A Note on Some New Bounds for Trigonometric Functions Using Infinite Products

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Abstract. In this note, new sharp bounds for trigonometric functions are proved. We provide alternative proofs to existing results on exponential bounds, with some improvements, by using infinite products and the so-called Bernoulli inequality.

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1 Introduction

The following two theorems have been proved by Yogesh J. Bagul [2]. The first result concerns bounds for $cos(x)$.

Theorem 1. *[2, Theorem 1] For $x \in (0,1)$, we have*

$$e^{-ax^2} < cos(x) < e^{-x^2/2},$$

with $a \approx 0.6156265$.

The second result is about bounds for $sin(x)/x$.

Theorem 2. *[2, Theorem 2] For $x \in (0,1)$, we have*

$$e^{-bx^2} < \frac{sin(x)}{x} < e^{-x^2/6},$$

with $b \approx 0.172604$.

Our main interest is to provide tight and tractable lower bounds for $cos(x)$ and $sin(x)/x$. The proofs of [2, Theorems 1 and 2] are based on the so-called l'Hospital's rule of monotonicity [1], with the use of non trivial derivative properties of involving functions.

In this paper, we first propose alternative proofs of these two results, with extensions, by the use of infinite product and the so-called Bernoulli inequality. Then new bounds for $cos(x)$ and $sin(x)/x$ defined with polynomial terms are proved. Their tightness are shown graphically.

2 Alternative proofs of existing bounds

The following version of the so-called Bernoulli inequality will be at the heart of our proofs.

Proposition 1. *[Bernoulli inequality] For $u, v \in (0,1)$, we have*

$$1 - uv \geqslant (1-v)^u.$$

We refer to [5, Theorem A] for the general version of the Bernoulli inequality (with less restriction on u and v). An elegant short proof for the considered version is given below.

Proof of Proposition 1. For $u, v \in (0,1)$ and $k \geqslant 1$, we have $u^k \leqslant u$. It follows from the logarithmic series expansion that

$$ln(1-uv) = -\sum_{k=1}^{+\infty} \frac{u^k v^k}{k} \geqslant u\left(-\sum_{k=1}^{+\infty} \frac{v^k}{k}\right) = uln(1-v).$$

Composing by the exponential function, we obtain the desired inequality. □

The following result is a slight generalization of [2, Theorem 1], but with a completely different proof.

Proposition 2. *For $\alpha \in (0, \pi/2)$ and $x \in (0, \alpha)$, we have*

$$e^{-\beta x^2} \leqslant cos(x) \leqslant e^{-x^2/2},$$

with $\beta = [-ln(cos(\alpha))]/\alpha^2$.

Proof of Proposition 2. The proof is centred around the infinite product of the cosine function: for all $x \in \mathbb{R}$, we have

$$cos(x) = \prod_{k=1}^{+\infty} \left(1 - \frac{4x^2}{\pi^2(2k-1)^2}\right).$$

- *Proof of the upper bound.* Using the known inequality : $e^y \geqslant 1+y$ for $y \in \mathbb{R}$ and $\sum_{k=1}^{+\infty} \frac{1}{(2k-1)^2} = \frac{\pi^2}{8}$, for $x \in (0, \pi/2)$ (such that all the terms in the product are strictly positive), we have

$$\begin{aligned} cos(x) &= \prod_{k=1}^{+\infty} \left(1 - \frac{4x^2}{\pi^2(2k-1)^2}\right) \leqslant \prod_{k=1}^{+\infty} exp\left(-\frac{4x^2}{\pi^2(2k-1)^2}\right) \\ &= exp\left(-\frac{4x^2}{\pi^2} \sum_{k=1}^{+\infty} \frac{1}{(2k-1)^2}\right) = exp\left(-\frac{4x^2}{\pi^2} \times \frac{\pi^2}{8}\right) = e^{-x^2/2}. \end{aligned}$$

