On the Null Controllability of the Lotka-Mckendrick System

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ABSTRACT. In this work, we study null-controllability of the Lotka-McKendrick system of population dynamics. The control is acting on the individuals in a given age range. The main novelty we bring in this work is that the age interval in which the control is active does not necessarily contain a neighbourhood of 0. The main result asserts the whole population can be steered into zero in large time. The proof is based on final-state observability estimates of the adjoint system with the use of characteristics.

Key words. Null Controllability, Population Dynamics, Observability Inequality
AMS subject classifications. 93B03, 93B05, 92D25.

1. Introduction

In this article, we study the null-controllability of an infinite dimensional linear system describing the dynamics of an age-structured single species population. To be specific, we study null-controllability of the classical Lotka-McKendrick system. Let, $p(t,a)$ be the distribution of individuals at age $a \geqslant 0$ and $t \geqslant 0$. Let $a_\dagger$ denotes the highest age attained by the individuals in the population and T be a positive constant. Let $\beta(a) \geqslant 0$ be the natural fertility rate and $\mu(a) \geqslant 0$ denotes the natural death-rate of the individuals at age $a \geqslant 0$. The system in consideration, already studied in [4] and [5], is described by the system

$$\begin{cases} \dfrac{\partial p}{\partial t}(t,a) + \dfrac{\partial p}{\partial a}(t,a) + \mu(a)p(t,a) = m(a)u(t,a), & (t,a) \in (0,T) \times (0,a_\dagger), \\ p(t,0) = \displaystyle\int_0^{a_\dagger} \beta(a)p(t,a)\ da, & t \in (0,T) \\ p(0,a) = p_0(a), & a \in (0,a_\dagger), \end{cases} \tag{1.1}$$

where u is the control function, $m = \chi_{(a_1,a_2)}$ is the characteristic function of the interval (a_1,a_2) $(0 \leqslant a_1 < a_2 \leqslant a_\dagger)$ and p_0 is the initial population density.

We assume that the fertility rate β and the mortality rate μ satisfy the following conditions:

(H1) $\beta \in L^\infty(0,a_\dagger)$, $\beta \geqslant 0$ for almost every $a \in (0,a_\dagger)$.
(H2) $\mu \in L^1[0,a^*]$ for every $a^* \in (0,a_\dagger)$, $\mu \geqslant 0$ for almost every $a \in (0,a_\dagger)$.
(H3) $\displaystyle\int_0^{a_\dagger} \mu(a)\ da = +\infty$.

These conditions have already been used in [4, 5]. For the biological significance of the hypotheses, we refer to [11].

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Our main result regarding the null controllability of the system (1.1) is the following

Theorem 1.1. *Assume that β and μ satisfy the the conditions $(H1)-(H3)$. Furthermore, suppose that the fertility rate β is such that*

$$\beta(a) = 0 \text{ for all } a \in (0, a_b). \tag{1.2}$$

for some $a_b \in (0, a_\dagger)$ and that $a_b > a_1$. Let us recall that m is a characteristic function of the interval (a_1, a_2) with $0 \leqslant a_1 < a_2 \leqslant a_\dagger$. Then for every $T > a_1 + \max\{a_1, a_\dagger - a_2\}$ and for every $p_0 \in L^2(0, a_\dagger)$ there exists a control $u \in L^2((0,T) \times (0, a_\dagger))$ such that the solution p of the system (1.1) satisfies

$$p(T, a) = 0 \text{ for all } a \in (0, a_\dagger). \tag{1.3}$$

Remark 1.2. *If $a_1 = 0$ and $a_2 < a_\dagger$, then from Theorem 1.1 we obtain (1.3) holds for $T > a_\dagger - a_2$. This result was obtained in* [5, Theorem 1.1].

