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Ofer Marmur, Boris Koichu

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Ofer Marmur[1] and Boris Koichu[2]

Technion - Israel Institute of Technology, Faculty of Education in Science and Technology, Israel; [1]omarmur@tx.technion.ac.il, [2]bkoichu@tx.technion.ac.il

This paper reports on part of a study regarding student learning-experiences and affective pathways in undergraduate calculus tutorials. The following question is pursued in this paper: How do the students' key affective states relate to the type of mathematical discourse conducted in class? We present and discuss two lessons where two similar problems were considered. The lessons were filmed and followed by stimulated recall interviews with nine students. Though the students in both lessons did not understand the solution to the challenging problem, they evaluated the lessons and subsequent learning experiences very differently. We suggest the difference was related to the type of discourse employed by the instructor. The lesson that evoked a negative reaction utilized only an object-level discourse. The lesson that evoked a positive reaction additionally utilized a meta-level discourse. We will call this heuristic-didactic discourse. Implications are drawn.

Keywords: Undergraduate calculus, emotional states, key affective events, discourse.

Introduction

Emotions have long been recognized to take an integral part in mathematical problem-solving activities, especially when coping with non-routine problems. However, relatively little is known on the role emotion plays in undergraduate student learning, and even more so in context of frontal tutorials. This paper is part of a wider research investigating student emotion and learning-experiences fostered by problem-solving explanations in calculus tutorials. In this paper we present and discuss 2 cases. The first case, which is at the focus of this paper, consists of students regarding a lesson containing a highly challenging problem rather positively, whilst not fully understanding the solution. Our interest in this case lies in the generally positive attitude towards this part of the lesson, accompanied by students admitting that key parts of the proof were incomprehensible, and showing disbelief in their ability to solve such problems on their own. This lesson will henceforth be referred to as Lesson-P (positive student attitude). Lesson-P especially stood out when juxtaposed with a lesson containing a similar challenging problem also not understood by students, yet their lack of understanding was accompanied by negative emotions of anger and frustration. This lesson will be briefly presented in the paper as a contrastive background and referred to as Lesson-N (negative student attitude). Thus, we were faced with the following question: how can it be that the students of Lesson-P described their learning experience in a rather positive manner, though not fully understanding the solution?

Theoretical background

Frontal teaching style of undergraduate mathematics

Undergraduate mathematics courses are typically comprised of lectures and tutorials. This paper focuses on large-group tutorials, which are lessons that present problems accompanying the theoretical material (presented in the lectures), and are taught in a traditional-frontal style (Marmur

& Koichu, 2016). The common practice of the frontal teaching style (henceforth referred to as FTS) in undergraduate mathematics education possesses pros and cons. On the one hand, there is evidence that FTS can be effective in modeling mathematical reasoning for students by "conceptual scaffolding through demonstration and worked examples" (Pritchard, 2010, p. 611). This modeling can be motivational to students, particularly when exposing the struggle that precedes the reaching of a solution (Pritchard, 2010). On the other hand, it has been argued that FTS at university level consists of a one-directional communication based on transmitting information (Biggs & Tang, 2011) and treating the students as "non-emotional audience" who are granted no room for individual difficulties (Alsina, 2002, pp. 5-6). It is not our intention to either support or oppose these claims. Rather, we recognize that FTS is widespread and will most likely not disappear in the near future. Therefore, it is vital to gain a better understanding of how students learn in this environment in order to be able to improve the system from "within", theoretically and practically, by identifying learning opportunities for students within the FTS paradigm. Lectures and tutorials comprise however only a certain percentage of the total time spent on an undergraduate mathematics course by students, and they are generally expected to spend many additional hours studying independently. Consequently, when we discuss the need to recognize and identify learning opportunities presented in the classroom, we mean not only those aspects related to the learning process in class, but also the aspects that support the learning that continues outside the classroom.

Emotions, learning, and discourse in the undergraduate classroom

In this paper we utilize Goldin's theory of local affect. Goldin (2000) defines *emotional states* as "the rapidly changing (and possibly very subtle) states of feeling that occur during problem solving" (p. 210). *Affective pathways* are regarded as a sequence of emotional states, and are linked by Goldin to mathematical cognition and heuristic processes students utilize at different stages of mathematical problem solving. Specifically, we choose to focus on what Goldin (2014) refers to as *key affective events* during mathematics learning, i.e., events "where strong emotion or change in emotion is expressed or inferred" (p. 404). Weber (2008) claims that emotional states may have a substantial impact on a student's failure or success in a high-level calculus course. In his paper, Weber demonstrates how a single and strong positive experience of success may alter a student's attitude and type of engagement with the material for the continuation of the course. Marmur and Koichu (2016) illustrate that also in a single lesson the creation of strong emotional experiences for students may significantly influence their level of focus, attention, and involvement in class.

