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Bayesian Nonparametric Priors for Hidden Markov Random Fields: Application to Image Segmentation

Hongliang Lü, Julyan Arbel, Florence Forbes

Inria Grenoble Rhône-Alpes and University Grenoble Alpes
Laboratoire Jean Kuntzmann
Mistis team
florence.forbes@inria.fr

July 2018
Unsupervised image segmentation

Challenges for mixture models (clustering)

inhomogeneities, noise

How many segments?

T1 gado 2 classes 4 classes

Extensions of Dirichlet Process mixture model with spatial regularization
Outline of the talk

1. Dirichlet process (DP)
2. Spatially-constrained mixture model: DP-Potts mixture model
   - Finite mixture model
   - Bayesian finite mixture model
   - DP mixture model
   - DP-Potts mixture model
3. Inference using variational approximation
4. Some image segmentation results
5. Conclusion and future work
The DP is a central Bayesian nonparametric (BNP) prior\(^1\).

**Definition (Dirichlet process)**

A **Dirichlet process** on the space \( \mathcal{Y} \) is a random process \( G \) such that there exist \( \alpha \) (concentration parameter) and \( G_0 \) (base distribution) such that for any finite partition \( \{A_1, \ldots, A_p\} \) of \( \mathcal{Y} \), the random vector \( (P(A_1), \ldots, P(A_p)) \) will be Dirichlet distributed:

\[
(P(A_1), \ldots, P(A_p)) \sim \text{Dir}(\alpha G_0(A_1), \ldots, \alpha G_0(A_p))
\]

Notation: \( G \sim \text{DP}(\alpha, G_0) \)

The DP is the infinite-dimensional generalization of the Dirichlet distribution.

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A DP prior $G$ can be constructed using three methods:

- The Blackwell-MacQueen urn scheme
- The Chinese Restaurant Process
- The Stick-Breaking construction

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The Dirichlet process (DP) construction

A DP prior $G$ can be constructed using three methods:

- The Blackwell-MacQueen urn scheme
- The Chinese Restaurant Process
- The Stick-Breaking construction

The DP has almost surely discrete realizations:

$$G = \sum_{k=1}^{\infty} \pi_k \delta_{\theta_k^*}$$

where $\theta_k^* \overset{iid}{\sim} G_0$ and $\pi_k = \tilde{\pi}_k \prod_{l<k} (1 - \tilde{\pi}_l)$ with $\tilde{\pi}_k \overset{iid}{\sim} Beta(1, \alpha)$.

---

Spatially-constrained mixture model: DP-Potts mixture model

Clustering/segmentation: Finite mixture models assume data are generated by a finite sum of probability distributions:

\[ y = (y_1, \ldots, y_N) \text{ with } y_i = (y_{i1}, \ldots, y_{iD}) \in \mathbb{R}^D \text{ i.i.d} \]

\[ p(y_i|\theta^*, \pi) = \sum_{k=1}^{K} \pi_k F(y_i|\theta_k^*) \]

where

- \( \theta^* = (\theta_1^*, \ldots, \theta_K^*) \) and \( \pi = (\pi_1, \ldots, \pi_K) \) with \( \theta^* \) class parameters and \( \pi \) mixture weights with \( \sum_{i=1}^{K} \pi_i = 1 \).
- \( \theta^* \) and \( \pi \) can be estimated using EM algorithm.

Equivalently

- \( G = \sum_{k=1}^{K} \pi_k \delta_{\theta_k^*} \) non random
- \( \theta_i \sim G \) and \( y_i|\theta_i \sim F(y_i|\theta_i) \).
In a Bayesian setting, a prior distribution is placed over $\theta^*$ and $\pi$.

Thus, the posterior distribution of parameters given the observations is

$$p(\theta^*, \pi | y) \propto p(y | \theta^*, \pi)p(\theta^*, \pi)$$

To generate a data point within a **Bayesian finite mixture model**:

- $\theta^*_k \sim G_0$
- $\pi_1, ..., \pi_K \sim \text{Dir}(\alpha/K, ..., \alpha/K)$
- $G = \sum_{k=1}^K \pi_k \delta_{\theta^*_k}$ is a random measure
- $\theta_i | G \sim G$, which means $\theta_i = \theta^*_k$ with probability $\pi_k$
- $y_i | \theta_i \sim F(y_i | \theta_i)$
In a Bayesian setting, a prior distribution is placed over $\theta^*$ and $\pi$.

