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Abstract:

This paper investigates the second order properties of a stationary continuous time process after random sampling. While a short memory process gives always rise to a short memory one, we prove that long-memory can disappear when the sampling law has very heavy tails. Despite the fact that the normality of the process is not maintained by random sampling, the normalized partial sum process converges to the fractional Brownian motion, at least when the long memory parameter is perserved.

Keywords: Gaussian process; Long memory; Partial sum; Random sampling; Regularly varying covariance.

1 Introduction

Long-range dependence (or long-memory) has diverse applications in many fields, including hydrology, economics and telecommunications (see Beran et al. (2013) ch.2). Most of the papers on this topic consider processes with discrete-time. However, some models and estimation methods have been extended to continuous-time processes (see Tsai and Chan (2005a); Viano et al. (1994); Comte and Renault (1996); Comte (1996)). Tsai and Chan (2005a) introduced the continuous-time autoregressive fractionally integrated moving average (CARFIMA(p,H,q)) model. Under the long-range dependence condition $H \in (1/2, 1)$, they calculate the auto-covariance function of the stationary CARFIMA process and its spectral density function (see Tsai and Chan (2005b)). Theses properties are extended to the case $H \in (0, 1)$ in Tsai (2009). In Viano et al. (1994), continuous-time fractional ARMA processes are constructed. They establish the L^2 properties (spectral density and auto covariance function) and the dependence structure. Comte and Renault (1996) study the continuous time moving average fractional process, a family of long memory model. The statistical inference for continuous-time processes is generally constructed from the sampled process (see Tsai and Chan (2005a,b); Chambers (1996); Comte (1996)). Different schemes of sampling can be considered. In Tsai and Chan (2005a), the estimation method is based on the maximum likelihood estimation for irregularly spaced deterministic time series data. Under the assumption of identifiability, Chambers (1996) considers the estimation of the long memory parameter of a continuous time fractional ARMA process with discrete time data using the low-frequency behaviour of the spectrum. Comte (1996) studied two methods for the estimation with regularly spaced data: Whittle likelihood method and the semiparametric approach of Geweke and Porter-Hudak. In the present paper we are interested in irregularly spaced data when the sampling intervals are independent and identically distributed positive random variables. In the light of previous results in discrete time, there was an effect of the random sampling on the dependence structure of the process. Indeed, Philippe and Viano (2010) show that the intensity of the long memory is preserved when the law of sampling intervals has finite first moment, but they also pointed out situations where a reduction of the long memory is observed.

We adopt the most usual definition of second order long memory process. Namely, a stationary process $\mathbf{U}$ has the long memory property if its auto-covariance function σ_U satisfies the condition

$$\int_{\mathbb{R}^+} |\sigma_U(x)| \, \mathrm{d}x = \infty \quad \text{in the continuous-time case,}$$
$$\sum_{h \geq 0} |\sigma_U(h)| = \infty \quad \text{in the discrete-time case.}$$

We study the effect of random sampling on the properties of a stationary continuous time process process. More precisely, we start with $\mathbf{X} = (X_t)_{t \in \mathbb{R}^+}$, a second-order stationary continuous time process. We assume that it is observed at random times $(T_n)_{n \geq 0}$ where $(T_n)_{n \geq 0}$ is a non-decreasing positive random walk independent of $\mathbf{X}$. We study the discrete-time process $\mathbf{Y}$ defined by

$$Y_n = X_{T_n}, \quad n \in \mathbb{N}. \tag{1.1}$$

The process $\mathbf{Y}$ obtained by random sampling is called the sampled process.

In this paper, we study the properties of this. In particular, we show that the results obtained by Philippe and Viano (2010) on the auto-covariance function are preserved for continuous time process $\mathbf{X}$. The large-sample statistical inference relies often on limit theorems of probability theory for partial sums. We show that Gaussianity is lost by random sampling. However, we prove that the asymptotic normality of the partial sum is preserved with the same standard normalization. (see Giraitis et al. (2012), Chapter 4 for a review).

In Section 2, we study the behavior of the sampled process (1.1) for the general case. We establish that Gaussianity of $\mathbf{X}$ is not transmitted to $\mathbf{Y}$. Under rather weak conditions on the covariance σ_X, the weak dependence is preserved. Howevr a stronger assumption on $T_1 : \mathbb{E}[T_{:1}] < \infty$ is necessary to preserve the long memory property.

In Section 3, we present the more specific situation of a regularly varying covariance where preservation or non-preservation of the memory can be quantified. In particular, we prove that for heavy tailed sampling distribution, a long memory process $\mathbf{X}$ can give raise to a short memory process $\mathbf{Y}$. In Section 4, we establish a Donsker's invariance principle when the initial process $\mathbf{X}$ is Gaussian and the long memory parameter is preserved.

2 General properties.

Throughout this document we assume that the following properties hold on the initial process $\mathbf{X}$ and the random sampling scheme:
Assumption $\mathcal{H}$:

1. $\mathbf{X} = (X_t)_{t \in \mathbb{R}^+}$ is a stationary continuous time process.
2. the random walk $(T_n)_{n \geq 0}$ is independent of $\mathbf{X}$
3. $T_0 = 0$
4. the increments $\Delta_j = T_{j+1} - T_j$ ($j \in \mathbb{N}$) are independent and identically distributed.
5. the distribution of T_1 admits a probability density function s (with respect to the Lebesgue measure) supported by $\mathbb{R}^+$.

If $\mathbf{X}$ is a second-order stationary process with zero mean and auto-covariance function σ_X and the assumption $\mathcal{H}$ holds then the discrete-time process $\mathbf{Y}$ defined in (1.1) is also second-order stationary with zero mean and its auto covariance sequence is

$$\begin{cases} \sigma_Y(0) = \sigma_X(0), \\ \sigma_Y(h) = \mathbb{E}\left[\sigma_X(T_h)\right], \quad h \geq 1. \end{cases} \tag{2.1}$$

2.1 Distribution of the sampled process

This part is devoted to the properties of the finite-dimensional distributions of the process $\mathbf{Y}$.

Proposition 2.1. *Let $\mathbf{X}$ be a strictly stationary process. Then, under Assumption $\mathcal{H}$, the sampled process $\mathbf{Y}$ is a strictly stationary discrete-time process.*

Proof. We arbitrarily fix $n \geq 1$, $p \in \mathbb{N}^*$ and $k_1, \dots k_n \in \mathbb{N}$ such that $0 \leq k_1 < \cdots < k_n$. We show that the joint distribution of $(Y_{k_1+p}, \dots, Y_{k_n+p})$ does not depend on $p \in \mathbb{N}$.

For $(y_1, \dots, y_n) \in \mathbb{R}^n$, we have

$$\begin{aligned} &P(Y_{k_1+p} \leq y_1, \dots, Y_{k_n+p} \leq y_n) = P\left(X_{T_{k_1+p}} \leq y_1, \ \dots, X_{T_{k_n+p}} \leq y_n\right) \\ &= \mathbb{E}\left[P\left(X_{\Delta_0+\cdots+\Delta_{k_1+p-1}} \leq y_1, \ \dots, X_{\Delta_0+\cdots+\Delta_{k_n+p-1}} \leq y_n | \Delta_0, \ \dots \ \Delta_{k_n+p-1}\right)\right]. \end{aligned}$$

By the strict stationarity of $\mathbf{X}$ the right-hand-side of the last equation is equal to

$$\begin{aligned} &\mathbb{E}\left[P\left(X_{\Delta_p+\cdots+\Delta_{k_1+p-1}} \leq y_1, \ \dots, X_{\Delta_p+\cdots+\Delta_{k_n+p-1}} \leq y_n | \Delta_0, \ \dots \ \Delta_{k_n+p-1}\right)\right] \\ &= P(X_{U_0+\cdots+U_{k_1-1}} \leq y_1, \ \dots, X_{U_0+\cdots+U_{k_n-1}} \leq y_n) = P(Y_{k_1} \leq y_1, \ \dots, Y_{k_n} \leq y_n), \end{aligned}$$

where $U_i = \Delta_{i+p}$ are i.i.d with density s. This concludes the proof. $\square$

The following proposition is devoted to the particular case of a Gaussian process. We establish that the Gaussianity is not preserved by random sampling.