Let us now mention some related works from the literature. The null controllability results of the system (1.1) were first obtained by Barbu, Iannelli and Martcheva [4]. They considered the case when the control is supported in the interval $(0, a_2)$, i.e when $a_1 = 0$. The main result of [4] asserts that the system can be steered to any steady state, in large time except for a small interval of ages near zero. Recently, Hegoburu, Magal and Tucsnak [5] considered the system (1.1) with distributed control supported in $(0, a_2)$. They proved that the restriction of [4] is unnecessary, i.e., the whole population can be steered into a steady state, provided the individuals do not reproduce at the age near zero. Moreover, they also showed that if initial and final states are positive then the control can be chosen such that the positivity of the state trajectory is preserved.

The main novelty we bring in this work, in contrast to the results of [4, 5], is that we do not need to apply control for arbitrary low ages. More precisely, in our case the control is active for ages $a \in (a_1, a_2)$, with arbitrary $a_1 \in [0, a_\dagger)$ and $a_2 \in (a_1, a_\dagger]$, provided that supp $\beta \cap [0, a_1] = \emptyset$. In other words, we control before the individuals start to reproduce. Moreover, we show our controllability result applies to individuals of all ages, without needing to exclude ages in a neighbourhood of zero. The methodology we employ in proving this result is quite classical. We exploit the fact that null controllability of a linear system is equivalent to the final-state observability inequality of the adjoint system.

Before ending this introduction let us mention some controllability results regarding population dynamics model (Lotka-McKendrik type) with spatial diffusion. Ainseba and Aniţa [2] considered the case when the control acts in a spatial subdomain ω and for all ages and initial data close to the target trajectory. In [3], they proved a similar result when the control acts in a spatial subdomain and only for small ages. In [1], Ainseba proved null controllability except for a small interval of ages near zero, with control acting everywhere in the ages but localized in a spatial subdomain. Hegoburu and Tucsnak [6] proved that this restriction is not necessary, i.e, the whole population can be driven to zero. In their case, the control is localized in space variable but active for all ages. Recently, Maity, Tucsnak and Zuazua showed [8] that the same result can be achieved by means of control localized in space variable as well as with respect to age, not necessarily containing a neighbourhood of zero.

The outline of the paper is as follows. In Section 2 we first recall some basic facts about Lotka-McKendrick semigroup. Next, we define the adjoint of the associated semigroup. In

Section 3, we prove the final-state observability for the adjoint system. As a consequence we get the proof of the main result.

2. Lotka-McKendrick Semigroup and its Adjoint.

In this section, with no claim of originality, we recall some existing results on the Lotka-McKendrick Semigroup and its adjoint. We define the operator $A:\mathcal{D}(A)\mapsto L^2(0,a_\dagger)$ defined by

$$
\begin{aligned}
\mathcal{D}(A)=\Big\{\varphi\in L^2(0,a_\dagger)\mid \varphi &\text{ is locally absolutely continuous on } [0,a_\dagger),\\
&\varphi(0)=\int_0^{a_\dagger}\beta(a)\varphi(a)\,da,\ -\frac{d\varphi}{da}-\mu\varphi\in L^2(0,a_\dagger)\Big\},\\
A\varphi=-\frac{d\varphi}{da}-\mu\varphi. &
\end{aligned}
\tag{2.1}
$$

We introduce the control operator $B\in\mathcal{L}(L^2(0,a_\dagger))$ defined by

$$Bu(t,\cdot)=mu(t,\cdot).$$

With the above notation, the system (1.1) can be rewritten as

$$\frac{d}{dt}p(t)=Ap(t)+Bu(t),\quad t\in[0,T],\qquad p(0)=p_0. \tag{2.2}$$

Let us first recall some well posedness result of the above system.

Lemma 2.1. *The operator $(A,\mathcal{D}(A))$ is the infinitesimal generator of a strongly continuous semigroup $(e^{tA})_{t\geqslant 0}$ on $L^2(0,a_\dagger)$.*

Proof. For a proof of this lemma we refer to Kappel and Zhang [7, Theorem 2.1] or Song et. al [9, Theorem 4]. □