Thus, the posterior distribution of parameters given the observations is

$$p(\theta^*, \pi|y) \propto p(y|\theta^*, \pi)p(\theta^*, \pi)$$

To generate a data point within a **Bayesian finite mixture model**:

- $\theta^*_k \sim G_0$
- $\pi_1, ..., \pi_K \sim \text{Dir}(\alpha / K, ..., \alpha / K)$
- $G = \sum_{k=1}^{K} \pi_k \delta_{\theta^*_k}$ is a random measure
- $\theta_i | G \sim G$, which means $\theta_i = \theta^*_k$ with probability $\pi_k$
- $y_i | \theta_i \sim F(y_i | \theta_i)$

**Limitation:**

Require specifying the number of components $K$ beforehand.

**Solution:**

Assume an infinite number of components using BNP priors.
To establish a DP mixture model, let $G$ be a DP prior ($K \to \infty$), namely

$$G \sim \text{DP}(\alpha, G_0)$$

and complement it with a likelihood associated to each $\theta_i$

To generate a data point within a DP mixture model:

- $G \sim \text{DP}(\alpha, G_0)$
- $\theta_i | G \sim G$
- $y_i | \theta_i \sim F(y_i | \theta_i)$
2D point clustering (unsupervised learning) based on the DP mixture model:

Let the data speak for themselves!
DP mixture model

Application to image segmentation:

Drawback:
Spatial constraints and dependencies are not considered.

Solution:
Combine the DP prior with a hidden Markov random field (HMRF).
To solve the issue, we introduce a spatial Potts model component:

\[
M(\theta) \propto \exp \left( \beta \sum_{i \sim j} \delta_{z(\theta_i) = z(\theta_j)} \right)
\]

with \( \theta = (\theta_1, ..., \theta_N) \) and \( \beta \) the interaction parameter.

The DP mixture model is thus extended:

- \( G \sim \text{DP}(\alpha, G_0) \)
- \( \theta | M, G \sim M(\theta) \times \prod_i G(\theta_i) \)
- \( y_i | \theta_i \sim F(y_i | \theta_i) \)
We propose a DP-Potts mixture model based on a general stick-breaking construction that allows a natural Full VB algorithm enabling scalable inference for large datasets and straightforward generalization to other priors.

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3 Orbanz & Buhmann (2008); Xu, Caron & Doucet (2016); Sodjo, Giremus, Dobigeon & Giovannelli (2017)
4 Albughdadi, Chaari, Tourneret, Forbes, Ciuciu (2017)
5 Chatzis & Tsechpenakis (2010); Chatzis (2013)
DP-Potts: Stick breaking construction

Stick breaking construction of DP: \( G \sim DP(\alpha, G_0) \)

- \( \theta^*_k | G_0 \sim G_0 \)
- \( \tau_k | \alpha \sim B(1, \alpha), k = 1, 2, \ldots \)
- \( \pi_k(\tau) = \tau_k \prod_{l=1}^{k-1} (1 - \tau_l), k = 1, 2, \ldots \)
- \( G = \sum_{k=1}^{\infty} \pi_k(\tau) \delta_{\theta^*_k} \)

\( + \)

- \( \theta_i | G \sim G \)
- \( y_i | \theta_i \sim F(y_i | \theta_i) \)

\( = \) Dirichlet Process Mixture Model (DPMM)
DP-Potts: Stick breaking construction

Stick breaking construction of DPMM

- $\theta^*_k | G_0 \sim G_0$
- $\tau_k | \alpha \sim B(1, \alpha), k = 1, 2, \ldots$
- $\pi_k(\tau) = \tau_k \prod_{l=1}^{k-1} (1 - \tau_l), k = 1, \ldots$
- $G = \sum_{k=1}^{\infty} \pi_k(\tau) \delta_{\theta^*_k}$
- $\theta_i | G \sim G$
- $y_i | \theta_i \sim F(y_i | \theta_i)$
DP-Potts: Stick breaking construction

**DPMM view**

- \( \theta^*_k | G_0 \sim G_0 \)
- \( \tau_k | \alpha \sim B(1, \alpha), k = 1, 2, \ldots \)
- \( \pi_k(\tau) = \tau_k \prod_{l=1}^{k-1} (1 - \tau_l), k = 1, \ldots \)
- \( \theta_i = \theta^*_k \) with probability \( \pi_k(\tau) \)
- \( y_i | \theta_i \sim F(y_i | \theta_i) \)  