Proposition 2.2. *Under Assumption $\mathcal{H}$, if $\mathbf{X}$ is a Gaussian process then the marginals of the sampled process $\mathbf{Y}$ are Gaussian. Furthermore, if σ_X is not almost everywhere constant on the support of s, then $\mathbf{Y}$ is not a Gaussian process.*

Proof. Let U be a random variable, we denote Φ_U its characteristic function. We have, for all $t \in \mathbb{R}$

$$\Phi_{Y_k}(t) = \mathbb{E}\left[\mathbb{E}[e^{itX_{T_k}} | T_k]\right].$$

Conditionally on T_k, the probability distribution of X_{T_k} is the Gaussian distribution with zero mean and variance $\sigma_X(0)$. We get

$$\Phi_{Y_k}(t) = e^{-\sigma_X(0)t^2/2}$$

and thus Y_k is a Gaussian variable with zero mean and variance $\sigma_X(0)$.

Now assume $\mathbf{Y}$ is a Gaussian process, then $Y_1 + Y_2$ is a Gaussian variable,

$$\Phi_{Y_1+Y_2}(t) = e^{-\mathrm{Var}(Y_1+Y_2)t^2/2} = e^{-\sigma_X(0)t^2} e^{-t^2\mathbb{E}[\sigma_X(T_2-T_1)]}$$

and

$$\begin{aligned} \Phi_{Y_1+Y_2}(t) &= \Phi_{X_{T_1}+X_{T_2}}(t) \\ &= \mathbb{E}\left[\exp\left\{-\frac{t^2}{2}\left(\begin{smallmatrix}1\\1\end{smallmatrix}\right)^T \left(\begin{smallmatrix}\sigma_X(0) & \sigma_X(T_2-T_1)\\ \sigma_X(T_2-T_1) & \sigma_X(0)\end{smallmatrix}\right)\left(\begin{smallmatrix}1\\1\end{smallmatrix}\right)\right\}\right] \\ &= e^{-\sigma_X(0)t^2}\mathbb{E}\left[e^{-t^2\sigma_X(T_2-T_1)}\right]. \end{aligned}$$

Then, for all $t \in \mathbb{R}$,

$$e^{-t^2 \mathbb{E}[\sigma_X(T_2 - T_1)]} = \mathbb{E}\left[e^{-t^2 \sigma_X(T_2 - T_1)}\right].$$

According to Jensen's inequality, this equality is achieved if and only if $\sigma_X(T_2 - T_1)$ is constant almost everywhere. $\square$

Example 1. In Figure 1, we illustrate the non-Gaussianity of the sampled process.
We take $\mathbf{X}$ a Gaussian process with autocovariance function $\sigma_X(t) = (1 + t^{0.9})^{-1}$ and $(T_i)_{i \in \mathbb{N}}$ is a homogeneous Poisson counting process with rate 1. To simulate a realization from the distribution of (Y_1, Y_2), we proceed as follows:

1. `Generate the time interval` $T_2 - T_1$ `according to an exponential distribution with mean 1.`
2. `Generate` (Y_1, Y_2) `as a Gaussian vector with zero mean and covariance`

$$\begin{pmatrix} \sigma_X(0) & \sigma_X(T_2 - T_1) \\ \sigma_X(T_2 - T_1) & \sigma_X(0) \end{pmatrix}.$$

We simulate a sample of size p. In Figure 1 (a) we represent the kernel estimate of the joint probability density function of (Y_1, Y_2). In order to compare the probability distribution of the sampled process with the corresponding Gaussian one. We simulate a sample of centered Gaussian vector (W_1, W_2) having the same variance matrix as (Y_1, Y_2) i.e.

$$\Sigma_{Y_1, Y_2} = \begin{pmatrix} \sigma_X(0) & \mathbb{E}[\sigma_X(T_1)] \\ \mathbb{E}[\sigma_X(T_1)] & \sigma_X(0) \end{pmatrix} = \begin{pmatrix} 1 & \Sigma_{1,2} \\ \Sigma_{1,2} & 1 \end{pmatrix},$$

where $\Sigma_{1,2} = \int_0^\infty \sigma_X(t) e^{-t} \, \mathrm{d}t = \int_0^\infty e^{-t}(1 + t^{.9})^{-1} \, \mathrm{d}t$ can be calculated numerically. In Figure 1 (b), we represent the kernel estimate of the density of (W_1, W_2). We see that the form of the distribution of sampled process differs widely from Gaussian distribution.

2.2 Dependence of the sampled process

We are interested in the dependence structure of the $\mathbf{Y}$ process. In the following propositions, we provide sufficient conditions to preserve the weak (respectively long) memory after sampling.

Proposition 2.3. *Assume Assumption $\mathcal{H}$ holds. Let p be a real greater than 1 ($p \geq 1$). If there is a positive bounded function $\sigma_*(.)$, non increasing on $\mathbb{R}^+$, such that*

1. $|\sigma_X(t)| \leq \sigma_*(t), \quad \forall t \in \mathbb{R}^+$
2. $\displaystyle\int_{\mathbb{R}^+} \sigma_*^p(t) dt < \infty$

then, the sampled process Y has an auto-covariance function (2.1) *in ℓ^p, i.e* $\displaystyle\sum_{h \geq 0} |\sigma_Y(h)|^p < \infty$.

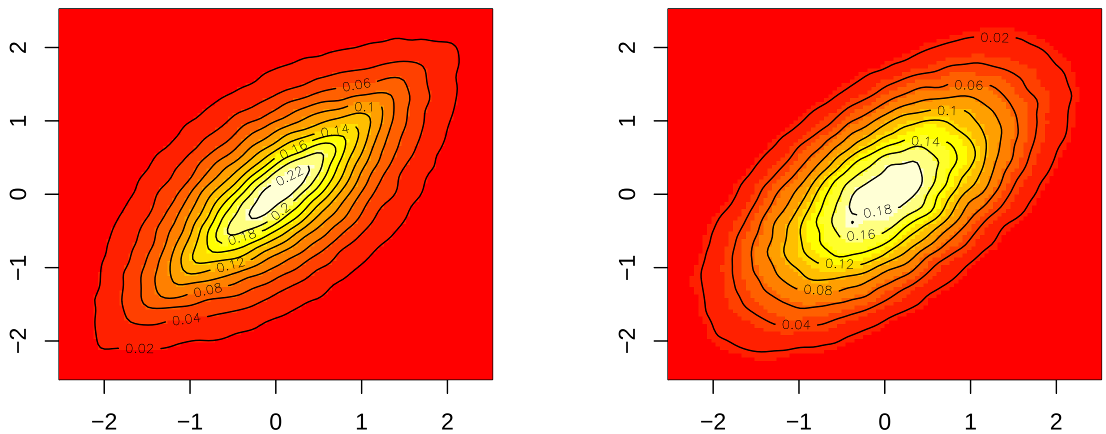
Remark 1. The proposition confirms an intuitive claim: random sampling cannot produce long memory from short memory. The particular case $p = 1$ implies that if $\mathbf{X}$ has short memory then, the sampled process $\mathbf{Y}$ has short memory too.

