$$\begin{aligned}
(\bigvee_{f \in \mathcal{F}} f)_{j,k} \circ (\bigvee_{f \in \mathcal{F}} f)_{i,j} &= \bigvee_{f,g \in \mathcal{F}} g_{j,k} \circ f_{i,j} \\
&\leq \bigvee_{h \in \mathcal{F}} h_{j,k} \circ h_{i,j}\,, && \text{using the fact that } \mathcal{F} \text{ is directed,} \\
&\leq \bigvee_{h \in \mathcal{F}} h_{i,k} = (\bigvee\nolimits_{\mathsf{Q}_\vee(\mathbb{I})^{[d]_2}} \mathcal{F})_{i,k}\,.
\end{aligned}$$

□

Proposition 76. *The lattice $\mathsf{L}(\mathbb{I}^d)$ has no compact elements.*

Proof. Suppose $c \in \mathsf{L}(\mathbb{I}^d)$ is a compact element. Recall from Lemma 69 and Corollary 70 that we can write

$$c = \bigvee_{x \in \mathbb{I}} \bigvee_{1 \leq m < M \leq d} e_{p(c,x,m,M)}\,,$$

where $p := p(c,x,m,M)$ is such that $m = \mu^\vee_p$, $M = M^\vee_p$, $p_m = x$ and $p_i = c^\wedge_{m,i}(x)$, for $i = m+1,\dots,M$. Since c is compact, there exists a finite set $P \subseteq \{ p(c,x,m,M) \mid x \in \mathbb{I}, 1 \leq m < M \leq d \}$ such that $c = \bigvee \{ e_p \mid p \in P \}$ and we can suppose that P is an antichain. Let $p^1 \in P$ be such that $m := \mu^\vee_{p^1}$ is minimal in $\{ \mu^\vee_p \mid p \in P \}$ and such that p^1_m is minimal in $\{ p_m \mid p \in P, \mu^\vee_p = m \}$. Let therefore $\{ p_1,\dots,p_n \} := P$.

Claim. For each $x \in \mathbb{I}$ such that $p^1_m < x \leq 1$, define p^1_x by $p^1_{x,m} := x$ and $p^1_{x,i} := p^1_i$ for $i \neq m$. Then $e_{p^1_x} \vee \bigvee_{i=2,\dots,n} e_{p^i} < \bigvee_{i=1,\dots,n} e_{p^i}$.

To ease reading of the proof of the claim, let $q_1 := p^1_x$ and let also $q^i := p^i$, for $i = 2,\dots,n$; notice that $q^1 < p^1$. Suppose $\bigvee_{i=1,\dots,n} e_{q^i} < \bigvee_{i=1,\dots,n} e_{p^i}$ does not hold. Then $e_{p^1} \leq c = \bigvee_{i=1,\dots,n} e_{q^i}$; by the formula for the join in equation (16) and by Lemma 17, there exists a subdivision $m = \ell_0 < \ell_1 < \dots < \ell_k = M$ of the interval $[m, M]$ such that

$$e_{p^1_m,p^1_M} \leq (\bigvee_{i=1,\dots,n} e_{q^i_{\ell_{k-1}},q^i_{\ell_k}}) \circ \dots \circ (\bigvee_{i=1,\dots,n} e_{q^i_{\ell_0},q^i_{\ell_1}})\,.$$

Considering that composition distributes over joins and that e_{p_m,p_M} is join-irreducible in $\mathsf{Q}_\vee(\mathbb{I})$, for each $u \in \{ 0,\dots,k-1 \}$, there exists $i_u \in \{ 1,\dots,n \}$ such that

$$e_{p^1_m,p^1_M} \leq e_{q^{i_{k-1}}_{\ell_{k-1}},q^{i_{k-1}}_{\ell_k}} \circ \dots \circ e_{q^{i_0}_{\ell_0},q^{i_0}_{\ell_1}}\,.$$

By Lemma 65 and since $\bot \neq e_{p_m,p_M}$, the expression above on the right equals to $e_{q^{i_0}_{\ell_0},q^{i_{k-1}}_{\ell_k}} = e_{q^{i_0}_m,q^{i_{k-1}}_M}$, so

$$e_{p^1_m,p^1_M} \leq e_{q^{i_0}_m,q^{i_{k-1}}_M}\,,$$

which, by Corollary 61, amounts to $q^{i_0}_m \leq p^1_m$ and $p^1_M \leq q^{i_{k-1}}_M$. Since $p^1_m < x = q^1_m$, $i_0 \neq 1$. It also implies that $\mu^{\vee}_{p^{i_0}} = \mu^{\vee}_{q^{i_0}} \leq \mu^{\vee}_{p^1} = m$ and, considering the minimality of $\mu^{\vee}_{p^1}$, $\mu^{\vee}_{p^{i_0}} = \mu^{\vee}_{p^1} = m$. Since p^1_m is minimal among elements of the form p^i_m, $i = 1, \ldots, n$, we also infer that $p^{i_0}_m = p^1_m$. Yet, this implies that, for $i = m+1, \ldots, \min(M^{\vee}_{p^1}, M^{\vee}_{p^{i_0}})$,

$$p^1_i = c^{\wedge}_{m,i}(p^1_m) = c^{\wedge}_{m,i}(p^1_m) = p^{i_0}_i\,.$$

Using Proposition 73, it immediately follows that p^1 and p^{i_0} are comparable, contradicting the assumption that P is an antichain. This ends the proof of the Claim.