As already mentioned in the introduction, we are going to use the duality between null controllability of a linear system and final-time observability of the associated adjoint system. Thus it is important to determine the adjoint of the linear operator A. To this aim, we first consider the unbounded operator $(A_0,\mathcal{D}(A_0))$ defined by

$$
\begin{aligned}
\mathcal{D}(A_0)=\Big\{\varphi\in L^2(0,a_\dagger)\mid \varphi &\text{ is locally absolutely continuous on } [0,a_\dagger),\\
&\lim_{a\to a_\dagger^-}\varphi(a)=0,\ \frac{d\varphi}{da}-\mu\varphi\in L^2(0,a_\dagger)\Big\},\\
A_0\varphi=\frac{d\varphi}{da}-\mu\varphi. &
\end{aligned}
\tag{2.3}
$$

We prove the following lemma

Lemma 2.2. *The operator $(A_0,D(A_0))$ is the infinitesimal generator of a strongly continuous semigroup $\left(e^{tA_0}\right)_{t\geqslant 0}$ on $L^2(0,a_\dagger)$.*

Proof. We first verify A_0 is dissipative. For $\varphi \in \mathcal{D}(A_0)$, we have

$$(A_0\varphi,\varphi) \;=\; \lim_{a\to a_\dagger^-}\int_0^a \left(\frac{d\varphi}{da}-\mu\varphi\right)\varphi \;=\; \lim_{a\to a_\dagger^-}\frac{\varphi^2(a)}{2} \;-\; \frac{\varphi^2(0)}{2} \;-\; \lim_{a\to a_\dagger^-}\int_0^a \mu\varphi^2 \;\leqslant\; 0.$$

Next, for any $f \in L^2(0,a_\dagger)$ we consider the following problem

$$\varphi - \frac{d\varphi}{da} + \mu\varphi = f, \quad \lim_{a\to a_\dagger^-}\varphi(a) = 0.$$

Then φ solves the above equation if and only if

$$\varphi(a) = \exp\left(a + \int_0^a \mu(\tau)\ d\tau\right)\left(\int_a^{a_\dagger} e^{-s} f(s)\exp\left(-\int_0^s \mu(\tau)\ d\tau\right)\ ds\right).$$

We can easily verify that $\varphi \in \mathcal{D}(A_0)$ and hence A_0 generates a C^0-semigroup on $L^2(0,a_\dagger)$. $\square$

We now define the adjoint of the unbounded operator A

Proposition 2.3. *The adjoint of $(A,\mathcal{D}(A))$ in $L^2(0,a_\dagger)$ is defined by*

$$\mathcal{D}(A^*) = \mathcal{D}(A_0), \quad A^*\psi = \frac{d\psi}{da} - \mu\psi + \beta\psi(0). \tag{2.4}$$

Proof. For any $\varphi \in \mathcal{D}(A)$ and $\psi \in \mathcal{D}(A_0)$ we have

$$\begin{aligned}
\langle A\varphi,\psi\rangle_{L^2(0,a_\dagger)} &= \lim_{a\to a_\dagger^-}\int_0^a -\left(\varphi'(s)+\mu(s)\varphi(s)\right)\psi(s)\ ds \\
&= \lim_{a\to a_\dagger^-}\int_0^a (\psi'(s)-\mu(s))\varphi(s)\ ds + \varphi(0)\psi(0) - \lim_{a\to a_\dagger^-}\varphi(a)\psi(a) \\
&= \lim_{a\to a_\dagger^-}\int_0^a (\psi'(s)-\mu(s))\varphi(s)\ ds + \int_0^{a_\dagger}\beta(s)\psi(0)\varphi(s).
\end{aligned}$$

This yields,

$$|\langle A\varphi,\psi\rangle_{L^2(0,a_\dagger)}| \leqslant C\|\varphi\|_X \text{ and } \langle A\varphi,\psi\rangle_X = \langle\varphi, A^*\psi\rangle_X \text{ for all } \varphi\in\mathcal{D}(A), \psi\in\mathcal{D}(A_0).$$

Therefore $\mathcal{D}(A_0) \subset \mathcal{D}(A^*)$. Now we prove the reverse inclusion. Let $\lambda > 0$ be such that

$$F(\lambda) = \int_0^{a_\dagger} e^{-\lambda a}\beta(a)\exp\left(-\int_0^a \mu(\tau)\ d\tau\right)\ da \neq 1.$$