Proof. It is clearly enough to prove that

$$\sum_{h \geq 1} \mathbb{E}\left[\sigma_*^p(T_h)\right] < \infty. \tag{2.2}$$

We use inequality

$$\Delta_h \sigma_*^p(T_h + \Delta_h) = (T_{h+1} - T_h)\sigma_*^p(T_{h+1}) \leq \int_{T_h}^{T_{h+1}} \sigma_*^p(t) dt, \quad \forall h \geq 0. \tag{2.3}$$



(a) (b)

Figure 1: In Figure (a) the estimated density of the centered couple (Y_1, Y_2) is represented for intervals Δ_j having an exponential distribution with mean 1 and Gaussian initial process with auto-covariance function $\sigma_X(t) = (1 + t^{0.9})^{-1}$. Figure (b) represent the estimated density of the centered Gaussian vector (W_1, W_2) with the same covariance matrix Σ_{Y_1,Y_2} as (Y_1, Y_2). Estimations are calculated on sample of size $p = 50000$

Taking the expectation of the left-hand-side and noting that Δ_h and T_h are independent, we obtain, for every $a > 0$,

$$
\begin{aligned}
\mathbb{E}\left[\Delta_h \sigma_*^p(T_h + \Delta_h)\right] &= \int_{\mathbb{R}^+} u\mathbb{E}\left[\sigma_*^p(T_h + u)\right] dS(u) \\
&= \int_0^a u\mathbb{E}\left[\sigma_*^p(T_h + u)\right] dS(u) + \int_a^{+\infty} u\mathbb{E}\left[\sigma_*^p(T_h + u)\right] dS(u) \\
&\geq \int_0^a u\mathbb{E}\left[\sigma_*^p(T_h + u)\right] dS(u) + a\int_a^{+\infty} \mathbb{E}\left[\sigma_*^p(T_h + u)\right] dS(u) \\
&= \int_0^a u\mathbb{E}\left[\sigma_*^p(T_h + u)\right] dS(u) + a\left(\int_{\mathbb{R}^+} \mathbb{E}\left[\sigma_*^p(T_h + u)\right] dS(u) - \int_0^a \mathbb{E}\left[\sigma_*^p(T_h + u)\right] dS(u)\right) \\
&= \int_0^a (u - a)\mathbb{E}\left[\sigma_*^p(T_h + u)\right] dS(u) + a\mathbb{E}\left[\sigma_*^p(T_{h+1})\right]
\end{aligned}
$$

Since $\sigma_*^p(T_h + u) \leq \sigma_*^p(T_h)$ and $u - a \leq 0$, we get

$$
\mathbb{E}\left[\Delta_h \sigma_*^p(T_h + \Delta_h)\right] \geq \left(\int_{[0,a[} (u - a) dS(u)\right) \mathbb{E}\left[\sigma_*^p(T_h)\right] + a\mathbb{E}\left[\sigma_*^p(T_{h+1})\right]. \tag{2.4}
$$

It is possible to choose a such that $S([0, a]) < 1$. For such a choice we obtain

$$
0 \leq -\int_{[0,a[} (u - a) dS(u) =: \ell(a) \leq aS([0, a]) < a.
$$

After summation, the inequalities (2.4) give, for every $K \geq 0$

$$\begin{aligned}
\mathbb{E}\left[\sum_{h=1}^{\infty} \Delta_h \sigma_*^p(T_{h+1})\right] &\geq \sum_{h=1}^{K} [-\ell(a)\mathbb{E}[\sigma_*^p(T_h)] + a\mathbb{E}[\sigma_*^p(T_{h+1})]] \\
&= a\left(\mathbb{E}[\sigma_*^p(T_{K+1})] - \mathbb{E}[\sigma_*^p(T_1)]\right) + (a-\ell(a))\sum_{h=1}^{K} \mathbb{E}\left[\sigma_*^p(T_h)\right] \\
&\geq -a\sigma_*^p(0) + (a-\ell(a))\sum_{h=1}^{K} \mathbb{E}\left[\sigma_*^p(T_h)\right],
\end{aligned}$$

which implies

$$\mathbb{E}\left[\sum_{h=1}^{\infty} \Delta_h \sigma_*^p(T_{h+1})\right] \geq -a\sigma_*^p(0) + (a-\ell(a))\sum_{h\geq 1} \mathbb{E}\left[\sigma_*^p(T_h)\right].$$

Then, using (2.3)

$$\mathbb{E}\left[\sum_{h\geq 1} \Delta_h \sigma_*^p(T_{h+1})\right] \leq \mathbb{E}\left[\sum_{h\geq 1} \int_{T_h}^{T_{h+1}} \sigma_*^p(t)dt\right] \leq \int_{\mathbb{R}^+} \sigma_*^p(t)dt < \infty$$

and consequently, as $a - \ell(a) > 0$

$$\sum_{h=1}^{\infty} \mathbb{E}\left[\sigma_*^p(T_h)\right] < \infty. \tag{2.5}$$

□

We now consider the case of long memory processes. We give conditions on T_1 that ensure the preservation of the long memory property.

Proposition 2.4. *Assume Assumption $\mathcal{H}$ holds. We suppose that $\sigma_X(.)$ is ultimately positive and non-increasing on $\mathbb{R}^+$, i.e there exists $t_0 \geq 0$ such that $\sigma_X(.)$ is positive and non increasing on the interval $[t_0, \infty)$. If $\mathbb{E}[T_1] < \infty$, then the long memory is preserved after the subsampling, i.e. $\int_{\mathbb{R}^+} |\sigma_X(x)|\, \mathrm{d}x = \infty$ implies $\sum_{h\geq 0} |\sigma_Y(h)| = \infty$.*

Remark 2. The assumptions on positivity and the decrease of the auto-covariance function are not too restrictive. They are satisfied in most of studied models. The condition of integrability of intervals Δ_j is the most difficult to verify since the underlying process is generally not observed.

Proof. Let h_0 be the (random) first index such that $T_{h_0} \geq t_0$. For every $h \geq h_0$,

$$\int_{T_h}^{T_{h+1}} \sigma_X(t)dt \leq (T_{h+1} - T_h)\sigma_X(T_h). \tag{2.6}$$

Summing up gives

$$\sum_{h\geq 1} \mathbb{I}_{h\geq h_0} \int_{T_h}^{T_{h+1}} \sigma_X(t)dt \quad \leq \quad \sum_{h\geq 1} \mathbb{I}_{h\geq h_0} \Delta_h \sigma_X(T_h).$$

Now, taking expectations, and noting that, since $\mathbb{E}[T_1] = \mathbb{E}[\Delta_1] > 0$, the law of large numbers implies that $T_h \xrightarrow{a.s.} \infty$, and in particular $h_0 < \infty$ a.s., whence

$$\mathbb{E}\left[\int_{T_{h_0}}^{\infty} \sigma_X(t)dt\right] \quad \leq \quad \mathbb{E}\left[\sum_{h=1}^{\infty} \Delta_h \sigma_X(T_h)\, \mathbb{I}_{h_0 \leq h}\right].$$

The left hand side is infinite. Since Δ_h is independent of $\sigma_X(T_h)\,\mathbb{I}_{h_0\le h}$, the right hand side is $\mathbb{E}[T_1]\sum_{h\ge 1}\mathbb{E}[\sigma_X(T_h)\,\mathbb{I}_{h_0\le h}]$. Consequently, since $\mathbb{E}[T_1]<\infty$, we have

$$\sum_{h\ge 1}\mathbb{E}[\sigma_X(T_h)\,\mathbb{I}_{h_0\le h}]=\infty. \tag{2.7}$$

It remains to be noted that $\mathbb{E}[h_0]<\infty$ (see for example Feller (1966) p.185), which implies

$$\sum_{h\ge 1}\mathbb{E}[|\sigma_X(T_h)|\,\mathbb{I}_{h_0>h}]\le\sigma_X(0)\sum_{h\ge 1}P(h_0\ge h)\le\sigma_X(0)\mathbb{E}[h_0]<\infty,$$

leading, via (2.7) to $\sum_{h\ge 1}|\mathbb{E}[\sigma_X(T_h)]|=\infty$.

□

3 Long memory processes

We consider a long memory process $\mathbf{X}$ and we impose a semi parametric form to auto-covariance function. We assume that the auto-covariance σ_X is regularly varying function at infinity of the form

$$\sigma_X(t)=t^{-1+2d}L(t),\qquad\forall t\ge 1 \tag{3.1}$$

where $0<d<1/2$ and L is ultimately non-increasing and slowly varying at infinity, in the sense that L is positive on $[t_0,\infty)$ for some $t_0>0$ and

$$\lim_{x\to+\infty}\frac{L(ax)}{L(x)}=1,\qquad\forall a>0.$$

This class of models contains for instance CARFIMA models.