Clearly, the following relations holds in $\mathsf{L}(\mathbb{I}^d)$:

$$\bigvee_{1 \geq x > p^1_m} (\, e_{p^1_x} \vee \bigvee_{i=2,\ldots,n} e_{p^i}\,) \leq c\,.$$

Let us argue that also the converse inclusion holds. Within $\mathsf{Q}_{\vee}(\mathbb{I})^{[d]_2}$, the following relation holds:

$$\bigvee_{i=1,\ldots,n} e_{p^i} \leq \bigvee_{1 \geq x > p^1_m} (\, e_{p^1_x} \vee \bigvee_{i=2,\ldots,n} e_{p^i}\,)\,.$$

Taking the closure, we have

$$\overline{\bigvee_{i=1,\ldots,n} e_{p^i}} \leq \overline{\bigvee_{1 \geq x > p^1_m} (\, e_{p^1_x} \vee \bigvee_{i=2,\ldots,n} e_{p^i}\,)} = \bigvee_{1 \geq x > p^1_m} \overline{(\, e_{p^1_x} \vee \bigvee_{i=2,\ldots,n} e_{p^i}\,)}\,.$$

where in the last equality we have used the fact that $\{\, \overline{e_{p^1_x} \vee \bigvee_{i=2,\ldots,n} e_{p^i}} \mid 1 \geq x > p_m\,\}$ is a directed set and Lemma 75.

Thus c is, within $\mathsf{L}(\mathbb{I}^d)$, an infinite join of a chain of elements that are strictly below it. This contradicts c being compact. ◻

11. Embeddings from multinomial lattices

The goal of this section is study how multinomial lattices [4, 37] embed into the lattices $\mathsf{L}(\mathbb{I}^d)$. We proceed by arguing that these lattices are of the form $\mathsf{L}_d(Q)$ for some mix $\star$-autonomous quantale Q such that Q embeds, as an ℓ-bisemigroup, into $\mathsf{Q}_{\vee}(\mathbb{I})$. By functoriality (Proposition 25), it follows that $\mathsf{L}_d(Q)$ embeds into $\mathsf{L}(\mathbb{I}^d) = \mathsf{L}_d(\mathsf{Q}_{\vee}(\mathbb{I}))$.

In the following let I_0, I_1 be two complete chains and let $\iota : I_0 \to I_1$ be a bi-continuous embedding (thus we ask that ι preserves all joins and all meets; in particular, ι preserves the bounds of the chains). Then ι has both a left adjoint $\ell : I_1 \to I_0$ and a right adjoint $\rho : I_1 \to I_0$. It useful, e.g. when I_0 is finite, to think of ℓ as the ceiling function and of ρ as the floor function.

Lemma 77. *The following statements hold:*

- $\ell \circ \iota = \rho \circ \iota = id_{I_0}$,
- $\rho \leq \ell$,
- *if* $y \in I_1$ *is such that* $\rho(y) = \ell(y)$, *then* $y = \iota(x)$, *with* $x = \ell(y) \in I_0$.

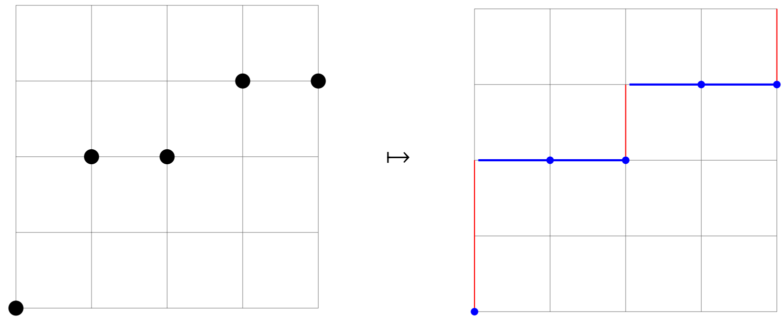

FIGURE 2. The correspondence sending f to $R_\iota(f)$

Proof. By standard laws of adjunctions, $\iota(x) = \iota(\ell(\iota(x)))$, for each $x \in I_0$. Since ι is an embedding, we deduce $x = \ell(\iota(x))$. The equality $x = \rho(\iota(x))$ is proved similarly.