Define $G : L^2(0,a_\dagger) \mapsto L^2(0,a_\dagger)$ by $\varphi = Gf$ where φ solves

$$\lambda\varphi - \frac{d\varphi}{da} + \mu\varphi - \beta\varphi(0) = f, \quad \lim_{a\to a_\dagger^-}\varphi(a) = 0. \tag{2.5}$$

φ solves the above equation if and only if

$$\varphi(a) = \exp\left(\lambda a + \int_0^a \mu(\tau)\ d\tau\right)\times \left(\int_a^{a_\dagger} e^{-\lambda s}\Big(f(s) + V_{\lambda,f}\beta(s)\Big)\exp\left(-\int_0^s \mu(\tau)\ d\tau\right)\ ds\right),$$

where

$$V_{\lambda,f} = (1-F(\lambda))^{-1}\left(\int_0^{a_\dagger} e^{-\lambda s} f(s) \exp\left(-\int_0^s \mu(\tau)\, d\tau\right)\, ds\right).$$

We can easily check that $\varphi \in \mathcal{D}(A_0)$.

Let us assume that $\psi \in \mathcal{D}((\lambda I - A)^*)$. Then there exists $f \in L^2(0, a_\dagger)$ such that

$$\int_0^{a_\dagger} \psi(\lambda I - A)\varphi = \int_0^{a_\dagger} f\varphi \text{ for all } \varphi \in \mathcal{D}(A).$$

Let $\eta = Gf$. Then

$$\int_0^{a_\dagger} f\varphi = \lim_{a\to a_\dagger^-}\int_0^a \left(\lambda\eta - \frac{d\eta}{da} + \mu\eta - \beta\eta(0)\right)\varphi = \lim_{a\to a_\dagger^-}\int_0^a \eta\left(\lambda\varphi + \frac{d\varphi}{da} + \mu\varphi\right)$$
$$= \int_0^{a_\dagger} \eta(\lambda I - A)\varphi.$$

Therefore

$$\int_0^{a_\dagger} (\psi - \eta)(\lambda I - A)\varphi = 0, \text{ for all } \varphi \in \mathcal{D}(A).$$

In particular, if we choose $\varphi = (\lambda I - A)^{-1}(\psi - \eta)$ we obtain

$$\int_0^{a_\dagger} |\psi - \eta|^2 = 0.$$

The invertibility of the operator $(\lambda I - A)$ follows from Song et. al [9, Theorem 1(i)]. Since $\eta \in \mathcal{D}(A_0)$, we get $\psi \in \mathcal{D}(A_0)$ and ψ solves (2.5). This completes the proof of the proposition. □

3. Observability inequality.

In this section we prove Theorem 1.1. With the notation introduced in Section 2, Theorem 1.1 can be stated as : the pair (A, B) is null controllable in time $T > a_1 + \max\{a_1, a_\dagger - a_2\}$. By classical duality argument, the null controllability of the pair (A, B) in time T is equivalent to the final-state observability of the pair (A^*, B^*) in time T (see for instance [10, Theorem 11.2.1]). In the remaining part of this section, we show that the pair (A^*, B^*) is final-state observable in time $T > a_1 + \max\{a_1, a_\dagger - a_2\}$.

For $t \geqslant 0$ and $q_0 \in L^2(0, a_\dagger)$, we set

$$q(t) = \left(e^{tA}\right)^* q_0, \tag{3.1}$$

where A is defined in (2.1). According to Proposition 2.3, q satisfies the following system:

$$\begin{cases} \dfrac{\partial q}{\partial t} - \dfrac{\partial q}{\partial a} + \mu(a)q(t,a) - \beta(a)q(t,0) = 0, & (t,a) \in (0,T)\times(0,a_\dagger),\\ q(t,a_\dagger) = 0, & t \in (0,T),\\ q(0,a) = q_0(a), & a \in (0,a_\dagger). \end{cases} \tag{3.2}$$

In view of [10, Theorem 11.2.1], to prove Theorem 1.1 it is enough to prove the following theorem:

Theorem 3.1. *Under the assumptions of Theorem 1.1, for every $T > a_1 + \max\{a_1, a_\dagger - a_2\}$ and for every $q_0 \in D(A^*)$ the solution q of the system (3.2), satisfies*

$$\int_0^{a_\dagger} q^2(T,a)\ da \leqslant C \int_0^T \int_{a_1}^{a_2} q^2(t,a)\ dadt. \tag{3.3}$$

Before we start the proof of the above theorem, let us briefly describe the essential steps. We rewrite the system (3.2) as follows:

$$\frac{d}{dt}q(t) = A_0 q(t) + V(t), \qquad q(0) = q_0, \tag{3.4}$$

where A_0 is defined in (2.3) and

$$V(t,a) = \beta(a) q(t,0), \qquad (t,a) \in (0,T) \times (0,a_\dagger). \tag{3.5}$$

With Duhamel's formula, it is easy to see that

$$\|q(T,\cdot)\|^2_{L^2(0,a_\dagger)} \leqslant C \left(\|e^{TA_0} q_0\|^2_{L^2(0,a_\dagger)} + \int_0^T q^2(t,0)\ dt \right), \tag{3.6}$$

with some constant C depending only on T. Thus to prove (3.3), we derive appropriate upper bounds for each of the terms in the right hand side of the above estimate.

Estimate of $\|e^{TA_0} q_0\|^2_{L^2(0,a_\dagger)}$: Recall that the operator A_0 is defined in (2.3). For $q_0 \in L^2(0,a_\dagger)$, let us set

$$z(t) = e^{tA_0} q_0, \qquad t \geqslant 0. \tag{3.7}$$

Then z solves the following system:

$$\begin{cases} \dfrac{\partial z}{\partial t} - \dfrac{\partial z}{\partial a} + \mu(a) z(t,a) = 0, & (t,a) \in (0,T) \times (0,a_\dagger) \\ z(t,a_\dagger) = 0, & t \in (0,T), \\ z(0,a) = q_0(a), & a \in (0,a_\dagger). \end{cases} \tag{3.8}$$

We prove the following proposition:

Proposition 3.2. *Let us assume that $0 \leqslant a_1 < a_2 \leqslant a_\dagger$. Then for every $T > \max\{a_1, a_\dagger - a_2\}$, there exits $C > 0$ such that, for every $q_0 \in D(A_0)$, the solution z of the system (3.20), satisfies*

$$\int_0^{a_\dagger} z^2(T,a)\ da \leqslant C \int_0^T \int_{a_1}^{a_2} z^2(t,a)\ dadt. \tag{3.9}$$

Proof. We set

$$\widetilde{z}(t,a) = e^{-\displaystyle\int_0^a \mu(\tau)\ d\tau} z(t,a), \qquad (t,a) \in (0,T) \times (0,a_\dagger).$$

Then $\widetilde{z}$ satisfies the adjoint of a transport equation:

$$\begin{cases} \dfrac{\partial \widetilde{z}}{\partial t} - \dfrac{\partial \widetilde{z}}{\partial a} = 0, & (t,a) \in (0,T) \times (0,a_\dagger) \\ \widetilde{z}(t,a_\dagger) = 0, & t \in (0,T), \\ \widetilde{z}(0,a) := \widetilde{z}_0(a), & a \in (0,a_\dagger), \end{cases} \tag{3.10}$$

with $\widetilde{z}_0(a) = e^{-\int_0^a \mu(\tau)\, d\tau} q_0$. Therefore $\widetilde{z}$ is given by

$$\widetilde{z}(t,a) = \begin{cases} \widetilde{z}_0(a+t) & \text{if } t \leqslant a_\dagger - a, \\ 0 & \text{if } t > a_\dagger - a. \end{cases} \tag{3.11}$$

With the above explicit expression, for $T > \max\{a_1, a_\dagger - a_2\}$, we have

