Maxima and minima of the displacement components for the Lamb modes

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This paper revisits the vanishing of the transverse component of the particle displacement vector in free surfaces of an isotropic homogeneous plate, for both symmetric and antisymmetric Lamb waves. Drawing on well-known analytical expressions from Viktorov's book [(1967) *Rayleigh and Lamb Waves: Physical Theory Applications*, Chap. II, pp. 67–121], two distinct frequency-thickness product expressions, in cases where this vanishing occurs, are derived: one for the symmetric modes and another for the antisymmetric modes. At these frequency-thickness products, phase and group velocities have appreciable values which are discussed herein. It appears that these velocities depend on the transverse bulk wave velocity only. This is the specific condition of the Lamé modes. Moreover, theoretical and experimental investigations of displacements in the surface of a plate in air have been carried out. The theoretical part shows that the normal and transverse displacements have, respectively, a local maximum and a local minimum in the vicinity of these frequency-thickness products. The experimental part corroborates the presence of the local maximum of the S_0 Lamb mode for various materials.

I. INTRODUCTION

Pilarsky *et al*. have demonstrated that the normal component of the particle displacement vector in free surfaces of a plate, for nonzero-order symmetric Lamb waves, vanishes when the phase velocity attains the velocity of the bulk longitudinal waves.[1] These authors have also shown that the group velocity is independent of the mode order. These features bear some significant practical importance in the nondestructive testing (NDT) of the fluid filled pipes. Indeed, energy leakage into the fluid can be substantially reduced due to the vanishing of the normal surface displacement. Certain authors like Hay *et al*. and Lowe *et al*. have extensively harnessed this specificity in their works devoted to pipe inspection.[2–4] In a recent paper,[5] Royer *et al*. have investigated the variation of dispersion curves of Lamb modes as a function of the Poisson ratio ν. They show that branches of curves for phase velocities higher than $C_T\sqrt{2}$ (where C_T is the transverse bulk wave velocity) are very sensitive to the Poisson ratio. Conversely, in the case where the phase velocity is lower than $C_T\sqrt{2}$, only a weak dependence on this ratio is observed.

In this paper we show that the transverse component of the particle displacement vector in free surfaces of a plate for both symmetric and antisymmetric Lamb modes can also vanish. Analytic expressions of the frequency-thickness products, f_se (for the symmetric modes) and f_ae (for the antisymmetric modes), are determined when the transverse component is nil. This specific condition is the one of the Lamé modes described in the Graff's book.[6] Just as in the works by Pilarsky *et al*., particular attention is paid herein on the group velocities and their expressions are written.

Further, theoretical and experimental investigations of the normal component of displacement in the surface of a plate in air have been carried out. The aim of these additional investigations is to demonstrate that the normal displacement in the surface reaches a maximum for specific values of frequency-thickness products f_se and f_ae. This phenomenon is of particular interest for the NDT achieved by means of a setup using a laser interferometer. This measurement method may be applied to estimate the transverse velocity, from the observation of the normal component in the surface of the plate.

II. THEORETICAL ANALYSIS: LAMÉ MODES IN A PLATE IN VACUUM

This section recalls the particular features of the Lamé modes described in the Graff's book.[6] Using Viktorov's notation,[7] expressions of the transverse components of the displacement of antisymmetric and symmetric Lamb waves can be written as,

$$U_s = \mathrm{D}\left(\frac{\left(k_s^2+s_s^2\right)\, sh(s_sd)}{2\; q_s\; sh(q_sd)}\; ch(q_sz) - s_s\; ch(s_sz)\right) e^{ik_sx}, \quad (1)$$

$$U_a = \mathrm{C}\left(\frac{\left(k_a^2+s_a^2\right)\, ch(s_ad)}{2\; q_a\; ch(q_ad)}\; sh(q_az) - s_a\; sh(s_az)\right) e^{ik_ax}, \quad (2)$$

where

$$q_{s,a}^2 = k_{s,a}^2 - k_L^2, \quad (3)$$

$$s_{s,a}^2 = k_{s,a}^2 - k_T^2, \quad (4)$$

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and d is the half thickness of the plate. k_L and k_T are, respectively, the longitudinal and transverse wavenumbers. C and D are the constants. The harmonic time dependence term $e^{-i\omega t}$ is omitted in the equations.