Let now $y \in I_1$ and suppose that $\ell(y) \leq \rho(y)$, then we have $y \leq \iota(\ell(y))$ as unit of the adjunction, $\iota(\ell(y)) \leq \iota(\rho(y))$ and $\iota(\rho(y)) \leq y$ as counit of the adjunction. Therefore $y = \iota(\ell(y)) = \iota(\rho(y))$ and $\ell(y) = \rho(x)$, since ι is an embedding.

From this it follows that, for $y \in I_1$, then either $\rho(y) = \ell(y)$, in which case $y = \iota(\ell(y))$, or $\rho(y) \neq \ell(y)$, in which case we cannot have $\ell(y) \leq \rho(y)$, so $\rho(y) < \ell(y)$. □

Lemma 78. *If $\ell(y) < x$, then $y < \iota(x)$.*

Proof. Assume $\ell(y) < x$. From $\ell(y) \leq x$ we deduce $y \leq \iota(x)$. If the latter inclusion is not strict, then $\iota(x) \leq y$ and $x \leq \rho(y)$, so $\ell(y) < x \leq \rho(y)$ yields the relation $\ell(y) < \rho(y)$, which contradicts $\rho \leq \ell$ established in Lemma 77. □

For each monotone $f : I_0 \longrightarrow I_0$, define $R_\iota(f) : I_1 \longrightarrow I_1$ by the formula

$$R_\iota(f) := \iota \circ f \circ \ell \,.$$

Figure 2 gives some hints on the geometric meaning of the correspondence $f \mapsto R_\iota(f)$. In the figure we have $I_0 = \mathbb{I}_4 = \{0, 1, 2, 3, 4, 5\}$, $I_1 = \mathbb{I}$, and $f(0) = 0$, $f(1) = f(2) = 2$, $f(3) = f(4) = 3$. In some sense, this correspondence the responsible for representing join-continuous functions from some $\mathbb{I}_n$ to itself as discrete paths in the plane. In the figure, the graph of the function $R_\iota(f)$ (in blue) is completed with the vertical intervals (in red), so to yield the path C_f, similarly to what we have done in Figure 8. From the figure it should also be clarified the recipe $R_\iota(f)(x) = (\iota \circ f \circ \ell)(x)$: give to x the same value of its ceiling $\ell(x)$ and then inject back this value back into $\mathbb{I}$ using ι.

Lemma 79. *For each monotone $h : I_1 \longrightarrow I_1$, $h \circ i \leq \iota \circ f$ if and only if $h \leq R_\iota(f)$. That is, $R_\iota(f)$ is the right Kan extension of $\iota \circ f : I_0 \longrightarrow I_1$ along $\iota : I_0 \longrightarrow I_1$.*

Proof. Indeed, observe that $R_\iota(f) \circ \iota = \iota \circ f \circ \ell \circ \iota = \iota \circ f$. Next, if $h \circ \iota \leq \iota \circ f$, then $h \leq h \circ \iota \circ \ell \leq \iota \circ f \circ \ell = R_\iota(f)$. □

Proposition 80. *R_ι is injective and restricts to a map from $\mathsf{Q}_\vee(I_0)$ to $\mathsf{Q}_\vee(I_1)$.*

Proof. R_ι is injective since ι is monic and ℓ is epic. For the second statement, notice that if $f \in \mathsf{Q}_\vee(I_0)$, then $R_\iota(f) \in \mathsf{Q}_\vee(I_1)$, since $R_\iota(f)$ is the composition of three join-continuous maps. □

We shall observe next that R_ι preserves part of the structure of $\mathsf{Q}_\vee(I_0)$, $\otimes, (-)^\star, \oplus$, as well as finite meets and infinite joins. On the other hand, it is easily seen that units are only semi-preserved.

Proposition 81. *For each $f, g \in \mathsf{Q}_\vee(I_0)$, the following relation holds*

$$R_\iota(f \otimes g) = R_\iota(f) \otimes R_\iota(g)\,, \qquad R_\iota(f)^\star = R_\iota(f^\star)\,,$$

and, consequently,

$$R_\iota(f \oplus g) = R_\iota(f) \oplus R_\iota(g)\,.$$

Proof. For the first relation we compute as follows:

$$\begin{aligned} R_\iota(f) \otimes R_\iota(g) &= R_\iota(g) \circ R_\iota(f) = \iota \circ g \circ \ell \circ \iota \circ f \circ \ell = \iota \circ g \circ f \circ \ell \\ &= R_\iota(g \circ f) = R_\iota(f \otimes g)\,. \end{aligned}$$

