High school teachers' choices concerning the teaching of real numbers: A case study

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The goal of this paper is to present a case study in which a high school teacher with PhD in Mathematics was asked to answer a questionnaire concerning the teaching and learning of real numbers and then he was interviewed in order to investigate the interplay between his resources, goals and orientations in the decision-making process. As a main result, I show how some orientations concerning the epistemology of real numbers, the goals of mathematics education in the high school and the students' conceptions and difficulties lead him to choose a very intuitive approach to the teaching of real numbers and to leave aside all his expertise as a mathematician.

Keywords: Real numbers, teaching, high school, teachers, tertiary education.

Epistemological issues concerning real numbers and continuum

The relation between the continuum and the real numbers is often considered as something intuitive and to be taken for granted (Lakoff & Nunez, 2000), but as it is well known to the experts in history and epistemology of mathematics, this is one of the most complex issues to face dealing with the foundations of mathematics. This topic deals indeed with very relevant challenges. I will just highlight some aspects that are relevant to characterize the orientations of a teacher concerning the epistemology of real numbers and continuum. In *Continuity and irrational numbers*, Dedekind (1872, transl. 1901) stated: "In discussing the notion of the approach of a variable magnitude to a fixed limiting value, and especially in proving the theorem that every magnitude which grows continually, but not beyond all limits, must certainly approach a limiting value, I had recourse to geometric evidences. [...] For myself this feeling of dissatisfaction was so overpowering that I made the fixed resolve to keep meditating on the question till I should find a purely arithmetic and perfectly rigorous foundation for the principles of infinitesimal analysis. The statement is so frequently made that the differential calculus deals with continuous magnitude, and yet an explanation of this continuity is nowhere given; even the most rigorous expositions of the differential calculus [...] depend upon theorems which are never established in a purely arithmetic manner" (p. 1-2). Dedekind came to the construction of R as the field of the rational cuts, stressing that the new numbers - irrationals - were creations necessary to identify the points of a line and the numbers and making explicit that the assumption of the property of continuity of the line is nothing else than an axiom. In this paper, I just report Dedekind's approach, since it's particularly relevant to analyse the case study I present here; for a complete dissertation see Bell (2014).

Challenges with teaching and learning of real numbers and continuum

The topic of teaching and learning real numbers and continuum in the high school and the university have been investigated in several countries and nowadays a lot of results are available; in particular, the researches concern the difficulties of high school and university students and prospective teachers. In this paper, I report just some examples, but the literature is very rich (for a complete review see Voskoglou & Kosyvas, 2012). First of all, cognitive issues "resonate" with the very relevant epistemological issues: space-temporal intuitions and metaphors (Bolzano, 1817), and the formal approach, based on static and rigorous should be clarified by teachers. Indeed, according to Lakoff and Nunez (2000), the identification between objects such as lines, points numbers and sequences, that is very usual at school and in the University, hide the intrinsically metaphorical nature of the relation between natural continuity and "formal continuity". The main students' difficulties reported in several studies concern: irrational numbers; infinity; points of a line; density and continuity; and, number line. Students from high school to university are often not able to define correctly the concepts of rational and irrational numbers, like if rational numbers in general remains isolated from the wider class of real numbers (ibid., 2012). In particular, students' ideas concerning the relation between 0.999… and 1 (Tall & Schwarzenberger, 1978) and, in general, concerning the meaning and the use of decimal representations (Margolinas, 1988) are usually very ingenuous and seem to be product rather of a spontaneous generalization from finite to infinite numbers than of a structured learning path. Tall (1980) observed a recurrent phenomenon, that was defined *dependence*: there are more points in a longer segment than in a shorter one, based on the generalization to infinite cases of what has been learnt of the biunivocal correspondence of finite cases. Tall (1980) also observed among the students an intuitive model of the line in which points are as much as they need to fill a segment with physical points, that are "non-overlapping marks" (p. 3) and the quantity of numbers is proportional to the length of the segment. This intuitive model, without a suitable elaboration, could influence the learning of properties like density and completeness if the model of real numbers considered is a line. Also, Bagni (2000) alerted from the false illusions concerning the introduction of properties of real numbers in the graphical domain. Bergé (2008) highlighted that in the transition from calculus to analysis in the university, it's necessary first to change the conception of the number line. Fischbein, Jehiam and Cohen (1995) carried out research based on the assumption that the irrational numbers could be counterintuitive because of their high complexity but, contrary to what they had hypothesized, they found out that many high school students and prospective teachers overcame the barrier quite easily in suitable contexts, paying attention to the potential difficulties and presenting them to the students.

