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Bayesian Nonparametric Priors for Hidden Markov Random Fields: Application to Image Segmentation

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July 2018
Unsupervised image segmentation

Challenges for mixture models (clustering)

inhomogeneities, noise

How many segments?

T1 gado 2 classes 4 classes

Extensions of Dirichlet Process mixture model with spatial regularization
Outline of the talk

1. Dirichlet process (DP)
2. Spatially-constrained mixture model: DP-Potts mixture model
   - Finite mixture model
   - Bayesian finite mixture model
   - DP mixture model
   - DP-Potts mixture model
3. Inference using variational approximation
4. Some image segmentation results
5. Conclusion and future work
Dirichlet process (DP)

The DP is a central Bayesian nonparametric (BNP) prior\(^1\).

**Definition (Dirichlet process)**

A **Dirichlet process** on the space \(\mathcal{Y}\) is a random process \(G\) such that there exist \(\alpha\) (concentration parameter) and \(G_0\) (base distribution) such that for any finite partition \(\{A_1, \ldots, A_p\}\) of \(\mathcal{Y}\), the random vector \((P(A_1), \ldots, P(A_p))\) will be Dirichlet distributed:

\[
(P(A_1), \ldots, P(A_p)) \sim \text{Dir}(\alpha G_0(A_1), \ldots, \alpha G_0(A_p))
\]

Notation: \(G \sim \text{DP}(\alpha, G_0)\)

The DP is the infinite-dimensional generalization of the Dirichlet distribution.

---

### Dirichlet process (DP) construction

A DP prior $G$ can be constructed using three methods:

- The Blackwell-MacQueen urn scheme
- The Chinese Restaurant Process
- The Stick-Breaking construction

\[ G = \sum_{k=1}^{\infty} \pi_k \delta_{\theta_k^*} \]

where $\theta_k^* \overset{iid}{\sim} G_0$ and $\pi_k = \tilde{\pi}_k \prod_{l<k}(1 - \tilde{\pi}_l)$ with $\tilde{\pi}_k \overset{iid}{\sim} \text{Beta}(1, \alpha)$.

---

A DP prior $G$ can be constructed using three methods:

- The Blackwell-MacQueen urn scheme
- The Chinese Restaurant Process
- The Stick-Breaking construction

The DP has almost surely discrete realizations\(^2\):

$$G = \sum_{k=1}^{\infty} \pi_k \delta_{\theta_k^*}$$

where $\theta_k^* \sim G_0$ and $\pi_k = \tilde{\pi}_k \prod_{l<k} (1 - \tilde{\pi}_l)$ with $\tilde{\pi}_k \sim \text{Beta}(1, \alpha)$.

Clustering/segmentation: Finite mixture models assume data are generated by a finite sum of probability distributions:

\[ y = (y_1, \ldots, y_N) \text{ with } y_i = (y_{i1}, \ldots, y_{iD}) \in \mathbb{R}^D \text{ i.i.d} \]

\[ p(y_i | \theta^*, \pi) = \sum_{k=1}^{K} \pi_k F(y_i | \theta_k^*) \]

where

- \( \theta^* = (\theta_1^*, \ldots, \theta_K^*) \) and \( \pi = (\pi_1, \ldots, \pi_K) \) with \( \theta^* \) class parameters and \( \pi \) mixture weights with \( \sum_{i=1}^{K} \pi_i = 1 \). 
- \( \theta^* \) and \( \pi \) can be estimated using EM algorithm.

Equivalently

- \( G = \sum_{k=1}^{K} \pi_k \delta_{\theta_k^*} \) non random
- \( \theta_i \sim G \) and \( y_i | \theta_i \sim F(y_i | \theta_i) \).
Bayesian finite mixture model

In a Bayesian setting, a prior distribution is placed over $\theta^*$ and $\pi$.

Thus, the posterior distribution of parameters given the observations is

$$p(\theta^*, \pi | y) \propto p(y | \theta^*, \pi) p(\theta^*, \pi)$$

To generate a data point within a **Bayesian finite mixture model**:

- $\theta^*_k \sim G_0$
- $\pi_1, ..., \pi_K \sim \text{Dir}(\alpha/K, ..., \alpha/K)$
- $G = \sum_{k=1}^{K} \pi_k \delta_{\theta^*_k}$ is a random measure
- $\theta_i | G \sim G$, which means $\theta_i = \theta^*_k$ with probability $\pi_k$
- $y_i | \theta_i \sim F(y_i | \theta_i)$
In a Bayesian setting, a prior distribution is placed over \( \theta^* \) and \( \pi \).