The parameter d characterizes the intensity of the memory of $\mathbf{X}$. In the following propositions, we evaluate the long memory parameter of the sampled process $\mathbf{Y}$ as a function of d and the probability distribution of T_1.

3.1 Preservation of the memory when $\mathbb{E}[T_1]<\infty$

Theorem 3.1. *Under Assumption $\mathcal{H}$ and (3.1), if $0<\mathbb{E}[T_1]<\infty$, the discrete time process $\mathbf{Y}$ has a long memory and its covariance function behaves as*

$$\sigma_Y(h)\sim(h\mathbb{E}[T_1])^{-1+2d}L(h),\qquad h\to\infty.$$

Remark 3. We can rewrite

$$\sigma_Y(h)=h^{-1+2d}\tilde{L}(h)$$

where $\tilde{L}$ is slowly varying at infinity and $\tilde{L}(h)\sim(\mathbb{E}[T_1])^{-1+2d}L(h)$ as $h\to\infty$. In particular, $\mathbf{X}$ and $\mathbf{Y}$ have the same memory parameter d.

Proof.

• We show first that

$$\liminf_{h\to\infty}\frac{\sigma_Y(h)}{(h\mathbb{E}[T_1])^{-1+2d}L(h)}\ge 1.$$

Let $0<c<\mathbb{E}[T_1]$, and $h\in\mathbb{N}$ such that $ch\ge 1$,

$$\sigma_Y(h)\quad\ge\quad\mathbb{E}\left[\sigma_X(T_h)\,\mathbb{I}_{T_h>ch}\right]\ge\inf_{t>ch}\{L(t)t^{2d}\}\mathbb{E}\left[\frac{\mathbb{I}_{T_h>ch}}{T_h}\right].$$

Thanks to Hölder inequality,

$$(P(T_h > ch))^2 \leq \mathbb{E}[T_h]\mathbb{E}\left[\frac{\mathbb{I}_{T_h>ch}}{T_h}\right],$$

that is

$$\mathbb{E}\left[\frac{\mathbb{I}_{T_h>ch}}{T_h}\right] \geq \frac{(P(T_h > ch))^2}{h\mathbb{E}[T_1]}.$$

Summarizing,

$$\sigma_Y(h) \geq \inf_{t>ch}\{L(t)t^{2d}\}\frac{(P(T_h > ch))^2}{h\mathbb{E}[T_1]}$$

$$\frac{\sigma_Y(h)}{(h\mathbb{E}[T_1])^{-1+2d}L(h)} \geq \inf_{t>ch}\{L(t)t^{2d}\}\frac{(P(T_h > ch))^2}{(h\mathbb{E}[T_1])^{2d}L(h)}. \tag{3.2}$$

Using Bingham et al. (1989) (Th 1.5.3, p23), we obtain, since $d > 0$

$$\inf_{t\geq ch}\{L(t)t^{2d}\} \sim L(ch)(ch)^{2d} \quad as \quad h \to \infty. \tag{3.3}$$

The law of large numbers implies that $T_h/h \xrightarrow{a.s.} \mathbb{E}[T_1]$. As $c < \mathbb{E}[T_1]$, we have $P(T_h > ch) \to 1$ and the r.h.s. of (3.2) tends to $(c/\mathbb{E}[T_1])^{2d}$ as $h \to \infty$. Finally, for all $c < \mathbb{E}[T_1]$,

$$\liminf_{h\to\infty} \frac{\sigma_Y(h)}{(h\mathbb{E}[T_1])^{-1+2d}L(h)} \geq \left(\frac{c}{\mathbb{E}[T_1]}\right)^{2d}$$

Taking the limit as $c \to \mathbb{E}[T_1]$, we get the lower bound.

• Let us now prove

$$\limsup_{h\to\infty} \frac{\sigma_Y(h)}{(h\mathbb{E}[T_1])^{-1+2d}L(h)} \leq 1.$$

We use a proof similar to that presented in Shi et al. (2010) (Theorem 1). We denote for $h \geq 1$ and $0 < s < 1$,

$$\begin{aligned}
\mu_h &= \mathbb{E}[T_h] = h\mathbb{E}[T_1] \\
T_{h,s} &= \sum_{j=0}^{h-1} \Delta_j\, \mathbb{I}_{\Delta_j \leq \mu_h^s/\sqrt{h}} \\
\mu_{h,s} &= \mathbb{E}\left[T_{h,s}\right] = h\mathbb{E}\left[\Delta_0\, \mathbb{I}_{\Delta_0 \leq \mu_h^s/\sqrt{h}}\right]
\end{aligned}$$

Since $\mathbb{E}[T_1] < \infty$, we have for $\frac{1}{2} < s < 1$, $\mu_{h,s} \sim \mu_h$ as $h \to \infty$.
Let $\frac{1}{2} < s < \tau < 1$, t_0 such that $L(.)$ is non-increasing on $[t_0, \infty)$ and h such that $\mu_{h,s} - \mu_{h,s}^\tau \geq t_0$,

$$\begin{aligned}
\sigma_Y(h) &= \mathbb{E}\left[T_h^{-1+2d}L(T_h)\,\mathbb{I}_{T_{h,s}\geq\mu_{h,s}-\mu_{h,s}^\tau}\right] + \mathbb{E}\left[T_h^{-1+2d}L(T_h)\,\mathbb{I}_{T_{h,s}<\mu_{h,s}-\mu_{h,s}^\tau}\right] \\
&= M_1 + M_2
\end{aligned}$$

$$\begin{aligned}
M_1 &\leq \mathbb{E}\left[T_{h,s}^{-1+2d}L(T_{h,s})\,\mathbb{I}_{T_{h,s}\geq\mu_{h,s}-\mu_{h,s}^\tau}\right] \leq \left(\mu_{h,s} - \mu_{h,s}^\tau\right)^{-1+2d} L(\mu_{h,s} - \mu_{h,s}^\tau) \\
&= (h\mathbb{E}[T_1])^{-1+2d}L(h)\left(\frac{\mu_{h,s} - \mu_{h,s}^\tau}{h\mathbb{E}[T_1]}\right)^{-1+2d} \frac{L(\mu_{h,s} - \mu_{h,s}^\tau)}{L(h)}
\end{aligned} \tag{3.4}$$

As $\tau < 1$ and $1/2 < s < 1$, $\left(\frac{\mu_{h,s}-\mu_{h,s}^\tau}{h\mathbb{E}[T_1]}\right)^{-1+2d} \to 1$ as $h \to \infty$. Then,

$$\frac{L(\mu_{h,s} - \mu_{h,s}^\tau)}{L(h)} = \frac{L\left(h\mathbb{E}[T_1]\frac{\mu_{h,s}-\mu_{h,s}^\tau}{h\mathbb{E}[T_1]}\right)}{L(h\mathbb{E}[T_1])}\frac{L(h\mathbb{E}[T_1])}{L(h)}$$

As we have uniform convergence of $\lambda \mapsto \frac{L(h\mathbb{E}[T_1]\lambda)}{L(h\mathbb{E}[T_1])}$ to 1 (as $h \to \infty$) in each interval $[a, b]$ and as $\frac{\mu_{h,s} - \mu^\tau_{h,s}}{h\mathbb{E}[T_1]} \to 1$, we get

$$\frac{L(\mu_{h,s} - \mu^\tau_{h,s})}{L(h)} \to 1,$$

as $h \to \infty$. We obtain

$$M_1 \leq \left(\mu_{h,s} - \mu^\tau_{h,s}\right)^{-1+2d} L(\mu_{h,s} - \mu^\tau_{h,s}) \sim (h\mathbb{E}[T_1])^{-1+2d} L(h). \tag{3.5}$$

