Non parametric Bayesian priors for hidden Markov random fields
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Bayesian Nonparametric Priors for Hidden Markov Random Fields: Application to Image Segmentation

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July 2018
Unsupervised image segmentation

Challenges for mixture models (clustering)

inhomogeneities, noise

How many segments?

- T1 gado
- 2 classes
- 4 classes

Extensions of Dirichlet Process mixture model with spatial regularization
Outline of the talk

1. Dirichlet process (DP)
2. Spatially-constrained mixture model: DP-Potts mixture model
   - Finite mixture model
   - Bayesian finite mixture model
   - DP mixture model
   - DP-Potts mixture model
3. Inference using variational approximation
4. Some image segmentation results
5. Conclusion and future work
The DP is a central Bayesian nonparametric (BNP) prior\(^1\).

### Definition (Dirichlet process)

A **Dirichlet process** on the space \( \mathcal{Y} \) is a **random process** \( G \) such that there exist \( \alpha \) (concentration parameter) and \( G_0 \) (base distribution) such that for any finite partition \( \{A_1, \ldots, A_p\} \) of \( \mathcal{Y} \), the random vector \( (P(A_1), \ldots, P(A_p)) \) will be Dirichlet distributed:

\[
(P(A_1), \ldots, P(A_p)) \sim \text{Dir}(\alpha G_0(A_1), \ldots, \alpha G_0(A_p))
\]

Notation: \( G \sim \text{DP}(\alpha, G_0) \)

The DP is the infinite-dimensional generalization of the Dirichlet distribution.

---

Dirichlet process (DP) construction

A DP prior $G$ can be constructed using three methods:

- The Blackwell-MacQueen urn scheme
- The Chinese Restaurant Process
- The Stick-Breaking construction

$G = \sum_{k=1}^{\infty} \pi_k \delta_{\theta^*_k}$

where $\theta^*_k iid \sim G_0$ and $\pi_k = \tilde{\pi}_k \prod_{l<k} (1 - \tilde{\pi}_l)$ with $\tilde{\pi}_k iid \sim \text{Beta}(1, \alpha)$.

--

A DP prior $G$ can be constructed using three methods:

- The Blackwell-MacQueen urn scheme
- The Chinese Restaurant Process
- The Stick-Breaking construction

The DP has almost surely discrete realizations$^2$:

$$G = \sum_{k=1}^{\infty} \pi_k \delta_{\theta^*_k}$$

where $\theta^*_k \overset{iid}{\sim} G_0$ and $\pi_k = \tilde{\pi}_k \prod_{l<k} (1 - \tilde{\pi}_l)$ with $\tilde{\pi}_k \overset{iid}{\sim} \text{Beta}(1, \alpha)$.

---

Spatially-constrained mixture model: DP-Potts mixture model

Clustering/segmentation: Finite mixture models assume data are generated by a finite sum of probability distributions:

\[ \mathbf{y} = (\mathbf{y}_1, \ldots, \mathbf{y}_N) \text{ with } \mathbf{y}_i = (y_{i1}, \ldots, y_{iD}) \in \mathbb{R}^D \text{ i.i.d} \]

\[ p(\mathbf{y}_i|\theta^*, \pi) = \sum_{k=1}^{K} \pi_k F(\mathbf{y}_i|\theta_k^*) \]

where

- \( \theta^* = (\theta_1^*, \ldots, \theta_K^*) \) and \( \pi = (\pi_1, \ldots, \pi_K) \) with \( \theta^* \) class parameters and \( \pi \) mixture weights with \( \sum_{i=1}^{K} \pi_i = 1 \).
- \( \theta^* \) and \( \pi \) can be estimated using EM algorithm.

Equivalently

- \( G = \sum_{k=1}^{K} \pi_k \delta_{\theta_k^*} \) non random
- \( \theta_i \sim G \) and \( \mathbf{y}_i|\theta_i \sim F(\mathbf{y}_i|\theta_i) \).
Bayesian finite mixture model

In a Bayesian setting, a prior distribution is placed over $\theta^*$ and $\pi$.

Thus, the posterior distribution of parameters given the observations is

$$p(\theta^*, \pi|y) \propto p(y|\theta^*, \pi)p(\theta^*, \pi)$$

To generate a data point within a Bayesian finite mixture model:

- $\theta^*_k \sim G_0$
- $\pi_1, ..., \pi_K \sim \text{Dir}(\alpha/K, ..., \alpha/K)$
- $G = \sum_{k=1}^{K} \pi_k \delta_{\theta^*_k}$ is a random measure
- $\theta_i|G \sim G$, which means $\theta_i = \theta^*_k$ with probability $\pi_k$
- $y_i|\theta_i \sim F(y_i|\theta_i)$

Limitation: Require specifying the number of components $K$ beforehand.