Thus, the posterior distribution of parameters given the observations is

\[
p(\theta^*, \pi | y) \propto p(y | \theta^*, \pi) p(\theta^*, \pi)
\]

To generate a data point within a **Bayesian finite mixture model**:

- \( \theta_k^* \sim G_0 \)
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- \( \theta_i | G \sim G \), which means \( \theta_i = \theta_k^* \) with probability \( \pi_k \)
- \( y_i | \theta_i \sim F(y_i | \theta_i) \)

**Limitation:**

Require specifying the number of components \( K \) beforehand.

**Solution:**

Assume an infinite number of components using BNP priors.
To establish a DP mixture model, let $G$ be a DP prior \((K \rightarrow \infty)\), namely
\[
G \sim \text{DP}(\alpha, G_0)
\]
and complement it with a likelihood associated to each $\theta_i$

To generate a data point within a **DP mixture model**: 
- $G \sim \text{DP}(\alpha, G_0)$
- $\theta_i | G \sim G$
- $y_i | \theta_i \sim F(y_i | \theta_i)$
DP mixture model

2D point clustering (unsupervised learning) based on the DP mixture model:

Let the data speak for themselves!
DP mixture model

Application to image segmentation:

Drawback:
Spatial constraints and dependencies are not considered.

Solution:
Combine the DP prior with a hidden Markov random field (HMRF).
To solve the issue, we introduce a spatial Potts model component:

\[ M(\theta) \propto \exp \left( \beta \sum_{i \sim j} \delta_{z(\theta_i) = z(\theta_j)} \right) \]

with \( \theta = (\theta_1, \ldots, \theta_N) \) and \( \beta \) the interaction parameter.

The DP mixture model is thus extended:

- \( G \sim \text{DP}(\alpha, G_0) \)
- \( \theta | M, G \sim M(\theta) \times \prod_i G(\theta_i) \)
- \( y_i | \theta_i \sim F(y_i | \theta_i) \)

4-neighbours

8-neighbours
We propose a **DP-Potts mixture model** based on a **general stick-breaking construction** that allows a **natural Full VB algorithm** enabling scalable inference for large datasets and straightforward generalization to other priors.

---

3 Orbanz & Buhmann (2008); Xu, Caron & Doucet (2016); Sodjo, Giremus, Dobigeon & Giovannelli (2017)
4 Albughdadi, Chaari, Tourneret, Forbes, Ciuciu (2017)
5 Chatzis & Tsechpenakis (2010); Chatzis (2013)
DP-Potts: Stick breaking construction

Stick breaking construction of DP: $G \sim DP(\alpha, G_0)$

- $\theta^*_k | G_0 \sim G_0$
- $\tau_k | \alpha \sim \mathcal{B}(1, \alpha), k = 1, 2, \ldots$
- $\pi_k(\tau) = \tau_k \prod_{l=1}^{k-1} (1 - \tau_l), k = 1, 2, \ldots$
- $G = \sum_{k=1}^{\infty} \pi_k(\tau) \delta_{\theta^*_k}$

+ 

- $\theta_i | G \sim G$
- $y_i | \theta_i \sim F(y_i | \theta_i)$

$= \text{Dirichlet Process Mixture Model (DPMM)}$
DP-Potts: Stick breaking construction

Stick breaking construction of DPMM

- $\theta^*_k | G_0 \sim G_0$
- $\tau_k | \alpha \sim \mathcal{B}(1, \alpha), k = 1, 2, \ldots$
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- $G = \sum_{k=1}^{\infty} \pi_k(\tau) \delta_{\theta^*_k}$
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Stick breaking construction of DPMM

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- $\pi_k(\tau) = \tau_k \prod_{l=1}^{k-1} (1 - \tau_l), k = 1, \ldots$
- $\theta_i = \theta^*_k$ with probability $\pi_k(\tau)$
- $y_i | \theta_i \sim F(y_i | \theta_i)$
DP-Potts: Stick breaking construction

Using assignment variables $z_i$

### DPMM view
- $\theta^*_k | G_0 \sim G_0$
- $\tau_k | \alpha \sim \mathcal{B}(1, \alpha), k = 1, 2, \ldots$
- $\pi_k(\tau) = \tau_k \prod_{l=1}^{k-1} (1 - \tau_l), k = 1, \ldots$
- $\theta_i = \theta^*_k$ with probability $\pi_k(\tau)$
- $y_i | \theta_i \sim F(y_i | \theta_i)$