Since $\sup_{t \in \mathbb{R}^+} |\sigma_X(t)| = \sigma_X(0) < \infty$, we have

$$M_2 \quad \leq \quad \sigma_X(0) P\left(T_{h,s} < \mu_{h,s} - \mu^\tau_{h,s}\right) = \sigma_X(0) P\left(-T_{h,s} + \mathbb{E}[T_{h,s}] > \mu^\tau_{h,s}\right).$$

We apply Hoeffding inequality to variables $Z_j = -\Delta_j \, \mathbb{I}_{\Delta_j \leq \mu^s_h / \sqrt{h}}$ which are a.s in $[-\frac{\mu^s_h}{\sqrt{h}}, 0]$ to get,

$$M_2 \leq \sigma_X(0) \exp\left(-2\left(\frac{\mu^\tau_{h,s}}{\mu^s_h}\right)^2\right)$$

and $\left(\frac{\mu^\tau_{h,s}}{\mu^s_h}\right)^2 \sim (h\mathbb{E}[T_1])^{2(\tau - s)}$. Finally

$$M_2 = o((h\mathbb{E}[T_1])^{-1+2d} L(h)). \tag{3.6}$$

With (3.5) and (3.6), we get the upper bound.

□

3.2 Decrease of memory when $\mathbb{E}[T_1] = \infty$

The phenomenon is the same as in the discrete case (see Philippe and Viano (2010)): starting from a long memory process, a heavy tailed sampling distribution can lead to a short memory process.

Proposition 3.2. *Assume that the covariance of* $\mathbf{X}$ *satisfies*

$$|\sigma_X(t)| \leq c \min(1, t^{-1+2d}) \qquad \forall t \in \mathbb{R}^+ \tag{3.7}$$

where $0 < d < 1/2$. *If for some* $\beta \in (0, 1)$

$$\liminf_{x \to \infty} \left(x^\beta P(T_1 > x)\right) > 0 \tag{3.8}$$

(implying $\mathbb{E}[T_1^\beta] = \infty$*) then*

$$|\sigma_Y(h)| \leq C h^{\frac{-1+2d}{\beta}}. \tag{3.9}$$

Proof. From hypothesis (3.7),

$$|\sigma_Y(h)| \leq \mathbb{E}[|\sigma_X(T_h)|] \leq c\mathbb{E}[\min\{1, T_h^{-1+2d}\}]$$

Then, denoting S^{*h} the distribution function of T_h and integrating by parts,

$$\begin{aligned}
\mathbb{E}[\min\{1, T_h^{-1+2d}\}] &= \int_0^1 dS^{*h}(x) + \int_1^\infty x^{-1+2d} dS^{*h}(x) \\
&= S^{*h}(1) + (1-2d) \int_1^\infty x^{-2+2d} S^{*h}(x) dx - S^{*h}(1) \\
&= (1-2d) \int_1^\infty x^{-2+2d} S^{*h}(x) dx. \qquad (3.10)
\end{aligned}$$

From hypothesis (3.8) on the tail of the sampling law, it follows that, there exists $C > 0$ and $x_0 \geq 1$ such that

$$\forall x \geq x_0, \quad P(T_1 > x) \geq C x^{-\beta}.$$

Furthermore for $x \in [1, x_0]$,

$$x^{\beta} P(T_1 > x) \geq P(T_1 > x_0) \geq C x_0^{-\beta}.$$

We obtain: $\forall x \geq 1,\ P(T_1 > x) \geq \tilde{C} x^{-\beta}$ with $\tilde{C} = C x_0^{-\beta}$.

$$\begin{aligned} S^{*h}(x) &= P(T_h \leq x) \leq P\left(\max_{0 \leq l \leq h-1} \Delta_l \leq x\right) = P\left(T_1 \leq x\right)^h \\ &\leq \left(1 - \tilde{C} x^{-\beta}\right)^h \leq e^{-\frac{\tilde{C} h}{x^{\beta}}}. \end{aligned} \tag{3.11}$$

Gathering (3.10) and (3.11) then gives

$$\begin{aligned} \mathbb{E}[\min\{1, T_h^{-1+2d}\}] &\leq (1-2d) \int_1^{\infty} x^{-2+2d} e^{-\frac{\tilde{C} h}{x^{\beta}}} dx \\ &= \frac{1-2d}{\beta} h^{-(1-2d)/\beta} \int_0^h u^{(1-2d)/\beta - 1} e^{-\tilde{C} u} du \end{aligned}$$

and the result follows since

$$\int_0^h u^{(1-2d)/\beta-1} e^{-\tilde{C} u} du \xrightarrow{h \to \infty} \int_0^{\infty} u^{(1-2d)/\beta-1} e^{-\tilde{C} u} du.$$

□

Under some additional assumptions, we show that the bound obtained in Proposition 3.2 is equal to the convergence rate (up to a multiplicative constant).

Proposition 3.3. *Assume that*

$$\sigma_X(t) = t^{-1+2d} L(t)$$

where $0 < d < 1/2$ *and where* L *is slowly varying at infinity and ultimately monotone.*

If

$$\beta := \sup\{\gamma : \mathbb{E}[T_1^{\gamma}] < \infty\} \in (0,1) \tag{3.12}$$

then, for every $\varepsilon > 0$*, there exists* $C_\varepsilon > 0$ *such that*

$$\sigma_Y(h) \geq C_\varepsilon h^{-\frac{1-2d}{\beta} - \varepsilon}, \qquad \forall h \geq 1. \tag{3.13}$$

Proof. Let $\varepsilon > 0$. We have

$$\frac{\sigma_X(T_h)}{h^{-\frac{1-2d}{\beta}-\varepsilon}} = \frac{T_h^{-1+2d}}{h^{-\frac{1-2d}{\beta}-\varepsilon}} L(T_h) = \frac{T_h^{-1+2d-\frac{\beta\varepsilon}{2}}}{h^{-\frac{1-2d}{\beta}-\varepsilon}} T_h^{\frac{\beta\varepsilon}{2}} L(T_h) = \left(\frac{T_h}{h^{\delta}}\right)^{-1+2d-\frac{\beta\varepsilon}{2}} T_h^{\frac{\beta\varepsilon}{2}} L(T_h)$$

where

$$\delta = \frac{(1-2d)/\beta + \varepsilon}{1-2d+\frac{\beta\varepsilon}{2}} = \frac{1}{\beta}\left(\frac{1-2d+\beta\varepsilon}{1-2d+\beta\varepsilon/2}\right).$$

Using Proposition 1.3.6 in Bingham et al. (1989),

$$T_h^{\frac{\beta\varepsilon}{2}} L(T_h) \xrightarrow{h \to \infty} \infty \quad \text{a.s}$$

Moreover $\delta > \frac{1}{\beta}$. From (3.12), this implies $\mathbb{E}[T_1^{1/\delta}] < \infty$. Then, the law of large numbers of Marcinkiewicz-Zygmund (see Stout (1974) Theorem 3.2.3) yields

$$\frac{T_h}{h^{\delta}} \xrightarrow{a.s.} 0 \quad \text{as} \quad h \to \infty. \tag{3.14}$$

Therefore by Fatou's Lemma, we get

$$\frac{\sigma_Y(h)}{h^{-\frac{1-2d}{\beta}-\varepsilon}} \xrightarrow{h\to\infty} \infty.$$

$\square$

Remark 4. In this context the long memory parameter d of the initial process $\mathbf{X}$ is not identifiable using the sampled process. Information on probability distribution of Δ_1 is required.

4 Limit theorems in semiparametric case

We consider the process of partial sums

$$S_n(\tau) = \sum_{j=1}^{[n\tau]} Y_j, \quad 0 \le \tau \le 1. \tag{4.1}$$

In Theorem 4.2, we show that if $\mathbf{X}$ is a Gaussian process and $\mathbf{X}$ and $\mathbf{Y}$ have the same long memory parameter, the normalized partial sum process converges to a fractional Brownian motion. According to Proposition 2.2, Gaussianity is lost after sampling, however we get the classical behavior obtained by Taqqu (1975); Davydov (1970)).