The transverse component of the displacement in free surfaces of an isotropic homogeneous plate vanishes when the phase velocity of Lamb waves reaches the value

$$C_{ph} = C_T\sqrt{2}, \tag{5}$$

where C_T is the celerity of the bulk transverse waves,

$$\lim_{k_{s,a}\to k_T/\sqrt{2}} U_{s,a}(z) = 0, \; for \; z = \pm d \; . \tag{6}$$

For this particular phase velocity, the Lamb modes are called the Lamé modes.

To prove the validity of expression in Eq. (6), it is necessary to determine the two frequency-thickness products $f_s e$ and $f_a e$ at which the pth antisymmetric and the pth symmetric modes have phase velocity equal to $C_T\sqrt{2}$ ($e = 2\,d$, e is the thickness of the plate). To this end, the characteristic equations $\Omega_s = 0$ and $\Omega_a = 0$, associated with these modes, are considered.[7]

The obtained frequency-thickness products $f_s e$ and $f_a e$ are expressed as follows:

$$(f_s e)_P = \frac{(2p+1)C_T}{\sqrt{2}}, \text{ for } p = 0, 1, 2, 3, ..., \tag{7}$$

$$(f_a e)_P = pC_T\sqrt{2}, \text{ for } p = 1, 2, 3, ... \tag{8}$$

Order $p = 0$ is excluded from expression in Eq. (8) because, at zero frequency, no mode with a phase velocity equal to $C_T\sqrt{2}$ exists.

Substituting Eq. (7) into Eq. (1) and Eq. (8) into Eq. (2), the limits of displacement of transverse components U_s and U_a when the phase velocity tends toward $C_T\sqrt{2}$ may be written as,

$$\lim_{k_{s,a}\to k_T/\sqrt{2}} U_s(z) = D\,\frac{k_T}{\sqrt{2}}\,\cos\left(\frac{(2p+1)\pi}{2d}z\right)\,e^{i(kx-\pi/2)}, \tag{9}$$

$$\lim_{k_{s,a}\to k_T/\sqrt{2}} U_a(z) = C\,\frac{k_T}{\sqrt{2}}\,\sin\left(\frac{p\pi}{d}z\right)\,e^{ikx}. \tag{10}$$

Therefore, at the surface of a plate (i.e., when $z = \pm d$), these two components become nil.

Group velocities can be found from the following implicit form,[1,8]

$$C_{gs,a} = -\frac{\partial\Omega_{s,a}/\partial k}{\partial\Omega_{s,a}/\partial\omega}. \tag{11}$$

As regards symmetric modes, the relation for the group velocity found in the limit case where $C_{ph} \to C_T\sqrt{2}$ and for $f_s e$ products given by Eq. (7), is expressed as follows:

$$\lim_{\substack{k\to k_T/\sqrt{2} \\ fe\to(f_s e)_p}} C_{gs} = \frac{C_T}{\sqrt{2}}\,\left(\frac{2 - k_T^2\,q_S\,d\,\mathrm{th}(q_S d)}{1 - k_T^2\,q_S\,d\,\mathrm{th}(q_S d)}\right) \tag{12}$$

and, for the antisymmetric modes, the relation in the same limit case for $f_a e$ products given by Eq. (8), is expressed as follows:

$$\lim_{\substack{k\to k_T/\sqrt{2} \\ fe\to(f_a e)_p}} C_{ga} = \frac{C_T}{\sqrt{2}}\,\left(\frac{2 - k_T^2\,q_a\,d\,\coth(q_a d)}{1 - k_T^2\,q_a\,d\,\coth(q_a d)}\right) \tag{13}$$

III. THEORETICAL AND EXPERIMENTAL ANALYSES: A PLATE IN AIR

In this section, we assume isotropic and homogeneous plates of thickness e placed in air. Two plates made of different materials are considered: Duraluminum and polymethyl methacrylate (PMMA). The standard physical parameters used in our computations are,[9–11]