### Mixture/Clustering view
- $\theta^*_k | G_0 \sim G_0$
- $\tau_k | \alpha \sim \mathcal{B}(1, \alpha), k = 1, 2, \ldots$
- $\pi_k(\tau) = \tau_k \prod_{l=1}^{k-1} (1 - \tau_l), k = 1, \ldots$
- $p(z_i = k | \tau) = \pi_k(\tau)$
- with $z_i = z(\theta_i) = k$ when $\theta_i = \theta^*_k$
- $y_i | z_i, \theta^* \sim F(y_i | \theta^*_z_i)$
DP-Potts: Stick breaking construction

Using assignment variables $z_i$

**Stick breaking of DPMM**
- $\theta^*_k | G_0 \sim G_0$
- $\tau_k | \alpha \sim B(1, \alpha), k = 1, 2, \ldots$
- $\pi_k (\tau) = \tau_k \prod_{l=1}^{k-1} (1 - \tau_l)$
- $p(z_i = k | \tau) = \pi_k (\tau)$
- $y_i | z_i, \theta^* \sim F(y_i | \theta^*_{z_i})$

**Stick breaking of DP-Potts**
- $\theta^*_k | G_0 \sim G_0$
- $\tau_k | \alpha \sim B(1, \alpha), k = 1, 2, \ldots$
- $\pi_k (\tau) = \tau_k \prod_{l=1}^{k-1} (1 - \tau_l)$
- $p(z | \tau, \beta) \propto \prod_i \pi_{z_i} (\tau) \exp(\beta \sum_{i \sim j} \delta_{z_i = z_j})$
  - $z = \{z_1, \ldots, z_N\}$
- $y_i | z_i, \theta^* \sim F(y_i | \theta^*_{z_i})$

**NB:** Well defined for every stick breaking construction ($\sum_{k=1}^{\infty} \pi_k = 1$):
- e.g. Pitman-Yor $(\tau_k | \alpha, \sigma) \sim B(1 - \sigma, \alpha + k\sigma)$
Clustering/segmentation task:
- Estimating $Z$
- While parameters $\Theta$ unknown, e.g., $\Theta = \{\tau, \alpha, \theta^*\}$

Bayesian setting
Access the intractable $p(Z, \Theta | y, \Phi)$ approximate as $q(z, \Theta) = q_z(z)q_\theta(\Theta)$

Variational Expectation-Maximization
Alternate maximization in $q_z$ and $q_\theta$ ($\phi$ are hyperparameters) of the Free Energy:

$$F(q_z, q_\theta, \phi) = E_{q_z q_\theta} \left[ \log \frac{p(y, Z, \Theta | \phi)}{q_z(z)q_\theta(\Theta)} \right]$$

$$= \log p(y | \phi) - KL(q_z q_\theta, p(Z, \Theta | y, \phi))$$
Joint DP-Potts (Gaussian) Mixture distribution

\[
p(y, z, \tau, \alpha, \theta^* | \phi) = \prod_{j=1}^{N} p(y_j | z_j, \theta^*) \ p(z | \tau, \beta) \ \prod_{k=1}^{\infty} p(\tau_k | \alpha) \ \prod_{k=1}^{\infty} p(\theta^*_k | \rho_k) \ p(\alpha | s_1, s_2)
\]

- \( p(y_j | z_j, \theta^*) = \mathcal{N}(y_j | \mu_{z_j}, \Sigma_{z_j}) \) is Gaussian
- \( p(z | \tau, \beta) \) is a DP-Potts model
- \( p(\tau_k | \alpha) \) is Beta \( B(1, \alpha) \)
- \( p(\theta^*_k | \rho_k) = \mathcal{NIW}(\mu_k, \Sigma_k | m_k, \lambda_k, \Psi_k, \nu_k) \) is Normal-inverse-Wishart
- \( p(\alpha | s_1, s_2) = \mathcal{G}(\alpha | s_1, s_2) \) is Gamma

Usual truncated variational posterior, \( q_{\tau_k} = \delta_1 \) for \( k \geq K \) (eg. \( K = 40 \))

\[
q(z, \Theta) = \prod_{j=1}^{N} q_{z_j}(z_j) \ q_\alpha(\alpha) \ \prod_{k=1}^{K-1} q_{\tau_k}(\tau_k) \ \prod_{k=1}^{K} q_{\theta^*_k}(\mu_k, \Sigma_k)
\]