Solution: Assume an infinite number of components using BNP priors.
In a Bayesian setting, a prior distribution is placed over $\theta^*$ and $\pi$.

Thus, the posterior distribution of parameters given the observations is:

$$p(\theta^*, \pi | y) \propto p(y | \theta^*, \pi) p(\theta^*, \pi)$$

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- $\theta_i | G \sim G$, which means $\theta_i = \theta^*_k$ with probability $\pi_k$
- $y_i | \theta_i \sim F(y_i | \theta_i)$

**Limitation:**

Require specifying the number of components $K$ beforehand.

**Solution:**

Assume an infinite number of components using BNP priors.
To establish a DP mixture model, let $G$ be a DP prior ($K \to \infty$), namely

$$G \sim \text{DP}(\alpha, G_0)$$

and complement it with a likelihood associated to each $\theta_i$

To generate a data point within a DP mixture model:

- $G \sim \text{DP}(\alpha, G_0)$
- $\theta_i | G \sim G$
- $y_i | \theta_i \sim F(y_i | \theta_i)$
2D point clustering (unsupervised learning) based on the DP mixture model:

Let the data speak for themselves!
DP mixture model

Application to image segmentation:

Drawback:
Spatial constraints and dependencies are not considered.

Solution:
Combine the DP prior with a hidden Markov random field (HMRF).
To solve the issue, we introduce a spatial Potts model component:

\[
M(\theta) \propto \exp \left( \beta \sum_{i \sim j} \delta_{z(\theta_i) = z(\theta_j)} \right)
\]

with \( \theta = (\theta_1, ..., \theta_N) \) and \( \beta \) the interaction parameter.

The DP mixture model is thus extended:

- \( G \sim \text{DP}(\alpha, G_0) \)
- \( \theta | M, G \sim M(\theta) \times \prod_i G(\theta_i) \)
- \( y_i | \theta_i \sim F(y_i | \theta_i) \)
DP-Potts mixture model

Other spatially-constrained BNP mixture models + inference algorithms:
- DP or PYP-Potts partition model + MCMC
- Hemodynamic brain parcellation (DP-Potts) + PARTIAL VB
- DP or PYP-Potts + Iterated Conditional Mode (ICM)

Markov chain Monte Carlo (MCMC):
- Advantage: asymptotically exact
- Drawback: computationally expensive

Variational Bayes (VB):
- Advantage: much faster
- Drawback: less accurate, no theoretical guarantee

We propose a DP-Potts mixture model based on a general stick-breaking construction that allows a natural Full VB algorithm enabling scalable inference for large datasets and straightforward generalization to other priors.

---

3 Orbanz & Buhmann (2008); Xu, Caron & Doucet (2016); Sodjo, Giremus, Dobigeon & Giovannelli (2017)
4 Albughdadi, Chaari, Tourneret, Forbes, Ciuciu (2017)
5 Chatzis & Tsechpenakis (2010); Chatzis (2013)
DP-Potts: Stick breaking construction

Stick breaking construction of DP: $G \sim DP(\alpha, G_0)$

- $\theta^*_k | G_0 \sim G_0$
- $\tau_k | \alpha \sim B(1, \alpha), k = 1, 2, \ldots$
- $\pi_k(\tau) = \tau_k \prod_{l=1}^{k-1} (1 - \tau_l), k = 1, 2, \ldots$
- $G = \sum_{k=1}^{\infty} \pi_k(\tau) \delta_{\theta^*_k}$

$\theta_i | G \sim G$

$y_i | \theta_i \sim F(y_i | \theta_i)$

= Dirichlet Process Mixture Model (DPMM)
DP-Potts: Stick breaking construction

**Stick breaking construction of DPMM**

- $\theta_k^* | G_0 \sim G_0$
- $\tau_k | \alpha \sim \mathcal{B}(1, \alpha), k = 1, 2, \ldots$
- $\pi_k(\tau) = \tau_k \prod_{l=1}^{k-1} (1 - \tau_l), k = 1, \ldots$
- $G = \sum_{k=1}^{\infty} \pi_k(\tau) \delta_{\theta_k^*}$
- $\theta_i | G \sim G$
- $y_i | \theta_i \sim F(y_i | \theta_i)$