−Duraluminum: $e = 5\,\text{mm}, \rho = 2765\,\text{kg/m}^3, C_L = 6440\,\text{m/s}, C_T = 3113\,\text{m/s}.$

−PMMA: $e = 4.2\,\text{mm}, \rho = 1180\,\text{kg/m}^3, C_L = 2690\,\text{m/s}, C_T = 1340\,\text{m/s}.$

−Air: $\rho = 1.293\,\text{kg/m}^3, C_L = 331.45\,\text{m/s}.$

For these two materials, we pay particular attention on the investigation of the first two Lamb modes which propagate along the plate in air, i.e. the A_1 mode and the S_0 mode. Contrary to the case of a plate in vacuum, it is possible to calculate the absolute displacements of these waves in the interface of a plate placed in a fluid, for example in air. These displacements are expressed by the following relations:

$$U_n = \frac{\partial\varphi^*_{s,a}}{\partial x} - \frac{\partial\psi^*_{s,a}}{\partial z}, \tag{14}$$

$$U_T = \frac{\partial\varphi^*_{s,a}}{\partial x} + \frac{\partial\psi^*_{s,a}}{\partial z}, \tag{15}$$

where the normal displacement component U_n and the transverse displacement component U_T are deduced from the scalar potentials $\varphi^*_{s,a}$ and the vector potentials $\psi^*_{s,a}$ defined by Izbicki *et al.*[12] Moreover, the frequency-thickness products of these two Lamb modes for a plate placed in vacuum are very close to those of a plate placed in air because of the low loading of this fluid.[12] Thus, analytical expressions in Eqs. (7) and (8) are applied to determine *fe* products for which the transverse motions of the Lamb modes are nil in the frequency range of 0–12 MHz mm. From these results, the elastic theory is applied to calculate transverse and normal displacements in the interface of the plate placed in air, in the vicinity of these *fe* products.

The experimental setup is presented in Fig. 1. Measurements are made on rectangular plates of 25 mm length, 20 mm width. The thicknesses of the duraluminum and the PMMA plates are, respectively, 5 and 4.2 mm. These plates are, in turn, placed vertically and fixed onto a piezoelectric transducer. The transducer excites a section of one of the edges of the plate to generate the S_0 Lamb mode. In order to

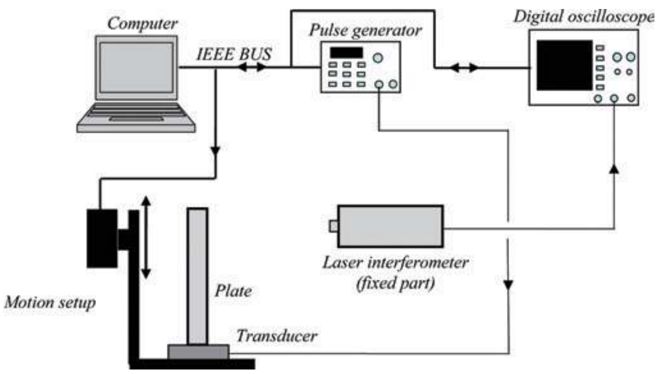

FIG. 1. (Color online) Experimental setup.

ensure proper ultrasound coupling, a *Metalscan gel* layer is applied on the transducer/plate contact surface. The setup is rendered vertically mobile by means of a motion mechanism, which allows the measuring of the normal displacement at various vertical positions in the plate surface. The excitation of the S_0 mode is achieved by using the broadband piezoelectric transducer Panametrics 401 with a central frequency of 500 kHz (see Fig. 1). The transducer is itself excited by a broadband short pulse delivered from a pulse generator. The detection of the signal is achieved using a laser interferometer (BMI heterodyne probe SH140) to measure the normal displacement in the surface of the plate. Measurements are made at a series of equally spaced positions along the plate with a step interval of 0.1 mm over a length of 50 mm. The recorded signals are averaged out and displayed on a Lecroy digital oscilloscope (Fig. 1). Thereafter, the signal obtained from these recordings is transmitted and saved in a computer via the IEEE bus, for further numerical processing. This computer also serves as a driving and controlling tool for the movements of the plate.