Student emotions are examined in this paper in relation to the discourse led by the instructor in class. Theoretically, Evans, Morgan, and Tsatsaroni (2006) link emotions with discourse by regarding emotions as a "socially organised phenomena which are constituted in discourse" (p. 209). According to Sfard (2008), learning is perceived as a change in the mathematical discourse, while distinguishing between a discourse on mathematical objects, called *object-level discourse*, and a "discourse about this discourse" (p. 300), referred to as *meta-level discourse*. In relation to the FTS, the focus on the instructor's discourse finds additional support in Sfard's (2014) claim that this teaching style allows an expert to teach students how to "talk mathematics" and thus promote student learning through their introduction to a new mathematical discourse (p. 201).

Research question

The study reported on in this paper is part of a broader research on the link between student emotions and learning during calculus tutorials. This broader research focuses on characterizing classroom events students respond to during calculus tutorials, students' affective pathways and learning experiences during tutorials, and classroom learning-opportunities as reflected by the students' own point of view. This paper addresses these issues by concentrating on the following question: How do students' key affective states relate to the type of mathematical discourse conducted in class?

Method

Context and participants

The two lessons reported on in this paper were of two separate tutorial groups that were part of the same second-semester calculus course. The course was highly demanding and challenging, and was attended by students from the computer science faculty. Both lessons were attended by approximately 50 students. The instructors (henceforth referred to as Instructor-P and Instructor-N) were both experienced instructors with a good reputation at the university.

The problems

Lesson-P took place during the second half of the semester and was regarding the topic of the two-variable Riemann integral. For this lesson the students were asked to prove that the function below is Riemann integrable in two variables on [0, 1] × [0, 1] (and the value of the integral is 0).

$$f(x,y) = \begin{cases} \frac{1}{q}, & x \in \mathbb{Q}\ and\ y = \frac{p}{q} \in \mathbb{Q}, \frac{p}{q}\ in\ lowest\ terms, q > 0 \\ 0, & x \notin \mathbb{Q}\ or\ y \notin \mathbb{Q} \end{cases}$$

Lesson-N took place during the first half of the semester and was regarding the one-variable Riemann integral. The problem of interest was to prove that the "popcorn function" below (also known as "Riemann's function") is Riemann integrable on [0, 1] (and the value of the integral is 0).

$$f(x) = \begin{cases} \frac{1}{q}, & x = \frac{p}{q} \in \mathbb{Q}, \frac{p}{q}\ in\ lowest\ terms, q > 0 \\ 0, & x \notin \mathbb{Q} \end{cases}$$

Both instructors referred to the definition of a Riemann-integrable function. The problem in Lesson-P was planned as a follow-up two-variable version of the "popcorn function".

Data collection and analysis

Both lessons were filmed by the first author of this paper who also took notes during the lessons. Subsequently, individual stimulated-recall interviews were conducted with nine volunteering students: five on Lesson-P and four on Lesson-N, each student participated in only one of the two lessons. The interviews were conducted over a nine-day period after the lessons. Stimulated recall was the chosen methodology as it presents a non-intrusive method to help students "relive" the lesson and reflect upon their thought processes during its course (Calderhead, 1981). During the interviews, the students were presented with an approximately 20-minute video excerpt of the filmed lesson in which the problem of interest was taught. They were explained that the video served as a tool to help them "relive" the lesson, and were instructed to stop the playback whenever they had a particular recollection of what they thought or felt at that moment. During this part of the interview the interviewer occasionally asked clarifying questions, mainly in the form of "can you explain *why* you thought/felt this way at that specific moment?" After watching the filmed episode, the students were asked follow-up questions regarding the problem, lesson, and course, the main ones being: a) Was the problem memorable for you, and if so, in what way? b) What were you pleased and displeased with during the lesson? and c) What is your general attitude towards the course? The interviews were audio-recorded and ranged in length from 40 to 65 minutes, depending on the level of detail shared by the student.