---

**Stick breaking construction of DPMM**

- $\theta_k^* | G_0 \sim G_0$
- $\tau_k | \alpha \sim \mathcal{B}(1, \alpha), k = 1, 2, \ldots$
- $\pi_k(\tau) = \tau_k \prod_{l=1}^{k-1} (1 - \tau_l), k = 1, \ldots$
- $\theta_i = \theta_k^*$ with probability $\pi_k(\tau)$
- $y_i | \theta_i \sim F(y_i | \theta_i)$
DP-Potts: Stick breaking construction

Using assignment variables $z_i$

**DPMM view**
- $\theta^*_k | G_0 \sim G_0$
- $\tau_k | \alpha \sim B(1, \alpha), k = 1, 2, \ldots$
- $\pi_k(\tau) = \tau_k \prod_{l=1}^{k-1} (1 - \tau_l), k = 1, \ldots$
- $\theta_i = \theta^*_k$ with probability $\pi_k(\tau)$
- $y_i | \theta_i \sim F(y_i | \theta_i)$

**Mixture/Clustering view**
- $\theta^*_k | G_0 \sim G_0$
- $\tau_k | \alpha \sim B(1, \alpha), k = 1, 2, \ldots$
- $\pi_k(\tau) = \tau_k \prod_{l=1}^{k-1} (1 - \tau_l), k = 1, \ldots$
- $p(z_i = k | \tau) = \pi_k(\tau)$
- with $z_i = z(\theta_i) = k$ when $\theta_i = \theta^*_k$
- $y_i | z_i, \theta^* \sim F(y_i | \theta^*_z)$
DP-Potts: Stick breaking construction

Using assignment variables $z_i$

**Stick breaking of DPMM**
- $\theta^*_k | G_0 \sim G_0$
- $\tau_k | \alpha \sim B(1, \alpha), k = 1, 2, \ldots$
- $\pi_k(\tau) = \tau_k \prod_{l=1}^{k-1} (1 - \tau_l)$
- $p(z_i = k | \tau) = \pi_k(\tau)$
- $y_i | z_i, \theta^* \sim F(y_i | \theta^*_z_i)$

**Stick breaking of DP-Potts**
- $\theta^*_k | G_0 \sim G_0$
- $\tau_k | \alpha \sim B(1, \alpha), k = 1, 2, \ldots$
- $\pi_k(\tau) = \tau_k \prod_{l=1}^{k-1} (1 - \tau_l)$
- $p(z | \tau, \beta) \propto \prod_{i} \pi_{z_i}(\tau) \exp(\beta \sum_{i \sim j} \delta_{z_i=z_j})$
- $z = \{z_1, \ldots, z_N\}$
- $y_i | z_i, \theta^* \sim F(y_i | \theta^*_z_i)$

**NB:** Well defined for every stick breaking construction ($\sum_{k=1}^{\infty} \pi_k = 1$):
- e.g. Pitman-Yor ($\tau_k | \alpha, \sigma \sim B(1 - \sigma, \alpha + k\sigma)$
Clustering/segmentation task:
- Estimating $Z$
- While parameters $\Theta$ unknown, eg. $\Theta = \{\tau, \alpha, \theta^*\}$

Bayesian setting
Access the intractable $p(Z, \Theta | y, \Phi)$ approximate as $q(z, \Theta) = q_z(z) q_\theta(\Theta)$

Variational Expectation-Maximization
Alternate maximization in $q_z$ and $q_\theta$ ($\phi$ are hyperparameters) of the Free Energy:

$$
\mathcal{F}(q_z, q_\theta, \phi) = E_{q_z q_\theta} \left[ \log \frac{p(y, Z, \Theta | \phi)}{q_z(z) q_\theta(\Theta)} \right] \\
= \log p(y | \phi) - KL(q_z q_\theta, p(Z, \Theta | y, \phi))
$$
Joint DP-Potts (Gaussian) Mixture distribution

\[ p(y, z, \tau, \alpha, \theta^* | \phi) = \prod_{j=1}^{N} p(y_j | z_j, \theta^*) \cdot p(z | \tau, \beta) \prod_{k=1}^{\infty} p(\tau_k | \alpha) \prod_{k=1}^{\infty} p(\theta^*_k | \rho_k) \cdot p(\alpha | s_1, s_2) \]

- \( p(y_j | z_j, \theta^*) = \mathcal{N}(y_j | \mu_{z_j}, \Sigma_{z_j}) \) is Gaussian
- \( p(z | \tau, \beta) \) is a DP-Potts model
- \( p(\tau_k | \alpha) \) is Beta \( \mathcal{B}(1, \alpha) \)
- \( p(\theta^*_k | \rho_k) = \mathcal{NIW}(\mu_k, \Sigma_k | m_k, \lambda_k, \Psi_k, \nu_k) \) is Normal-inverse-Wishart
- \( p(\alpha | s_1, s_2) = \mathcal{G}(\alpha | s_1, s_2) \) is Gamma