- E-steps: VE-Z, VE-\( \alpha \), VE-\( \tau \) and VE-\( \theta^* \)
- M-step: \( \phi \) updating straightforward except for \( \beta \)
Some image segmentation results

Model validation and verification:

Segmented image using DP-Potts model with $\beta = 2.5$. 
Some image segmentation results

Convergence of the VB algorithm initialized by the k-means++ clustering:
Some image segmentation results

Segmentation results for Berkeley Segmentation Dataset:

The segmentation results obtained by DP-Potts model with $\beta = 0, 1, 5$. 
Some image segmentation results

Segmentation with estimated $\beta = 1.66$
Quantitative evaluation of the segmentations

**Probabilistic Rand Index** on 154 color (RGB) images with ground truth (several) from Berkeley dataset (1000 superpixels). But Manual ground truth segmentations are subjective!

PRI results with DP-Potts model

<table>
<thead>
<tr>
<th>K</th>
<th>Mean</th>
<th>Median</th>
<th>St.D.</th>
</tr>
</thead>
<tbody>
<tr>
<td>10</td>
<td>71.48</td>
<td>72.54</td>
<td>0.1040</td>
</tr>
<tr>
<td>20</td>
<td>73.64</td>
<td>73.42</td>
<td>0.0935</td>
</tr>
<tr>
<td>40</td>
<td>75.33</td>
<td>76.47</td>
<td>0.0853</td>
</tr>
<tr>
<td>50</td>
<td>75.81</td>
<td>76.31</td>
<td>0.0873</td>
</tr>
<tr>
<td>60</td>
<td>76.55</td>
<td>77.12</td>
<td>0.0848</td>
</tr>
<tr>
<td>80</td>
<td><strong>77.06</strong></td>
<td><strong>78.30</strong></td>
<td><strong>0.0835</strong></td>
</tr>
</tbody>
</table>

PRI results from Chatzis 2013

<table>
<thead>
<tr>
<th>PRI (%)</th>
<th>DPM</th>
<th>iHMRF</th>
<th>MRF-PYP</th>
<th>GC</th>
</tr>
</thead>
<tbody>
<tr>
<td>Mean</td>
<td>74.15</td>
<td>75.50</td>
<td>76.49</td>
<td>76.10</td>
</tr>
<tr>
<td>Median</td>
<td>75.49</td>
<td>76.89</td>
<td>78.08</td>
<td>77.59</td>
</tr>
<tr>
<td>St.D.</td>
<td>0.084</td>
<td>0.082</td>
<td>0.079</td>
<td>0.083</td>
</tr>
</tbody>
</table>

**Computation time**: Berkeley 321x481 image reduced to 1000 superpixels takes **10-30 s** on a PC with CPU Intel(R) Core(TM) i7-5500U CPU 2.40GHz and 8GB RAM
A general DP-Potts model and the associated VB algorithm were built. The DP-Potts model was applied to image segmentation and tested on different types of datasets. Impact of the interaction parameter $\beta$ on the final results is significant. An estimation procedure was proposed for $\beta$. Extend the model with other priors (Pitman-Y or process, Gibbs-type priors, etc.). Try other variational approximations (truncation-free). Investigate theoretical properties of BNP priors under structural constraints (time, spatial).... for other applications, such as discovery probabilities, etc.
A general DP-Potts model and the associated VB algorithm were built. The DP-Potts model was applied to image segmentation and tested on different types of datasets. Impact of the interaction parameter $\beta$ on the final results is significant. An estimation procedure was proposed for $\beta$.

- How does $\beta$ influence the number of components?
- Extend the model with other priors (Pitman-Yor process, Gibbs-type priors, etc.).
- Try other variational approximations (truncation-free).
- Investigate theoretical properties of BNP priors under structural constraints (time, spatial) ....
- ... for other applications, such as discovery probabilities, etc.
References


Thank you for your attention!

contact: florence.forbes@inria.fr
Université Grenoble Alpes invites applications for a 2-year junior research chair (post-doc) in Data Science for Life Sciences and Health

- Starting in October 2018
- Data science methodology and machine learning to Life Sciences and Health
- Application deadline: August, 31 2018
- Website: https://data-institute.univ-grenoble-alpes.fr/
- Contact: florence.forbes@inria.fr
Stick breaking construction

DP simulations with $G_0$ being a standard normal distribution $\mathcal{N}(0, 1)$ and $\alpha = 1, 10$ using the Stick-Breaking representation.
Variational EM

