A Nonlinear Liouville Property for the Generalized Kawahara Equation

André Kabakouala

People's Friendship University of Russia (RUDN University),
Miklukho-Maklaya str. 6, Moscow, 117198 Russian Federation.

kadodo1987@yahoo.fr

Abstract

In this paper, by applying the method of Martel and Merle [9], we prove that if the global solution of the generalized Kawahara equation (gKW) is close to the soliton (constructed by Kabakouala and Molinet [4]) at initial time in the energy space, moreover if this solution is uniformly localized, then it is identically equal to a soliton all time.

1 Introduction

The generalized Kawahara equation is given by:

$$u_t + u^p \partial_x u + \partial_x^3 u - \mu \partial_x^5 u = 0, \quad (t, x) \in \mathbb{R}_+^* \times \mathbb{R}, \quad (1.1)$$

where $p \in \mathbb{N}^*$ denotes the power of nonlinearity, and $\mu > 0$ the parameter which control the fifth-order dispersion term. For $p = 1$ and 2 , the gKW equation has applications for instance in fluid mechanics and plasma physics. For $p \geq 3$, what interests us is the balance between the nonlinear effect and the scattering effect, this leads to the formation of solitary waves.

The local Cauchy problem associated to (1.1) is locally well-posed in $H^2(\mathbb{R})$ (see for instance Abdelouhab et al. [1]). The following conservation laws hold for the $H^2(\mathbb{R})$ solutions:

$$E_{p,\mu}(u(t)) = \int_{\mathbb{R}} \left[\frac{\mu}{2} (\partial_x^2 u)^2(t) + \frac{1}{2} (\partial_x u)^2(t) - \frac{1}{(p+1)(p+2)} u^{p+2}(t) \right] = E_{p,\mu}(u_0) \quad (\text{energy}) \quad (1.2)$$

and

$$V(u(t)) = \frac{1}{2} \int_{\mathbb{R}} u^2(t) = V(u_0) \quad (\text{mass}). \quad (1.3)$$

The global existence of all solutions in $H^2(\mathbb{R})$, as well as uniform bound in $H^2(\mathbb{R})$, follow from these conserved quantities. Moreover, thanks to the conservation laws one can rewrite equation (1.1) in the Hamiltonian form:

$$\partial_t V'(u) = \partial_x E'(u), \quad (1.4)$$

where $V'(u)$ and $E'(u)$ denote respectively the Fréchet derivative of $V(u)$ and $E(u)$.

The solitons of gKW are solution of (1.1) of the form $u(t, x) = \varphi_{c,p,\mu}(x - ct)$, which travels to the right direction with the constant speed $c > 0$. Substituting u by $\varphi_{c,p,\mu}$ in (1.1), integrating on $\mathbb{R}$ with

the assumption $\partial_x^k \varphi_{c,p,\mu}(\pm\infty) = 0$ for $k = 0, \dots, 4$, we obtain the equation of gKW-solitons:

$$\mu \partial_x^4 \varphi_{c,p,\mu}(x) - \partial_x^2 \varphi_{c,p,\mu}(x) + c \varphi_{c,p,\mu}(x) = \frac{1}{p+1} \varphi_{c,p,\mu}^{p+1}(x), \quad \forall x \in \mathbb{R}. \quad (1.5)$$

In 1996 Dey, Khare and Kumar [2] compute the explicit solitons of gKW, they found:

$$\varphi_{c,p,\mu}(x) = \left[\frac{(p+1)(p+4)(3p+4)c}{8(p+2)} \right]^{1/p} \text{sech}^{4/p} \left[\frac{p\sqrt{(p^2+4p+8)c}}{4(p+2)} x \right], \quad \forall x \in \mathbb{R}, \quad (1.6)$$

with a speed completely determined when we fix the value of μ :

$$c = \frac{2^2(p+2)^2}{(p^2+4p+8)^2\mu}. \quad (1.7)$$

Due to the necessary condition of stability of the soliton $\varphi_{c,p,\mu}$ introduced by Karpman [5]:

$$\frac{\mu \int_{\mathbb{R}} (\partial_x^2 \varphi_{c,p,\mu}(x))^2}{c \int_{\mathbb{R}} \varphi_{c,p,\mu}^2(x)} > \frac{p(p-4)}{(p^2+4p+32)}, \quad (1.8)$$

Dey et al.[2] obtain that the single soliton $\varphi_{c,p,\mu}$ is unstable with respect to the small perturbation in the case $p \geq 5$.

Now we fix $\mu_p = \frac{2^2(p+2)^2}{(p^2+4p+8)^2}$ then the speed of the explicit soliton (1.6)-(1.7) is $c_0 = 1$. In Kabakouala and Molinet [4], by applying the Implicit Function Theorem in the neighborhood of φ_{1,p,μ_p} , we construct a continuous in $H^4(\mathbb{R})$ branch $\{\varphi_{c_0,p,\mu_p}, c_0 \in]1-\delta_p, 1+\delta_p[\}$, with $0 < \delta_p \ll 1$, of even solutions to equation (1.5). Moreover, for each $c_0 \in]1-\delta_p, 1+\delta_p[$, φ_{c_0,p,μ_p} is the unique even $H^4(\mathbb{R})$ solution of (1.5) in some H^4 -neighborhood of φ_{1,p,μ_p} (see Lemma 2.2). Next, we prove the orbital stability of these family by combining the continuity arguments: for c_0 close to 1, we have φ_{c_0,p,μ_p} is close to φ_{1,p,μ_p} , and the well-known sufficient condition of the stability of the soliton φ_{1,p,μ_p} introduced by Benjamin: $\langle \mathcal{L}_{1,p,\mu_p}^{-1} \varphi_{1,p,\mu_p}, \varphi_{1,p,\mu_p} \rangle_{L^2} < 0$, where $\mathcal{L}_{1,p,\mu_p} u = \mu_p \partial_x^4 u - \partial_x^2 u + u - \varphi_{1,p,\mu_p}^p u$. This stability condition has been verified numerically and is satisfied for the nonlinearity $p \in \{1, 2, 3, 4\}$ (see [4] Subsection 2.3 and Subsection 2.4).

In a short note Kabakouala [3], by using the method of Martel [7] applied on the generalized Korteweg-de Vries equation (gKdV), we prove the linear Liouville property for the gKW equation: if $u(t, x)$ is solution to the linearized gKW equation around φ_{c_0,p,μ_p} : $\partial_t u = \partial_x (\mathcal{L}_{c_0,p,\mu_p} u)$, where $\mathcal{L}_{c_0,p,\mu_p} u = \mu_p \partial_x^4 u - \partial_x^2 u + c_0 u - \varphi_{c_0,p,\mu_p}^p u$, and moreover if $u(t, x)$ is uniformly localized: $|u(t, x)| \lesssim e^{-\sigma|x|}$, then $u(t, x)$ is static and coincides with the unique element of the kernel of $\mathcal{L}_{c_0,p,\mu_p}$ which is φ'_{c_0,p,μ_p} .

In this paper, following the method of Martel and Merle [9] on the gKdV equation, we prove the nonlinear Liouville property for the gKW equation: if $u(t, x)$ is a H^2 -global solution of gKW equation close to the soliton $\varphi_{c_0,p,\mu}$ (up to a translation) in the energy space: $\|u(t, \cdot + \xi(t)) - \varphi_{c_0,p,\mu_p}\|_{H^2(\mathbb{R})} \leq \gamma$, with $\gamma \ll 1$, moreover if $u(t, x)$ is uniformly localized: $|u(t, x + \xi(t))| \leq e^{-\sigma|x|}$, then $u(t, x)$ is identically equal to a soliton $\varphi_{c_1,p,\mu_p}(x - \xi_1 - c_1 t)$, where $|c_1 - c_0| \leq \gamma$ and $|\xi_1| \leq \gamma$.

Theorem 1.1 (Nonlinear Liouville property). *Assume that $p \in \{1, 2\}$, $c_0 \in]1-\delta_p, 1+\delta_p[$, with $\delta_p \ll 1$, and $\mu_p = \frac{2^2(p+2)^2}{(p^2+4p+8)^2}$. There exists $\gamma_0 > 0$ such that if $0 < \gamma < \gamma_0$ and $u \in C(\mathbb{R}; H^2(\mathbb{R}))$ is a solution of gKW equation satisfying for some function $\xi(t)$ and $\sigma > 0$,*

$$\|u(t, \cdot + \xi(t)) - \varphi_{c_0,p,\mu_p}\|_{H^2(\mathbb{R})} \leq \gamma, \quad \forall t \in \mathbb{R}, \quad (1.9)$$

$$|u(t, x + \xi(t))| \lesssim e^{-\sigma|x|}, \quad \forall (t, x) \in \mathbb{R} \times \mathbb{R}, \quad (1.10)$$

then, there exist $c_1 > 0$ (close to 1) and $\xi_1 \in \mathbb{R}$ (close to 0), such that

$$u(t, x) = \varphi_{c_1,p,\mu_p}(x - \xi_1 - c_1 t), \quad \forall (t, x) \in \mathbb{R} \times \mathbb{R}. \quad (1.11)$$

Remark 1.1 (Stability hypothesis). Due to the stability result of Kabakouala and Molinet [4], Theorem 1.1 remains true if we assume that: $\|u_0 - \varphi_{c_0,p,\mu_p}\|_{H^2(\mathbb{R})} \leq \gamma$ instead of (1.9).

2 Nonlinear Liouville property

2.1 Preliminaries

We start by recalling the spectral properties of the operator $\mathcal{L}_{1,p,\mu_p}$, the result of existence of a family of solitons (solutions to (1.5)) in a neighborhood of the explicit soliton φ_{1,p,μ_p} of velocity 1 (defined in (1.6)-(1.7)), also we give the decomposition of the solution $u(t, x)$ around the soliton φ_{c_0,p,μ_p} .

Lemma 2.1 (Spectral properties of $\mathcal{L}_{1,p,\mu_p}$; see [4]). *Let $p \in \mathbb{N}^*$. We consider the unbounded operator $\mathcal{L}_{1,p,\mu_p} : L^2(\mathbb{R}) \rightarrow L^2(\mathbb{R})$, defined by: $u \mapsto \mu_p \partial_x^4 u - \partial_x^2 u + u - \varphi_{1,p,\mu_p}^p u$. We claim that $\mathcal{L}_{1,p,\mu_p}$ possesses, among others, the following three crucial properties:*

- (P1) *The essential spectrum of $\mathcal{L}_{1,p,\mu_p}$ is $[1, +\infty[$;*
- (P2) *$\mathcal{L}_{1,p,\mu_p}$ has only one negative eigenvalue λ_{1,p,μ_p} which is simple;*
- (P3) *The kernel of $\mathcal{L}_{1,p,\mu_p}$ is spanned by φ'_{1,p,μ_p} .*

Lemma 2.2 (Existence of solitons φ_{c_0,p,μ_p} for c_0 close to 1; see [4]). *There exist $\delta_p > 0$ and $\tilde{\delta}_p > 0$ such that for any $c_0 > 0$ with $|c_0 - 1| < \delta_p$, there exists a unique $H^4(\mathbb{R})$ even solution φ_{c_0,p,μ_p} of (1.5) in the ball of $H^4(\mathbb{R})$ centered at φ_{1,p,μ_p} with radius $\tilde{\delta}_p > 0$. Moreover, the function $c_0 \mapsto \varphi_{c_0,p,\mu_p}$ is of class C^1 from $]1 - \delta_p, 1 + \delta_p[$ into $B_{H^4}(\varphi_{1,p,\mu_p})$.*

Remark 2.1 (Important properties on φ_{c_0,p,μ_p} and $\mathcal{L}_{c_0,p,\mu_p}$ for c_0 close to 1; see [4] and [3]). We recall the following important properties and notations: $\varphi_{c_0,p,\mu_p} \in H^k(\mathbb{R})$ for all $k \in \mathbb{N}$; $|\partial_x^k \varphi_{c_0,p,\mu_p}(x)| \lesssim e^{-\sqrt{c_0}|x|}$, for all $x \in \mathbb{R}$ and $\forall k \in \mathbb{N}$; $\mathcal{L}_{c,p,\mu_p}(-\frac{d}{dc}\varphi_{c,p,\mu_p}) = \varphi_{c,p,\mu_p}$ and we set $\rho_{c_0,p,\mu_p} = -(\frac{d}{dc}\varphi_{c,p,\mu_p})_{c=c_0}$; $\ker(\mathcal{L}_{c_0,p,\mu_p}) = \{0, \varphi'_{c_0,p,\mu_p}\}$; $\langle \mathcal{L}_{1,p,\mu_p}^{-1} \varphi_{1,p,\mu_p}, \varphi_{1,p,\mu_p} \rangle_{L^2} < 0$ for $p \in \{1, 2, 3, 4\}$; and $\langle \mathcal{L}_{1,p,\mu_p}^{-2} \varphi_{1,p,\mu_p}, \mathcal{L}_{1,p,\mu_p}^{-1} \varphi_{1,p,\mu_p} \rangle_{L^2} < 0$ for $p \in \{1, 2\}$.

Lemma 2.3 (Decomposition of u around φ_{c_0,p,μ_p}). *Assume that $p = \{1, 2, 3, 4\}$, $c_0 \in]1 - \delta_p, 1 + \delta_p[$ with $\delta_p \ll 1$, and $\mu_p = \frac{2^2(p+2)^2}{(p^2+4p+8)^2}$. There exist $\gamma_0 > 0$ and $K_0 > 0$ such that for $0 < \gamma < \gamma_0$ the following is true. For any $u \in C(\mathbb{R}; H^2(\mathbb{R}))$ solution of gKW equation satisfying:*

$$\inf_{z \in \mathbb{R}} \|u(t, \cdot) - \varphi_{c_0,p,\mu_p}(\cdot - z)\|_{H^2(\mathbb{R})} \leq \gamma, \quad \forall t \in \mathbb{R}, \quad (2.1)$$

there exist $c(t) \in C^1(\mathbb{R})$ and $\xi(t) \in C^1(\mathbb{R})$ such that

$$\eta(t, x) = u(t, x + \xi(t)) - \varphi_{c(t),p,\mu_p}(x), \quad (2.2)$$

satisfies for all $t \in \mathbb{R}$,

$$|c(t) - 1| + \|\eta(t, \cdot)\|_{H^2(\mathbb{R})} \leq K_0 \gamma, \quad (2.3)$$

$$\langle \eta(t, \cdot), \varphi'_{c(t)} \rangle_{L^2} = \langle \eta(t, \cdot), \varphi_{c(t)} \rangle_{L^2} = 0, \quad (2.4)$$

and

$$|c'(t)|^{1/2} + |\xi'(t) - c(t)| \leq K_0 \|\eta\|_{L^2(\mathbb{R})}. \quad (2.5)$$

Sketch of proof. The proof of Lemma 2.3 is a classical application of the Implicit Function Theorem (see for instance: Martel and Merle [8]). We just give the nondegeneracy conditions to satisfy the orthogonality conditions (2.4). Since c_0 is close to 1, then by Lemma 2.2 we have

$$\|\varphi'_{c_0,p,\mu_p}\|_{L^2(\mathbb{R})}^2 \sim \|\varphi'_{1,p,\mu_p}\|_{L^2(\mathbb{R})}^2 > 0, \quad (2.6)$$

and

$$\langle \rho_{c_0,p,\mu_p}, \varphi_{c_0,p,\mu_p} \rangle_{L^2} \sim \langle \rho_{1,p,\mu_p}, \varphi_{1,p,\mu_p} \rangle_{L^2} < 0 \quad \text{for } 1 \leq p \leq 4, \quad (2.7)$$

where ρ_{c_0,p,μ_p} is such that $\mathcal{L}_{c_0,p,\mu_p} \rho_{c_0,p,\mu_p} = \varphi_{c_0,p,\mu_p}$.