For the data analysis we utilized a "general inductive approach" (Thomas, 2006) that allowed us to coordinate the raw data into a brief summary that addresses and explains the "underlying structure of experiences or processes" (p. 238) most apparent in the data. The goal of the analysis was to identify: 1) students' key affective states as indicators of potential-learning or obstacle-for-learning episodes; and 2) types of mathematical discourse in the classroom. Specifically, we focused on: 1) episodes where all students stopped the video to reflect on the lesson; and 2) repeated statements or themes (whether within a specific interview or between interviews). Subsequently, we continued with a recursive process of going back and forth between the video observations and the student interviews in order to refine our conclusions. Although students were asked in each interview to express their emotional states *during the lesson itself*, it should be recognized that the accounts shared by the students might have been of their emotions *during the interview*. However, we considered this issue as a point of strength for the research, rather than a limitation. Such a selective recollection of emotions may shed light on the process, addressed by Goldin (2014), of how in-the-moment emotional states transform into longer-term attitudes and beliefs, and on how this process shapes the mathematical learning. Accordingly, while adopting Goldin's (2014) terminology of "key affective events", and in line with Marmur and Koichu (2016), we regarded: a) the most *memorable* emotional states of students as key emotional states that shape their overall learning experience; and b) student expressions of *strong* emotions as indicators of potential-learning or obstacle-for-learning episodes in class.

Findings

Due to the scope limitations of this paper, in the Findings section we will focus on: a) the instructional episodes most prominently addressed in the student interviews; and b) student thoughts and emotions regarding these episodes. Additionally, the Findings section will predominantly focus on Lesson-P as the main explored phenomenon, utilizing Lesson-N as a contrastive background to illustrate and emphasize certain aspects of the findings.

Lesson-P

After having presented the problem to the students, Instructor-P said: "Let us first try to understand what's going on here. [...] I want us to make some observations." The instructor reminded the students of the one-variable Riemann (popcorn) function, after which the following 10 minutes were focused on what was titled on the board as "Observation" and "Observation no. 2". The first "Observation" entailed that for a fixed x we get $\int_0^1 f(x,y)dy = 0$ and therefore the following iterated integral equals zero: $\int_0^1 \left(\int_0^1 f(x,y)dy\right)dx = 0$. After having written the title "Observation no. 2", Instructor-P asked the students: "What happens if I fix y?" The students participated in the discussion regarding a fixed $y \notin \mathbb{Q}$ and a fixed $y \in \mathbb{Q}$, the latter giving the Dirichlet function $D(x) = \begin{cases} \frac{1}{q}, & x \in \mathbb{Q} \\ 0, & x \notin \mathbb{Q} \end{cases}$. This led to the conclusion that the integral $\int_0^1 f(x,y)dx$ does not exist and therefore it is impossible to calculate the (opposite-direction) iterated integral $\int_0^1 \left(\int_0^1 f(x,y)dx\right) dy$.

These observations led the class to two conclusions regarding the function: 1) Its *double* integral exists, yet the *iterated* integral (in one of the directions) does not; 2) It demonstrates the necessity of the *continuity* assumption in Fubini's theorem which allows us to calculate the double integral $\iint_D f(x,y)dxdy, D = [0,1] \times [0,1]$, as the iterated integral $\int_0^1 \left(\int_0^1 f(x,y)dx\right) dy$. These conclusions were in fact surprising for the students, as conveyed in their interviews. It was only then that the instructor admitted that everything discussed so far "does not yet answer our problem". At that point he wrote: 'So how do we solve?', and said: "In such a case we need to follow the definition."

We interpreted the students' general attitudes towards that part of the lesson dealing with this problem, as rather positive in the following manner. At different levels of explicitness, all students claimed that the lesson and teaching were good, while mainly pointing at the problem presented above. For example, even before knowing what the interview was about exactly, Student A said: "You came to a very special lesson […] The instructor chose a non-standard problem [the discussed one] to convey his messages. I really loved it". Student B said: "You came to a good lesson, really!", and referred to the discussed problem as "*the* problem" of the lesson (emphasis in intonation). Student C said: "The lesson was interesting. The lesson was clear. The first function [the discussed problem] was different and new." Students A, C, and D called the problem beautiful. Students D and E claimed the problem was good since it prepared them for similar problems that may appear in the exam. However, all students admitted the problem was difficult and challenging, and the unexpected impression we got was that the students did not fully understand the solution, nor expressed confidence in their abilities to solve similar problems independently. For example: Student A admitted that a key line in the (actual) solution seemed to him like gibberish; Student C found the same line to be full of incomprehensible transitions which she referred to as "jumps"; Student B referred to the same line with: "What??? He said it, so it is probably true", and later in the interview admitted: "If this problem was in the exam, I wouldn't have succeeded solving it"; and Student E hoped such a problem would not appear in the exam and hoped the lecturer did not have such a "dark heart". However, these statements were not accompanied by any explicit expressions conveying negative emotions. Additionally, such statements barely appeared in other parts of the interviews, and did not even appear at all when the students were explicitly asked what they were displeased with during the lesson.