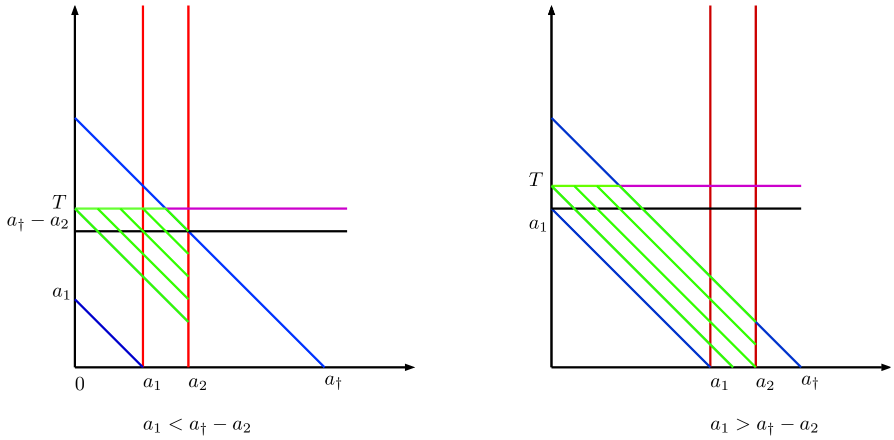

Figure 1. Minimal time required for observability inequality to hold for the transport equation. For both cases, $\widetilde{q}(T,\cdot) = 0$ on the purple region.

$$\int_0^{a_\dagger} \widetilde{z}^2(T,a)\, da \leqslant C \int_0^T \int_{a_1}^{a_2} \widetilde{z}^2(t,a)\, dadt. \tag{3.12}$$

which is the standard observability estimate for the one dimensional transport equation. Let us briefly explain how one can obtain the estimate (3.12).

- First of all, since $T > \max\{a_1, a_\dagger - a_2\}$, we can always find $a_* < a_2$ such that $\widetilde{z}(T,a) = 0$ for all $a \in (a_*, a_2)$. Thus we need to estimate $\displaystyle\int_0^{\widetilde{a}} \widetilde{z}^2(T,a)\, da$.
- Next, for $a \in (0, a_*)$ we note that the trajectory $\gamma(s) = (T-s, a+s)$, $s \in [0,T]$ (or equivalently the backward characteristics starting from (T,a)) always enters the observation region $(a_1,a_2)\times(0,T)$ (see Fig. 1). Since $\widetilde{z}$ is constant along the trajectory γ, one can easily estimate $\displaystyle\int_0^{\widetilde{a}} \widetilde{z}^2(T,a)\, da$ by the right hand side of (3.12).

Finally (3.9) follows from (3.12). Indeed using the characteristics, we also have

$$z(T,a) = 0 \text{ for all } a \in (a_*, a_\dagger).$$

Therefore, using (3.12) we obtain

$$\int_0^{a_\dagger} z^2(T,a)\, da = \int_0^{a_*} z^2(T,a)\, da \leqslant \left(e^{2\displaystyle\int_0^{a_*} \mu(\tau)\, d\tau} \right) \int_0^{a_*} \widetilde{z}^2(T,a)\, da$$

$$\leqslant C\int_0^{a_\dagger} \widetilde{z}^2(T,a)\ da \leqslant C\int_0^T\int_{a_1}^{a_2} \widetilde{z}^2(t,a)\ dadt \leqslant C\int_0^T\int_{a_1}^{a_2} z^2(t,a)\ dadt. \quad (3.13)$$

This completes the proof of the proposition. □

Estimate of $q(t,0)$: In the following proposition, we provide an estimate of $q(t,0)$:

Proposition 3.3. *Let us assume the hypothesis of Theorem 1.1, $T > a_1$ and $\eta \in (a_1, T)$. Then there exists a constant $C > 0$ such that for every $q_0 \in D(A^*)$ the solution q of the system (3.2), satisfies*

$$\int_\eta^T q^2(t,0)\ dt \leqslant C\int_0^T\int_{a_1}^{a_2} q^2(t,a)\ dadt. \quad (3.14)$$