Let us prove the control of modulation parameters (2.5). We set $f(u) = [1/(p+1)]u^{p+1}$ and we recall that $\rho_{c,p,\mu_p} = -\partial_c \varphi_{c,p,\mu_p}$. By differentiating (2.2) with respect to time and using equation of $u(t, x)$ and φ_{c,p,μ_p} , we get the time variation of $\eta(t, x)$:

$$\begin{aligned} \partial_t \eta &= \partial_t u + \xi' \partial_x u + c' \rho_{c,p,\mu_p} \\ &= \partial_x [\mu_p \partial_x^4 \eta - \partial_x^2 \eta + c\eta - (f(\varphi_{c,p,\mu_p} + \eta) - f(\varphi_{c,p,\mu_p}))] + (\xi' - c) \partial_x (\varphi_{c,p,\mu_p} + \eta) + c' \rho_{c,p,\mu_p} \\ &= \partial_x [\mathcal{L}_{c,p,\mu_p} \eta - (f(\varphi_{c,p,\mu_p} + \eta) - f(\varphi_{c,p,\mu_p}) - f'(\varphi_{c,p,\mu_p})\eta)] + (\xi' - c) \partial_x (\varphi_{c,p,\mu_p} + \eta) + c' \rho_{c,p,\mu_p}. \end{aligned} \quad (2.8)$$

Now, differentiating equation $\langle \eta, \varphi_{c,p,\mu_p} \rangle_{L^2} = 0$ with respect to time, using (2.8) and integration by parts, we have

$$\begin{aligned} & - \left\langle \mathcal{L}_{c,p,\mu_p} \eta - (f(\varphi_{c,p,\mu_p} + \eta) - f(\varphi_{c,p,\mu_p}) - f'(\varphi_{c,p,\mu_p})\eta), \varphi'_{c,p,\mu_p} \right\rangle_{L^2} \\ & - (\xi' - c) \left\langle \varphi_{c,p,\mu_p} + \eta, \varphi'_{c,p,\mu_p} \right\rangle_{L^2} + c' \left\langle \varphi_{c,p,\mu_p}, \rho_{c,p,\mu_p} \right\rangle_{L^2} - c' \left\langle \eta, \rho_{c,p,\mu_p} \right\rangle_{L^2} = 0. \end{aligned} \quad (2.9)$$

One can notice that

$$f(\varphi_{c,p,\mu_p} + \eta) - f(\varphi_{c,p,\mu_p}) - f'(\varphi_{c,p,\mu_p})\eta = [1/(p+1)] \sum_{k=2}^{p+1} C_k^{p+1} \varphi_{c,p,\mu_p}^{p-k+1} \eta^k. \quad (2.10)$$

Then, using that $\mathcal{L}_{c,p,\mu_p}(\varphi'_{c,p,\mu_p}) = 0$, $\langle \varphi_{c,p,\mu_p} + \eta, \varphi'_{c,p,\mu_p} \rangle_{L^2} = 0$, and combining (2.3), (2.7), (2.9) and (2.10), we obtain

$$\begin{aligned} |c'| &\leq \frac{\frac{p}{2} \left| \left\langle \varphi_{c,p,\mu_p}^{p-1} \varphi'_{c,p,\mu_p}, \eta^2 \right\rangle_{L^2} \right| + [1/(p+1)] \left| \sum_{k=3}^{p+1} C_k^{p+1} \left\langle \varphi_{c,p,\mu_p}^{p-k+1} \varphi'_{c,p,\mu_p}, \eta^k \right\rangle_{L^2} \right|}{\left| \left\langle \varphi_{c,p,\mu_p}, \rho_{c,p,\mu_p} \right\rangle_{L^2} - \left\langle \eta, \rho_{c,p,\mu_p} \right\rangle_{L^2} \right|} \\ &\leq \frac{\left(\frac{p}{2} \|\varphi_{c,p,\mu_p}\|_{L^\infty(\mathbb{R})}^{p-1} \|\varphi'_{c,p,\mu_p}\|_{L^\infty(\mathbb{R})} + K_0 \gamma_0 \right) \|\eta\|_{L^2(\mathbb{R})}^2}{-\left\langle \varphi_{c,p,\mu_p}, \rho_{c,p,\mu_p} \right\rangle_{L^2} - K_0 \gamma_0} \\ &\leq K_0 \|\eta\|_{L^2(\mathbb{R})}^2. \end{aligned} \quad (2.11)$$

Similarly, differentiating equation $\langle \eta, \varphi'_{c,p,\mu_p} \rangle_{L^2} = 0$ with respect to time, using (2.8) and integration by parts, we have

$$\begin{aligned} & - \left\langle \mathcal{L}_{c,p,\mu_p} \eta - (f(\varphi_{c,p,\mu_p} + \eta) - f(\varphi_{c,p,\mu_p}) - f'(\varphi_{c,p,\mu_p})\eta), \varphi''_{c,p,\mu_p} \right\rangle_{L^2} \\ & - (\xi' - c) \left\langle (\varphi_{c,p,\mu_p} + \eta), \varphi''_{c,p,\mu_p} \right\rangle_{L^2} + c' \left\langle \varphi'_{c,p,\mu_p}, \rho_{c,p,\mu_p} \right\rangle_{L^2} - c' \left\langle \eta, \rho'_{c,p,\mu_p} \right\rangle_{L^2} = 0. \end{aligned} \quad (2.12)$$

Next, using that $\langle \varphi'_{c,p,\mu_p}, \rho_{c,p,\mu_p} \rangle_{L^2} = 0$, (2.3), (2.6) and (2.11), we get

$$\begin{aligned}
|\xi' - c| &\leq \frac{\left| \langle \eta, \mathcal{L}_{c,p,\mu_p} \varphi''_{c,p,\mu_p} \rangle_{L^2} \right| + \left| c' \langle \eta, \rho'_{c,p,\mu_p} \rangle_{L^2} \right|}{\left| \langle \varphi_{c,p,\mu_p} + \eta, \varphi''_{c,p,\mu_p} \rangle_{L^2} \right|} \\
&\quad + \frac{[1/(p+1)] \sum_{k=2}^{p+1} C_k^{p+1} \left| \langle \varphi_{c,p,\mu_p}^{p-k+1} \varphi''_{c,p,\mu_p}, \eta^k \rangle_{L^2} \right|}{\left| \langle \varphi_{c,p,\mu_p} + \eta, \varphi''_{c,p,\mu_p} \rangle_{L^2} \right|} \\
&\leq \frac{\|\mathcal{L}_{c,p,\mu_p} \varphi''_{c,p,\mu_p}\|_{L^2(\mathbb{R})} \|\eta\|_{L^2(\mathbb{R})} + |c'| \|\rho'_{c,p,\mu_p}\|_{L^2(\mathbb{R})} \|\eta\|_{L^2(\mathbb{R})}}{\|\varphi'_{c,p,\mu_p}\|_{L^2(\mathbb{R})}^2 - K_0 \gamma_0} \\
&\quad + \frac{\left(\frac{p}{2} \|\varphi_{c,p,\mu_p}\|_{L^\infty(\mathbb{R})}^{p-1} \|\varphi''_{c,p,\mu_p}\|_{L^\infty(\mathbb{R})} + K_0 \gamma_0 \right) \|\eta\|_{L^2(\mathbb{R})}^2}{\|\varphi'_{c,p,\mu_p}\|_{L^2(\mathbb{R})}^2 - K_0 \gamma_0} \\
&\leq K_0 \|\eta\|_{L^2(\mathbb{R})}, \tag{2.13}
\end{aligned}$$

and this prove (2.5).

Lemma 2.4 (Smoothness and Exponential Decay). *Let $u \in C(\mathbb{R}; H^2(\mathbb{R}))$ be a solution of gKW equation satisfying (2.1)-(2.5) and the decay property (1.10). Then $u \in C^\infty(\mathbb{R} \times \mathbb{R})$. Moreover, there exists $\sigma > 0$ such that for all $k \in \mathbb{N}$,*

$$|\partial_x^k u(t, x + \xi(t))| \lesssim e^{-\sigma|x|}, \quad \forall (t, x) \in \mathbb{R} \times \mathbb{R}. \tag{2.14}$$

The proof of Lemma 2.4 is based on a monotonicity property and is given in the Subsection 2.3.

2.2 Proof of Theorem 1.1

We divide the proof in four steps and we argue as Martel and Merle [9].

Step 1. Nonlinear dual problem related to gKW equation.

We set

$$\begin{aligned}
v &= \mu_p \partial_x^4 \eta - \partial_x^2 \eta + c\eta - (f(\varphi_{c,p,\mu_p} + \eta) - f(\varphi_{c,p,\mu_p})) \\
&= \mathcal{L}_{c,p,\mu_p} \eta - (f(\varphi_{c,p,\mu_p} + \eta) - f(\varphi_{c,p,\mu_p}) - f'(\varphi_{c,p,\mu_p}) \eta). \tag{2.15}
\end{aligned}$$

By differentiating $v(x, t)$ with respect to x , we get

$$\partial_x v = \mu_p \partial_x^5 \eta - \partial_x^3 \eta + c \partial_x v - \partial_x \eta f'(\varphi_{c,p,\mu_p} + \eta) - \varphi'_{c,p,\mu_p} (f'(\varphi_{c,p,\mu_p} + \eta) - f'(\varphi_{c,p,\mu_p})). \tag{2.16}$$

Now, using (2.8), the time variation of $v(x, t)$ gives us

$$\begin{aligned}
\partial_t v &= \mu_p \partial_x^4 \eta_t - \partial_x^2 \eta_t + c \eta_t - \eta_t f'(\varphi_{c,p,\mu_p} + \eta) + c' \eta + c' \rho_{c,p,\mu_p} (f'(\varphi_{c,p,\mu_p} + \eta) - f'(\varphi_{c,p,\mu_p})) \\
&= \mu_p \partial_x^5 v - \partial_x^3 v + c \partial_x v - \partial_x v f'(\varphi_{c,p,\mu_p} + \eta) \\
&\quad + (\xi' - c) (\mu_p \partial_x^4 (\varphi_{c,p,\mu_p} + \eta) - \partial_x^3 (\varphi_{c,p,\mu_p} + \eta) + c \partial_x (\varphi_{c,p,\mu_p} + \eta) - \partial_x (\varphi_{c,p,\mu_p} + \eta) f'(\varphi_{c,p,\mu_p} + \eta)) \\
&\quad + c' (\mu_p \partial_x^4 \rho_{c,p,\mu_p} - \partial_x^2 \rho_{c,p,\mu_p} + c \rho_{c,p,\mu_p} - \rho_{c,p,\mu_p} f'(\varphi_{c,p,\mu_p})) + c' \eta. \tag{2.17}
\end{aligned}$$

Then, using that $\mathcal{L}_{c,p,\mu_p} \rho_{c,p,\mu_p} = \varphi_{c,p,\mu_p}$, $\mathcal{L}_{c,p,\mu_p} \varphi'_{c,p,\mu_p} = 0$ and (2.16), it holds

$$\partial_t v = \mathcal{L}_{c,p,\mu_p} (\partial_x v) - \partial_x v (f'(\varphi_{c,p,\mu_p} + \eta) - f'(\varphi_{c,p,\mu_p})) + (\xi' - c) \partial_x v + c' (\varphi_{c,p,\mu_p} + \eta). \tag{2.18}$$

Step 2. Almost orthogonality conditions of $v(t, x)$ with respect to $\varphi'_{1,p,\mu_p}(x)$ and $\rho_{1,p,\mu_p}(x)$, and positivity property of the quadratic form $\mathcal{L}_{c,p,\mu_p}$ with respect to $v(t, x)$.