Research problem and framework

In this work, I face the problem from the point of view of teaching in the high school, focusing the attention on high school mathematics teachers' intended reported choices concerning the teaching of real numbers in the high school; in particular the focus is on what considerations and necessities guide their decisional process. Schoenfeld's model (2010) provides a tool to distinguish between three main factors that may influence the teachers' choices: *resources* (mathematical and pedagogical knowledge); *goals* (educational, instructional and social aims); *orientations* (beliefs concerning knowledge, concerning teaching-learning processes). I used this model to design a written questionnaire answered by the teachers in the first part of this research. Orientations may be very general (what mathematics is, what learning is) or more specific and may concern epistemological (what is the role of real numbers in the history of mathematics, what are real numbers necessary for), cognitive (what is difficult for the students, what is a good choice to help students) or ecological aspects of the didactical activities (what teachers must do within an institution, what are the aims of the teaching in high school). Some of these orientations act as criteria to make choices. Since real numbers and continuum both in the history and in the teaching-learning processes oscillates between the two poles of intuition and formalization, I owe special attention to the orientations concerning rigor and intuition.

Methodology

This *case study* is part of a research carried out for a PhD dissertation concerning the teaching and learning of real numbers in the high school, in which I involved 89 Italian high school teachers with very different backgrounds. I will discuss in a *case study* the role of teachers' resources, orientations and goals in a teacher's decision making intended process, according to Schoenfeld's model (2010). In Italy, according to the national curricula, an introduction to calculus and some theorems of analysis are proposed to students in the end of high school. According to Bergé (2008) what characterize more the transition from calculus to analysis is the transformation of intuitive models of the line into a formal construction of R. Usually, in high school, teachers are asked to introduce: limits, continuous functions and related theorems (Bolzano, Weierstrass), derivatives, Rolle and Lagrange theorems, examples of computations of limits and derivatives, Riemann integral, examples of integration based on Torricelli-Barrow theorem and examples of differential equations. Italian textbooks are usually based on formal definitions and intuitive examples that are not suitably connected each other. Usually intuitive approaches to the definition and use of real numbers are proposed in the first 4 years of high school (roots, points, decimal numbers). Then, in the last year, definition and procedures of calculus and theorems are introduced through formal expressions and using concepts such as limits, convergence towards a point and open intervals. The research question in this study is: *how does the interplay between resources, goals and orientations about teaching and learning of real numbers and continuum in the high school affect a high school teacher's intended choices?* I designed a study with a questionnaire and follow up interviews. The teacher first was asked to answer an online questionnaire structured in order to investigate his knowledge, goals and orientations. The first questions concerned the teacher's background and training. The teacher was asked to answer questions about the main properties of the set R, the construction of R starting from Q, the definition of limit points. In the next section of the questionnaire the teacher was asked what they thought to introduce R was necessary for and to comment on teaching materials or parts of lessons concerning real numbers and the line, from different points of view (construction of $\sqrt{2}$, correspondence between points of a line and numbers, algebraic and graphic approach to inequalities). Then the teacher was interviewed in order to make him declare his choices concerning teaching and learning of real numbers in depth and to follow the thread of his thoughts, in order to make the orientations emerge in relation to the epistemological, cognitive and institutional issues that were emerging time after time. The interview was a semi-guided one: a general question concerning the way the teacher introduces usually real numbers in the high school and the motivations of the choices; a particular question concerning the relation between numbers and points of a line; a particular question concerning the way the teacher presents the enlargement from *Q* to *R i*n relation with the line; a general question concerning the relevance in the last year of high school (in particular for calculus and analysis) of the previous knowledge concerning real numbers and the way connect the two.

I identified the following *a priori categories*:

1. *Resources*
   a. Mathematical knowledge
      i. The teacher knows the main properties of R (complete, ordered, Archimedean field)

ii. The teacher knows at least one construction of R (Dedekind, Cantor, Hilbert, …)

iii. The teacher knows that a limit point of a set A can be defined in every dense set

iv. R as a complete ordered field is necessary for Analysis

2. *Goals*

a. Institutional

i. To introduce intuitively some real numbers, after providing some examples and proofs of irrationality of some numbers introduced in geometrical constructions

ii. To formalize real numbers in order to give foundations to the theorems of Analysis

b. Personal

i. To construct a set in which it is possible to define the most used continuous functions (exponential, logarithmic, …)

ii. To construct a set in which it's possible to solve equations (but the ones with complex solutions) and inequalities

iii. To construct a set in which it's possible to formulate some theorems of Analysis and to define limits, integrals and derivatives