4.1 Convergence of the partial sum process

To prove the convergence of the normalized partial sum process, we first need a result on the convergence in probability of conditional variance of S_n.

Lemma 4.1. *If* $\mathbf{X}$ *is a Gaussian process with regularly varying covariance function* $\sigma_X(t) = L(t)t^{-1+2d}$, *with* $0 < d < 1/2$ *and* L *is slowly varying at infinity and ultimately non-increasing.*
Then, if $\mathbb{E}[T_1] < \infty$, *we have*

$$L(n)^{-1}n^{-1-2d}\mathrm{Var}(X_{T_1} + \cdots + X_{T_n}|T_1\ ,\ldots,\ T_n) \xrightarrow[n\to\infty]{p} \gamma_d, \tag{4.2}$$

where $\gamma_d := \frac{\mathbb{E}[T_1]^{-1+2d}}{d(1+2d)}$.

Proof. See Appendix. $\square$

Theorem 4.2. *Assume Assumption* $\mathcal{H}$ *holds. If* $\mathbf{X}$ *is a Gaussian process with regularly varying covariance function* $\sigma_X(t) = L(t)t^{-1+2d}$, *with* $0 < d < 1/2$ *and* L *slowly varying at infinity and ultimately non increasing (Hypothesis 3.1). Then, if* $\mathbb{E}[T_1] < \infty$, *we get*

$$\gamma_d^{-1/2}L(n)^{-1/2}n^{-1/2-d}S_n(.) \Rightarrow B_{\frac{1}{2}+d}(.), \quad \text{in } \mathcal{D}[0,1] \text{ with the uniform metric.} \tag{4.3}$$

where $B_{\frac{1}{2}+d}$ *is the fractional Brownian motion with parameter* $\frac{1}{2}+d$ *and* $\gamma_d := \frac{\mathbb{E}[T_1]^{-1+2d}}{d(1+2d)}$.

Proof. We first prove the weak convergence in finite-dimensional distributions of

$$\gamma_d^{-1/2}L(n)^{-1/2}n^{-1/2-d}S_n(.)$$

to the corresponding finite-dimensional distributions of $B_{\frac{1}{2}+d}(.)$.
It suffices to show that for every $k \ge 1$, $(b_1,\ \ldots\ ,b_k) \in \mathbb{R}^k$, $0 \le t_1, \ldots, t_k \le 1$,

$$A_n := \gamma_d^{-1/2}L(n)^{-1/2}n^{-1/2-d}\sum_{i=1}^{k} b_i S_n(t_i)$$

satisfies

$$A_n \xrightarrow{d} \sum_{i=1}^k b_i B_{\frac{1}{2}+d}(t_i).$$

If $t_1 = \cdots = t_k = 0$, then $\gamma_d^{-1/2} L(n)^{-1/2} n^{-1/2-d} \sum_{i=1}^k b_i S_n(t_i) = \sum_{i=1}^k b_i B_{\frac{1}{2}+d}(t_i) = 0$. So we fix n large enough to have $[n \max_i(t_i)] \geq 1$ and denote $T^{(n)} = (T_1, \ldots, T_{[n \max_i(t_i)]})$. The characteristic function of A_n is

$$\Phi_{A_n}(t) = \mathbb{E}[e^{itA_n}] = \mathbb{E}[e^{-\frac{t^2}{2} \mathrm{Var}(A_n | T^{(n)})}].$$

Moreover we have

$$\begin{aligned}
&\mathrm{Var}(A_n | T^{(n)}) \\
&= \sum_{i,j=1}^k b_i b_j \gamma_d^{-1} L(n)^{-1} n^{-1-2d} \mathbb{E}[S_n(t_i) S_n(t_j) | T^{(n)}] \\
&= \sum_{i,j=1}^k \frac{b_i b_j \gamma_d^{-1} L(n)^{-1} n^{-1-2d}}{2} \left[\mathrm{Var}(S_n(t_i) | T^{(n)}) + \mathrm{Var}(S_n(t_j) | T^{(n)}) - \mathrm{Var}(S_n(t_i) - S_n(t_j) | T^{(n)}) \right]
\end{aligned}$$

By Lemma 4.1,

$$L(n)^{-1} n^{-1-2d} \mathrm{Var}(Y_1 + \cdots + Y_n | T_1 \ , \ldots , \ T_n) \xrightarrow[n \to \infty]{p} \gamma_d.$$

therefore

$$\gamma_d^{-1} L(n)^{-1} n^{-1-2d} \mathrm{Var}(S_n(t_i) | T^{(n)}) \xrightarrow[n \to \infty]{p} t_i^{1+2d}$$

and for $t_i > t_j$

$$\begin{aligned}
\gamma_d^{-1} L(n)^{-1} n^{-1-2d} \mathrm{Var}(S_n(t_i) - S_n(t_j) | T^{(n)}) &= \gamma_d^{-1} L(n)^{-1} n^{-1-2d} \mathrm{Var}(Y_{[nt_i]+1} + \cdots + Y_{[nt_j]} | T^{(n)}) \\
&\xrightarrow[n \to \infty]{p} (t_i - t_j)^{1+2d}.
\end{aligned}$$

Finally, we have

$$\mathrm{Var}(A_n | T^{(n)}) \xrightarrow[n \to \infty]{p} \sum_{i,j=1}^k b_i b_j r_{\frac{1}{2}+d}(t_i, t_j)$$

where $r_{\frac{1}{2}+d}$ is the covariance function of a fractional Brownian motion, and hence

$$\exp\left(-\frac{t^2}{2} \mathrm{Var}(A_n | T^{(n)}) \right) \xrightarrow[n \to \infty]{p} \exp\left(-\frac{t^2}{2} \sum_{i,j=1}^k b_i b_j r_{\frac{1}{2}+d}(t_i, t_j) \right)$$

Therefore, applying bounded convergence theorem, we get

$$\Phi_{A_n}(t) \xrightarrow[n \to \infty]{} \exp\left(-\frac{t^2}{2} \sum_{i,j=1}^k b_i b_j r_{\frac{1}{2}+d}(t_i, t_j) \right) = \Phi_{\sum_{i=1}^k b_i B_{\frac{1}{2}+d}(t_i)}(t).$$

The sequence of partial-sum processes $L(n)^{-1/2} n^{-1/2-d} S_n(.)$ is tight with respect to the uniform norm (see Giraitis et al. (2012) Prop 4.4.2 p78, for the proof of the tightness) and then we get the convergence in $\mathcal{D}[0,1]$ with the uniform metric. □

4.2 Estimation of the long memory parameter

An immediate consequence of this limit theorem is to provide a nonparametric estimation of the long memory parameter d using the well-known R/S statistics. It is quite obvious that this is a heuristic method for estimating the long memory parameter. A more efficient estimate can be obtained by using Whittle's estimate or estimators based on the spectral approach (see Giraitis et al. (2012); Beran et al. (2013)). But the sampled process does not satisfy the assumptions under which these estimation methods are asymptotically validated.