Usual truncated variational posterior, \( q_{\tau_k} = \delta_1 \) for \( k \geq K \) (eg. \( K = 40 \))

\[ q(z, \Theta) = \prod_{j=1}^{N} q_{z_j}(z_j) \cdot q_{\alpha}(\alpha) \prod_{k=1}^{K-1} q_{\tau_k}(\tau_k) \prod_{k=1}^{K} q_{\theta^*_k}(\mu_k, \Sigma_k) \]

- E-steps: VE-Z, VE-\( \alpha \), VE-\( \tau \) and VE-\( \theta^* \)
- M-step: \( \phi \) updating straightforward except for \( \beta \)
Some image segmentation results

Model validation and verification:

Segmented image using DP-Potts model with $\beta = 2.5$. 
Some image segmentation results

Convergence of the VB algorithm initialized by the k-means++ clustering:

![Segmentation by DP-Potts (iteration=01)](image-url)
Some image segmentation results

Segmentation results for Berkeley Segmentation Dataset:

The segmentation results obtained by DP-Potts model with $\beta = 0, 1, 5$. 

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Some image segmentation results

Segmentation results for Berkeley Segmentation Dataset:

The segmentation results obtained by DP-Potts model with $\beta = 0, 1, 5$. 

---
Some image segmentation results

Segmentation with estimated $\beta = 1.66$
Quantitative evaluation of the segmentations

Probabilistic Rand Index on 154 color (RGB) images with ground truth (several) from Berkeley dataset (1000 superpixels). But Manual ground truth segmentations are subjective!

PRI results with DP-Potts model

<table>
<thead>
<tr>
<th>K</th>
<th>Mean</th>
<th>Median</th>
<th>St.D.</th>
</tr>
</thead>
<tbody>
<tr>
<td>10</td>
<td>71.48</td>
<td>72.54</td>
<td>0.1040</td>
</tr>
<tr>
<td>20</td>
<td>73.64</td>
<td>73.42</td>
<td>0.0935</td>
</tr>
<tr>
<td>40</td>
<td>75.33</td>
<td>76.47</td>
<td>0.0853</td>
</tr>
<tr>
<td>50</td>
<td>75.81</td>
<td>76.31</td>
<td>0.0873</td>
</tr>
<tr>
<td>60</td>
<td>76.55</td>
<td>77.12</td>
<td>0.0848</td>
</tr>
<tr>
<td>80</td>
<td>77.06</td>
<td>78.30</td>
<td>0.0835</td>
</tr>
</tbody>
</table>

PRI results from Chatzis 2013

<table>
<thead>
<tr>
<th>PRI (%)</th>
<th>DPM</th>
<th>iHMRF</th>
<th>MRF-PYP</th>
<th>GC</th>
</tr>
</thead>
<tbody>
<tr>
<td>Mean</td>
<td>74.15</td>
<td>75.50</td>
<td>76.49</td>
<td>76.10</td>
</tr>
<tr>
<td>Median</td>
<td>75.49</td>
<td>76.89</td>
<td>78.08</td>
<td>77.59</td>
</tr>
<tr>
<td>St.D.</td>
<td>0.084</td>
<td>0.082</td>
<td>0.079</td>
<td>0.083</td>
</tr>
</tbody>
</table>

Computation time: Berkeley 321x481 image reduced to 1000 superpixels takes **10-30 s** on a PC with CPU Intel(R) Core(TM) i7-5500U CPU 2.40GHz and 8GB RAM
A general DP-Potts model and the associated VB algorithm were built. The DP-Potts model was applied to image segmentation and tested on different types of datasets. Impact of the interaction parameter $\beta$ on the final results is significant. An estimation procedure was proposed for $\beta$. Extend the model with other priors (Pitman-Y or process, Gibbs-type priors, etc.). Try other variational approximations (truncation-free). Investigate theoretical properties of BNP priors under structural constraints (time, spatial) for other applications, such as discovery probabilities, etc.
Conclusion and future work

- A general DP-Potts model and the associated VB algorithm were built.
- The DP-Potts model was applied to image segmentation and tested on different types of datasets.
- Impact of the interaction parameter $\beta$ on the final results is significant.
- An estimation procedure was proposed for $\beta$.

- How does $\beta$ influence the number of components?
- Extend the model with other priors (Pitman-Yor process, Gibbs-type priors, etc.).
- Try other variational approximations (truncation-free).
- Investigate theoretical properties of BNP priors under structural constraints (time, spatial) ....
- ... for other applications, such as discovery probabilities, etc.