For the second relation, we first establish that $R_\iota(f)^\star \leq R_\iota(f^\star)$. In view of Lemma 79, it is enough to prove $R_\iota(f)^\star \circ \iota \leq \iota \circ f^\star$. This is accomplished as follows:

$$\begin{aligned} R_\iota(f)^\star(\iota(x)) &= \bigvee\{\, y \in I_1 \mid (\iota \circ f)(\ell(y)) < \iota(x)\,\}\,, && \text{by equation (14),} \\ &= \bigvee\{\, y \in I_1 \mid f(\ell(y)) < x\,\}\,, && \text{since } \iota \text{ is an embedding,} \\ &\leq \bigvee\{\, \iota(x') \mid x' \in I_0\,,\ f(x') < x\,\}\,, \end{aligned}$$

since if $y \in I_1$ is such that $f(\ell(y)) < x$, then, by letting $x' := \ell(y)$, $f(x') < x$ and $y \leq \iota(\ell(y)) = \iota(x')$,

$$= \iota(\bigvee\{\, x' \in I_0 \mid f(x') < x\,\}) = (\iota \circ f^\star)(x)\,.$$

Next we establish that $R_\iota(0) \leq 0$. Let us recall that, for each $y \in I_1$,

$$R_\iota(0)(y) = \bigvee_{x < \ell(y)} \iota(x)\,, \quad 0(y) = \bigvee_{z < y} z\,.$$

Therefore, to prove $R_\iota(0) \leq 0$, it is enough to argue that $x < \ell(y)$ implies $\iota(x) < y$. Now, if $x < \ell(y)$, then $\ell(y) \not\leq x$, so $y \not\leq \iota(x)$, that is, $\iota(x) < y$.

We can now argue that $R_\iota(f^\star) \leq R_\iota(f)^\star$. This relation is equivalent to $R_\iota(f^\star) \otimes R_\iota(f) \leq 0$ which can be derived as follows:

$$R_\iota(f^\star) \otimes R_\iota(f) = R_\iota(f^\star \otimes f) \leq R_\iota(0) \leq 0\,.$$

Therefore $R_\iota(f)^\star = R_\iota(f^\star)$. For the last statement, recall that $f \oplus g = (g^\star \otimes f^\star)^\star$, so preservation of $\oplus$ follows from preservation of $\otimes$ and $(-)^\star$. $\square$

Proposition 82. *We have*

$$R_\iota(\bigvee_{i \in I} f_i) = \bigvee_{i \in I} R_\iota(f_i)\,, \qquad R_\iota(\bigwedge_{i=1,..,n} f_i) = \bigwedge_{i=1,..,n} R_\iota(f_i)\,.$$

Proof.

$$R_\iota(\bigvee_{i \in I} f_i)(x) = \iota((\bigvee_{i \in I} f_i)(\ell(x))) = \iota(\bigvee_{i \in I} (f_i(\ell(x)))\,) = \bigvee_{i \in I} \iota(f_i(\ell(x))) = (\bigvee_{i \in I} R_\iota(f_i))(x)\,.$$

In a similar way, considering that finite meets in $\mathsf{Q}_\vee(I)$ are computed pointwise, we have

$$R_\iota(\bigwedge_{i \in I} f_i)(x) = \iota((\bigwedge_{i \in I} f_i)(\ell(x))) = \iota(\bigwedge_{i \in I} (f_i(\ell(x)))\,) = \bigwedge_{i \in I} \iota(f_i(\ell(x))) = (\bigwedge_{i \in I} R_\iota(f_i))(x)\,. \quad \square$$

We can state now our main result.

Theorem 83. *For each pair of perfect chains I_0, I_1 and each bi-continuous embedding $\iota : I_0 \to I_1$, the map $R_\iota : \mathsf{Q}_\vee(I_0) \to \mathsf{Q}_\vee(I_1)$ is an ℓ-bisemigroup embedding. Together with $R_{(\_)}$, $\mathsf{Q}_\vee(\_)$ is a functor from the category of perfect chains and bi-continuous embeddings to the category of ℓ-bisemigroups.*

Proof. The first statement of the Theorem just summarizes the observations made up to now. The expression R_ι is functorial in ι, since if $\iota = \iota_2 \circ \iota_1$, then $\iota_\ell = (\iota_1)_\ell \circ (\iota_2)_\ell$. Therefore

$$R_{\iota_2 \circ \iota_1}(f) = \iota_2 \circ \iota_1 \circ f \circ (\iota_1)_\ell \circ (\iota_2)_\ell = \iota_2 \circ R_{\iota_1}(f) \circ (\iota_2)_\ell = R_{\iota_2}(R_{\iota_1}(f)) .$$

In a similar way, $R_{id_{I_0}} = id_{\mathsf{Q}_\vee(I_0)}$. □

Definition 84. For each $n \geq 1$ and each $x \in \mathbb{I}_n$, define $j_n(x) := \frac{x}{n} \in \mathbb{I}$. For each $n, m \geq 1$ and each $x \in \mathbb{I}_n$, let $j_{n,m}(x) := mx \in \mathbb{I}_{nm}$.