**Mixture/Clustering view**

- \( \theta^*_k | G_0 \sim G_0 \)
- \( \tau_k | \alpha \sim B(1, \alpha), k = 1, 2, \ldots \)
- \( \pi_k(\tau) = \tau_k \prod_{l=1}^{k-1} (1 - \tau_l), k = 1, \ldots \)
- \( p(z_i = k | \tau) = \pi_k(\tau) \)
- with \( z_i = z(\theta_i) = k \) when \( \theta_i = \theta^*_k \)
- \( y_i | z_i, \theta^* \sim F(y_i | \theta^*_z) \)
**DP-Potts: Stick breaking construction**

Using assignment variables $z_i$

**Stick breaking of DP-Potts**
- $\theta^*_k|G_0 \sim G_0$
- $\tau_k|\alpha \sim \mathcal{B}(1, \alpha), k = 1, 2, \ldots$
- $\pi_k(\tau) = \tau_k \prod_{l=1}^{k-1} (1 - \tau_l)$
- $p(z_i = k|\tau) = \pi_k(\tau)$
- $y_i|z_i, \theta^* \sim F(y_i|\theta^*_{z_i})$

**Stick breaking of DP-Potts**
- $\theta^*_k|G_0 \sim G_0$
- $\tau_k|\alpha \sim \mathcal{B}(1, \alpha), k = 1, 2, \ldots$
- $\pi_k(\tau) = \tau_k \prod_{l=1}^{k-1} (1 - \tau_l)$
- $p(z|\tau, \beta) \propto \prod_{i} \pi_{z_i}(\tau) \exp(\beta \sum_{i\sim j} \delta_{z_i=z_j})$
- $z = \{z_1, \ldots, z_N\}$
- $y_i|z_i, \theta^* \sim F(y_i|\theta^*_{z_i})$

**NB:** Well defined for every stick breaking construction ($\sum_{k=1}^{\infty} \pi_k = 1$):

*e.g.* Pitman-Yor $(\tau_k|\alpha, \sigma) \sim \mathcal{B}(1 - \sigma, \alpha + k\sigma)$
Clustering/segmentation task:
- Estimating $Z$
- while parameters $\Theta$ unknown, eg. $\Theta = \{\tau, \alpha, \theta^*\}$

Bayesian setting
Access the intractable $p(Z, \Theta|y, \Phi)$ approximate as $q(z, \Theta) = q_z(z)q_\theta(\Theta)$

Variational Expectation-Maximization
Alternate maximization in $q_z$ and $q_\theta$ ($\phi$ are hyperparameters) of the Free Energy:

$$\mathcal{F}(q_z, q_\theta, \phi) = E_{q_z q_\theta} \left[ \log \frac{p(y, Z, \Theta|\phi)}{q_z(z)q_\theta(\Theta)} \right]$$
$$= \log p(y|\phi) - KL(q_z q_\theta, p(Z, \Theta|y, \phi))$$
Inference using variational approximation

**DP-Potts Variational EM procedure**

Joint DP-Potts (Gaussian) Mixture distribution

\[
p(y, z, \tau, \alpha, \theta^*|\phi) = \prod_{j=1}^{N} p(y_j|z_j, \theta^*) \ p(z|\tau, \beta) \prod_{k=1}^{\infty} p(\tau_k|\alpha) \prod_{k=1}^{\infty} p(\theta^*_k|\rho_k) \ p(\alpha|s_1, s_2)
\]

- \( p(y_j|z_j, \theta^*) = \mathcal{N}(y_j|\mu_{z_j}, \Sigma_{z_j}) \) is Gaussian
- \( p(z|\tau, \beta) \) is a DP-Potts model
- \( p(\tau_k|\alpha) \) is Beta \( \mathcal{B}(1, \alpha) \)
- \( p(\theta^*_k|\rho_k) = \mathcal{NIW}(\mu_k, \Sigma_k|m_k, \lambda_k, \Psi_k, \nu_k) \) is Normal-inverse-Wishart
- \( p(\alpha|s_1, s_2) = \mathcal{G}(\alpha|s_1, s_2) \) is Gamma