We start with the almost orthogonality conditions. Using that $\mathcal{L}_{c,p,\mu_p} \varphi'_{c,p,\mu_p} = 0$, (2.10) and (2.15), we have

$$\begin{aligned} |\langle v, \varphi'_{c,p,\mu_p} \rangle_{L^2}| &= |\langle \mathcal{L}_{c,p,\mu_p} \eta - (f(\varphi_{c,p,\mu_p} + \eta) - f(\varphi_{c,p,\mu_p}) - f'(\varphi_{c,p,\mu_p})\eta), \varphi'_{c,p,\mu_p} \rangle_{L^2}| \\ &\leq \left(\frac{p}{2} \|\varphi_{c,p,\mu_p}\|_{L^\infty(\mathbb{R})}^{p-1} \|\varphi'_{c,p,\mu_p}\|_{L^\infty(\mathbb{R})} + K_0 \gamma_0 \right) \|\eta\|_{L^2(\mathbb{R})}^2 \\ &\leq K_0 \|\eta\|_{L^2(\mathbb{R})}^2. \end{aligned} \quad (2.19)$$

Next, using $\langle \eta, \varphi_{c,p,\mu_p} \rangle_{L^2} = 0$, $\mathcal{L}_{c,p,\mu_p} \rho_{c,p,\mu_p} = \varphi_{c,p,\mu_p}$, (2.10) and (2.15), we have

$$\begin{aligned} |\langle v, \rho_{c,p,\mu_p} \rangle_{L^2}| &= |\langle \mathcal{L}_{c,p,\mu_p} \eta - (f(\varphi_{c,p,\mu_p} + \eta) - f(\varphi_{c,p,\mu_p}) - f'(\varphi_{c,p,\mu_p})\eta), \rho_{c,p,\mu_p} \rangle_{L^2}| \\ &\leq \left(\frac{p}{2} \|\varphi_{c,p,\mu_p}\|_{L^\infty(\mathbb{R})}^{p-1} \|\rho_{c,p,\mu_p}\|_{L^\infty(\mathbb{R})} + K_0 \gamma_0 \right) \|\eta\|_{L^2(\mathbb{R})}^2 \\ &\leq K_0 \|\eta\|_{L^2(\mathbb{R})}^2. \end{aligned} \quad (2.20)$$

From (2.4), we know that $\eta \in \left\{ \varphi'_{c,p,\mu_p}, \varphi_{c,p,\mu_p} \right\}^\perp$, then $|\langle \eta, \varphi'_{1,p,\mu_p} \rangle_{L^2}| + |\langle \eta, \varphi_{1,p,\mu_p} \rangle_{L^2}| \lesssim \gamma_0 \|\eta\|_{L^2(\mathbb{R})}$ (because $|c-1| \leq \gamma_0$). Since $\langle \rho_{1,p,\mu_p}, \varphi_{1,p,\mu_p} \rangle_{L^2} < 0$ for $1 \leq p \leq 4$, arguing as Kabakouala and Molinet [4] (see Subsections 2.3 and 2.4), we infer that there exists $C_0 > 0$ such that

$$\langle \mathcal{L}_{c,p,\mu_p} \eta, \eta \rangle_{L^2} \sim \langle \mathcal{L}_{1,p,\mu_p} \eta, \eta \rangle_{L^2} \geq C_0 \|\eta\|_{L^2(\mathbb{R})}^2. \quad (2.21)$$

Now, combining (2.3) and (2.21), we have

$$\begin{aligned} \langle v, \eta \rangle_{L^2} &= \langle \mathcal{L}_{c,p,\mu_p} \eta, \eta \rangle_{L^2} - \langle f(\varphi_{c,p,\mu_p} + \eta) - f(\varphi_{c,p,\mu_p}) - f'(\varphi_{c,p,\mu_p})\eta, \eta \rangle_{L^2} \\ &\geq C_0 \|\eta\|_{L^2(\mathbb{R})}^2 - \left(\frac{p}{2} \|\varphi_{c,p,\mu_p}\|_{L^\infty(\mathbb{R})}^{p-1} + K_0 \gamma_0 \right) \|\eta\|_{L^3(\mathbb{R})}^3 \\ &\geq K_0 \|\eta\|_{L^2(\mathbb{R})}^2. \end{aligned} \quad (2.22)$$

Thus, applying Cauchy-Schwarz inequality to $\langle v, \eta \rangle_{L^2}$ and using (2.22), we obtain

$$\|\eta\|_{L^2(\mathbb{R})} \leq K_0 \|v\|_{L^2(\mathbb{R})}. \quad (2.23)$$

Finally, combining (2.19)-(2.20) and (2.23), and using that by continuity φ_{c,p,μ_p} (respectively ρ_{c,p,μ_p}) is close to φ_{1,p,μ_p} in $H^4(\mathbb{R})$ (respectively to ρ_{c,p,μ_p}) when c is close to 1, we infer that $|\langle v, \varphi'_{1,p,\mu_p} \rangle_{L^2}| + |\langle v, \rho_{1,p,\mu_p} \rangle_{L^2}| \lesssim \gamma_0 \|v\|_{L^2(\mathbb{R})}$. Since $\langle \mathcal{L}_{1,p,\mu_p}^{-1} \rho_{1,p,\mu_p}, \rho_{1,p,\mu_p} \rangle < 0$ for $1 \leq p \leq 2$, once again, proceeding as Kabakouala and Molinet [4] (see Subsection 2.3 and 2.4), we deduce that there exists $C_1 > 0$ such that

$$\langle \mathcal{L}_{c,p,\mu_p} v, v \rangle_{L^2} \sim \langle \mathcal{L}_{1,p,\mu_p} v, v \rangle_{L^2} \geq C_1 \|v\|_{L^2(\mathbb{R})}^2. \quad (2.24)$$

Step 3. Virial identity related to (2.18).

By multiplying (2.18) with the function $v(t, x)x$, and using the computations done on the Linear Dual Problem (see Kabakouala [3], Subsection 2.2, p. 5 – 6), we establish that

$$\begin{aligned} -\frac{d}{dt} \int_{\mathbb{R}} v^2 x &= -2 \int_{\mathbb{R}} \mathcal{L}_{c,p,\mu_p} (\partial_x v) v x + \mathcal{R}(\eta, v) \\ &= 5\mu_p \int_{\mathbb{R}} (\partial_x^2 v)^2 + 3 \int_{\mathbb{R}} (\partial_x v)^2 + c \int_{\mathbb{R}} v^2 - \int_{\mathbb{R}} v^2 \varphi_{c,p,\mu_p}^p - p \int_{\mathbb{R}} v^2 \varphi_{c,p,\mu_p}^{p-1} \varphi'_{c,p,\mu_p} x + \mathcal{R}(\eta, v), \end{aligned} \quad (2.25)$$

with

$$\begin{aligned} \mathcal{R}(\eta, v) &= - \int_{\mathbb{R}} v^2 (f'(\varphi_{c,p,\mu_p} + \eta) - f'(\varphi_{c,p,\mu_p})) - \int_{\mathbb{R}} v^2 (f''(\varphi_{c,p,\mu_p} + \eta) - f''(\varphi_{c,p,\mu_p})) \varphi'_{c,p,\mu_p} x \\ &\quad - \int_{\mathbb{R}} v^2 f''(\varphi_{c,p,\mu_p} + \eta) \eta' x + (\xi' - c) \int_{\mathbb{R}} v^2 - 2c' \int_{\mathbb{R}} v(\varphi_{c,p,\mu_p} + \eta) x. \end{aligned} \quad (2.26)$$

Let us estimate the term $\mathcal{R}(v, \eta)$. Using that $\|\eta\|_{C^1(\mathbb{R})} \leq C_S \|\eta\|_{H^2(\mathbb{R})}$ (by Sobolev embedding), $f'(\varphi_{c,p,\mu_p} + \eta) - f'(\varphi_{c,p,\mu_p}) = \sum_{k=1}^p C_k^p \varphi_{c,p,\mu_p}^{p-k} \eta^k = 0$, (2.5) and (2.23), we have the following estimates:

$$\begin{aligned} \left| \int_{\mathbb{R}} v^2 (f'(\varphi_{c,p,\mu_p} + \eta) - f'(\varphi_{c,p,\mu_p})) \right| &\leq \|v\|_{L^2(\mathbb{R})}^2 \sum_{k=1}^p C_k^p \|\eta^k\|_{L^\infty(\mathbb{R})} \|\varphi_{c,p,\mu_p}^{p-k}\|_{L^\infty(\mathbb{R})} \\ &\leq K_0 \|\eta\|_{L^2(\mathbb{R})} \|v\|_{H^2(\mathbb{R})}^2 \end{aligned} \quad (2.27)$$

and

$$\left| (\xi' - c) \int_{\mathbb{R}} v^2 \right| \leq K_0 \|\eta\|_{L^2(\mathbb{R})} \|v\|_{L^2(\mathbb{R})}^2, \quad (2.28)$$

$$\begin{aligned} \left| c' \int_{\mathbb{R}} v(\varphi_{c,p,\mu_p} + \eta)x \right| &\leq |c'| \|(\varphi_{c,p,\mu_p} + \eta)x\|_{L^2(\mathbb{R})} \|v\|_{L^2(\mathbb{R})} \\ &\leq K_0 \|\eta\|_{L^2(\mathbb{R})} \|v\|_{L^2(\mathbb{R})}^2. \end{aligned} \quad (2.29)$$

Similarly for the other terms. Thus, we deduce that

$$|\mathcal{R}(v, \eta)| \leq K_0 \|\eta\|_{L^2(\mathbb{R})} \|v\|_{H^2(\mathbb{R})}^2. \quad (2.30)$$

Step 4. Conclusion of the proof of Theorem 1.1.

At this stage, we have all the ingredients to prove Theorem 1.1. First, using the definition of $\mathcal{L}_{c,p,\mu_p}$, one can rewrite identity (2.25) as

$$-\frac{d}{dt} \int_{\mathbb{R}} v^2 x = \langle \mathcal{L}_{c,p,\mu_p} v, v \rangle_{L^2} + 4\mu_p \int_{\mathbb{R}} (\partial_x^2 v)^2 + 2 \int_{\mathbb{R}} (\partial_x v)^2 - p \int_{\mathbb{R}} v^2 \varphi_{c,p,\mu_p}^{p-1} \varphi'_{c,p,\mu_p} x + \mathcal{R}(v, \eta). \quad (2.31)$$

With the last integral we deal as follow, let us set $\varphi_{c_0,p,\mu_p}(x) = \varphi_{1,p,\mu_p}(x) + (\varphi_{c_0,p,\mu_p}(x) - \varphi_{1,p,\mu_p}(x)) = \varphi_{c_0,p,\mu_p}(x) + \gamma(x)$. For c_0 close to 1, we have $\sup_{x \in \mathbb{R}} |\gamma(x)| + \sup_{x \in \mathbb{R}} |\gamma'(x)| \leq \gamma$, with $0 < \gamma \ll 1$, and $|\gamma(x)| + |\gamma'(x)| \lesssim e^{-\sqrt{\min(c_0,1)}|x|}$ for all $x \in \mathbb{R}$. Then we deduce the following estimate:

$$-p \int_{\mathbb{R}} v^2 \varphi_{c_0,p,\mu_p}^{p-1} \varphi'_{c_0,p,\mu_p} x \geq -p \int_{\mathbb{R}} v^2 \varphi_{1,p,\mu_p}^{p-1} \varphi'_{1,p,\mu_p} x - \gamma \int_{\mathbb{R}} v^2. \quad (2.32)$$

Since $\varphi_{c_0,p,\mu_p}(x) > 0$ and $-\varphi'_{c_0,p,\mu_p}(x)x > 0$ for all $x \in \mathbb{R}$, we get

$$-p \int_{\mathbb{R}} v^2 \varphi_{c_0,p,\mu_p}^{p-1} \varphi'_{c_0,p,\mu_p} x \geq -\gamma \int_{\mathbb{R}} v^2. \quad (2.33)$$

Next, combining (2.3), (2.24), (2.30), (2.31) and (2.33), we infer the following inequality:

$$-\frac{d}{dt} \int_{\mathbb{R}} v^2 x \geq (C_1 - K_0 \gamma_0) \|v\|_{H^2(\mathbb{R})}^2 \geq C_2 \|v\|_{H^2(\mathbb{R})}^2. \quad (2.34)$$

Now, Lemma 2.4, the definition of $\mathcal{L}_{c_0,p,\mu_p}$, the smoothness and exponential decay properties of φ_{c_0,p,μ_p} , imply that $v \in C^\infty(\mathbb{R} \times \mathbb{R})$, and there exists $\sigma_0 > 0$ such that

$$|v(t, x)| \lesssim e^{-\sigma_0|x|}, \quad \forall (t, x) \in \mathbb{R} \times \mathbb{R}. \quad (2.35)$$

For all $t \in \mathbb{R}$, we set

$$\mathcal{J}(t) = \int_{\mathbb{R}} v^2(t, x)x. \quad (2.36)$$

Thanks to (2.35), one can check that for all $t \in \mathbb{R}$,

$$|\mathcal{J}(t)| \leq \int_{\mathbb{R}} v^2(t, x)|x| \lesssim \int_{\mathbb{R}} e^{-2\sigma|x|}|x| = \frac{1}{2\sigma_1^2}. \quad (2.37)$$

Now, integrating (2.34) with respect to time and using (2.37), we obtain

$$\int_{-\infty}^{+\infty} \|v(t)\|_{H^2(\mathbb{R})}^2 dt \leq C_1^{-1}(\mathcal{J}(-\infty) - \mathcal{J}(+\infty)) < +\infty. \quad (2.38)$$

It follows that for a time sequence $t_n \rightarrow +\infty$, we have $v(t_n) \rightarrow 0$ in $H^2(\mathbb{R})$. In particular, we have

$$|\mathcal{J}(t_n)| \leq \|v(t_n)x\|_{L^2(\mathbb{R})} \|v(t_n)\|_{L^2(\mathbb{R})} \lesssim \frac{1}{\sqrt{2\sigma^3_1}} \|v(t_n)\|_{L^2(\mathbb{R})} \rightarrow 0 \quad \text{as } t_n \rightarrow +\infty. \quad (2.39)$$

We infer that $\mathcal{J}(+\infty) = 0$. Similarly, $\mathcal{J}(-\infty) = 0$. Thus (2.38) becomes

$$\int_{-\infty}^{+\infty} \|v(t)\|_{H^2(\mathbb{R})}^2 dt = 0,$$

which implies that $v(t, x) \equiv 0$ for all $(t, x) \in \mathbb{R} \times \mathbb{R}$. Then, from (2.23), we infer that $\eta(t, x) = 0$, and from (2.5), we have $c(t) = c(0)$ and $\xi(t) = \xi(0) + c(0)t$. We set $c(0) = c_1$ and $\xi(0) = \xi_1$, thus (2.2) leads to $u(t, x) = \varphi_{c_1}(x - \xi_1 - c_1 t)$ for all $(t, x) \in \mathbb{R} \times \mathbb{R}$. This proves (1.11).

2.3 Monotonicity property related to the gKW equation

In this section we give the proof of Lemma 2.4. We follow the arguments of Laurent and Martel [6] with the gKdV equation, and we present the proof in six steps.

We consider the same test function: $\phi(x) = \frac{1}{2} + \frac{1}{2} \tanh(\frac{x}{2})$ for $x \in \mathbb{R}$, as in Kabakouala [3] (Subsection 2.3). One can easily check the following properties: $\phi'(x) = \frac{1}{4} \text{sech}^2(\frac{x}{2})$ and $|\phi^{(j+1)}(x)| \lesssim \phi'(x)$, for all $j \in \mathbb{N}^*$, and for all $x \in \mathbb{R}$. Let $K > 0$, we set $\phi_K(x) = \phi(x/K)$. Then we deduce that $|\phi_K^{(j+1)}(x)| \leq \frac{1}{K^j} \phi'_K(x)$.