Clearly, j_n and $j_{n,m}$ are complete embeddings; observe also that

$$j_{mn}(j_{n,m}(x)) = \frac{mx}{nm} = j_n(x) ,$$

and that

Fact. *The diagram $j_{n,m} : \mathbb{I}_n \to \mathbb{I}_m$ is directed and $j_n : \mathbb{I}_n \to \mathbb{I}$ is a cocone.*

The following statement is a consequence of functoriality of the constructions $R_{(\_)}$ and $\mathsf{L}_d(\_)$, see Proposition 25 and Theorem 83.

Proposition 85. *The diagram $R_{j_{n,m}} : \mathsf{Q}_\vee(\mathbb{I}_n) \to \mathsf{Q}_\vee(\mathbb{I}_m), m \geq n \geq 1$ is directed and $R_{j_n} : \mathsf{Q}_\vee(\mathbb{I}_n) \to \mathsf{Q}_\vee(\mathbb{I})$ is a cocone. For each $d \geq 2$, there is a directed diagram in the category of lattices $\mathsf{L}_d(R_{j_{n,m}}) : \mathsf{L}_d(\mathsf{Q}_\vee(\mathbb{I}_n)) \to \mathsf{L}_d(\mathsf{Q}_\vee(\mathbb{I}_m)), m \geq n \geq 1$ is directed and $\mathsf{L}_d(R_{j_n}) : \mathsf{L}_d(\mathsf{Q}_\vee(\mathbb{I}_n)) \to \mathsf{L}_d(\mathsf{Q}_\vee(\mathbb{I}))$ is a cocone.*

Definition 86. We let $\mathsf{L}_{\mathcal{R}}(\mathbb{I}^d)$ be the image of all the mappings $\mathsf{L}_d(R_{j_n}) : \mathsf{L}_d(\mathsf{Q}_\vee(\mathbb{I}_n)) \to \mathsf{L}_d(\mathsf{Q}_\vee(\mathbb{I}))$.

By general facts, $\mathsf{L}_{\mathcal{R}}(\mathbb{I}^d)$ yields an explicit representation of the colimit of the directed diagram $\mathsf{L}_d(R_{j_n}) : \mathsf{L}_d(\mathsf{Q}_\vee(\mathbb{I}_n)) \to \mathsf{L}_d(\mathsf{Q}_\vee(\mathbb{I}))$; in particular it is a sublattice of $\mathsf{L}(\mathbb{I}^d)$. Observe that $f \in \mathsf{L}_{\mathcal{R}}(\mathbb{I}^d)$ if and only if f is clopen and, for each $(i, j) \in [d]_2$, $f_{i,j}$ is a finite join of rational one step functions.

12. Generation from rational one step functions

As a first application of the characterization of join-irreducible elements and of their order, we show that if $d \geq 3$ $\mathsf{L}(\mathbb{I}^d)$ is not the Dedekind-MacNeille completion of $\mathsf{L}_{\mathcal{R}}(\mathbb{I}^d)$, see definition 86. This is the sublattice if $\mathsf{L}(\mathbb{I}^d)$ of those $f \in \mathsf{L}(\mathbb{I}^d)$ such that each $f_{i,j}$ is a finite join of rational one-step functions. This contrasts with the case where $d = 2$, when $\mathsf{L}(\mathbb{I}^d) = \mathsf{Q}_\vee(\mathbb{I})$, cf. Remark 64.

Theorem 87. *For $d \geq 3$, the lattice $\mathsf{L}(\mathbb{I}^d)$ is not (isomorphic to) the Dedekind-MacNeille completion of $\mathsf{L}_{\mathcal{R}}(\mathbb{I}^d)$.*

Proof. We need to find an element of $\mathsf{L}(\mathbb{I}^d)$ which is not an infinite join of elements of $\mathsf{L}_{\mathcal{R}}(\mathbb{I}^d)$. For example, let $d = 3$ and choose $p \in \mathbb{I}^3$ such that $p_1 < 1$, p_2 is irrational, and $0 < p_3$ (so $\mu_p^\vee = 1$ and $M_p^\vee = 3$). If e_p can be written as an infinite join of elements from $\mathsf{L}_{\mathcal{R}}(\mathbb{I}^d)$, then it can also be written as an infinite join of join-irreducible elements from $\mathsf{L}_{\mathcal{R}}(\mathbb{I}^d)$ below it, and these are of the form e_r with $r \in (\mathbb{I} \cap \mathbb{Q})^3$. We can therefore write

$$e_p = \bigvee \{ e_r \in \mathsf{L}_{\mathcal{R}}(\mathbb{I}^3) \mid \perp < e_r \leq e_p \} = \bigvee_{(i,j) \in [3]_2} \bigvee J_{\mathcal{R}}(e_p, i, j) ,$$

where

$$J_{\mathcal{R}}(e_p, i, j) := \{\, e_r \mid r \in (\mathbb{I} \cap \mathbb{Q})^3,\ \perp < e_r \leq e_p,\ \mu_r^{\vee} = i,\ M_r^{\vee} = j \,\}\,.$$