- *Proof of the lower bound.* Using the infinite product expression of the cosine function and Proposition 1 (with respect to the considered u and v that satisfy $u, v \in (0,1)$), for $x \in (0, \alpha)$, we have

$$\begin{aligned} cos(x) &= \prod_{k=1}^{+\infty} \left(1 - \frac{4\alpha^2}{\pi^2(2k-1)^2} \frac{x^2}{\alpha^2}\right) \geqslant \prod_{k=1}^{+\infty} \left(1 - \frac{4\alpha^2}{\pi^2(2k-1)^2}\right)^{x^2/\alpha^2} \\ &= \left(\prod_{k=1}^{+\infty} \left(1 - \frac{4\alpha^2}{\pi^2(2k-1)^2}\right)\right)^{x^2/\alpha^2} = (cos(\alpha))^{x^2/\alpha^2} = e^{-\beta x^2}, \end{aligned}$$

  with $\beta = [-ln(cos(\alpha))]/\alpha^2$.

By combining the obtained upper and lower bounds, we end the proof of Proposition 2. □

Note: Taking $\alpha = 1$, we obtain $\beta = -ln(cos(1)) \approx 0.6156265$, and Proposition 2 becomes [2, Theorem 1].

Proposition 3 below gives a generalization of [2, Theorem 2].

Proposition 3. *For $\alpha \in (0, \pi)$ and $x \in (0, \alpha)$, we have*

$$e^{-\gamma x^2} \leqslant \frac{sin(x)}{x} \leqslant e^{-x^2/6},$$

with $\gamma = [-ln(sin(\alpha)/\alpha)]/\alpha^2$.

Proof of Proposition 3. The proof is centred around the infinite product of the sinc function, i.e. $sinc(x) = sin(x)/x$ for $x \neq 0$, the so-called Euler formula: for all $x \in \mathbb{R} - \{0\}$, we have

$$\frac{sin(x)}{x} = \prod_{k=1}^{+\infty} \left(1 - \frac{x^2}{\pi^2 k^2}\right).$$

- *Proof of the upper bound.* Using the inequality : $e^y \geqslant 1+y$ for $y \in \mathbb{R}$ and $\sum\limits_{k=1}^{+\infty} \frac{1}{k^2} = \frac{\pi^2}{6}$, for $x \in (0,\pi)$, we have

$$\frac{sin(x)}{x} = \prod_{k=1}^{+\infty}\left(1-\frac{x^2}{\pi^2 k^2}\right) \leqslant \prod_{k=1}^{+\infty} exp\left(-\frac{x^2}{\pi^2 k^2}\right)$$
$$= exp\left(-\frac{x^2}{\pi^2}\sum_{k=1}^{+\infty}\frac{1}{k^2}\right) = exp\left(-\frac{x^2}{\pi^2}\times\frac{\pi^2}{6}\right) = e^{-x^2/6}.$$

- *Proof of the lower bound.* Using the infinite product expression of the sinc function and Proposition 1, for $x \in (0,\alpha)$, we have

$$\frac{sin(x)}{x} = \prod_{k=1}^{+\infty}\left(1-\frac{\alpha^2}{\pi^2 k^2}\frac{x^2}{\alpha^2}\right) \geqslant \prod_{k=1}^{+\infty}\left(1-\frac{\alpha^2}{\pi^2 k^2}\right)^{x^2/\alpha^2}$$
$$= \left(\prod_{k=1}^{+\infty}\left(1-\frac{\alpha^2}{\pi^2 k^2}\right)\right)^{x^2/\alpha^2} = \left(\frac{sin(\alpha)}{\alpha}\right)^{x^2/\alpha^2} = e^{-\gamma x^2},$$

with $\gamma = [-ln(sin(\alpha)/\alpha)]/\alpha^2$.

□

Note: Taking $\alpha = 1$, we obtain $\gamma = -ln(sin(1)) \approx 0.1726037$, and Proposition 3 becomes [2, Theorem 2].