Let $x_0 > 0$ and $t_0 \in \mathbb{R}$. For all $t \leq t_0$, we set $\bar{x} = x - \xi(t_0) + \frac{c_0}{2}(t_0 - t) - x_0$, and we define the following L^2 -localized energy:

$$I_{x_0, t_0}(t) = \int_{\mathbb{R}} u^2(t, x) \phi_K(\bar{x}) dx. \quad (2.40)$$

We start with the almost monotonicity property of the functional $I_{x_0, t_0}(t)$.

Step 1. For $x_0 > 0$ and $t \leq t_0$, we claim that:

$$I_{x_0, t_0}(t_0) - I_{x_0, t_0}(t) \lesssim e^{-x_0/K}. \quad (2.41)$$

Proof of (2.41). Applying the Virial Type Identity I (see Subsection 2.4), the time variation of $I_{x_0, t_0}(t)$ leads to

$$\begin{aligned} \frac{d}{dt} I_{x_0, t_0}(t) &= -5\mu_p \int_{\mathbb{R}} (\partial_x^2 u)^2 \phi'_K(\bar{x}) - 3 \int_{\mathbb{R}} (\partial_x u)^2 \phi'_K(\bar{x}) - \frac{c_0}{2} \int_{\mathbb{R}} u^2 \phi'_K(\bar{x}) \\ &\quad + 5\mu_p \int_{\mathbb{R}} (\partial_x u)^2 \phi_K'''(\bar{x}) - \mu_p \int_{\mathbb{R}} u^2 \phi_K^{(5)}(\bar{x}) + \int_{\mathbb{R}} u^2 \phi_K'''(\bar{x}) \\ &\quad + \frac{2}{p+2} \int_{\mathbb{R}} u^{p+2} \phi'_K(\bar{x}). \end{aligned} \quad (2.42)$$

Note that the negative term: $-5\mu_p \int_{\mathbb{R}} (\partial_x^2 u)^2 \phi'_K(\bar{x})$ will permit us to gain regularity in the next steps.

Let us make the estimate of the terms in the second line of (2.42). Using that $|\phi_K^{(j+1)}| \lesssim \frac{1}{K^j} \phi'_K$ on $\mathbb{R}$, for $j = 1, \dots, 4$, and $\phi'_K > 0$ on $\mathbb{R}$, we get:

$$\begin{aligned} &\left| 5\mu_p \int_{\mathbb{R}} (\partial_x u)^2 \phi_K'''(\bar{x}) - \mu_p \int_{\mathbb{R}} u^2 \phi_K^{(5)}(\bar{x}) + \int_{\mathbb{R}} u^2 \phi_K'''(\bar{x}) \right| \\ &\lesssim \frac{1}{K^2} \int_{\mathbb{R}} ((\partial_x u)^2 + u^2) \phi'_K(\bar{x}). \end{aligned} \quad (2.43)$$

Now, let us make the estimate of the nonlinear term: $\int_{\mathbb{R}} u^{p+2} \phi'_K(\bar{x})$. Let $R_0 > 0$, we split the line in two regions:

$$\begin{aligned} \frac{2}{p+2} \int_{\mathbb{R}} u^{p+2} \phi'_K(\bar{x}) &= \frac{2}{p+2} \int_{|x-\xi(t)| > R_0} u^{p+2} \phi'_K(\bar{x}) + \frac{2}{p+2} \int_{|x-\xi(t)| < R_0} u^{p+2} \phi'_K(\bar{x}) \\ &= A + B. \end{aligned} \quad (2.44)$$

First, since $\|u((t, \cdot) - \varphi_{c_0, p, \mu_p}(\cdot - \xi(t)))\|_{C^0(\mathbb{R})} \leq \gamma_0$ (by the Sobolev embedding of $H^1(\mathbb{R})$ into $C^0(\mathbb{R})$), we have $\sup_{x \in \mathbb{R}} |u(t, x)| \leq \sup_{x \in \mathbb{R}} |\varphi_{c_0, p, \mu_p}(x - \xi(t))| + \gamma_0$. Then it holds

$$|A| \lesssim (e^{-\sqrt{c_0} R_0} + \gamma_0)^p \int_{\mathbb{R}} u^2 \phi'_K(\bar{x}). \quad (2.45)$$

Second, using that $\dot{\xi}(t) \geq 3c_0/4$, one can see that: $|\bar{x}| \geq -|x - \xi(t)| + |(\xi(t_0) - \xi(t)) - \frac{c_0}{2}(t_0 - t) + x_0| \geq -R_0 + \frac{c_0}{4}(t_0 - t) + x_0$. Then, using that $\phi'_K \lesssim e^{-|\cdot|/K}$ on $\mathbb{R}$, and that $u(t)$ is uniformly bounded in $H^1(\mathbb{R})$, it holds

$$|B| \lesssim e^{-x_0/K} e^{-c_0(t_0 - t)/4K}. \quad (2.46)$$

Finally, combining (2.42)-(2.46), taking $R_0 \gg 1$ and $\gamma \ll 1$, and integrating $\frac{d}{dt} I_{x_0, t_0}(t)$ between $[t, t_0]$, we obtain

$$I_{x_0, t_0}(t_0) + C_0 \int_t^{t_0} \int_{\mathbb{R}} ((\partial_x^2 u)^2 + (\partial_x u)^2 + u^2) (s, x) \phi'_K(\bar{x}) dx ds \lesssim e^{-x_0/K} + I_{x_0, t_0}(t), \quad (2.47)$$

where C_0 is a positive constant. This gives in particular (2.41).

We present now an immediate consequence of Step 1 and the exponential decay assumption on $u(t, x)$.

Step 2. For $x_0 > 0$ and $t_0 \in \mathbb{R}$, we claim that:

$$\int_{\mathbb{R}} u^2(t_0, x) \phi_K(x - \xi(t_0) - x_0) dx + C_0 \int_{-\infty}^{t_0} \int_{\mathbb{R}} ((\partial_x^2 u)^2 + (\partial_x u)^2 + u^2) (s, x) \phi'_K(\bar{x}) dx ds \lesssim e^{-x_0/K}. \quad (2.48)$$

Proof of (2.48). In view of Step 1, it suffices to prove that: $I_{x_0, t_0}(t) \rightarrow 0$ as $t \rightarrow -\infty$. Let $R_1 > 0$, we divide $\mathbb{R}$ in two parts: $\{x < R_1\}$ and $\{x > R_1\}$, and we proceed as follow:

$$I_{x_0, t_0}(t) = \int_{x > R_1} u^2(t, x + \xi(t)) \phi_K(\bar{x} + \xi(t)) + \int_{x < R_1} u^2(t, x + \xi(t)) \phi_K(\bar{x} + \xi(t)) = A + B. \quad (2.49)$$

First, using that $|u(t, x + \xi(t))| \leq e^{-\sigma|x|}$ and $\|\phi_K\|_{L^\infty(\mathbb{R})} \leq 1$, it holds:

$$A \lesssim e^{-2\sigma R_1}. \quad (2.50)$$

Second, using that $\dot{\xi}(t) \geq 3c_0/4$, one can check that: $\bar{x} + \xi(t) \leq x - \frac{c_0}{4}(t_0 - t) - x_0$. Then, since ϕ_K is increasing on $\mathbb{R}$, and the L^2 -norm of $u(t, x)$ is conserved, it holds:

$$B \lesssim \phi_K(R_1 - \frac{c_0}{4}(t_0 - t) - x_0). \quad (2.51)$$

Letting $t \rightarrow -\infty$ in (2.51), and taking R_1 large, we obtain that: $\lim_{t \rightarrow -\infty} I_{x_0, t_0}(t) = 0$, and claim (2.48) follows.

Next, we define the H^2 -localized energy. Let $x_0 > 0$ and $t_0 \in \mathbb{R}$. For all $t \leq t_0$, we define:

$$J_{x_0, t_0}(t) = \int_{\mathbb{R}} \left(\frac{1}{2} (\partial_x^2 u)^2 + \frac{1}{2} (\partial_x u)^2 - \frac{1}{(p+1)(p+2)} u^{p+2} \right) (t, x) \phi_K(\bar{x}) dx. \quad (2.52)$$

As in Step 1, we establish the monotonicity property of the functional $J_{x_0, t_0}(t)$.

Step 3. For $x_0 > 0$ and $t \leq t_0$, we claim that:

$$J_{t_0, x_0}(t_0) - J_{x_0, t_0}(t) \lesssim e^{-x_0/K}. \quad (2.53)$$

Proof of (2.53). We just present the strategy since the proof is identical to (2.41). First, we differentiate $J_{x_0, t_0}(t)$ with respect to time, we apply the Virial Type Identity II (see Subsection 2.5), and we order the terms following the same logic as in (2.42). Observe that, we just need to estimate carefully the term: $|\frac{5\mu_p}{2} \int_{\mathbb{R}} (\partial_x^3 u)^2 \phi_K'''(\bar{x})| \lesssim \frac{1}{K^2} \int_{\mathbb{R}} (\partial_x^3 u)^2 \phi_K'(\bar{x})$. The rest of the term in Virial Identity II are bounded by: $\sum_{j=0}^2 \int_{\mathbb{R}} (\partial_x^j u)^2 \phi_K'(\bar{x})$ (we use that $u(t)$ is H^2 -bounded). Then, integration $\frac{d}{dt} J_{x_0, t_0}(t)$ between $[t, t_0]$, using (2.48), and choosing $K > 0$ correctly, we get

$$J_{x_0, t_0}(t_0) + C_1 \int_t^{t_0} \int_{\mathbb{R}} ((\partial_x^4 u)^2 + (\partial_x^3 u)^2 + (\partial_x^2 u)^2 + (\partial_x u)^2 + u^{2p+2})(s, x) \phi_K'(\bar{x}) dx ds \lesssim e^{-x_0/K} + J_{x_0, t_0}(t), \quad (2.54)$$

with $C_1 > 0$. This implies (2.53).

Estimate (2.54) and the decay assumption on $u(t, x)$ lead to the following control.

Step 4. For $x_0 > 0$ and $t_0 \in \mathbb{R}$, we claim that:

$$\begin{aligned} & \int_{\mathbb{R}} ((\partial_x^2 u)^2 + (\partial_x u)^2)(t_0, x) \phi_K(x - \xi(t_0) - x_0) dx \\ & + \int_{-\infty}^{t_0} \int_{\mathbb{R}} ((\partial_x^4 u)^2 + (\partial_x^3 u)^2 + (\partial_x^2 u)^2 + (\partial_x u)^2 + u^{2p+2})(s, x) \phi_K'(\bar{x}) dx ds \lesssim e^{-x_0/K}. \end{aligned} \quad (2.55)$$

Let us prove (2.55). In view of (2.54), it suffices to prove that $\lim_{t \rightarrow -\infty} J_{x_0, t_0}(t) = 0$. Remark that $J_{x_0, t_0}(t) \lesssim \sum_{j=0}^2 \int_{\mathbb{R}} (\partial_x^j u)^2(t, x) \phi_K(\bar{x})$ (since $u(t)$ is bounded in $H^1(\mathbb{R})$). From (2.48), we know that

$$\sum_{k=0}^2 \int_{-\infty}^{t_0} \int_{\mathbb{R}} (\partial_x^k u)^2 \phi_K'(\bar{x}) \lesssim e^{-x_0/K}. \quad (2.56)$$

Then

$$\sum_{k=0}^2 \int_{-\infty}^{t_0} \int_{x < \xi(t_0) - \frac{c_0}{2}(t_0 - t) + x_0} (\partial_x^k u)^2 e^{(x - \xi(t_0) + \frac{c_0}{2}(t_0 - t))/K} \lesssim 1, \quad (2.57)$$

since $e^{\bar{x}/K} \lesssim \phi_K'(\bar{x})$ for $\bar{x} < 0$. Thus, passing to the limit as x_0 tends to $+\infty$ in (2.57) and multiplying by $e^{-x_0/K}$, we get

$$\sum_{k=0}^2 \int_{-\infty}^{t_0} \int_{\mathbb{R}} (\partial_x^k u)^2 e^{\bar{x}/K} \lesssim e^{-x_0/K}, \quad (2.58)$$

and then, since $\phi(\bar{x}) \lesssim e^{\bar{x}/K}$ for all $x \in \mathbb{R}$,

$$\sum_{k=0}^2 \int_{-\infty}^{t_0} \int_{\mathbb{R}} (\partial_x^k u)^2 \phi_K(\bar{x}) \lesssim e^{-x_0/K}. \quad (2.59)$$

Thus, we infer that

$$\liminf_{t \rightarrow -\infty} \sum_{k=0}^2 \int_{\mathbb{R}} (\partial_x^k u)^2 \phi_K(\bar{x}) = 0. \quad (2.60)$$

We prove now that all derivatives of $u(t, x)$ are uniformly localized in $\mathbb{R}$.