The R/S statistic is defined as the quotient between R_n and S_n where

$$R_n := \max_{1\le k\le n}\sum_{j=1}^{k}(Y_j-\overline{Y_n}) - \min_{1\le k\le n}\sum_{j=1}^{k}(Y_j-\overline{Y_n}) \tag{4.4}$$

and

$$S_n := \left(\frac{1}{n}\sum_{j=1}^{n}(Y_j-\overline{Y_n})^2\right)^{1/2}. \tag{4.5}$$

Proposition 4.3. *Under the same assumptions as Theorem 4.2, we have*

$$\frac{1}{L(n)^{1/2}n^{1/2+d}}\frac{R_n}{S_n} \xrightarrow[n\to\infty]{d} \mathcal{R}(1) := \sqrt{\frac{\gamma_d}{\sigma_X(0)}}\left(\max_{0\le t\le 1} B^0_{\frac{1}{2}+d}(t) - \min_{0\le t\le 1} B^0_{\frac{1}{2}+d}(t)\right) \tag{4.6}$$

where $B^0_{\frac{1}{2}+d}(t) = B_{\frac{1}{2}+d}(t) - tB_{\frac{1}{2}+d}(1)$ *is a fractional Brownian bridge and* γ_d *is a constant defined in Lemma 4.1.*

Proof. Using the equality

$$\sum_{j=1}^{k}(Y_j-\overline{Y_n}) = \sum_{j=1}^{k}Y_j - \frac{k}{n}\sum_{j=1}^{n}Y_j = S_n\left(\frac{k}{n}\right) - \frac{k}{n}S_n(1)$$

and the convergence of the partial-sum process given in Theorem 4.2, we get

$$\frac{R_n}{L(n)^{1/2}n^{1/2+d}} \xrightarrow[n\to\infty]{d} \sqrt{\gamma_d}\left(\max_{0\le t\le 1} B^0_{\frac{1}{2}+d}(t) - \min_{0\le t\le 1} B^0_{\frac{1}{2}+d}(t)\right).$$

Then, we establish the convergence in probability of S_n^2 defined in (4.5). As

$$\mathrm{Var}\left(\sum_{j=1}^{n}Y_j\right) \sim Cn^{1+2d},$$

we have for $\varepsilon > 0$

$$P\left(\left|\frac{1}{n}\sum_{j=1}^{n}Y_j\right| > \varepsilon\right) \le \frac{1}{n^2\varepsilon^2}\mathrm{Var}\left(\sum_{j=1}^{n}Y_j\right) \xrightarrow[n\to\infty]{} 0$$

and

$$\begin{aligned} P\left(\left|\frac{1}{n}\sum_{j=1}^{n}Y_j^2 - \sigma_X(0)\right| > \varepsilon\right) &\le \frac{1}{n^2\varepsilon^2}\mathrm{Var}\left(\sum_{j=1}^{n}Y_j^2\right) \\ &= \frac{1}{n^2\varepsilon^2}\sum_{j=1}^{n}\sum_{k=1}^{n}\mathrm{Cov}\left(Y_j^2, Y_k^2\right) \\ &= \frac{1}{n^2\varepsilon^2}\sum_{j=1}^{n}\sum_{k=1}^{n}\left(\mathbb{E}\left[\mathbb{E}[X_{T_j}^2X_{T_k}^2|T_j,T_k]\right] - \sigma_X(0)^2\right) \end{aligned}$$

For $(s,t) \in (\mathbb{R}^+)^2$, we decompose X_s^2 and X_t^2 in the complete orthogonal system of Hermite polynomials $(H_k)_{k\ge 0}$:

$$\left(\frac{X_s}{\sqrt{\sigma_X(0)}}\right)^2 = H_0\left(\frac{X_s}{\sqrt{\sigma_X(0)}}\right) + H_2\left(\frac{X_s}{\sqrt{\sigma_X(0)}}\right),$$

thus, we get

$$\begin{aligned}\frac{\mathbb{E}[X_s^2X_t^2]}{\sigma_X(0)^2} &= \mathbb{E}\left[H_0\left(\frac{X_s}{\sqrt{\sigma_X(0)}}\right)H_0\left(\frac{X_t}{\sqrt{\sigma_X(0)}}\right)\right] + \mathbb{E}\left[H_2\left(\frac{X_s}{\sqrt{\sigma_X(0)}}\right)H_0\left(\frac{X_t}{\sqrt{\sigma_X(0)}}\right)\right]\\ &+ \mathbb{E}\left[H_0\left(\frac{X_s}{\sqrt{\sigma_X(0)}}\right)H_2\left(\frac{X_t}{\sqrt{\sigma_X(0)}}\right)\right] + \mathbb{E}\left[H_2\left(\frac{X_s}{\sqrt{\sigma_X(0)}}\right)H_2\left(\frac{X_t}{\sqrt{\sigma_X(0)}}\right)\right]\end{aligned}$$

Using the orthogonality property of Hermite polynomials for a bivariate normal density with unit variances (see Giraitis et al. (2012), Prop 2.4.1), we obtain

$$\begin{aligned}\mathbb{E}[X_s^2X_t^2] &= \sigma_X^2(0)\left[1+2\mathrm{Cov}^2\left(\frac{X_s}{\sqrt{\sigma_X(0)}},\frac{X_t}{\sqrt{\sigma_X(0)}}\right)\right]\\ &= \sigma_X^2(0)+2\sigma_X^2(t-s)\end{aligned}$$

Finally,

$$\begin{aligned}P\left(\left|\frac{1}{n}\sum_{j=1}^{n}Y_j^2-\sigma_X(0)\right|>\varepsilon\right) &\leq \frac{2}{n^2\varepsilon^2}\sum_{j=1}^{n}\sum_{k=1}^{n}\mathbb{E}\left[\sigma_X^2(T_j-T_k)\right]\\ &= \frac{4}{n^2\varepsilon^2}\sum_{j=0}^{n-1}(n-j)\mathbb{E}\left[\sigma_X^2(T_j)\right]\end{aligned}$$

If $0\leq d\leq 1/4$, we apply Proposition 2.3 with $p=1$ and the function σ_X^2 to obtain

$$\frac{1}{n^2}\sum_{j=0}^{n-1}(n-j)\mathbb{E}\left[\sigma_X^2(T_j)\right]\leq\frac{1}{n}\sum_{j=0}^{\infty}\mathbb{E}\left[\sigma_X^2(T_j)\right]\xrightarrow[n\to\infty]{}0.$$

If $1/4<d<1/2$, Theorem 3.1 ensures that

$$\frac{1}{n^2}\sum_{j=0}^{n-1}(n-j)\mathbb{E}\left[\sigma_X^2(T_j)\right]\sim Cn^{-2+4d}.$$

Therefore, we get in both cases

$$P\left(\left|\frac{1}{n}\sum_{j=1}^{n}Y_j^2-\sigma_X(0)\right|>\varepsilon\right)\xrightarrow[n\to\infty]{}0.$$

We conclude that $S_n \xrightarrow[n\to\infty]{p} \sqrt{\sigma_X(0)}$ and

$$\frac{1}{L(n)^{1/2}n^{1/2+d}}\frac{R_n}{S_n}\xrightarrow[n\to\infty]{d}\mathcal{R}(1):=\sqrt{\frac{\gamma_d}{\sigma_X(0)}}\left(\max_{0\leq t\leq 1}B^0_{\frac{1}{2}+d}(t)-\min_{0\leq t\leq 1}B^0_{\frac{1}{2}+d}(t)\right)$$

. □

In the case $L(t)=c>0$ for all $t>t_0$, taking logarithms of both sizes, we obtain from 4.6 a heuristic identity

$$\log\left(\frac{R_n}{S_n}\right)\sim(1/2+d)\log(n)+\log(\sqrt{c}\mathcal{R}(1))$$

We estimate the slope of the regression line of $(\log(n),\log(R_n/S_n))$ which provides an R/S estimate of d. Remark that for the more general case with L slowly varying at infinity and ultimately non increasing, we have $\log\left(\frac{R_n}{S_n}\right)\sim(1/2+d)\log(n)+\log(L(n))/2+\log(\mathcal{R}(1))$ and $\log(L(n))$ is negligible compared to $\log(n)$.