Note: Similar results to Propositions 2 and 3 can be obtained with hyperbolic functions instead of trigonometric functions. Indeed, the following Bernoulli inequality exists: for $u \in (0,1)$ and $v > 0$, we have $1+uv \geqslant (1+v)^u$ (see [5, Theorem A]). Therefore, using the same arguments to the proofs of Propositions 2 and 3 with the infinite products for $cosh(x)$ and $sinh(x)/x$, i.e. $cosh(x) = \prod\limits_{k=1}^{+\infty}\left(1+\frac{4x^2}{\pi^2(2k-1)^2}\right)$ and $sinh(x)/x = \prod\limits_{k=1}^{+\infty}\left(1+\frac{x^2}{\pi^2 k^2}\right)$, we establish that, for $\alpha > 0$ and $x \in (0,\alpha)$,

- $e^{\theta x^2} \leqslant cosh(x) \leqslant e^{x^2/2}$, with $\theta = [ln(cosh(\alpha))]/\alpha^2$, which is appeared in [3, Remark 2.1] and
- $e^{\zeta x^2} \leqslant sinh(x)/x \leqslant e^{x^2/6}$, with $\zeta = [ln(sinh(\alpha)/\alpha)]/\alpha^2$.

3 Polynomial bounds for trigonometric functions

This section is devoted to the proof of new sharp polynomial lower bounds or new proofs, for trigonometric functions.

Proposition 4. *For $x \in (0, \pi/2)$, we have*

$$cos(x) \geqslant \left(1 - \frac{4x^2}{\pi^2}\right)^{\pi^2/8}. \tag{3.1}$$

Proof of Proposition 4. The proof combines the infinite product of the cosine function, Proposition 1 and $\sum_{k=1}^{+\infty} \frac{1}{(2k-1)^2} = \frac{\pi^2}{8}$, but in a different way to the proof of Proposition 2. We have

$$\begin{aligned} cos(x) &= \prod_{k=1}^{+\infty} \left(1 - \frac{4x^2}{\pi^2} \frac{1}{(2k-1)^2}\right) \geqslant \prod_{k=1}^{+\infty} \left(1 - \frac{4x^2}{\pi^2}\right)^{1/(2k-1)^2} \\ &= \left(1 - \frac{4x^2}{\pi^2}\right)^{\sum_{k=1}^{+\infty} 1/(2k-1)^2} = \left(1 - \frac{4x^2}{\pi^2}\right)^{\pi^2/8}. \end{aligned}$$

□

We provide a graphical illustration of the bounds (3.1) in Figure 1.

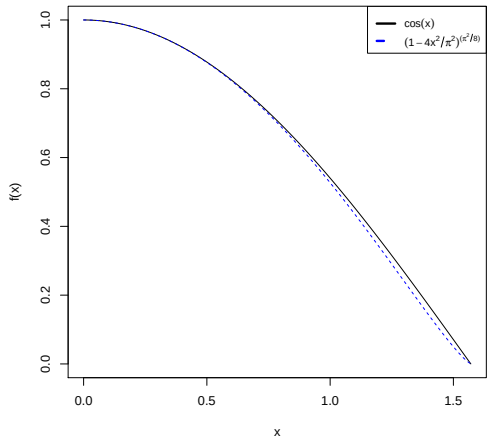


Figure 1: Graphs of the functions in (3.1) for $x \in (0, \pi/2)$.

Proposition 5 below corresponds to [4, Lower bound in Theorem 1.26]. The proof in [4] is based on the study of the function $f(x) = log(x/sin(x)) - (\pi^2/6)log(1/(1 - x^2/\pi^2))$. Here we give a more direct proof using infinite products.