Proof. We set

$$\widetilde{q}(t,a) = e^{-\int_0^a \mu(\tau)\ d\tau} q(t,a). \quad (3.15)$$

Since $\beta(a) = 0$ for all $a \in (0, a_b)$, $\widetilde{q}$ satisfies

$$\frac{\partial \widetilde{q}}{\partial t} - \frac{\partial \widetilde{q}}{\partial a} = 0 \text{ for all } (t,a) \in (0,T)\times(0,a_b). \quad (3.16)$$

We are going to estimate $\widetilde{q}(t,0)$. We exploit the fact that for $(t,a) \in (0,T)\times(0,a_b)$, $\widetilde{q}$ remains constant along the characteristics $\gamma(s) = (t-s, s)$, $s \leqslant t \leqslant T$ (or equivalently the backward characteristic starting from $(t,0)$). If $T > a_1$, the trajectory $\gamma(s)$ always reaches the observation region $(0,T)\times(a_1,a_2)$ (see the green region in Fig. 2). Without loss of generality, let us assume that $T > a_b$, $\eta < a_b$ and $a_2 \leqslant a_b$.

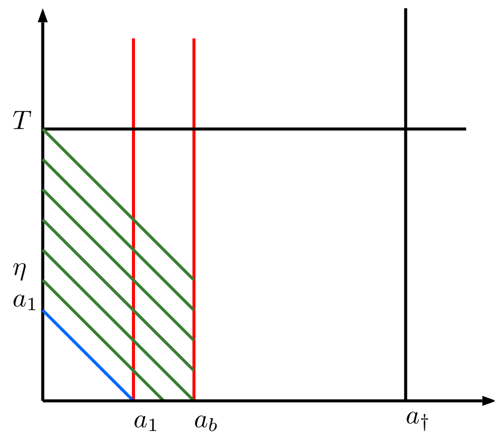

FIGURE 2. An illustration of estimate of $q(t,0)$ with $a_2 = a_b$.

Case 1: Let us fix $t \in [a_b, T]$. We define

$$w(s) = \widetilde{q}(s, t-s), \quad s \in (t-a_b, t). \quad (3.17)$$

Then $\dfrac{\partial w}{\partial s} = 0$ for all $s \in (t - a_b, t)$. In particular, $w(s) = \text{constant}$ for all $s \in (t - a_b, t)$. Since $a_1 < a_b$ we have

$$w(t) = \frac{1}{a_b - a_1} \int_{t-a_b}^{t-a_1} w(s)\, ds.$$

Therefore,

$$\widetilde{q}(t,0) = \frac{1}{a_b - a_1} \int_{t-a_b}^{t-a_1} \widetilde{q}(s, t-s)\, ds = \frac{1}{a_b - a_1} \int_{a_1}^{a_b} \widetilde{q}(t-s, s)\, ds.$$

Integrating with respect to t over $[a_b, T]$ we obtain

$$\begin{aligned} \int_{a_b}^{T} \widetilde{q}^2(t,0)\, dt \leqslant C \int_{a_b}^{T} \int_{a_1}^{a_b} \widetilde{q}^2(t-s, s)\, dsdt = C \int_{a_1}^{a_b} \int_{a_b}^{T} \widetilde{q}^2(t-s,s)\, dtds \\ = C \int_{a_1}^{a_b} \int_{a_b - s}^{T-s} \widetilde{q}^2(\tau, s)\, d\tau ds \leqslant C \int_0^T \int_{a_1}^{a_b} \widetilde{q}^2(t,a)\, dadt. \end{aligned} \tag{3.18}$$

Case 2: Let us fix $t \in (\eta, a_b)$. We define

$$w_1(s) = \widetilde{q}(s, t-s), \quad s \in (0,t).$$

Then $\dfrac{\partial w_1}{\partial s} = 0$ for all $s \in (0,t)$. In particular, $w(s) = \text{constant}$ for all $s \in (0,t)$. Since $t > a_1$ we obtain

$$w(t) = \frac{1}{t - a_1} \int_0^{t-a_1} w(s)\, ds.$$

Therefore,

$$\widetilde{q}(t,0) = \frac{1}{t - a_1} \int_0^{t-a_1} \widetilde{q}(s, t-s)\, ds = \frac{1}{t-a_1} \int_{a_1}^{t} \widetilde{q}(t-s, s)\, ds.$$