Thank you for your attention!

contact: florence.forbes@inria.fr
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- Contact: florence.forbes@inria.fr
Stick breaking construction

DP simulations with $G_0$ being a standard normal distribution $\mathcal{N}(0, 1)$ and $\alpha = 1, 10$ using the Stick-Breaking representation.
Variational EM

General formulation, at iteration \( (r) \)

\[
\begin{align*}
E-Z & \quad q^{(r)}_z(z) \propto \exp \left( E_{q^{(r-1)}_\theta} \left[ \log p(y, z, \Theta | \phi^{(r-1)}) \right] \right) \\
E-\Theta & \quad q^{(r)}_\theta(\Theta) \propto \exp \left( E_{q^{(r)}_z} \left[ \log p(y, Z, \Theta | \phi^{(r-1)}) \right] \right) \\
M-\phi & \quad \phi^{(r)} = \arg \max_\phi \ E_{q^{(r)}_z q^{(r)}_\theta} \left[ \log p(y, Z, \Theta | \phi) \right]
\end{align*}
\]

VE-Z, VE-\( \alpha \), VE-\( \tau \), and VE-\( \theta^* \)

e.g. VE-Z step divides into \( N \) VE-Z\( j \) steps \( (q_{z_j}(z_j) = 0 \text{ for } z_j > K) \)

\[
q_{z_j}(z_j) \propto \exp \left( E_{q^{(r)}_{\theta^*}} \left[ \log p(y_j | \theta_{z_j}^*) \right] + E_{q_{\tau}} \left[ \log \pi_{z_j}(\tau) \right] + \beta \sum_{i \sim j} q_{z_i}(z_j) \right)
\]
Estimation of $\beta$

M-$\beta$ step: involves $p(z|\tau, \beta) = K(\beta, \tau)^{-1} \exp(V(z; \tau, \beta))$

with $V(z; \tau, \beta) = \sum_i \log \pi_{zi}(\tau) + \beta \sum_{i \sim j} \delta(z_i = z_j)$

$\hat{\beta} = \arg \max_{\beta} E_{q_z q_\tau} [\log p(z|\tau; \beta)]$

$= \arg \max_{\beta} E_{q_z q_\tau} [V(z; \tau, \beta)] - E_{q_\tau} [\log K(\beta, \tau)]$

Two difficulties

1. $p(z|\tau, \beta)$ is intractable (normalizing constant $K(\beta, \tau)$, typical of MRF)
2. it depends on $\tau$ (typical of DP)

Two approximations

1. "standard" Mean Field like approximation$^a$
2. Replace the random $\tau$ by a fixed $\tilde{\tau} = E_{q_\tau} [\tau]$

$^a$Forbes & Peyrard 2003
Approximation of $p(z|\tau; \beta)$

$$
p(z|\tau, \beta) \approx \tilde{q}_z(z|\beta) = \prod_{j=1}^{N} \tilde{q}_{z_j}(z_j|\beta)
$$

$$
\tilde{q}_{z_j}(z_j = k|\beta) = \frac{\exp(\log \pi_k(\tilde{\tau}) + \beta \sum_{i \in N(j)} q_{zi}(k))}{\sum_{l=1}^{\infty} \exp(\log \pi_l(\tilde{\tau}) + \beta \sum_{i \in N(j)} q_{zi}(l))}
$$

and

$$
\tilde{\tau} = E_{q_{\tau}}[\tau]
$$

$\beta$ is estimated at each iteration by setting the approximate gradient to 0

$$
E_{qz q_{\tau}}[\nabla_\beta V(z; \tau, \beta)] = \sum_{k=1}^{K} \sum_{i \sim j} q_{z_j}(k) q_{zi}(k)
$$

$$
\nabla_\beta E_{q_{\tau}}[\log K(\beta, \tau)] = E_{p(z|\tau, \beta)q_{\tau}}[\nabla_\beta V(z; \tau, \beta)] \approx \sum_{k=1}^{K} \sum_{i \sim j} \tilde{q}_{z_j}(k|\beta) \tilde{q}_{zi}(k|\beta)
$$
Some image segmentation results

Segmentation results for medical images: all hyperparameters fixed

The segmentation results obtained by DP-Potts model with $\beta = 0, 1, 5$. 
Some image segmentation results

Segmentation with estimated hyperparameters ($\beta = 0.75$)
Some image segmentation results

Segmentation with estimated $\beta = 0.96$ (pixels with partial volume)
The segmentation results obtained by DP-Potts model with $\beta = 0, 1, 5$. 

Segmentation results for SAR images:
Some image segmentation results

Segmentation results with estimated $\beta$

$\beta = 1.11$

$\beta = 1.02$