Step 5. For $x_0 > 0$, $t_0 \in \mathbb{R}$ and $k \in \mathbb{N}$, we claim that:

$$\int_{\mathbb{R}} (\partial_x^k u)^2(t, x) \phi_K(x - \xi(t_0) - x_0) dx + \sum_{j=0}^2 \int_{-\infty}^{t_0} \int_{\mathbb{R}} (\partial_x^{k+j} u)^2(s, x) \phi_K'(\bar{x}) dx ds \lesssim e^{-x_0/K}. \quad (2.61)$$

Proof of (2.61). We set: $I_{x_0, t_0}^k(t) = \int_{\mathbb{R}} (\partial_x^k u)^2(t, x) \phi_K(\bar{x})$ and we proceed by induction on k . First, from Step 2 and Step 4, the estimate (2.61) holds for $k = 0, 1, 2$. We assume that (2.61) still true for $0 \leq j \leq k-1$. From the induction hypothesis one can deduce that $\int_{\mathbb{R}} (\partial_x^{k-1} u)^2(t_0, x + \xi(t_0)) e^{|x|/K} \lesssim 1$ (in particular $u(t)$ is bounded in $H^{k-1}(\mathbb{R})$, see Step 6 below). Also, $\int_{-\infty}^{t_0} \int_{\mathbb{R}} (\partial_x^k u)^2 \phi_K(\bar{x}) \lesssim e^{-x_0/K}$, which implies that $\lim_{x \rightarrow -\infty} I_{x_0, t_0}^k(t) = 0$. Now, by computing the time variation of $I_{x_0, t_0}^{(k)}(t)$, with the aid of Virial Type Identity I applied with $\partial_x^k u(t, x)$, we get:

$$\begin{aligned} \frac{d}{dt} I_{x_0, t_0}^k(t) = & -5 \int_{\mathbb{R}} (\partial_x^{k+2} u)^2 \phi'_K(\bar{x}) - 3 \int_{\mathbb{R}} (\partial_x^{k+1} u)^2 \phi'_K(\bar{x}) - \frac{c_0}{2} \int_{\mathbb{R}} (\partial_x^k u)^2 \phi'_K(\bar{x}) \\ & + 5\mu_p \int_{\mathbb{R}} (\partial_x^{k+1} u)^2 \phi_K^{(5)}(\bar{x}) - \mu_p \int_{\mathbb{R}} (\partial_x^k u)^2 \phi_K^{(5)}(\bar{x}) + \int_{\mathbb{R}} (\partial_x^k u)^2 \phi_K^{(5)}(\bar{x}) \\ & - \frac{2}{p+1} \int_{\mathbb{R}} (\partial_x^k u) (\partial_x^{k+1} u^{p+1}) \phi_K(\bar{x}). \end{aligned} \quad (2.62)$$

The terms in the second line of (2.62) are clearly bounded by: $\int_{\mathbb{R}} ((\partial_x^{k+1} u)^2 + (\partial_x^k u)^2) \phi'_K(\bar{x})$. Let us estimate the nonlinear term: $N = -\frac{2}{p+1} \int_{\mathbb{R}} (\partial_x^k u) (\partial_x^{k+1} u^{p+1}) \phi_K(\bar{x})$. By differentiating thrice u^{p+1} , we get

$$\begin{aligned} N = & -2 \int_{\mathbb{R}} \partial_x^{k-2} [(\partial_x^3 u) u^p] (\partial_x^k u) \phi_K(\bar{x}) - 6p \int_{\mathbb{R}} \partial_x^{k-2} [(\partial_x^2 u) (\partial_x u) u^{p-1}] (\partial_x^k u) \phi_K(\bar{x}) \\ & - 2(p-1) \int_{\mathbb{R}} \partial_x^{k-2} [(\partial_x u)^3 u^{p-2}] (\partial_x^k u) \phi_K(\bar{x}) \\ = & N_1 + N_2 + N_3. \end{aligned} \quad (2.63)$$

We claim that:

$$N \lesssim \sum_{j=1}^{k+1} \int_{\mathbb{R}} (\partial_x^j u)^2 \phi_K(\bar{x}). \quad (2.64)$$

Let us estimate N_1 . By applying the Leibniz formula on $\partial_x^{k-2} [(\partial_x^3 u) u^p]$, we can rewrite N_1 as

$$N_1 = -2 \sum_{j=0}^{k-2} C_{k-2}^j \int_{\mathbb{R}} (\partial_x^{k-j+1} u) (\partial_x^j u) (\partial_x^j u^p) \phi_K(\bar{x}). \quad (2.65)$$

Now, using that $\|\partial_x^j u^p\|_{L^\infty(\mathbb{R})} \lesssim 1$ for $0 \leq j \leq k-2$, and applying the Young's inequality, we infer that

$$N_1 \lesssim \sum_{j=3}^{k+1} \int_{\mathbb{R}} (\partial_x^j u)^2 \phi_K(\bar{x}). \quad (2.66)$$

The estimates of N_2 and N_3 follow by applying the same arguments. When estimating N_2 , you just have to pay attention to two terms: $\int_{\mathbb{R}} (\partial_x^2 u)^2 (\partial_x^k u) (\partial_x^{k-3} u^{p-1}) \phi_K(\bar{x})$ and $\int_{\mathbb{R}} (\partial_x^2 u) (\partial_x^{k-1} u) (\partial_x^k u) u^{p-1} \phi_K(\bar{x})$, which require one integration by part.

Finally, by integrating $I_{x_0, t_0}^k(t)$ between $[-\infty, t_0]$, and using that $\sum_{j=0}^{k+1} \int_{-\infty}^{t_0} \int_{\mathbb{R}} (\partial_x^j u)^2 \phi_K(\bar{x}) \lesssim e^{-x_0/K}$ and $I_{x_0, t_0}^k(-\infty) = 0$, we get (2.61).

Step 6. For all $k \in \mathbb{N}$, we claim that:

$$\int_{\mathbb{R}} [\partial_x^k u(t_0, x + \xi(t_0))]^2(t, x) e^{|x|/K} \lesssim 1. \quad (2.67)$$

Proof of (2.67). From Step 5, we know that:

$$\int_{\mathbb{R}} [\partial_x^k u(t_0, x + \xi(t_0))]^2 \phi_K(x - x_0) \lesssim e^{-x_0/K}, \quad (2.68)$$

and then

$$\int_{x < x_0} [\partial_x^k u(t_0, x + \xi(t_0))]^2 e^{x/K} \lesssim 1, \quad (2.69)$$

since $e^{(x-x_0)/K} \lesssim \phi_K(x-x_0)$ for $x < x_0$. Now, passing to the limit as x_0 tends to $+\infty$ in (2.69), we get the exponential decay property by the right:

$$\int_{\mathbb{R}} [\partial_x^k u(t_0, x + \xi(t_0))]^2 e^{x/K} \lesssim 1. \quad (2.70)$$

For the left exponential decay property, one can remark that the equation of gKW is invariant by the transformation $u(t, x) \mapsto u(-t, -x)$, moreover $u(-t, -x)$ is uniformly localized. Then, repeating the same analysis with $u(-t, -x)$, we get the claim (2.67). In particular, applying the Gagliardo-Nirenberg inequality, we infer that for all $k \in \mathbb{N}$,

$$\begin{aligned} \left\| [\partial_x^k u(t_0, \cdot + \xi(t_0))] e^{|\cdot|/2K} \right\|_{L^\infty(\mathbb{R})} &\leq C_G \left\| [\partial_x^k u(t_0, \cdot + \xi(t_0))] e^{|\cdot|/2K} \right\|_{L^2(\mathbb{R})}^{1/2} \times \\ &\quad \left(\left\| [\partial_x^k u(t_0, \cdot + \xi(t_0))] e^{|\cdot|/2K} \right\|_{L^2(\mathbb{R})}^2 + \left\| [\partial_x^{k+1} u(t_0, \cdot + \xi(t_0))] e^{|\cdot|/2K} \right\|_{L^2(\mathbb{R})}^2 \right)^{1/4} \lesssim 1, \end{aligned} \quad (2.71)$$

and this prove (2.14).

2.4 Time variation of the localized mass

In this section, we compute the time variation of L^2 -localized norm.

Lemma 2.5 (Virial Type Identity I). *Assume that $\mu_0 > 0$ and $p \in \mathbb{N}^*$. Let $u(t) \in H^2(\mathbb{R})$ be a solution of μ_0 -gKW equation. Then, for any smooth (C^5) space function $g : \mathbb{R} \rightarrow \mathbb{R}$, it holds*

$$\begin{aligned} \frac{d}{dt} \int_{\mathbb{R}} u^2 g &= -5\mu_0 \int_{\mathbb{R}} (\partial_x^2 u)^2 g' - 3 \int_{\mathbb{R}} (\partial_x u)^2 g' + \int_{\mathbb{R}} u^2 g''' \\ &\quad + 5\mu_0 \int_{\mathbb{R}} (\partial_x u)^2 g''' - \mu_0 \int_{\mathbb{R}} u^2 g^{(5)} + \frac{2}{p+2} \int_{\mathbb{R}} u^{p+2} g'. \end{aligned} \quad (2.72)$$

Proof. We have

$$\frac{d}{dt} \int_{\mathbb{R}} u^2 g = 2 \int_{\mathbb{R}} (\partial_t u) u g, \quad (2.73)$$

and using that $u(t, x)$ is a solution of μ_0 -gKW equation, we obtain

$$\begin{aligned} \frac{d}{dt} \int_{\mathbb{R}} u^2 g &= -\frac{2}{p+1} \int_{\mathbb{R}} (\partial_x u^{p+1}) u g - 2 \int_{\mathbb{R}} (\partial_x^3 u) u g + 2\mu_0 \int_{\mathbb{R}} (\partial_x^5 u) u g \\ &= I + J + K. \end{aligned} \quad (2.74)$$

Let us compute I :

$$\begin{aligned} I &= \frac{2}{p+1} \int_{\mathbb{R}} u^{p+1} \partial_x (u g) \\ &= \frac{2}{(p+1)(p+2)} \int_{\mathbb{R}} (\partial_x u^{p+2}) g + \frac{2}{p+1} \int_{\mathbb{R}} u^{p+2} g' \\ &= \frac{2}{p+2} \int_{\mathbb{R}} u^{p+2} g'. \end{aligned} \quad (2.75)$$

Let us compute J :

$$\begin{aligned}
J &= 2 \int_{\mathbb{R}} (\partial_x^2 u) \partial_x (ug) \\
&= \int_{\mathbb{R}} \partial_x [(\partial_x u)^2] g + 2 \int_{\mathbb{R}} (\partial_x^2 u) u g' \\
&= - \int_{\mathbb{R}} (\partial_x u)^2 g' + 2 \int_{\mathbb{R}} (\partial_x^2 u) u g',
\end{aligned} \tag{2.76}$$

with

$$\begin{aligned}
2 \int_{\mathbb{R}} (\partial_x^2 u) u g' &= -2 \int_{\mathbb{R}} (\partial_x u) \partial_x (u g') \\
&= -2 \int_{\mathbb{R}} (\partial_x u)^2 g' - \int_{\mathbb{R}} (\partial_x u^2) g'' \\
&= -2 \int_{\mathbb{R}} (\partial_x u)^2 g' + \int_{\mathbb{R}} u^2 g''',
\end{aligned} \tag{2.77}$$

and then summing (2.76) and (2.77), we get

$$J = -3 \int_{\mathbb{R}} (\partial_x u)^2 g' + \int_{\mathbb{R}} u^2 g'''. \tag{2.78}$$

Let us compute K :

$$\begin{aligned}
K &= -2\mu_0 \int_{\mathbb{R}} (\partial_x^4 u) \partial_x (ug) \\
&= -2\mu_0 \int_{\mathbb{R}} (\partial_x^4 u) (\partial_x u) g - 2\mu_0 \int_{\mathbb{R}} (\partial_x^4 u) u g' = K_1 + K_2.
\end{aligned} \tag{2.79}$$

Let us tackle K_1 :

$$\begin{aligned}
K_1 &= 2\mu_0 \int_{\mathbb{R}} (\partial_x^3 u) \partial_x [(\partial_x u) g] \\
&= \mu_0 \int_{\mathbb{R}} \partial_x [(\partial_x^2 u)^2] g + 2\mu_0 \int_{\mathbb{R}} (\partial_x^3 u) (\partial_x u) g' \\
&= -\mu_0 \int_{\mathbb{R}} (\partial_x^2 u)^2 g' + K_{11},
\end{aligned} \tag{2.80}$$

with

$$\begin{aligned}
K_{11} &= -2\mu_0 \int_{\mathbb{R}} (\partial_x^2 u) [(\partial_x^2 u) g' + (\partial_x u) g''] \\
&= -2\mu_0 \int_{\mathbb{R}} (\partial_x^2 u)^2 g' - \mu_0 \int_{\mathbb{R}} \partial_x [(\partial_x u)^2] g'' \\
&= -2\mu_0 \int_{\mathbb{R}} (\partial_x^2 u)^2 g' + \mu_0 \int_{\mathbb{R}} (\partial_x u)^2 g''',
\end{aligned} \tag{2.81}$$

and then summing (2.80) and (2.81), we get

$$K_1 = -3\mu_0 \int_{\mathbb{R}} (\partial_x^2 u)^2 g' + \mu_0 \int_{\mathbb{R}} (\partial_x u)^2 g'''. \tag{2.82}$$

Let us tackle K_2 :

$$\begin{aligned} K_2 &= 2\mu_0 \int_{\mathbb{R}} (\partial_x^3 u) \partial_x (ug') \\ &= 2\mu_0 \int_{\mathbb{R}} (\partial_x^3 u) (\partial_x u) g' + 2\mu_0 \int_{\mathbb{R}} (\partial_x^3 u) u g'' = K_{21} + K_{22}, \end{aligned} \quad (2.83)$$

with

$$\begin{aligned} K_{21} &= -2\mu_0 \int_{\mathbb{R}} (\partial_x^2 u) \partial_x [(\partial_x u) g'] \\ &= -2\mu_0 \int_{\mathbb{R}} (\partial_x^2 u)^2 g' - \mu_0 \int_{\mathbb{R}} \partial_x [(\partial_x u)^2] g'' \\ &= -2\mu_0 \int_{\mathbb{R}} (\partial_x^2 u)^2 g' + \mu_0 \int_{\mathbb{R}} (\partial_x u)^2 g''', \end{aligned} \quad (2.84)$$

and

$$\begin{aligned} K_{22} &= -2\mu_0 \int_{\mathbb{R}} (\partial_x^2 u) \partial_x (ug'') \\ &= -\mu_0 \int_{\mathbb{R}} \partial_x [(\partial_x u)^2] g'' + 2\mu_0 \int_{\mathbb{R}} (\partial_x u) \partial_x (ug''') \\ &= 3\mu_0 \int_{\mathbb{R}} (\partial_x u)^2 g''' + \mu_0 \int_{\mathbb{R}} \partial_x (u^2) g^{(4)} \\ &= 3\mu_0 \int_{\mathbb{R}} (\partial_x u)^2 g''' - \mu_0 \int_{\mathbb{R}} u^2 g^{(5)}. \end{aligned} \quad (2.85)$$

Then, summing (2.85) and (2.84), we get

$$K_2 = -2\mu_0 \int_{\mathbb{R}} (\partial_x^2 u)^2 g' + 4\mu_0 \int_{\mathbb{R}} (\partial_x u)^2 g''' - \mu_0 \int_{\mathbb{R}} u^2 g^{(5)}, \quad (2.86)$$

and summing (2.82) and (2.86), we find

$$K = -5\mu_0 \int_{\mathbb{R}} (\partial_x^2 u)^2 g' + 5\mu_0 \int_{\mathbb{R}} (\partial_x u)^2 g''' - \mu_0 \int_{\mathbb{R}} u^2 g^{(5)}. \quad (2.87)$$

Finally, by adding (2.75), (2.78) and (2.87), we obtain (2.72), and this prove the lemma. $\square$

2.5 Time variation of the localized energy

In this section, we compute the time variation of H^2 -localized norm.