It is towards the opening "observations" part of the lesson that the students mainly expressed positive opinions on what had happened. Additionally, all students claimed that even though the "observations" part was not directly utilized in the actual solution, it was nonetheless an *indispensable* part in support of their learning. They supplied us with a variety of reasons: it exposed the thinking process of how to reach a solution; it allowed time to think about the problem; it included a counter-example for claims they thought were true; it imitated what they would actually do if they were to start solving the problem on their own (i.e., try to calculate the iterated integral); it helped them understand the problem through step-by-step analysis; it gave room for "mathematical play" where the goal was not merely to solve a problem; and it demonstrated that even if an attempt for a solution did not succeed, they should just try again in a different way. Some students clearly linked their positive attitude with the following didactic aspect of the mathematical discourse led by the instructor in class. Student A shared that he really loved the approach taken by Instructor-P during the “observations” part. The student described that the instructor put himself in the position of a student, approaching the problem through their eyes, and instead of immediately solving the problem because he was already familiar with the solution, he started “playing” with it with the aim of seeing where this will lead them. A supportive angle is given by Student D who said that during the lesson Instructor-P really tried to give the impression that he did not already know the answer, but rather was trying to solve the problem with them. She said that only once she was convinced he was not “fooling” them, she started thinking with him. The instructor, however, was indeed familiar with the solution, and Student D admitted that only when watching the lesson again during the interview, she realized how planned and structured the lesson was.

Lesson-N

After having written what needs to be proven according to the definition of a Riemann integrable function, Instructor-N told the students: “At the beginning you may experience some lack of understanding. Once we reach the end [of the solution] you’ll understand where I took the numbers from that initially might have looked a bit weird.” Then he wrote the following: Choose n_0 such that $\frac{1}{n_0}<\frac{\epsilon}{2}$. This is a key moment where all 4 interviewed students stopped the video and expressed similar thoughts and strong dissatisfactions. The main criticism the students conveyed is expressed in the following interview excerpt: “It really bothers me that he reads the solution by the order of the proof and not by the order of how you think about the proof. [...] At the end it all works out. But it doesn’t help me with how to solve a problem.” The student then continues while expressing her anger: “It really pissed me off. He pulls the answer out of a hat, and I don’t know how he got to it.”

These negative opinions towards the lesson, while pinpointing the underlying reason to the key moment presented above, continued and repeated throughout all interviews. The students claimed that also at the end of the lesson they did not understand the solution, and that the promise made by the instructor at the beginning was left unfulfilled.

Discussion

While the case of Lesson-N demonstrates that students can possess negative emotions towards a solution they did not understand, the case of Lesson-P, containing a similar problem, demonstrates that a lack of understanding can still be accompanied by positive student emotions. Both lessons contained episodes focused on the *solving* of a challenging problem, which we suggest to regard as an object-level type of discourse. However, the positive emotions in Lesson-P were mainly directed towards that part of the lesson focused on *how* to approach a challenging problem, which we regard as a meta-level type of discourse. While other explanations for the students’ positive attitude towards Lesson-P are certainly possible, our interpretation is based on what we found to be most prominently conveyed by students during the interviews. Furthermore, we suggest that not only did students appreciate this meta-level discourse, as expressed in their interviews, but that this discourse may have also had a neutralizing effect on the potential negative emotions related to not understanding the solution.

The meta-level discourse in Lesson-P revealed a heuristic approach on how to tackle a challenging problem. On the one hand, the discourse was planned and monitored from an expert’s point of view, which may be viewed as a teacher's learning goal (Simon, 1995) that did not coincide with the declared main goal of solving the problem. In the case of Lesson-P, the "observations" part did not constitute a directionless exploration, but rather led to the conclusions mentioned in the findings. On the other hand, as also regarded by the students themselves, the discourse was led by the instructor through a student's point of view, considering students' cognitive and affective needs, their ways of thinking, their assumed misconceptions, and the steps they would most likely take. We call such a discourse, presenting heuristics monitored from an expert's point of view yet derived from a student's point of view, a *heuristic-didactic discourse*. In the case of Lesson-P, the heuristic aspect of the discourse may be viewed in line with what Featherstone (2000) refers to as "mathematical play", which puts emphasis on the act of exploring rather than solving, and may support the creation of a zone of proximal development, giving guidance to the learning student. The didactic aspect of the discourse may be viewed in line with what Jaworski (2002) refers to as "harmony" between "mathematical challenge" and "sensitivity to students" (both their cognitive and affective needs) in order to help students make mathematical progress. This is one example of a heuristic-didactic type of discourse, and we call for further research on characterizing different types of heuristic-didactic discourses in the undergraduate classroom.