3. *Orientations*

a. Epistemological

i. R is complete in the sense of the continuity of the line

ii. A theoretical construction of R is not necessary to develop calculus and formulate theorems of Analysis, since it was constructed after the theorems

iii. R is a set of points of a line

iv. The representations of real numbers (points, decimal numbers, …) are all equivalent (framed in the same theory)

v. A postulate is necessary in the constructions or axiomatization of R

b. Cognitive

i. R is intuitive for students; students have preconceptions of real numbers

ii. The construction of R is too abstract for students

iii. Students prefer simple and concrete things, even if they don't understand everything

iv. It's important be make the lessons intuitive for students

v. It's important to be rigorous and consistent during the lessons

c. Ecological

i. It's important to respect institutional constraints

I analyzed data of the questionnaire using *a priori* categories concerning the different dimensions of the teacher's profile - goals, dimensions and knowledge - and then I looked for further emerging, unexpected phenomena to frame in this research background and to compare them with further literature review. Using a qualitative analysis, I labeled the relevant features concerning the three dimension and the declared choices. Then, I looked at the interview searching for sentences that could confirm the teacher's belonging to the categories I used in the questionnaire and to look for new relevant elements emerged in the interview. Finally, I looked for a relations between the different categories in order to interpret the teacher's choices in terms of the interplay between resources, goals and orientations.

Data analysis

I report first the background and the teacher's answers in the first part of the questionnaire, to show that the teacher showed advanced mathematical knowledge. I label the sentences with the codes presented before.

Background: Master, PhD in Mathematics and National qualification for Mathematics and Physics high school teachers*;* 5 years of experience as a teacher

He studied real numbers: at the University in a course of Analysis and at school

Properties: two operations make real numbers a field (with characteristic 0); total order, compatible with the operations; complete [R-a-i]

Construction: Two equivalent constructions: 1) the method of Dedekind's cuts (separating elements of two sets whose union is Q and that have no maximum or minimum in Q, e.g. $\{x \in Q : x^2 < 2\}$ and $\{x \in Q : x^2 > 2\}$; 2) quotient of set of the Cauchy's sequences with convergent ones [R-a-ii]

Limit point: It's possible to define it also in Q*;* the set $\{x \in Q : x < 0\}$ has 0 as a limit point [R-a-iii]

Then, I report the most relevant results of the data analysis carried out in the second stage:

1. the properties of real numbers are necessary to introduce only differential and integral Calculus, sequences and series. [G-b-iii]
2. a video in which the graphic and algebraic solution of linear inequalities are presented as two different solutions should be changed because the solution is the set of numbers that satisfy the equation and only the representation may be graphic or algebraic [O-a-iv]
3. a tutorial in which a concrete problem involving measures of courtyard containment is used to present the "reality of irrational numbers" helps the student to create good images of real numbers, even if something is not convincing [O-b-iii]
4. a video in which the correspondence between R and points of a line is showed using a point moving on a line, with the extreme indicated by a decimal number with one decimal digit, can't help to grasp the correspondence between real numbers and points of a line, because only an origin is fixed and not a unit and it's difficult to justify negative numbers [O-b-v]

In the interview, some further relevant aspects emerged concerning the teacher's knowledge and cognitive and epistemological orientations and goals. Since new categories emerged, I label the synopsis with a priori but also with a further category (NEW_C_i).

To present the following synopsis, I use a chronological criterion:

1. students have preconceptions of the relations between numbers and point of a line. He presents real numbers intuitively, as points of a line, and to base on this intuition all the definitions, also

very formal (e.g. limits, Cauchy-Weierstrass continuity, decimal numbers, ...) [O-b-i]
2. this is enough to "do what we have to do", a "pseudo-mathematics" [NEW_G_1]
3. real numbers are imagined as the real line, with an abuse of language, that makes a few damages at this level but may have many advantages [G-b-i&ii]
4. it's important to be coherent with mathematics [O-b-v] without saying it to the students [O-b-iii]
5. to introduce analysis "seriously" R is needed [R-a-iv]
6. formal definitions are not useful but were only useful to clear "the conscience of Dedekind" and that "Euler did so many good things without formalizing R" [O-a-ii]
7. history confuses students [NEW_O_1]
8. students can't understand very much of real numbers in the high school [O-b-ii]
9. asked to declare his choices concerning the introduction of continuous functions he referred to formal definitions (Cauchy-Weierstrass approach) that are traditional in Italy [G-a-ii]
10. there is a "parallelism between geometrical and algebraic postulates" [O-a-iii]
11. R and the line are the same object [O-a-iv]
12. we live between two truths, the 'pure mathematical' and the 'operational' one [NEW_O_2].
13. none really use R and the line is really strange; maybe teachers are disappointed because no one really knows what numbers are, thinking at infinite convergent sequences and the definition of new numbers (irrationals) that are limits of convergent rational sequences [NEW_O_3]
14. he uses representations like decimal numbers, the line, the roots only in order to make operations with them [O-a-iv] but never deepen their meaning and mutual relations [O-b-iii]
15. it's simpler for the students and for the teachers to "sneak off the theoretical crevices" [O-b-iii]