Lemma 2.6 (Virial Type Identity II). *Assume that $\mu_0 > 0$ and $p \in \mathbb{N}^*$. Let $u(t) \in H^4(\mathbb{R})$ be a solution of μ_0 -gKW equation. Then, for any smooth (C^5) space function: $g : \mathbb{R} \rightarrow \mathbb{R}$, it holds*

$$\begin{aligned} &\frac{d}{dt} \int_{\mathbb{R}} \left(\frac{\mu_0}{2} (\partial_x^2 u)^2 + \frac{1}{2} (\partial_x u)^2 - \frac{1}{(p+1)(p+2)} u^{p+2} \right) g \\ &= -\frac{5\mu_0^2}{2} \int_{\mathbb{R}} (\partial_x^4 u)^2 g' - 4\mu_0 \int_{\mathbb{R}} (\partial_x^3 u)^2 g' - \frac{3}{2} \int_{\mathbb{R}} (\partial_x^2 u)^2 g' - \frac{1}{2(p+1)^2} \int_{\mathbb{R}} u^{2p+2} g' \\ &\quad + \frac{5\mu_0^2}{2} \int_{\mathbb{R}} (\partial_x^3 u)^2 g''' - \frac{\mu_0^2}{2} \int_{\mathbb{R}} (\partial_x^2 u)^2 g^{(5)} + 3\mu_0 \int_{\mathbb{R}} (\partial_x^2 u)^2 g''' - \frac{\mu_0}{2} \int_{\mathbb{R}} (\partial_x u)^2 g^{(5)} + \frac{1}{2} \int_{\mathbb{R}} (\partial_x u)^2 g''' \\ &\quad + 3\mu_0 \int_{\mathbb{R}} (\partial_x^2 u)^2 u^p g' - \mu_0 p(p-1) \int_{\mathbb{R}} (\partial_x u)^4 u^{p-2} g' - \frac{15\mu_0 p}{6} \int_{\mathbb{R}} (\partial_x u)^3 u^{p-1} g'' + 2 \int_{\mathbb{R}} (\partial_x u)^2 u^p g' \\ &\quad - \frac{5\mu_0}{2} \int_{\mathbb{R}} (\partial_x u)^2 u^p g''' - \frac{1}{(p+1)(p+2)} \int_{\mathbb{R}} u^{p+2} g''' + \frac{\mu_0}{(p+1)(p+2)} \int_{\mathbb{R}} u^{p+2} g^{(5)}. \end{aligned} \quad (2.88)$$

Proof. We set:

$$I = \frac{\mu_0}{2} \frac{d}{dt} \int_{\mathbb{R}} (\partial_x^2 u)^2 g, \quad J = \frac{1}{2} \frac{d}{dt} \int_{\mathbb{R}} (\partial_x u)^2 g, \quad \text{and} \quad K = -\frac{1}{(p+1)(p+2)} \frac{d}{dt} \int_{\mathbb{R}} u^{p+2} g. \quad (2.89)$$

Let us start with K :

$$K = -\frac{1}{p+1} \int_{\mathbb{R}} u^{p+1} (\partial_t u) g, \quad (2.90)$$

using that $u(t, x)$ satisfies the μ_0 -gKW equation, we get

$$\begin{aligned} K &= \frac{1}{p+1} \int_{\mathbb{R}} (\partial_x u) u^{2p+1} g + \frac{1}{p+1} \int_{\mathbb{R}} (\partial_x^3 u) u^{p+1} g - \frac{\mu_0}{p+1} \int_{\mathbb{R}} (\partial_x^5 u) u^{p+1} g \\ &= -\frac{1}{2(p+1)^2} \int_{\mathbb{R}} u^{2p+2} g' + K_1 + K_2. \end{aligned} \quad (2.91)$$

Let us compute K_1 :

$$\begin{aligned} K_1 &= -\frac{1}{p+1} \int_{\mathbb{R}} (\partial_x^2 u) \partial_x (u^{p+1} g) \\ &= -\int_{\mathbb{R}} (\partial_x^2 u) (\partial_x u) u^p g - \frac{1}{p+1} \int_{\mathbb{R}} (\partial_x^2 u) u^{p+1} g' = K_{11} + K_{12}, \end{aligned} \quad (2.92)$$

with

$$K_{11} = -\frac{1}{2} \int_{\mathbb{R}} \partial_x [(\partial_x u)^2] u^p g = \frac{p}{2} \int_{\mathbb{R}} (\partial_x u)^3 u^{p-1} g + \frac{1}{2} \int_{\mathbb{R}} (\partial_x u)^2 u^p g', \quad (2.93)$$

and

$$\begin{aligned} K_{12} &= \frac{1}{p+1} \int_{\mathbb{R}} (\partial_x u) \partial_x (u^{p+1} g') \\ &= \int_{\mathbb{R}} (\partial_x u)^2 u^p g' + \frac{1}{(p+1)(p+2)} \int_{\mathbb{R}} \partial_x (u^{p+2}) g'' \\ &= \int_{\mathbb{R}} (\partial_x u)^2 u^p g' - \frac{1}{(p+1)(p+2)} \int_{\mathbb{R}} u^{p+2} g''', \end{aligned} \quad (2.94)$$

and summing (2.94) and (2.93), we get

$$K_1 = \frac{p}{2} \int_{\mathbb{R}} (\partial_x u)^3 u^{p+1} g + \frac{3}{2} \int_{\mathbb{R}} (\partial_x u)^2 u^p g' - \frac{1}{(p+1)(p+2)} \int_{\mathbb{R}} u^{p+2} g'''. \quad (2.95)$$

Now, let us compute K_2 :

$$\begin{aligned} K_2 &= \frac{\mu_0}{p+1} \int_{\mathbb{R}} (\partial_x^4 u) \partial_x (u^{p+1} g) \\ &= \mu_0 \int_{\mathbb{R}} (\partial_x^4 u) (\partial_x u) u^p g + \frac{\mu_0}{p+1} \int_{\mathbb{R}} (\partial_x^4 u) u^{p+1} g' = K_{21} + K_{22}. \end{aligned} \quad (2.96)$$

K_{21} give us

$$\begin{aligned} K_{21} &= -\mu_0 \int_{\mathbb{R}} (\partial_x^3 u) \partial_x [(\partial_x u) u^p g] \\ &= -\mu_0 \int_{\mathbb{R}} (\partial_x^3 u) (\partial_x^2 u) u^p g - \mu_0 p \int_{\mathbb{R}} (\partial_x^3 u) (\partial_x u)^2 u^{p-1} g - \mu_0 \int_{\mathbb{R}} (\partial_x^3 u) (\partial_x u) u^p g' \\ &= -\mu_0 \int_{\mathbb{R}} (\partial_x^3 u) (\partial_x^2 u) u^p g + K_{211} + K_{212}, \end{aligned} \quad (2.97)$$

with

$$\begin{aligned}
& -\mu_0 \int_{\mathbb{R}} (\partial_x^3 u)(\partial_x^2 u) u^p g \\
& = -\frac{\mu_0}{2} \int_{\mathbb{R}} \partial_x [(\partial_x^2 u)^2] u^p g \\
& = \frac{\mu_0 p}{2} \int_{\mathbb{R}} (\partial_x^2 u)^2 (\partial_x u) u^{p-1} g + \frac{\mu_0}{2} \int_{\mathbb{R}} (\partial_x^2 u)^2 u^p g',
\end{aligned} \tag{2.98}$$

and

$$\begin{aligned}
K_{211} &= \mu_0 p \int_{\mathbb{R}} (\partial_x^2 u) \partial_x [(\partial_x u)^2 u^{p-1} g] \\
&= 2\mu_0 p \int_{\mathbb{R}} (\partial_x^2 u)^2 (\partial_x u) u^{p-1} g + \mu_0 p(p-1) \int_{\mathbb{R}} (\partial_x^2 u) (\partial_x u)^3 u^{p-2} g \\
&\quad + \mu_0 p \int_{\mathbb{R}} (\partial_x^2 u) (\partial_x u)^2 u^{p-1} g',
\end{aligned} \tag{2.99}$$

with

$$\begin{aligned}
& \mu_0 p(p-1) \int_{\mathbb{R}} (\partial_x^2 u) (\partial_x u)^3 u^{p-2} g \\
&= \frac{\mu_0 p(p-1)}{4} \int_{\mathbb{R}} \partial_x [(\partial_x u)^4] u^{p-2} g \\
&= -\frac{\mu_0 p(p-1)(p-2)}{4} \int_{\mathbb{R}} (\partial_x u)^5 u^{p-3} g - \frac{\mu_0 p(p-1)}{4} \int_{\mathbb{R}} (\partial_x u)^4 u^{p-2} g',
\end{aligned} \tag{2.100}$$

and

$$\begin{aligned}
& \mu_0 p \int_{\mathbb{R}} (\partial_x^2 u) (\partial_x u)^2 u^{p-1} g' \\
&= \frac{\mu_0 p}{3} \int_{\mathbb{R}} \partial_x [(\partial_x u)^3] u^{p-1} g' \\
&= -\frac{\mu_0 p(p-1)}{3} \int_{\mathbb{R}} (\partial_x u)^4 u^{p-2} g' - \frac{\mu_0 p}{3} \int_{\mathbb{R}} (\partial_x u)^3 u^{p-1} g''.
\end{aligned} \tag{2.101}$$

Then summing (2.99), (2.100) and (2.101), we get

$$\begin{aligned}
K_{211} &= 2\mu_0 p \int_{\mathbb{R}} (\partial_x^2 u)^2 (\partial_x u) u^{p-1} g - \frac{\mu_0 p(p-1)(p-2)}{4} \int_{\mathbb{R}} (\partial_x u)^5 u^{p-3} g \\
&\quad - \frac{7\mu_0 p(p-1)}{12} \int_{\mathbb{R}} (\partial_x u)^4 u^{p-2} g' - \frac{\mu_0 p}{3} \int_{\mathbb{R}} (\partial_x u)^3 u^{p-2} g''.
\end{aligned} \tag{2.102}$$

K_{212} give us

$$\begin{aligned}
K_{212} &= \mu_0 \int_{\mathbb{R}} (\partial_x^2 u) \partial_x [(\partial_x u) u^p g'] \\
&= \mu_0 \int_{\mathbb{R}} (\partial_x^2 u)^2 u^p g' + \mu_0 p \int_{\mathbb{R}} (\partial_x^2 u) (\partial_x u)^2 u^{p-1} g' + \mu_0 \int_{\mathbb{R}} (\partial_x^2 u) (\partial_x u) u^p g'',
\end{aligned} \tag{2.103}$$

with

$$\begin{aligned}
& \mu_0 p \int_{\mathbb{R}} (\partial_x^2 u) (\partial_x u)^2 u^{p-1} g' \\
&= \frac{\mu_0 p}{3} \int_{\mathbb{R}} \partial_x [(\partial_x u)^3] u^{p-1} g' \\
&= -\frac{\mu_0 p(p-1)}{3} \int_{\mathbb{R}} (\partial_x u)^4 u^{p-2} g' - \frac{\mu_0 p}{3} \int_{\mathbb{R}} (\partial_x u)^3 u^{p-1} g'',
\end{aligned} \tag{2.104}$$

and

$$\begin{aligned}
& \mu_0 \int_{\mathbb{R}} (\partial_x^2 u) (\partial_x u) u^p g'' \\
&= \frac{\mu_0}{2} \int_{\mathbb{R}} \partial_x [(\partial_x u)^2] u^p g'' \\
&= -\frac{\mu_0 p}{2} \int_{\mathbb{R}} (\partial_x u)^3 u^{p-1} g'' - \frac{\mu_0}{2} \int_{\mathbb{R}} (\partial_x u)^2 u^p g'''.
\end{aligned} \tag{2.105}$$

Then, summing (2.103), (2.104) and (2.105), we get

$$\begin{aligned}
K_{212} &= \mu_0 \int_{\mathbb{R}} (\partial_x^2 u)^2 u^p g' - \frac{\mu_0 p(p-1)}{3} \int_{\mathbb{R}} (\partial_x u)^4 u^{p-2} g' \\
&\quad - \frac{5\mu_0 p}{6} \int_{\mathbb{R}} (\partial_x u)^3 u^{p-1} g'' - \frac{\mu_0}{2} \int_{\mathbb{R}} (\partial_x u)^2 u^p g''',
\end{aligned} \tag{2.106}$$

and summing (2.98), (2.102) and (2.106), we obtain

$$\begin{aligned}
K_{21} &= \frac{5\mu_0 p}{2} \int_{\mathbb{R}} (\partial_x^2 u)^2 (\partial_x u) u^{p-1} g - \frac{\mu_0 p(p-1)(p-2)}{4} \int_{\mathbb{R}} (\partial_x u)^5 u^{p-3} g + \frac{3\mu_0}{2} \int_{\mathbb{R}} (\partial_x^2 u) u^p g' \\
&\quad - \frac{11\mu_0 p(p-1)}{12} \int_{\mathbb{R}} (\partial_x u)^4 u^{p-2} g' - \frac{7\mu_0 p}{6} \int_{\mathbb{R}} (\partial_x u)^3 u^{p-1} g'' - \frac{\mu_0}{2} \int_{\mathbb{R}} (\partial_x u)^2 u^p g'''.
\end{aligned} \tag{2.107}$$

Let us compute K_{22} :

$$\begin{aligned}
K_{22} &= -\frac{\mu_0}{p+1} \int_{\mathbb{R}} (\partial_x^3 u) \partial_x (u^{p+1} g') \\
&= -\mu_0 \int_{\mathbb{R}} (\partial_x^3 u) (\partial_x u) u^p g' - \frac{\mu_0}{p+1} \int_{\mathbb{R}} (\partial_x^3 u) u^{p+1} g'' = K_{221} + K_{222}.
\end{aligned} \tag{2.108}$$

We check that

$$\begin{aligned}
K_{221} &= \mu_0 \int_{\mathbb{R}} (\partial_x^2 u) \partial_x [(\partial_x u) u^p g'] \\
&= \mu_0 \int_{\mathbb{R}} (\partial_x^2 u)^2 u^p g' + \mu_0 p \int_{\mathbb{R}} (\partial_x^2 u) (\partial_x u)^2 u^{p-1} g' + \mu_0 \int_{\mathbb{R}} (\partial_x^2 u) (\partial_x u) u^p g'',
\end{aligned} \tag{2.109}$$

with

$$\begin{aligned}
& \mu_0 p \int_{\mathbb{R}} (\partial_x^2 u) (\partial_x u)^2 u^{p-1} g' \\
&= \frac{\mu_0 p}{3} \int_{\mathbb{R}} \partial_x [(\partial_x u)^3] u^{p-1} g' \\
&= -\frac{\mu_0 p(p-1)}{3} \int_{\mathbb{R}} (\partial_x u)^4 u^{p-2} g' - \frac{\mu_0 p}{3} \int_{\mathbb{R}} (\partial_x u)^3 u^{p-1} g'',
\end{aligned} \tag{2.110}$$

and

$$\begin{aligned}
\mu_0 \int_{\mathbb{R}} (\partial_x^2 u)(\partial_x u) u^p g'' &= \frac{\mu_0}{2} \int_{\mathbb{R}} \partial_x [(\partial_x u)^2] u^p g'' \\
&= -\frac{\mu_0 p}{2} \int_{\mathbb{R}} (\partial_x u)^3 u^{p-1} g'' - \frac{\mu_0}{2} \int_{\mathbb{R}} (\partial_x u)^2 u^p g'''.
\end{aligned} \tag{2.111}$$