5 Appendix

To prove Lemma 4.1, we need the following intermediate result:

Lemma 5.1. *If* $\mathbb{E}[T_1] < \infty$ *and* $\mathbf{X}$ *has a regularly varying covariance function*

$$\sigma_X(t) = L(t)t^{-1+2d}$$

with $0 < d < 1/2$ *and* L *slowly varying at infinity and ultimately non-increasing. Then,*

$$\mathrm{Var}(\sigma_X(T_h)) = \circ(L(h)^2 h^{-2+4d}) \qquad \textit{as } h \to \infty \tag{5.1}$$

Proof. By theorem 3.1, we have $\mathbb{E}[\sigma_X(T_h)] \underset{h\to\infty}{\sim} L(h)(h\mathbb{E}[T_1])^{-1+2d}$. To get the result, it is enough to prove that

$$\mathbb{E}[\sigma_X(T_h)^2] \underset{h\to\infty}{\sim} L(h)^2(h\mathbb{E}[T_1])^{-2+4d}.$$

To prove the asymptotic behavior of $\mathbb{E}[\sigma_X(T_h)^2]$, we will follow a similar proof as theorem 3.1:

• Let $0 < c < \mathbb{E}[T_1]$, and $h \in \mathbb{N}$ such that $ch \geq 1$,

$$\mathbb{E}[\sigma_X(T_h)^2] \quad \geq \quad \mathbb{E}\left[\sigma_X(T_h)^2\,\mathbb{I}_{T_h>ch}\right] \geq \mathbb{E}\left[L(T_h)^2 T_h^{-2+4d}\,\mathbb{I}_{T_h>ch}\right] \geq \inf_{t>ch}\{L(t)^2t^{4d}\}\mathbb{E}\left[\frac{\mathbb{I}_{T_h>ch}}{T_h^2}\right]$$

Thanks to Jensen and Hölder inequalities,

$$\mathbb{E}\left[\frac{\mathbb{I}_{T_h>ch}}{T_h^2}\right] \geq \mathbb{E}\left[\frac{\mathbb{I}_{T_h>ch}}{T_h}\right]^2 \text{ and } P(T_h > ch)^2 \leq \mathbb{E}[T_h]\mathbb{E}\left[\frac{\mathbb{I}_{T_h>ch}}{T_h}\right],$$

that is

$$\mathbb{E}\left[\frac{\mathbb{I}_{T_h>ch}}{T_h^2}\right] \geq \frac{P(T_h > ch)^4}{\mathbb{E}[T_h]^2}.$$

Summarizing,

$$\frac{\mathbb{E}[\sigma_X(T_h)^2]}{L(h)^2(h\mathbb{E}[T_1])^{-2+4d}} \geq \frac{\inf_{t>ch}\{L(t)^2t^{4d}\}}{L(h)^2h^{4d}\mathbb{E}[T_1]^{4d}}P(T_h > ch)^4 \tag{5.2}$$

Then, for $c < \mathbb{E}[T_1]$, we have $P(T_h > ch) \to 1$ and $\inf_{t>ch}\{L(t)^2t^{4d}\} \sim L(ch)^2(ch)^{4d}$. Finally, for all $c < \mathbb{E}[T_1]$,

$$\liminf_{h\to\infty} \frac{\mathbb{E}[\sigma_X(T_h)^2]}{L(h)^2(h\mathbb{E}[T_1])^{-2+4d}} \geq \left(\frac{c}{\mathbb{E}[T_1]}\right)^{4d}$$

Taking the limit as $c \to \mathbb{E}[T_1]$, we get

$$\liminf_{h\to\infty} \frac{\mathbb{E}[\sigma_X(T_h)^2]}{L(h)^2(h\mathbb{E}[T_1])^{-2+4d}} \geq 1$$

• Let $\frac{1}{2} < s < \tau < 1$, t_0 such that $L(.)$ is non-increasing and positive on $[t_0, \infty)$ and h such that $\mu_{h,s} - \mu_{h,s}^\tau \geq t_0$, with the same notation as Theorem 3.1,

$$\begin{aligned}
\mathbb{E}[\sigma_X(T_h)^2] &= \mathbb{E}\left[L(T_h)^2T_h^{-2+4d}\,\mathbb{I}_{T_{h,s}\geq\mu_{h,s}-\mu_{h,s}^\tau}\right] + \mathbb{E}\left[\sigma(T_h)^2\,\mathbb{I}_{T_{h,s}<\mu_{h,s}-\mu_{h,s}^\tau}\right] \\
&\leq L(\mu_{h,s} - \mu_{h,s}^\tau)^2\left(\mu_{h,s} - \mu_{h,s}^\tau\right)^{-2+4d} + \sigma_X(0)^2P\left(T_{h,s} < \mu_{h,s} - \mu_{h,s}^\tau\right)
\end{aligned}$$

$$\begin{aligned}
\frac{\mathbb{E}[\sigma_X(T_h)^2]}{L(h)^2(h\mathbb{E}[T_1])^{-2+4d}} \leq \left(\frac{L(\mu_{h,s} - \mu_{h,s}^\tau)}{L(h)}\right)^2\left(\frac{\mu_{h,s} - \mu_{h,s}^\tau}{h\mathbb{E}[T_1]}\right)^{-2+4d} \\
+ \sigma_X(0)^2\frac{P\left(T_{h,s} < \mu_{h,s} - \mu_{h,s}^\tau\right)}{L(h)^2(h\mathbb{E}[T_1])^{-2+4d}}
\end{aligned}$$

Finally

$$\limsup_{h\to\infty} \frac{\mathbb{E}[\sigma_X(T_h)^2]}{L(h)^2(h\mathbb{E}[T_1])^{-2+4d}} \leq 1$$

□

Proof of Lemma 4.1:

Denote

$$W_n = L(n)^{-1} n^{-1-2d} \sum_{i=1}^{n} \sum_{j=1}^{n} \sigma_X(T_j - T_i) = L(n)^{-1} n^{-1-2d} \mathrm{Var}(X_{T_1} + \cdots + X_{T_n} | T_1 \ , \dots , \ T_n).$$

We want to prove that W_n converges in probability to γ_d. To do this, we will show that $\mathbb{E}[W_n] \xrightarrow[n\to\infty]{} \gamma_d$ and $\mathrm{Var}(W_n) \xrightarrow[n\to\infty]{} 0$.

• As $\mathbf{X}$ is a centered process $E[W_n] = L(n)^{-1} n^{-1-2d} \mathrm{Var}(Y_1 + \cdots + Y_n)$. By theorem 3.1, we have

$$\sigma_Y(h) \sim L(h)(h\mathbb{E}[T_1])^{-1+2d} \qquad h \to \infty,$$

then

$$L(n)^{-1} n^{-1-2d} \mathrm{Var}(Y_1 + \cdots + Y_n) \xrightarrow[n\to\infty]{} \gamma_d \tag{5.3}$$

(see for instance Giraitis et al. (2012) Prop 3.3.1 p.43).
and we obtain

$$E[W_n] \xrightarrow[n\to\infty]{} \gamma_d.$$

• Furthermore,

$$\begin{aligned} \mathrm{Var}(W_n) \quad &= L(n)^{-2} n^{-2-4d} \mathrm{Var} \left(\sum_{i=1}^{n} \sum_{j=1}^{n} \sigma_X(T_j - T_i) \right) \\ &\le L(n)^{-2} n^{-2-4d} \left(\sum_{i=1}^{n} \sum_{j=1}^{n} \sqrt{\mathrm{Var}(\sigma_X(T_j - T_i))} \right)^2 \\ &= \left(2n^{-1-2d} L(n)^{-1} \sum_{h=1}^{n} (n-h) \sqrt{\mathrm{Var}(\sigma_X(T_h))} \right)^2 \end{aligned}$$

Then, by Lemma 5.1, $\sqrt{\mathrm{Var}(\sigma_X(T_h))} = \circ(L(h)h^{-1+2d})$ and $2\sum_{h=1}^{n}(n-h)L(h)h^{-1+2d} \sim \dfrac{L(n)n^{1+2d}}{d(1+2d)}$.
We get

$$2\sum_{h=1}^{n} (n-h)\sqrt{\mathrm{Var}(\sigma_X(T_h))} = \circ(L(n)n^{1+2d})$$

Finally, $\mathrm{Var}(W_n) = \circ(1)$ which means that $\mathrm{Var}(W_n) \xrightarrow[n\to\infty]{} 0$. We obtain

$$W_n \xrightarrow[n\to\infty]{L^2, \ p} \gamma_d.$$

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