General formulation, at iteration \((r)\)

\[
\text{E-Z} \quad q_z^{(r)}(z) \propto \exp \left( E_{q_\theta^{(r-1)}} [\log p(y, z, \Theta|\phi^{(r-1)})] \right)
\]

\[
\text{E-\Theta} \quad q_\theta^{(r)}(\Theta) \propto \exp \left( E_{q_z^{(r)}} [\log p(y, Z, \Theta|\phi^{(r-1)})] \right)
\]

\[
\text{M-\phi} \quad \phi^{(r)} = \arg \max_\phi E_{q_z^{(r)} q_\theta^{(r)}} [\log p(y, Z, \Theta|\phi)]
\]

VE-Z, VE-\(\alpha\), VE-\(\tau\), and VE-\(\theta^*\)

e.g. VE-Z step divides into \(N\) VE-\(Z_j\) steps \((q_{z_j}(z_j) = 0 \text{ for } z_j > K)\)

\[
q_{z_j}(z_j) \propto \exp \left( E_{q_{\theta_{z_j}^*}} [\log p(y_j|\theta_{z_j}^*)] + E_{q_\tau} [\log \pi_{z_j}(\tau)] + \beta \sum_{i \sim j} q_{z_i}(z_j) \right)
\]
**Estimation of $\beta$**

M-$\beta$ step: involves $p(z|\tau, \beta) = K(\beta, \tau)^{-1} \exp(V(z; \tau, \beta))$

with $V(z; \tau, \beta) = \sum_i \log \pi_{z_i}(\tau) + \beta \sum_{i \sim j} \delta(z_i = z_j)$

$$\hat{\beta} = \arg \max_{\beta} E_{q_{z|\tau}} \left[ \log p(z|\tau; \beta) \right]$$

$$= \arg \max_{\beta} E_{q_{z|\tau}} \left[ V(z; \tau, \beta) \right] - E_{q_{\tau}} \left[ \log K(\beta, \tau) \right]$$

**Two difficulties**

1. $p(z|\tau, \beta)$ is intractable (normalizing constant $K(\beta, \tau)$, typical of MRF)
2. it depends on $\tau$ (typical of DP)

**Two approximations**

1. "standard" Mean Field like approximation
2. Replace the random $\tau$ by a fixed $\tilde{\tau} = E_{q_{\tau}}[\tau]

---

\(^{a}\)Forbes & Peyrard 2003
Approximation of $p(z|\tau; \beta)$

\[
p(z|\tau, \beta) \approx \tilde{q}_z(z|\beta) = \prod_{j=1}^{N} \tilde{q}_{z_j}(z_j|\beta)
\]

\[
\tilde{q}_{z_j}(z_j = k|\beta) = \frac{\exp(\log \pi_k(\tilde{\tau}) + \beta \sum_{i \in N(j)} q_{zi}(k))}{\sum_{l=1}^{\infty} \exp(\log \pi_l(\tilde{\tau}) + \beta \sum_{i \in N(j)} q_{zi}(l))}
\]

\[
\text{and} \quad \tilde{\tau} = \mathbb{E}_{q_\tau}[\tau]
\]

$\beta$ is estimated at each iteration by setting the approximate gradient to 0

\[
\mathbb{E}_{q_\tau q_\tau}[\nabla_\beta V(z; \tau, \beta)] = \sum_{k=1}^{K} \sum_{i \sim j} q_{z_j}(k) q_{zi}(k)
\]

\[
\nabla_\beta \mathbb{E}_{q_\tau}[\log K(\beta, \tau)] = \mathbb{E}_{p(z|\tau, \beta)q_\tau}[\nabla_\beta V(z; \tau, \beta)] \approx \sum_{k=1}^{K} \sum_{i \sim j} \tilde{q}_{z_j}(k|\beta) \tilde{q}_{zi}(k|\beta)
\]
Some image segmentation results

Segmentation results for medical images: all hyperparameters fixed

The segmentation results obtained by DP-Potts model with $\beta = 0, 1, 5$. 
Some image segmentation results

Segmentation with estimated hyperparameters ($\beta = 0.75$)
Some image segmentation results

Segmentation with estimated $\beta = 0.96$ (pixels with partial volume)
Some image segmentation results

Segmentation results for SAR images:

The segmentation results obtained by DP-Potts model with $\beta = 0, 1, 5$. 
Some image segmentation results

Segmentation results with estimated $\beta$

$\beta = 1.11$

$\beta = 1.02$