Then, summing (2.109)-(2.111), we get

$$\begin{aligned}
K_{221} &= \mu_0 \int_{\mathbb{R}} (\partial_x^2 u)^2 u^p g' - \frac{\mu_0 p(p-1)}{3} \int_{\mathbb{R}} (\partial_x u)^4 u^{p-2} g' \\
&\quad - \frac{5\mu_0 p}{6} \int_{\mathbb{R}} (\partial_x u)^3 u^{p-1} g'' - \frac{\mu_0}{2} \int_{\mathbb{R}} (\partial_x u)^2 u^p g'''.
\end{aligned} \tag{2.112}$$

We check that

$$\begin{aligned}
K_{222} &= \frac{\mu_0}{p+1} \int_{\mathbb{R}} (\partial_x^2 u) \partial_x (u^{p+1} g'') \\
&= \mu_0 \int_{\mathbb{R}} (\partial_x^2 u)(\partial_x u) u^p g'' + \frac{\mu_0}{p+1} \int_{\mathbb{R}} (\partial_x^2 u) u^{p+1} g''',
\end{aligned} \tag{2.113}$$

with

$$\begin{aligned}
\mu_0 \int_{\mathbb{R}} (\partial_x^2 u)(\partial_x u) u^p g'' &= \frac{\mu_0}{2} \int_{\mathbb{R}} \partial_x (\partial_x u)^2 u^p g'' \\
&= -\frac{\mu_0 p}{2} \int_{\mathbb{R}} (\partial_x u)^3 u^{p-1} g'' - \frac{\mu_0}{2} \int_{\mathbb{R}} (\partial_x u)^2 u^p g''',
\end{aligned} \tag{2.114}$$

and

$$\begin{aligned}
\frac{\mu_0}{p+1} \int_{\mathbb{R}} (\partial_x^2 u) u^{p+1} g''' &= -\frac{\mu_0}{p+1} \int_{\mathbb{R}} (\partial_x u) \partial_x (u^{p+1} g''') \\
&= -\mu_0 \int_{\mathbb{R}} (\partial_x u)^2 u^p g''' - \frac{\mu_0}{(p+1)(p+2)} \int_{\mathbb{R}} \partial_x (u^{p+2}) g^{(4)} \\
&= -\mu_0 \int_{\mathbb{R}} (\partial_x u)^2 u^p g''' + \frac{\mu_0}{(p+1)(p+2)} \int_{\mathbb{R}} u^{p+2} g^{(5)},
\end{aligned} \tag{2.115}$$

and then summing (2.114) and (2.115), we get

$$K_{222} = -\frac{\mu_0 p}{2} \int_{\mathbb{R}} (\partial_x u)^3 u^{p-1} g'' - \frac{3\mu_0}{2} \int_{\mathbb{R}} (\partial_x u)^2 u^p g''' + \frac{\mu_0}{(p+1)(p+2)} \int_{\mathbb{R}} u^{p+2} g^{(5)}. \tag{2.116}$$

Summing (2.112) and (2.116), we get

$$\begin{aligned}
K_{22} &= \mu_0 \int_{\mathbb{R}} (\partial_x^2 u)^2 u^p g' - \frac{\mu_0 p(p-1)}{3} \int_{\mathbb{R}} (\partial_x u)^4 u^{p-2} g' - \frac{4\mu_0 p}{3} \int_{\mathbb{R}} (\partial_x u)^3 u^{p-1} g'' \\
&\quad - 2\mu_0 \int_{\mathbb{R}} (\partial_x u)^2 u^p g''' + \frac{\mu_0}{(p+1)(p+2)} \int_{\mathbb{R}} u^{p+2} g^{(5)},
\end{aligned} \tag{2.117}$$

and summing (2.107) and (2.117), we get

$$\begin{aligned}
K_2 = & \frac{5\mu_0 p}{2} \int_{\mathbb{R}} (\partial_x^2 u)^2 (\partial_x u) u^{p-1} g - \frac{\mu_0 p(p-1)(p-2)}{4} \int_{\mathbb{R}} (\partial_x u)^5 u^{p-3} g + \frac{5\mu_0}{2} \int_{\mathbb{R}} (\partial_x^2 u)^2 u^p g' \\
& - \frac{15\mu_0 p(p-1)}{12} \int_{\mathbb{R}} (\partial_x u)^4 u^{p-2} g' - \frac{15\mu_0 p}{6} \int_{\mathbb{R}} (\partial_x u)^3 u^{p-1} g'' \\
& - \frac{5\mu_0}{2} \int_{\mathbb{R}} (\partial_x u)^2 u^p g''' + \frac{\mu_0}{(p+1)(p+2)} \int_{\mathbb{R}} u^{p+2} g^{(5)}. \tag{2.118}
\end{aligned}$$

Finally, summing (2.91), (2.95) and (2.118), we obtain

$$\begin{aligned}
K = & \frac{p}{2} \int_{\mathbb{R}} (\partial_x u)^3 u^{p-1} g + \frac{5\mu_0 p}{2} \int_{\mathbb{R}} (\partial_x^2 u)^2 (\partial_x u) u^{p-1} g - \frac{\mu_0 p(p-1)(p-2)}{4} \int_{\mathbb{R}} (\partial_x u)^5 u^{p-3} g \\
& - \frac{1}{2(p+1)^2} \int_{\mathbb{R}} u^{2p+2} g' + \frac{3}{2} \int_{\mathbb{R}} (\partial_x u)^2 u^p g' - \frac{1}{(p+1)(p+2)} \int_{\mathbb{R}} u^{p+2} g''' \\
& + \frac{5\mu_0}{2} \int_{\mathbb{R}} (\partial_x^2 u)^2 u^p g' - \frac{15\mu_0 p(p-1)}{12} \int_{\mathbb{R}} (\partial_x u)^4 u^{p-2} g' - \frac{15\mu_0 p}{6} \int_{\mathbb{R}} (\partial_x u)^3 u^{p-1} g'' \\
& - \frac{5\mu_0}{2} \int_{\mathbb{R}} (\partial_x u)^2 u^p g''' + \frac{\mu_0}{(p+1)(p+2)} \int_{\mathbb{R}} u^{p+2} g^{(5)}. \tag{2.119}
\end{aligned}$$

Let us compute J :

$$J = \int_{\mathbb{R}} [\partial_x (\partial_t u)] (\partial_x u) g, \tag{2.120}$$

and using that $u(t, x)$ satisfies μ_0 -gKW equation, we have

$$\begin{aligned}
J = & -p \int_{\mathbb{R}} (\partial_x u)^3 u^{p-1} g - \int_{\mathbb{R}} (\partial_x^2 u) (\partial_x u) u^p g - \int_{\mathbb{R}} (\partial_x^4 u) (\partial_x u) g + \mu_0 \int_{\mathbb{R}} (\partial_x^6 u) (\partial_x u) g \\
= & -p \int_{\mathbb{R}} (\partial_x u)^3 u^{p-1} g + J_1 + J_2 + J_3. \tag{2.121}
\end{aligned}$$

Let us compute J_1 :

$$\begin{aligned}
J_1 = & -\frac{1}{2} \int_{\mathbb{R}} \partial_x [(\partial_x u)^2] u^p g \\
= & \frac{p}{2} \int_{\mathbb{R}} (\partial_x u)^3 u^{p-1} g + \frac{1}{2} \int_{\mathbb{R}} (\partial_x u)^2 u^p g'. \tag{2.122}
\end{aligned}$$

Let us compute J_2 :

$$\begin{aligned}
J_2 = & \int_{\mathbb{R}} (\partial_x^3 u) \partial_x [(\partial_x u) g] \\
= & \frac{1}{2} \int_{\mathbb{R}} \partial_x [(\partial_x^2 u)^2] g + \int_{\mathbb{R}} (\partial_x^3 u) (\partial_x u) g' \\
= & -\frac{1}{2} \int_{\mathbb{R}} (\partial_x^2 u)^2 g' + \int_{\mathbb{R}} (\partial_x^3 u) (\partial_x u) g', \tag{2.123}
\end{aligned}$$

with

$$\begin{aligned}
& \int_{\mathbb{R}} (\partial_x^3 u)(\partial_x u) g' \\
&= - \int_{\mathbb{R}} (\partial_x^2 u) \partial_x [(\partial_x u) g'] \\
&= - \int_{\mathbb{R}} (\partial_x^2 u)^2 g' - \frac{1}{2} \int_{\mathbb{R}} \partial_x (\partial_x u)^2 g'' \\
&= - \int_{\mathbb{R}} (\partial_x^2 u)^2 g' + \frac{1}{2} \int_{\mathbb{R}} (\partial_x u)^2 g''',
\end{aligned} \tag{2.124}$$

and then summing (2.123) and (2.124), we get

$$J_2 = -\frac{3}{2} \int_{\mathbb{R}} (\partial_x^2 u)^2 g' + \frac{1}{2} \int_{\mathbb{R}} (\partial_x u)^2 g'''. \tag{2.125}$$

Let us compute J_3 :

$$\begin{aligned}
J_3 &= -\mu_0 \int_{\mathbb{R}} (\partial_x^5 u) \partial_x [(\partial_x u) g] \\
&= -\mu_0 \int_{\mathbb{R}} (\partial_x^5 u) (\partial_x^2 u) g - \mu_0 \int_{\mathbb{R}} (\partial_x^5 u) (\partial_x u) g' = J_{31} + J_{32}.
\end{aligned} \tag{2.126}$$

Let us compute J_{31} :

$$\begin{aligned}
J_{31} &= \mu_0 \int_{\mathbb{R}} (\partial_x^4 u) \partial_x [(\partial_x^2 u) g] \\
&= \frac{\mu_0}{2} \int_{\mathbb{R}} \partial_x [(\partial_x^3 u)^2] g + \mu_0 \int_{\mathbb{R}} (\partial_x^4 u) (\partial_x^2 u) g' \\
&= -\frac{\mu_0}{2} \int_{\mathbb{R}} (\partial_x^3 u)^2 g' + \mu_0 \int_{\mathbb{R}} (\partial_x^4 u) (\partial_x^2 u) g',
\end{aligned} \tag{2.127}$$

with

$$\begin{aligned}
& \mu_0 \int_{\mathbb{R}} (\partial_x^4 u) (\partial_x^2 u) g' \\
&= -\mu_0 \int_{\mathbb{R}} (\partial_x^3 u) \partial_x [(\partial_x^2 u) g'] \\
&= -\mu_0 \int_{\mathbb{R}} (\partial_x^3 u)^2 g' - \frac{\mu_0}{2} \int_{\mathbb{R}} \partial_x [(\partial_x^2 u)^2] g'' \\
&= -\mu_0 \int_{\mathbb{R}} (\partial_x^3 u)^2 g' + \frac{\mu_0}{2} \int_{\mathbb{R}} (\partial_x^2 u)^2 g''',
\end{aligned} \tag{2.128}$$

and then summing (2.127) and (2.128), we get

$$J_{31} = -\frac{3\mu_0}{2} \int_{\mathbb{R}} (\partial_x^3 u)^2 g' + \frac{\mu_0}{2} \int_{\mathbb{R}} (\partial_x^2 u)^2 g'''. \tag{2.129}$$

Let us compute J_{32} :

$$\begin{aligned}
J_{32} &= \mu_0 \int_{\mathbb{R}} (\partial_x^4 u) \partial_x [(\partial_x u) g'] \\
&= \mu_0 \int_{\mathbb{R}} (\partial_x^4 u) (\partial_x^2 u) g' + \mu_0 \int_{\mathbb{R}} (\partial_x^4 u) (\partial_x u) g'' = J_{321} + J_{322},
\end{aligned} \tag{2.130}$$

with

$$\begin{aligned}
J_{321} &= -\mu_0 \int_{\mathbb{R}} (\partial_x^3 u) \partial_x [(\partial_x^2 u) g'] \\
&= -\mu_0 \int_{\mathbb{R}} (\partial_x^3 u)^2 g' - \frac{\mu_0}{2} \int_{\mathbb{R}} \partial_x [(\partial_x^2 u)^2] g'' \\
&= -\mu_0 \int_{\mathbb{R}} (\partial_x^3 u)^2 g' + \frac{\mu_0}{2} \int_{\mathbb{R}} (\partial_x^2 u)^2 g''',
\end{aligned} \tag{2.131}$$

and

$$\begin{aligned}
J_{322} &= -\mu_0 \int_{\mathbb{R}} (\partial_x^3 u) \partial_x [(\partial_x u) g''] \\
&= -\frac{\mu_0}{2} \int_{\mathbb{R}} \partial_x [(\partial_x^2 u)^2] g'' - \mu_0 \int_{\mathbb{R}} (\partial_x^3 u) (\partial_x u) g''' \\
&= \frac{\mu_0}{2} \int_{\mathbb{R}} (\partial_x^2 u)^2 g''' - \mu_0 \int_{\mathbb{R}} (\partial_x^3 u) (\partial_x u) g''',
\end{aligned} \tag{2.132}$$

with

$$\begin{aligned}
& -\mu_0 \int_{\mathbb{R}} (\partial_x^3 u) (\partial_x u) g''' \\
&= \mu_0 \int_{\mathbb{R}} (\partial_x^2 u) \partial_x [(\partial_x u) g'''] \\
&= \mu_0 \int_{\mathbb{R}} (\partial_x^2 u)^2 g''' + \frac{\mu_0}{2} \int_{\mathbb{R}} \partial_x [(\partial_x u)^2] g^{(4)} \\
&= \mu_0 \int_{\mathbb{R}} (\partial_x^2 u)^2 g''' - \frac{\mu_0}{2} \int_{\mathbb{R}} (\partial_x u)^2 g^{(5)},
\end{aligned} \tag{2.133}$$

and summing (2.132) and (2.133), we get

$$J_{322} = \frac{3\mu_0}{2} \int_{\mathbb{R}} (\partial_x^2 u)^2 g''' - \frac{\mu_0}{2} \int_{\mathbb{R}} (\partial_x u)^2 g^{(5)}. \tag{2.134}$$