Since e_p is join-irreducible, then we have $e_p = \bigvee J_{\mathcal{R}}(e_p, i, j)$ for some $(i, j) \in [3]_2$. If $(i, j) = (1, 2)$, then we deduce that $p_3 = 0$, and if $(i, j) = (2, 3)$, then we deduce that $p_1 = 1$; these are contradictions. Therefore we have $(i, j) = (1, 3)$. Yet, by Proposition 73, $J_{\mathcal{R}}(e_p, 1, 3) = \emptyset$, since if $\perp < e_r \leq e_p$, then $r_2 = p_2$ is irrational. We deduce therefore $e_p = \perp$, a contradiction. □

To understand how the lattice $\mathsf{L}(\mathbb{I}^d)$ is generated from $\mathsf{L}_{\mathcal{R}}(\mathbb{I}^d)$, we need to study its meet-irreducible elements. For $x, y \in \mathbb{I}$, we define $m_{x,y} \in \mathsf{Q}_{\vee}(\mathbb{I})$ as follows:

$$m_{x,y}(t) = \begin{cases} 0\,, & t = 0\,, \\ y & 0 < t \leq x\,, \\ 1 & x < t \leq 1\,. \end{cases}$$

Observe that $m_{x,y} = e^{\star}_{y,x}$ and, therefore, meet-irreducible elements of $\mathsf{Q}_{\vee}(\mathbb{I})$ are, by duality, exactly those of the form $m_{x,y}$ for $x, y \in \mathbb{I}$ such that $0 < x$ and $y < 1$. Notice also that

$$m_{x,y} = e_{0,y} \vee e_{x,1}\,. \tag{25}$$

Let now $d \geq 3$; for each $p \in \mathbb{I}^d$, let in the following

$$m_p := \langle\, m_{p_i, p_j} \mid (i, j) \in [d]_2 \,\rangle\,,$$

as well as

$$\mu_p^{\wedge} := \min\{\, i \in [d] \mid 0 < p_i \,\}\,, \qquad M_p^{\wedge} := \max\{\, j \in [d] \mid p_j < 1 \,\}\,,$$

where, by convention, $\min \emptyset = d + 1$ and $\max \emptyset = 0$. For $p \in \mathbb{I}^d$, let

$$\dim^{\wedge}(p) := M_p^{\wedge} - \mu_p^{\wedge}\,.$$

Notice that, since we assume $d \geq 1$, we cannot have $M_p^{\wedge} = 0$ and $\mu_p^{\wedge} = d + 1$, so $\dim(m_p) \in \{-d, \ldots, d\}$.

Proposition 88. *The meet-irreducible elements of* $\mathsf{L}(\mathbb{I}^d)$ *are exactly the elements of the form* m_p *for some* $p \in \mathbb{I}^d$ *such that* $\dim^{\wedge}(p) > 0$.

Proof. It is enough to verify that these elements of $\mathsf{L}(\mathbb{I}^d)$ correspond, under the duality, to join-irreducible elements. Indeed we have

$$\begin{aligned} m^{\star}_{(p_1,\ldots,p_d)} &= \langle\, m^{\star}_{p_{\sigma(j)}, p_{\sigma(i)}} \mid (i, j) \in [d]_2 \,\rangle \\ &= \langle\, e_{p_{\sigma(i)}, p_{\sigma(j)}} \,\rangle = e_{p_d,\ldots,p_1}\,. \end{aligned}$$

Moreover, writing $\sigma(p)$ for $(p_d, \ldots, p_1)$, we have $\dim^{\wedge}(p) = \dim^{\vee}(\sigma(p))$. The statement of the proposition follows now by the previous characterization of join-irreducible elements of $\mathsf{L}(\mathbb{I}^d)$, see Propositions 71 and 74. □

We find next an analogous of equation (25) for higher dimensions. Such an analogous will allow us to argue that every $f \in \mathsf{L}(\mathbb{I}^d)$ is a meet of joins (and, dually, a join of meets) of elements from $\mathsf{L}_{\mathcal{R}}(\mathbb{I}^d)$. Let $M^{i,j}_{x,y} \in \mathsf{Q}_{\vee}(\mathbb{I})^{[d]_2}$ be the tuple that has $m_{x,y}$ in coordinate (i, j) and $\perp$ in the other coordinates. Similarly, $E^{i,j}_{x,y} \in \mathsf{Q}_{\vee}(\mathbb{I})^{[d]_2}$ denotes the tuple that has $e_{x,y}$

in coordinate (i, j) and $\perp$ in the other coordinates. The following relations hold within $\mathsf{Q}_\vee(\mathbb{I})^{[d]_2}$:

