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A new location-scale model for conditional heavy-tailed distributions
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1. Abstract
We are interested in a location-scale model for heavy-tailed distributions where the covariate is deterministic. We first address the nonparametric estimation of the location and scale functions and derive an estimator of the conditional extreme-value index.

Second, new estimators of the extreme conditional quantiles are introduced. The asymptotic properties of the estimators are established under mild assumptions.

2. Model
\[ Y = a(x) + b(x)Z, \]
where \( x \in \mathbb{R} \) is a nonrandom covariate, \( Y \in \mathbb{R} \) a random variable that depends on \( x \) and \( Z \) an independent random variable of \( x \). The location function \( a(\cdot) \) and the scale function \( b(\cdot) \) are unknown. We assume that the survival function of \( Z \) denoted by \( F_Z \) belongs to the class of regular varying functions at infinity.

\[ F_Z(z) = z^{-1/\gamma}(z), \quad \gamma > 0, \]
where \( \ell \) is a slowly-varying function at infinity and \( \gamma \) is the conditional extreme-value index.

3. Estimators
Let \( \{x_1, y_1\}, \ldots, \{x_n, y_n\} \) be a \( n \)-sample of observations such that \( Y_j = a(x_j) + b(x_j)Z_j \), \( j = 1, \ldots, n \), where \( Z_j \) are independent and identically distributed. For simplicity, the design points are of the form \( x_i = i/n, i = 1, \ldots, n \) and \( x_0 = 0 \) by convention.

**Notations**
- \( \hat{F}_Y(y | x) \): the conditional survival function of \( Y \) given \( x \).
- \( q_\alpha(\cdot) \): the \( \alpha \)-th quantile of \( Z \).
- \( q_\alpha(\cdot | x) \): the conditional \( \alpha \)-th quantile of \( Y \) given \( x \).

**Kernel estimator of \( F_Y(y | x) \)**
\[ \hat{F}_Y(y | x) = \frac{1}{n} \sum_{i=1}^{n} \mathbb{I}(y_{i} - y, y_{i} + y) \]
where the bandwidth \( h \) is a nonrandom sequence such that \( h \to 0 \) as \( n \to \infty \).

**Kernel estimator of \( q_\alpha(\cdot | x) \)**
\[ \hat{q}_\alpha(x | y) = \text{inf}\{ y | \hat{F}_Y(y | x) \leq \alpha \} \]
where \( \hat{F}_Y(\cdot | x) \) is the generalized inverse of \( F_Y(\cdot | x) \).

**Estimators of \( \alpha \) and \( b(\cdot) \)**
Under this assumption : \( \beta_1, \beta_2, \beta_3 \in (0, 1) \), \( \beta_1 > \beta_2 > \beta_3 \) such that \( q_\beta(x) = 0 \) and \( q_\beta(y) - q_\beta(z) = 1 \) we propose as :
- An estimator of \( a(\cdot) \):
  \[ \hat{a}_n(x) = \hat{q}_\beta(x / x \mid \).
- An estimator of \( b(\cdot) \):
  \[ \hat{b}_n(x) = \hat{q}_\beta(x / x \mid | x \mid \).

**Estimator of \( \gamma \)**
- **Pseudo-observations** \( \hat{Z}_i \), associated with \( Z_i \), \( |Z_i| \leq i \leq n - |nh| \) defined by :
  \[ \hat{Z}_i = \frac{y_i - \hat{a}_n(x)}{b_n(x)} \]
- **Number of pseudo-observations** :
  \[ m = \#(i/|nh| \leq i \leq n - |nh|) \]
- **Associated order-estimates** :
  \[ \hat{Z}_{1,m} = \hat{Z}_{1,m} \]
- **Hill-type estimator** of \( \gamma \):
  \[ \gamma_n = \frac{1}{k} \log \frac{\hat{Z}_{1,m}}{\hat{Z}_{1,m} - k} = \log \frac{\hat{Z}_{1,m} - k}{\hat{Z}_{1,m} - k} \]

where \( k \in \{1, \ldots, m\} \) is an intermediate sequence, i.e \( k \to +\infty \) as \( k \to 0 \).

5. Conclusion
The study of the problem in fixed design, which we have made, reveals interesting theoretical results. As a perspective, we are considering the conditional extreme-value index estimation and a validation of our results on simulations. We also plan to extend these results to the random framework.

6. References

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