Then, summing (2.131) and (2.134), we get

$$J_{32} = -\mu_0 \int_{\mathbb{R}} (\partial_x^3 u)^2 g' + 2\mu_0 \int_{\mathbb{R}} (\partial_x^2 u)^2 g''' - \frac{\mu_0}{2} \int_{\mathbb{R}} (\partial_x u)^2 g^{(5)}, \tag{2.135}$$

and summing (2.129) and (2.135), we get

$$J_3 = -\frac{5\mu_0}{2} \int_{\mathbb{R}} (\partial_x^3 u)^2 g' + \frac{5\mu_0}{2} \int_{\mathbb{R}} (\partial_x^2 u)^2 g''' - \frac{\mu_0}{2} \int_{\mathbb{R}} (\partial_x u)^2 g^{(5)}. \tag{2.136}$$

Finally, summing (2.121), (2.122), (2.125) and (2.136), we obtain

$$\begin{aligned}
J &= -\frac{p}{2} \int_{\mathbb{R}} (\partial_x u)^3 u^{p-1} g + \frac{1}{2} \int_{\mathbb{R}} (\partial_x u)^2 u^p g' - \frac{3}{2} \int_{\mathbb{R}} (\partial_x^2 u)^2 g' + \frac{1}{2} \int_{\mathbb{R}} (\partial_x u)^2 g''' \\
&\quad - \frac{5\mu_0}{2} \int_{\mathbb{R}} (\partial_x^3 u)^2 g' + \frac{5\mu_0}{2} \int_{\mathbb{R}} (\partial_x^2 u)^2 g''' - \frac{\mu_0}{2} \int_{\mathbb{R}} (\partial_x u)^2 g^{(5)}.
\end{aligned} \tag{2.137}$$

Let us compute I :

$$I = \mu_0 \int_{\mathbb{R}} [\partial_x^2 (\partial_t u)] (\partial_x^2 u) g, \tag{2.138}$$

and using that $u(t, x)$ satisfies μ_0 -gKW equation, we have

$$\begin{aligned}
I &= -3p\mu_0 \int_{\mathbb{R}} (\partial_x^2 u)^2 (\partial_x u) u^{p-1} g - \mu_0 p(p-1) \int_{\mathbb{R}} (\partial_x^2 u) (\partial_x u)^3 u^{p-2} g - \mu_0 \int_{\mathbb{R}} (\partial_x^3 u) (\partial_x^2 u) u^p g \\
&\quad - \mu_0 \int_{\mathbb{R}} (\partial_x^5 u) (\partial_x^2 u) g + \mu_0^2 \int_{\mathbb{R}} (\partial_x^7 u) (\partial_x^2 u) g \\
&= -3p\mu_0 \int_{\mathbb{R}} (\partial_x^2 u)^2 (\partial_x u) u^{p-1} g + \sum_{j=1}^4 I_j.
\end{aligned} \tag{2.139}$$

Let us compute I_1 :

$$\begin{aligned}
I_1 &= -\frac{\mu_0 p(p-1)}{4} \int_{\mathbb{R}} \partial_x [(\partial_x u)^4] u^{p-2} g \\
&= \frac{\mu_0 p(p-1)(p-2)}{4} \int_{\mathbb{R}} (\partial_x u)^5 u^{p-3} g + \frac{\mu_0 p(p-1)}{4} \int_{\mathbb{R}} (\partial_x u)^4 u^{p-2} g'.
\end{aligned} \tag{2.140}$$

Let us compute I_2 :

$$\begin{aligned}
I_2 &= -\frac{\mu_0}{2} \int_{\mathbb{R}} \partial_x [(\partial_x^2 u)^2] u^p g \\
&= \frac{\mu_0 p}{2} \int_{\mathbb{R}} (\partial_x^2 u)^2 (\partial_x u) u^{p-1} g + \frac{\mu_0}{2} \int_{\mathbb{R}} (\partial_x^2 u)^2 u^p g'.
\end{aligned} \tag{2.141}$$

Let us compute I_3 :

$$\begin{aligned}
I_3 &= \mu_0 \int_{\mathbb{R}} (\partial_x^4 u) \partial_x [(\partial_x^2 u) g] \\
&= \frac{\mu_0}{2} \int_{\mathbb{R}} \partial_x [(\partial_x^3 u)^2] g + \mu_0 \int_{\mathbb{R}} (\partial_x^4 u) (\partial_x^2 u) g' \\
&= -\frac{\mu_0}{2} \int_{\mathbb{R}} (\partial_x^3 u)^2 g' + I_{31},
\end{aligned} \tag{2.142}$$

with

$$\begin{aligned}
I_{31} &= -\mu_0 \int_{\mathbb{R}} (\partial_x^3 u) \partial_x [(\partial_x^2 u) g'] \\
&= -\mu_0 \int_{\mathbb{R}} (\partial_x^3 u)^2 g' - \frac{\mu_0}{2} \int_{\mathbb{R}} \partial_x [(\partial_x^2 u)^2] g'' \\
&= -\mu_0 \int_{\mathbb{R}} (\partial_x^3 u)^2 g' + \frac{\mu_0}{2} \int_{\mathbb{R}} (\partial_x^2 u)^2 g''',
\end{aligned} \tag{2.143}$$

and summing (2.142) and (2.143), we get

$$I_3 = -\frac{3\mu_0}{2} \int_{\mathbb{R}} (\partial_x^3 u)^2 g' + \frac{\mu_0}{2} \int_{\mathbb{R}} (\partial_x^2 u)^2 g'''. \tag{2.144}$$

Let us compute I_4 :

$$\begin{aligned}
I_4 &= -\mu_0^2 \int_{\mathbb{R}} (\partial_x^6 u) \partial_x [(\partial_x^2 u) g] \\
&= -\mu_0^2 \int_{\mathbb{R}} (\partial_x^6 u) (\partial_x^3 u) g - \mu_0^2 \int_{\mathbb{R}} (\partial_x^6 u) (\partial_x^2 u) g' = I_{41} + I_{42}.
\end{aligned} \tag{2.145}$$

Let us compute I_{41} :

$$\begin{aligned}
I_{41} &= \mu_0^2 \int_{\mathbb{R}} (\partial_x^5 u) \partial_x [(\partial_x^3 u)g] \\
&= \frac{\mu_0^2}{2} \int_{\mathbb{R}} \partial_x [(\partial_x^4 u)^2]g + \mu_0^2 \int_{\mathbb{R}} (\partial_x^5 u)(\partial_x^3 u)g' \\
&= -\frac{\mu_0^2}{2} \int_{\mathbb{R}} (\partial_x^4 u)^2 g' + I_{411},
\end{aligned} \tag{2.146}$$

with

$$\begin{aligned}
I_{411} &= -\mu_0^2 \int_{\mathbb{R}} (\partial_x^4 u) \partial_x [(\partial_x^3 u)g'] \\
&= -\mu_0^2 \int_{\mathbb{R}} (\partial_x^4 u)^2 g' - \frac{\mu_0^2}{2} \int_{\mathbb{R}} \partial_x [(\partial_x^3 u)^2]g'' \\
&= -\mu_0^2 \int_{\mathbb{R}} (\partial_x^4 u)^2 g' + \frac{\mu_0^2}{2} \int_{\mathbb{R}} (\partial_x^3 u)^2 g''',
\end{aligned} \tag{2.147}$$

and summing (2.146) and (2.147), we get

$$I_{41} = -\frac{3\mu_0^2}{2} \int_{\mathbb{R}} (\partial_x^4 u)^2 g' + \frac{\mu_0^2}{2} \int_{\mathbb{R}} (\partial_x^3 u)^2 g'''. \tag{2.148}$$

Let us compute I_{42} :

$$\begin{aligned}
I_{42} &= \mu_0^2 \int_{\mathbb{R}} (\partial_x^5 u) \partial_x [(\partial_x^2 u)g'] \\
&= \mu_0^2 \int_{\mathbb{R}} (\partial_x^5 u)(\partial_x^3 u)g' + \mu_0^2 \int_{\mathbb{R}} (\partial_x^5 u)(\partial_x^2 u)g'' = I_{421} + I_{422}.
\end{aligned} \tag{2.149}$$

Let us compute I_{421} :

$$\begin{aligned}
I_{421} &= -\mu_0^2 \int_{\mathbb{R}} (\partial_x^4 u) \partial_x [(\partial_x^3 u)g'] \\
&= -\mu_0^2 \int_{\mathbb{R}} (\partial_x^4 u)^2 g' - \frac{\mu_0^2}{2} \int_{\mathbb{R}} \partial_x [(\partial_x^3 u)^2]g'' \\
&= -\mu_0^2 \int_{\mathbb{R}} (\partial_x^4 u)^2 g' + \frac{\mu_0^2}{2} \int_{\mathbb{R}} (\partial_x^3 u)^2 g'''.
\end{aligned} \tag{2.150}$$

Let us compute I_{422} :

$$\begin{aligned}
I_{422} &= -\mu_0^2 \int_{\mathbb{R}} (\partial_x^4 u) \partial_x [(\partial_x^2 u)g''] \\
&= -\frac{\mu_0^2}{2} \int_{\mathbb{R}} \partial_x [(\partial_x^3 u)^2]g'' - \mu_0^2 \int_{\mathbb{R}} (\partial_x^4 u)(\partial_x^2 u)g''' \\
&= \frac{\mu_0^2}{2} \int_{\mathbb{R}} (\partial_x^3 u)^2 g''' + I_{4221},
\end{aligned} \tag{2.151}$$

with

$$\begin{aligned}
I_{4221} &= \mu_0^2 \int_{\mathbb{R}} (\partial_x^3 u) \partial_x [(\partial_x^2 u)g'''] \\
&= \mu_0^2 \int_{\mathbb{R}} (\partial_x^3 u)^2 g''' + \frac{\mu_0^2}{2} \int_{\mathbb{R}} \partial_x [(\partial_x^2 u)^2]g^{(4)} \\
&= \mu_0^2 \int_{\mathbb{R}} (\partial_x^3 u)^2 g''' - \frac{\mu_0^2}{2} \int_{\mathbb{R}} (\partial_x^2 u)^2 g^{(5)},
\end{aligned} \tag{2.152}$$

and summing (2.151) and (2.152), we get

$$I_{422} = \frac{3\mu_0^2}{2} \int_{\mathbb{R}} (\partial_x^3 u)^2 g''' - \frac{\mu_0^2}{2} \int_{\mathbb{R}} (\partial_x^2 u)^2 g^{(5)}. \quad (2.153)$$

Then, summing (2.150) and (2.153), we get

$$I_{42} = -\mu_0^2 \int_{\mathbb{R}} (\partial_x^4 u)^2 g' + 2\mu_0^2 \int_{\mathbb{R}} (\partial_x^3 u)^2 g''' - \frac{\mu_0^2}{2} \int_{\mathbb{R}} (\partial_x^2 u)^2 g^{(5)}. \quad (2.154)$$

Now, summing (2.148) and (2.154), we get

$$I_4 = -\frac{5\mu_0^2}{2} \int_{\mathbb{R}} (\partial_x^4 u)^2 g' + \frac{5\mu_0^2}{2} \int_{\mathbb{R}} (\partial_x^3 u)^2 g''' - \frac{\mu_0^2}{2} \int_{\mathbb{R}} (\partial_x^2 u)^2 g^{(5)}. \quad (2.155)$$

Finally, we obtain

$$\begin{aligned} I = & -\frac{5\mu_0 p}{2} \int_{\mathbb{R}} (\partial_x^2 u)^2 (\partial_x u) u^{p-1} g + \frac{\mu_0 p(p-1)(p-2)}{4} \int_{\mathbb{R}} (\partial_x u)^5 u^{p-3} g + \frac{\mu_0 p(p-1)}{4} \int_{\mathbb{R}} (\partial_x u)^4 u^{p-2} g' \\ & + \frac{\mu_0}{2} \int_{\mathbb{R}} (\partial_x^2 u)^2 u^p g' - \frac{3\mu_0}{2} \int_{\mathbb{R}} (\partial_x^3 u)^2 g' + \frac{\mu_0}{2} \int_{\mathbb{R}} (\partial_x^2 u)^2 g''' \\ & - \frac{5\mu_0^2}{2} \int_{\mathbb{R}} (\partial_x^4 u)^2 g' + \frac{5\mu_0^2}{2} \int_{\mathbb{R}} (\partial_x^3 u)^2 g''' - \frac{\mu_0^2}{2} \int_{\mathbb{R}} (\partial_x^2 u)^2 g^{(5)}. \end{aligned} \quad (2.156)$$

Identity (2.88) follows by combining (2.119), (2.137) and (2.156). $\square$

Acknowledgements. The publication has been prepared with the support of the RUDN University Program 5-100.

References

- [1] L. Abdelouhab, J. L. Bona, M. Felland, and J.-C. Saut. Nonlocal models for nonlinear, dispersive waves. *Phys. D*, 40(3):360–392, 1989.
- [2] B. Dey, A. Khare, and C. N. Kumar. Stationary solitons of the fifth order KdV-type. Equations and their stabilization. *Phys. Lett. A*, 223(6):449–452, 1996.
- [3] André Kabakouala. A Linear Liouville Property for the Generalized Kawahara Equation. working paper or preprint, October 2018.
- [4] André Kabakouala and Luc Molinet. On the stability of the solitary waves to the (generalized) Kawahara equation. *J. Math. Anal. Appl.*, 457(1):478–497, 2018.
- [5] V. I. Karpman. Stabilization of soliton instabilities by higher order dispersion: KdV-type equations. *Phys. Lett. A*, 210(1-2):77–84, 1996.
- [6] Celine Laurent and Yvan Martel. Smoothness and exponential decay of L^2 -compact solutions of the generalized KdV equations. *Comm. Partial Differential Equations*, 28(11-12):2093–2107, 2003.
- [7] Yvan Martel. Linear problems related to asymptotic stability of solitons of the generalized KdV equations. *SIAM J. Math. Anal.*, 38(3):759–781, 2006.
- [8] Yvan Martel and Frank Merle. Asymptotic stability of solitons for subcritical generalized KdV equations. *Arch. Ration. Mech. Anal.*, 157(3):219–254, 2001.
- [9] Yvan Martel and Frank Merle. Asymptotic stability of solitons of the gKdV equations with general nonlinearity. *Math. Ann.*, 341(2):391–427, 2008.