Usual truncated variational posterior, \( q_{\tau_k} = \delta_1 \) for \( k \geq K \) (eg. \( K = 40 \))

\[
q(z, \Theta) = \prod_{j=1}^{N} q_{z_j}(z_j) \ q_\alpha(\alpha) \prod_{k=1}^{K-1} q_{\tau_k}(\tau_k) \prod_{k=1}^{K} q_{\theta^*_k}(\mu_k, \Sigma_k)
\]

- E-steps: VE-Z, VE-\( \alpha \), VE-\( \tau \) and VE-\( \theta^* \)
- M-step: \( \phi \) updating straightforward except for \( \beta \)
Some image segmentation results

Model validation and verification:

Segmented image using DP-Potts model with $\beta = 2.5$. 
Some image segmentation results

Convergence of the VB algorithm initialized by the k-means++ clustering:

![Segmentation by DP-Potts (iteration=01)](image-url)
Some image segmentation results

Segmentation results for Berkeley Segmentation Dataset:

- Original image
- Segmentation by DP-Potts (K=40, $\beta = 0$)
- Segmentation by DP-Potts (K=40, $\beta = 2$)
- Segmentation by DP-Potts (K=40, $\beta = 10$)

The segmentation results obtained by DP-Potts model with $\beta = 0, 1, 5$. 
Some image segmentation results

Segmentation with estimated $\beta = 1.66$
Quantitative evaluation of the segmentations

**Probabilistic Rand Index** on 154 color (RGB) images with ground truth (several) from Berkeley dataset (1000 superpixels). But Manual ground truth segmentations are subjective!

<table>
<thead>
<tr>
<th>PRI results with DP-Potts model</th>
</tr>
</thead>
<tbody>
<tr>
<td>$K$</td>
</tr>
<tr>
<td>------</td>
</tr>
<tr>
<td>10</td>
</tr>
<tr>
<td>20</td>
</tr>
<tr>
<td>40</td>
</tr>
<tr>
<td>50</td>
</tr>
<tr>
<td>60</td>
</tr>
<tr>
<td>80</td>
</tr>
</tbody>
</table>

**PRI results from Chatzis 2013**

<table>
<thead>
<tr>
<th>PRI (%)</th>
<th>DPM</th>
<th>iHMRF</th>
<th>MRF-PYP</th>
<th>GC</th>
</tr>
</thead>
<tbody>
<tr>
<td>Mean</td>
<td>74.15</td>
<td>75.50</td>
<td>76.49</td>
<td>76.10</td>
</tr>
<tr>
<td>Median</td>
<td>75.49</td>
<td>76.89</td>
<td>78.08</td>
<td>77.59</td>
</tr>
<tr>
<td>St.D.</td>
<td>0.084</td>
<td>0.082</td>
<td>0.079</td>
<td>0.083</td>
</tr>
</tbody>
</table>

**Computation time**: Berkeley 321x481 image reduced to 1000 superpixels takes **10-30 s** on a PC with CPU Intel(R) Core(TM) i7-5500U CPU 2.40GHz and 8GB RAM
A general DP-Potts model and the associated VB algorithm were built. The DP-Potts model was applied to image segmentation and tested on different types of datasets. Impact of the interaction parameter $\beta$ on the final results is significant. An estimation procedure was proposed for $\beta$. Extend the model with other priors (Pitman-Y or process, Gibbs-type priors, etc.). Try other variational approximations (truncation-free). Investigate theoretical properties of BNP priors under structural constraints (time, spatial) ... for other applications, such as discovery probabilities, etc.
A general DP-Potts model and the associated VB algorithm were built. The DP-Potts model was applied to image segmentation and tested on different types of datasets. Impact of the interaction parameter $\beta$ on the final results is significant. An estimation procedure was proposed for $\beta$.

- How does $\beta$ influence the number of components?
- Extend the model with other priors (Pitman-Yor process, Gibbs-type priors, etc.).
- Try other variational approximations (truncation-free)
- Investigate theoretical properties of BNP priors under structural constraints (time, spatial) ....
- ... for other applications, such as discovery probabilities, etc.