$$m_p = \bigvee_{(i,j)\in[d]_2} M^{i,j}_{p_i,p_j}\,, \qquad\qquad M^{i,j}_{p_i,p_j} = E^{i,j}_{0,p_j} \vee E^{i,j}_{p_i,1}\,,$$

and therefore

$$m_p = \bigvee_{(i,j)\in[d]_2} M^{i,j}_{p_i,p_j} = \bigvee_{(i,j)\in[d]_2} E^{i,j}_{0,p_j} \vee \bigvee_{(i,j)\in[d]_2} E^{i,j}_{p_i,1} = E_{0,p} \vee E_{p,1}\,,$$

where

$$E_{0,p} := \bigvee_{(i,j)\in[d]_2} E^{i,j}_{0,p_j} \quad \text{and} \quad E_{p,1} := \bigvee_{(i,j)\in[d]_2} E^{i,j}_{p_i,1}.$$

Lemma 89. *For each $p \in \mathbb{I}^d$, both $E_{0,p}$ and $E_{p,1}$ belong to $\mathsf{L}(\mathbb{I}^d)$.*

Proof. We firstly consider $E_{0,p}$, observing that $E_{0,p} = \langle f_{i,j} \mid (i, j) \in [d]_2 \rangle$ with $f_{i,j} = e_{0,p_j}$. We argue that $E_{0,p}$ is clopen relying on Remark 29. Let $i, j, k \in [d]$ with $i < j < k$. If $0 < p_j$, then $f_{j,k} \circ f_{i,j} = e_{0,p_k} \circ e_{0,p_j} = e_{0,p_k} = f_{i,k}$, by Lemma 65. If $p_j = 0$, then $f^\wedge_{j,k} \circ f^\wedge_{i,j} = e^\wedge_{0,p_k} \circ e^\wedge_{0,p_j} = e^\wedge_{0,p_k} \circ e^\wedge_{0,0} = e^\wedge_{0,p_k} = f^\wedge_{i,k}$, by Lemma 66.

Next, we observe that $E_{p,1} = \langle f_{i,j} \mid (i, j) \in [d]_2 \rangle$ with $f_{i,j} = e_{p_i,1}$. We use again Remark 29 to verify that $E_{p,1}$ is clopen. Let $i, j, k \in [d]$ with $i < j < k$; if $p_j < 1$, then $f_{j,k} \circ f_{i,j} = e_{p_j,1} \circ e_{p_i,1} = e_{p_i,1} = f_{i,k}$, by Lemma 65; if $p_j = 1$, then $f^\wedge_{j,k} \circ f^\wedge_{i,j} = e^\wedge_{p_j,1} \circ e^\wedge_{p_i,1} = e^\wedge_{1,1} \circ e^\wedge_{p_i,1} = e^\wedge_{p_i,1} = f^\wedge_{i,k}$, using Lemma 66. ◻

Remark 90. Let L be a complete lattice and let M be a subset of L which is itself a complete lattice w.r.t. the order inherited from L. If $Q \subseteq M$, $q \in M$ and the relation $\bigvee Q = q$ holds in L, then the same relation holds in M.

In view of the remark, we have achieved generalizing equation 25 to higher dimensions:

Corollary 91. *The relation*

$$m_p = E_{0,p} \vee E_{p,1}$$

holds in $\mathsf{L}(\mathbb{I}^d)$.

For $i \in [d]$, $x \in \mathbb{I}$ and $y \in \{0, 1\}$, let us use $|i, x, y|$ to denote the point of $\mathbb{I}^d$ that has x in position i and y in all the other coordinates. For an example with $d = 3$, consider $|2, x, 1| = (1, x, 1)$; notice that $e_{|2,x,1|} = \langle e_{1,x}, e_{1,1}, e_{x,1}\rangle = \langle \perp, \perp, e_{x,1}\rangle$.

Lemma 92. *The relations*

$$E_{0,p} = \bigvee_{1<j\leq d} e_{|j,p_j,0|}\,, \qquad\qquad E_{p,1} = \bigvee_{1\leq i<d} e_{|i,p_i,1|}\,,$$

hold in $\mathsf{L}(\mathbb{I}^d)$.