Integrating with respect to t over $[\eta, a_b]$ we get

$$\begin{aligned} \int_{\eta}^{a_b} \widetilde{q}^2(t,0)\, dt \leqslant \frac{1}{(\eta - a_1)^2} \int_{\eta}^{a_b} \int_{a_1}^{t} \widetilde{q}^2(t-s,s)\, dsdt \leqslant C \int_0^{a_b} \int_{a_1}^{t} \widetilde{q}^2(t-s,s)\, dsdt \\ = C \int_{a_1}^{a_b} \int_{s}^{a_b} \widetilde{q}^2(t-s,s)\, dtds = C \int_{a_1}^{a_b} \int_0^{a_b - s} \widetilde{q}^2(\tau, s)\, d\tau ds \leqslant C \int_0^T \int_{a_1}^{a_b} \widetilde{q}^2(t,a)\, dadt. \end{aligned} \tag{3.19}$$

Combining, (3.18) and (3.19) we obtain

$$\int_{\eta}^{T} \widetilde{q}^2(t,0)\, dt \leqslant C \int_0^T \int_{a_1}^{a_2} \widetilde{q}^2(t,a)\, dadt.$$

Note that, $\widetilde{q}(t,0) = q(t,0)$. Thus from the above estimate we clearly get (3.14).

□

Now, we are in a position to prove Theorem 3.1 and consequently our main result in Theorem 1.1.

Proof of Theorem 3.1. Since $T > a_1 + \max\{a_1, a_\dagger - a_2\}$, we can choose $\eta > a_1$ such that $T - \eta > \max\{a_1, a_\dagger - a_2\}$. For $t \in (\eta, T)$ and $a \in (0, a_\dagger)$, we define

$$V(t,a) = \beta(a)q(t,0).$$

Since $q(\cdot, 0) \in L^2(0,T)$, we have that $V \in L^2(\eta, T; L^2(0, a_\dagger))$. Then q satisfies

$$\begin{cases} \dfrac{\partial q}{\partial t}(t,a) - \dfrac{\partial q}{\partial a}(t,a) + \mu(a)q(t,a) = V(t,a), & (t,a) \in (\eta, T) \times (0, a_\dagger), \\ q(t, a_\dagger) = 0, & t \in (0,T). \end{cases} \tag{3.20}$$

Therefore

$$q(t,a) = e^{(t-\eta)A_0} q_\eta(a) + \int_\eta^t e^{(t-s)A_0} V(s)\, ds, \tag{3.21}$$

where $q_\eta = q(\eta, \cdot)$ and A_0 is the unbounded operator defined as in (2.3). The above representation formula for q yields

$$\int_0^{a_\dagger} q^2(T,a)\, da \leqslant C \left(\left\| e^{(T-\eta)A} q_\eta \right\|^2_{L^2(0,a_\dagger)} + \int_\eta^T q^2(t,0)\, dt \right). \tag{3.22}$$

Since $T - \eta > \max\{a_1, a_\dagger - a_2\}$, using Proposition 3.2 we have

$$\left\| e^{(T-\eta)A} q_\eta \right\|^2_{L^2(0,a_\dagger)} \leqslant C \int_\eta^T \int_{a_1}^{a_2} q^2(t,a)\, dadt \leqslant C \int_0^T \int_{a_1}^{a_2} q^2(t,a)\, dadt.$$

On the other hand, since $T > a_1$, applying Proposition 3.3 we get

$$\int_\eta^T q^2(t,0)\, dt \leqslant C \int_0^T \int_{a_1}^{a_2} q^2(t,a)\, dadt.$$

Finally combining the above two estimates together with (3.22), we obtain

$$\int_0^{a_\dagger} q^2(T,a) \leqslant C \int_0^T \int_{a_1}^{a_2} q^2(t,a)\, dadt. \tag{3.23}$$

This completes the proof of the theorem. $\square$

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