Thank you for your attention!

contact: florence.forbes@inria.fr
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- Contact: florence.forbes@inria.fr
DP simulations with $G_0$ being a standard normal distribution $\mathcal{N}(0, 1)$ and $\alpha = 1, 10$ using the Stick-Breaking representation.
Variational EM

General formulation, at iteration \((r)\)

\[
\begin{align*}
E\text{-}Z \quad & q_z^{(r)}(z) \propto \exp \left( E_{q_{\Theta}^{(r-1)}} [\log p(y, z, \Theta | \phi^{(r-1)})] \right) \\
E\text{-}\Theta \quad & q_{\theta}^{(r)}(\Theta) \propto \exp \left( E_{q_z^{(r)}} [\log p(y, Z, \Theta | \phi^{(r-1)})] \right) \\
M\text{-}\phi \quad & \phi^{(r)} = \arg \max_{\phi} E_{q_{z}^{(r)} q_{\theta}^{(r)}} [\log p(y, Z, \Theta | \phi)]
\end{align*}
\]

VE-Z, VE-\(\alpha\), VE-\(\tau\), and VE-\(\theta^*\)

e.g. VE-Z step divides into \(N\) VE-\(Z_j\) steps \((q_{z_j}(z_j) = 0 \text{ for } z_j > K)\)

\[
q_{z_j}(z_j) \propto \exp \left( E_{q_{\theta_{z_j}^*}} [\log p(y_j | \theta_{z_j}^*)] + E_{q_\tau} [\log \pi_{z_j}(\tau)] + \beta \sum_{i \sim j} q_{z_i}(z_j) \right)
\]
Estimation of $\beta$

**M-$\beta$ step:** involves $p(z|\tau, \beta) = K(\beta, \tau)^{-1} \exp(V(z; \tau, \beta))$

with $V(z; \tau, \beta) = \sum_i \log \pi_{z_i}(\tau) + \beta \sum_{i \sim j} \delta(z_i = z_j)$

$$\hat{\beta} = \arg \max_\beta E_{qzq\tau} [\log p(z|\tau; \beta)]$$

$$= \arg \max_\beta E_{qzq\tau} [V(z; \tau, \beta)] - E_{q\tau} [\log K(\beta, \tau)]$$

**Two difficulties**

1. $p(z|\tau, \beta)$ is intractable (normalizing constant $K(\beta, \tau)$, typical of MRF)
2. It depends on $\tau$ (typical of DP)

**Two approximations**

1. "standard" Mean Field like approximation
2. Replace the random $\tau$ by a fixed $\tilde{\tau} = E_{q\tau}[\tau]$  

---

\(^a\)Forbes & Peyrard 2003
Approximation of $p(z|\tau; \beta)$

\[
p(z|\tau, \beta) \approx \tilde{q}_z(z|\beta) = \prod_{j=1}^{N} \tilde{q}_{z_j}(z_j|\beta)
\]

\[
\tilde{q}_{z_j}(z_j = k|\beta) = \frac{\exp(\log \pi_k(\tilde{\tau}) + \beta \sum_{i \in N(j)} q_{zi}(k))}{\sum_{l=1}^{\infty} \exp(\log \pi_l(\tilde{\tau}) + \beta \sum_{i \in N(j)} q_{zi}(l))}
\]

\[
\tilde{\tau} = E_{q_{\tau}}[\tau]
\]

\textbf{β is estimated at each iteration by setting the approximate gradient to 0}

\[
E_{q_z q_{\tau}}[\nabla_\beta V(z; \tau, \beta)] = \sum_{k=1}^{K} \sum_{i \sim j} q_{z_j}(k) q_{zi}(k)
\]

\[
\nabla_\beta E_{q_{\tau}}[\log K(\beta, \tau)] = E_{p(z|\tau, \beta)q_{\tau}}[\nabla_\beta V(z; \tau, \beta)] \approx \sum_{k=1}^{K} \sum_{i \sim j} \tilde{q}_{z_j}(k|\beta) \tilde{q}_{zi}(k|\beta)
\]
Some image segmentation results

Segmentation results for medical images: all hyperparameters fixed

The segmentation results obtained by DP-Potts model with $\beta = 0, 1, 5$. 
Some image segmentation results

Segmentation with estimated hyperparameters ($\beta = 0.75$)
Some image segmentation results

Segmentation with estimated $\beta = 0.96$ (pixels with partial volume)
Some image segmentation results

Segmentation results for SAR images:

The segmentation results obtained by DP-Potts model with $\beta = 0, 1, 5$. 
Some image segmentation results

Segmentation results with estimated $\beta$

$\beta = 1.11$

$\beta = 1.02$