In practice, the presented study suggests that university students wish for a more heuristic-didactic discourse to be held in the undergraduate mathematics classroom. In simple terms this means that it is necessary for students to get "tools" on how to approach a challenging problem on their own. In the presented lessons, not only were students satisfied when a heuristic-didactic discourse took place, students also showed strong emotional responses of anger and frustration when this need was not fulfilled. Furthermore, even though Lesson-P could have been improved by the students also understanding the solution better, it clearly demonstrated that the learning induced by the heuristic-didactic discourse was perceived by the students as the most valuable kind of learning, even at the expense of not fully understanding a solution. Sfard (2008) regards meta-level learning as a change in meta-rules of the discourse, while claiming that this change is not likely to be initiated by students on their own. Accordingly, meta-level discourse in class may serve as an initial point of aid for students to continue a meta-level learning-process at home. All this implies that lecturers and instructors should consider paying more didactic attention in revealing to students *how* they came up with their solutions and proofs. This learning-opportunity may be implemented in the common undergraduate frontal teaching style and could supply valuable tools for the learning process that the students are required to continue independently.

References

Alsina, C. (2002). Why the Professor Must be a Stimulating Teacher: Towards a New Paradigm of Teaching Mathematics at University Level. In D. Holton, M. Artigue, U. Kirchgraber, J. Hillel, M. Niss, & A. H. Schoenfeld (Eds.), *The teaching and learning of mathematics at university level* (pp. 3–12). New York: Springer Science & Business Media.

Biggs, J., & Tang, C. (2011). *Teaching for Quality Learning at University*. Maidenhead, England: Open University Press.

Calderhead, J. (1981). Stimulated Recall: A Method for Research on Teaching. *British Journal of Educational Psychology*, *51*, 211–217.

Evans, J., Morgan, C., & Tsatsaroni, A. (2006). Discursive Positioning and Emotion in School Mathematics Practices. *Educational Studies in Mathematics, 63*(2), 209– 226.

Featherstone, H. (2000). ”-Pat + Pat = 0”: Intellectual Play in Elementary Mathematics. *For the Learning of Mathematics, 20*(2), 14–23.

Goldin, G. A. (2000). Affective Pathways and Representation in Mathematical Problem Solving. *Mathematical Thinking and Learning, 2*(3), 209–219.

Goldin, G. A. (2014). Perspectives on Emotion in Mathematical Engagement, Learning, and Problem Solving. In R. Pekrun & L. Linnenbrink-Garcia (Eds.), *International handbook of emotions in education* (pp. 391–414). New York: Routledge.

Jaworski, B. (2002). Sensitivity and Challenge in University Mathematics Tutorial Teaching. *Educational Studies in Mathematics, 51*, 71–94.

Marmur, O., & Koichu, B. (2016). Surprise and the Aesthetic Experience of University Students: A Design Experiment. *Journal of Humanistic Mathematics, 6*(1), 127– 151.

Pritchard, D. (2010). Where learning starts? A framework for thinking about lectures in university mathematics. *International Journal of Mathematical Education in Science and Technology, 41*(5), 609–623.

Sfard, A. (2008). *Thinking as Communicating: Human Development, the Growth of Discourses, and Mathematizing*. New York: Cambridge University Press.

Sfard, A. (2014). University mathematics as a discourse – why, how, and what for? *Research in Mathematics Education, 16*(2), 199–203.

Simon, M. A. (1995). Reconstructing Mathematics Pedagogy from a Constructivist Perspective. *Journal for Research in Mathematics Education, 26*(2), 114–145.

Thomas, D. R. (2006). A General Inductive Approach for Analyzing Qualitative Evaluation Data. *American Journal of Evaluation, 27*(2), 237–246.

Weber, K. (2008). The role of affect in learning Real Analysis: a case study. *Research in Mathematics Education, 10*(1), 71–85.