Discussion and conclusions

The teacher is a PhD in Mathematics (Analysis) and attended teachers' training courses. The knowledge he showed about the topic is advanced. The pedagogical knowledge has never been taken in account by the teacher to support argumentations, while his orientations, reflections and experiences are used to motivate his statements during the interview. Also, he never quoted explicitly the institutional constraints. He declared to choose usually to avoid completely the formal introductions of real numbers and the historical issues and to simplify as much as possible. Even if he's aware also of some epistemological issues, his choices are very traditional and are suitable calculus but not for analysis (Bergé, 2008). Asked to declare his choices concerning the introduction of continuous functions he referred to formal definitions (Cauchy-Weierstrass approach) that are traditional in Italy. He declared to switch suddenly from intuitions of continuity and a set of numbers with different representations to a formal implicit meaning of R, used in the hypothesis of theorems without a contextualization and without stressing the epistemological implications of such a step. The teacher conflicted with the true relevance of formal constructions of real numbers: sometimes he said it's necessary, sometimes it seemed just a fancy of some mathematicians. He's convinced that some representations of real numbers can't be interpreted in high schools – even if he never considers not to use them – so he prefers the students to use them without being aware of their complexity. Moreover, he's convinced that not only the students have limitations dealing with real numbers but there is an epistemological issue: there are two truths, the 'pure mathematical' and the 'operational' one. Furthermore, he showed "epistemological doubts" concerning the deep meaning and the existence of irrational numbers. To sum up what could seem to be only didactical and cognitive motivations (he wants the students to understand; simplifying and omitting is always better for students) hide – or at least are accompanied by – deep epistemological unsolved doubts and noisy ambiguities highlighted by the teacher, declared several times during the interview, both spontaneously and answering the interviewer's questions, as confirmed by relevant sentences like "The line is… is perceptual. No.. it's not perceptual, is stem from... you don't see. But … what is the line?"; "In practice is it useful for anything? It was useful for the purpose of a clear conscience for Dedekind but it's not useful at all"; "I take this point, limit of a function. A bit an approaching to the border of the abyss, keep the feet ... approach something that doesn't exist ... infinite rational paths doesn't imply to be rational. It's something that maybe we don't understand very well too ...". The teacher's orientations concerning the cognitive aspects of teaching and learning real numbers (intuition and preconceptions) that seemed in the beginning the most relevant motivations towards an intuitive oversimplification, considered helpful for students, are thus deeply intertwined with his epistemological orientations. Firstly, his orientations concerning the uselessness of formal definitions do not motivate him to look for suitable teaching strategies and, on the contrary, act as factor that reinforces his naive orientations towards what is better to foster in students' learning processes. Secondly, his epistemological doubts, hidden under perfect formal definitions, encourage him to keep the "Pandora's vase" closed and to avoid to face his own uncertainties, thinking that for the students it's absolutely better not to know them in order to keep on trusting him and let him going on presenting the "pseudo-mathematics" that is enough in the high school. His orientations and his decision to use a very traditional, internally disconnected and full of "theoretical crevices" approach to the teaching of real numbers have been proved to be unsuitable by a lot of researchers both from a general point of view and for the specific problem of real numbers. I can state that, in this investigation, it emerges that an advanced mathematical knowledge, even very significant, doesn't imply the use of this knowledge in teachers' choices: doubts and personal orientation can lead the teacher to use a trivial and sterile approach for all the complex issues that characterize real numbers and continuum from an epistemological point of view, taking the risk to create at least the same problems to the students that teachers with a weaker background in mathematics would create. The teachers, without suitable reflections and teacher training courses, could also reinforce their motivation towards such a choice mixing in their mind personal orientations and expected students' cognitive features, justifying and hiding with the last ones the epistemological uncertainty. The main implications of the study are the following:

1) mathematicians, even with a PhD in Analysis, in their transition to a teaching profession may miss the opportunity to benefit from their knowledge by not being completely aware of the epistemological issues of the importance of formalizations into teaching;

2) mathematicians who become high school teachers, in order to become able to design good teaching and learning activities for their students concerning real numbers, should be trained not only from the disciplinary point of view, but also from the epistemological and the didactical one.

These observations are particularly relevant in the country in which I carried out my investigation, from an institutional point of view, where often the significance of the epistemological and the didactical background of teachers is a central in the debate between policy makers (and some teachers) and the community of researchers in mathematics education.

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