Proposition 5. *[4, Lower bound in Theorem 1.26] For $x \in (0, \pi)$, we have*

$$\frac{sin(x)}{x} \geqslant \left(1 - \frac{x^2}{\pi^2}\right)^{\pi^2/6}. \tag{3.2}$$

Proof of Proposition 5. The proof combines the infinite product of the sinc function, Proposition 1 and $\sum_{k=1}^{+\infty} \frac{1}{k^2} = \frac{\pi^2}{6}$:

$$\frac{sin(x)}{x} = \prod_{k=1}^{+\infty} \left(1 - \frac{x^2}{\pi^2}\frac{1}{k^2}\right) \geqslant \prod_{k=1}^{+\infty} \left(1 - \frac{x^2}{\pi^2}\right)^{1/k^2}$$
$$= \left(1 - \frac{x^2}{\pi^2}\right)^{\sum_{k=1}^{+\infty} 1/k^2} = \left(1 - \frac{x^2}{\pi^2}\right)^{\pi^2/6}.$$

□

In the following Corollary, we give tight upper bound for $cos(x)$.

Corollary 1. *For $x \in (0, \pi)$ we have*

$$cos(x) \leqslant 1 + \frac{3\pi^2}{\pi^2 + 6}\left[\left(1 - \frac{x^2}{\pi^2}\right)^{(\pi^2+6)/6} - 1\right]. \tag{3.3}$$

Proof of Corollary 1. Using (3.2) and integrating we have

$$\int_0^x sin(t)\, dt \geqslant \int_0^x \left(1 - \frac{t^2}{\pi^2}\right)^{\pi^2/6} t\, dt$$

which turns into

$$-cos(x) \geqslant -1 - \frac{3\pi^2}{\pi^2 + 6}\left[\left(1 - \frac{x^2}{\pi^2}\right)^{(\pi^2+6)/6} - 1\right]$$

proving desired result. □

The upper bound of Corollary 1 can be seen in Figure 2.

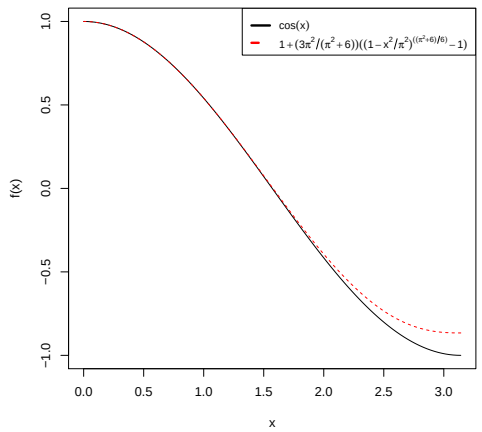


Figure 2: Graphs of the functions in (3.3) for $x \in (0, \pi)$.

Note: Similar results to Propositions 4 and 5 can be obtained with hyperbolic functions instead of trigonometric functions. Indeed, using the following Bernoulli inequality: for $u \in (0,1)$ and $v > 0$, we have $1 + uv \geqslant (1+v)^u$ and the infinite products for $cosh(x)$ and $sinh(x)/x$, we prove that, for $x > 0$,

- $cosh(x) \geqslant \left(1 + 4x^2/\pi^2\right)^{\pi^2/8}$,
- $sinh(x)/x \geqslant \left(1 + x^2/\pi^2\right)^{\pi^2/6}$.

References

[1] G. D. Anderson, M. K. Vamanamurthy and M. Vuorinen, *Conformal Invarients, Inequalities and Quasiconformal maps*, John Wiley and Sons, New York, 1997.

[2] Y. J. Bagul, *Inequalities involving circular, hyperbolic and exponential functions*, Journal of Mathematical Inequalities, Volume 11, Number 3, pp. 695-699, 2017, doi: 10.7153/jmi-2017-11-55.

[3] Y. J. Bagul, *On Exponential Bounds of Hyperbolic Cosine*, Bulletin of The International Mathematical Virtual Institute, Vol. 8, No. 2, pp. 365-367, 2018.

[4] B. A. Bhayo and J. Sándor, *On jordan's, redheffer's and Wilker's inequality*, Mathematical Inequalities and Applications, Volume 19, Number 3, pp. 823-839, 2016. doi:10.7153/mia-19-60

[5] H . N. Shi, *Generalizations of Bernoulli's inequality with applications*, Journal of Mathematical Inequalities, Volume 2, Number 1, pp. 101-107, 2008.