Proof. Recalling that $E_{p,1} = \langle f_{i,j} \mid (i, j) \in [d]_2 \rangle$ with $f_{i,j} = e_{p_i,1}$, we can compute within $\mathsf{Q}_\vee(\mathbb{I})^{[d]_2}$ as follows:

$$\begin{aligned} E_{p,1} &= \bigvee_{(i,j)\in[d]_2} E^{i,j}_{p_i,1} = \bigvee_{1\leq i_0<d} \; \bigvee_{i_0<j\leq d} E^{i_0,j}_{p_i,1} \\ &= \bigvee_{1\leq i_0<d} \Big(\bigvee_{i_0<j\leq d} E^{i_0,j}_{p_{i_0},1} \vee \bigvee_{(i,j)\in[d]_2, i\neq i_0} E^{i,j}_{1,1} \Big) = \bigvee_{1\leq i_0<d} e_{|i_0,p_{i_0},1|}\,. \end{aligned}$$

Again, Remark 90 ensures that the relation so derived holds in $\mathsf{L}(\mathbb{I}^d)$ as well. The proof that $E_{0,p} = \bigvee_{1<j\leq d} e_{|j,p_j,0|}$ is analogous. ◻

Lemma 93. *For each $i \in [d]$, $x \in \mathbb{I}$ and $y \in \{0, 1\}$, $e_{|i,x,y|}$ is a join of elements in $\mathsf{L}_{\mathcal{R}}(\mathbb{I}^d)$.*

Proof. Let us consider the case where $y = 1$ (the proof when $y = 0$ is similar).

If $x = 1$, then $e_{|i,x,y|}$ already belongs to $\mathsf{L}_{\mathcal{R}}(\mathbb{I}^d)$. If $x \neq 1$, then all the coordinates different from i are rational. If x is not rational, then we can choose a descending sequence r_n of rational numbers such that $\bigwedge_{n\geq 0} r_n = x$. Then, using the characterization of the order given in Corollary 61, we see that the relation $\bigvee_{n\geq 0} e_{|i,r_n,1|} = e_{|i,x,1|}$ holds in $\mathsf{Q}_{\vee}(\mathbb{I})^{[d]_2}$. A fortiori, the same relation holds in $\mathsf{L}(\mathbb{I}^d)$. ◻

We can summarize our observations with the following statement:

Proposition 94. *Every meet-irreducible element of $\mathsf{L}(\mathbb{I}^d)$ is a join of elements from $\mathsf{L}_{\mathcal{R}}(\mathbb{I}^d)$.*

Let in the following $\Sigma_0(\mathsf{L}_{\mathcal{R}}(\mathbb{I}^d)) = \Pi_0(\mathsf{L}_{\mathcal{R}}(\mathbb{I}^d)) = \mathsf{L}_{\mathcal{R}}(\mathbb{I}^d)$ be the set of tuples that have discrete rational functions as components. Let $\Sigma_{n+1}(\mathsf{L}_{\mathcal{R}}(\mathbb{I}^d))$ be the closure under joins of $\Pi_n(\mathsf{L}_{\mathcal{R}}(\mathbb{I}^d))$; let $\Pi_{n+1}(\mathsf{L}_{\mathcal{R}}(\mathbb{I}^d))$ be the closure under meets of $\Sigma_n(\mathsf{L}_{\mathcal{R}}(\mathbb{I}^d))$.

Theorem 95. *Every element of $\mathsf{L}(\mathbb{I}^d)$ belongs both to $\Sigma_2(\mathsf{L}_{\mathcal{R}}(\mathbb{I}^d))$ and $\Pi_2(\mathsf{L}_{\mathcal{R}}(\mathbb{I}^d))$.*

Proof. By Corollary 70 and the fact that $\mathsf{L}(\mathbb{I}^d)$ is autodual, every element of $\mathsf{L}(\mathbb{I}^d)$ is a meet of meet-irreducible elements. We have seen above that each meet-irreducible element is a join of elements from $\mathsf{L}_{\mathcal{R}}(\mathbb{I}^d)$ so it belongs to $\Sigma_1(\mathsf{L}_{\mathcal{R}}(\mathbb{I}^d))$. It follows that every element of $\mathsf{L}(\mathbb{I}^d)$ is an element of $\Pi_2(\mathsf{L}_{\mathcal{R}}(\mathbb{I}^d))$. Since $\mathsf{L}(\mathbb{I}^d)$ and $\mathsf{L}_{\mathcal{R}}(\mathbb{I}^d)$ are autodual, this also proves that every element of $\mathsf{L}(\mathbb{I}^d)$ is an element of $\Sigma_2(\mathsf{L}_{\mathcal{R}}(\mathbb{I}^d))$. ◻

Using the terminology of [20], the previous theorem states that $\mathsf{L}_{\mathcal{R}}(\mathbb{I}^d)$ is dense in $\mathsf{L}(\mathbb{I}^d)$. Yet, $\mathsf{L}(\mathbb{I}^d)$ is not a canonical extension of $\mathsf{L}_{\mathcal{R}}(\mathbb{I}^d)$. A canonical extension of a lattice is a complete spatial lattice, meaning that every element is the infinite join of the completely join-irreducible elements below it, see [20, Lemma 3.4.]. As argued in Section 10.3, there are no completely join-irreducible elements in $\mathsf{L}(\mathbb{I}^d)$, in particular the lattices $\mathsf{L}(\mathbb{I}^d)$ are not spatial.

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