IV. RESULTS AND DISCUSSION

This section presents some distinctive features of the symmetric and antisymmetric Lamb modes when phase velocity attains $C_T\sqrt{2}$. This particular condition of mode propagating in the free isotropic plate is the one of the Lamé modes. The studied characteristics are the frequency-thickness product, the group velocity, and the amplitudes of displacement in the surface of the duraluminum plate.

Figure 2 presents a typical plot of phase velocity dispersions of the Lamb modes as a function of the frequency-thickness product in the range of 0–12 MHz mm. The symmetric modes are labeled as S_0, S_1, S_2 and the antisymmetric modes are labeled as A_1, A_2. The frequency-thickness products are labeled as $(f_se)_0$, $(f_ae)_1$, $(f_se)_1$, $(f_ae)_2$, $(f_se)_2$ which correspond to values where the mode phase velocity is equal to $C_T\sqrt{2}$. The interval between two successive frequency-thickness products is equal to $C_T/\sqrt{2}$. This first particularity is easily verified by computing the difference between Eqs. (7) and (8) as illustrated in Sec. II.

Figures 3 and 4 represent, respectively, the dispersion curves of phase and group velocities of the S_0 and A_1 Lamb modes for the frequency-thickness products $(f_se)_0$ and $(f_ae)_1$ for a duraluminum plate. The two figures show that the group velocity of these two modes is very close to $C_T/\sqrt{2}$. This second feature is generalized to any other mode order by using the two dispersion group relations presented in Eqs. (12) and (13), for *fe* products given by Eqs. (7) and (8). Indeed, it appears, clearly, that the group velocities C_{ga} and C_{gs} in the expressions of Eqs. (12) and (13) tend toward $C_T/\sqrt{2}$ as the mode order rises. As regards the lowest orders such as the modes S_0 and A_1, this remains true insofar as the thickness of the plate is not too big. In the case of the studied plate of thickness 5 mm, the absolute relative error of

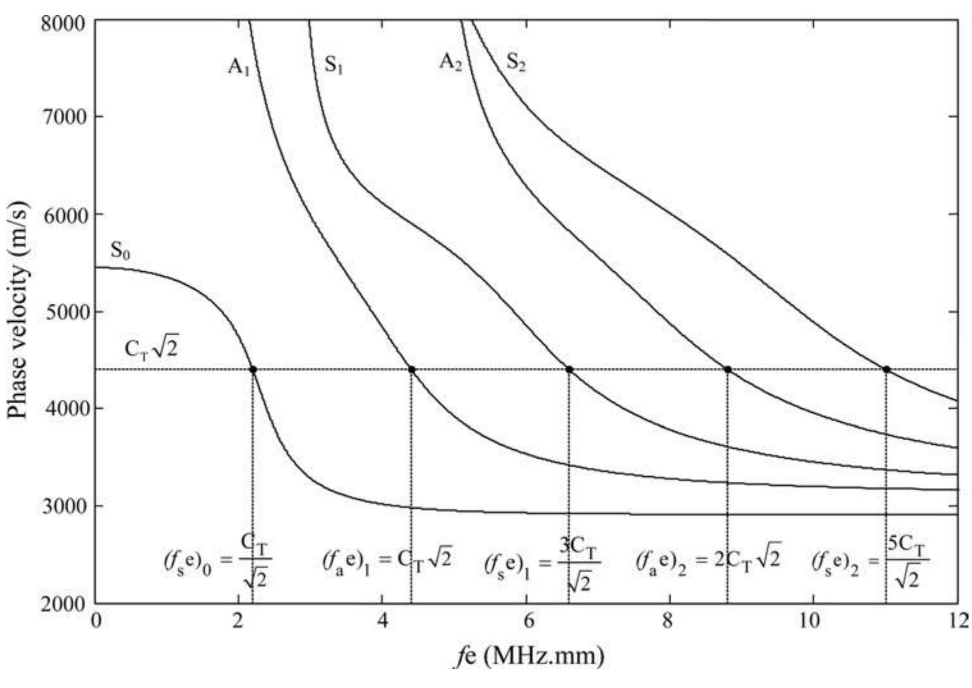

FIG. 2. Dispersion curves of the phase velocity of Lamb modes in a duraluminum plate of thickness 5 mm.

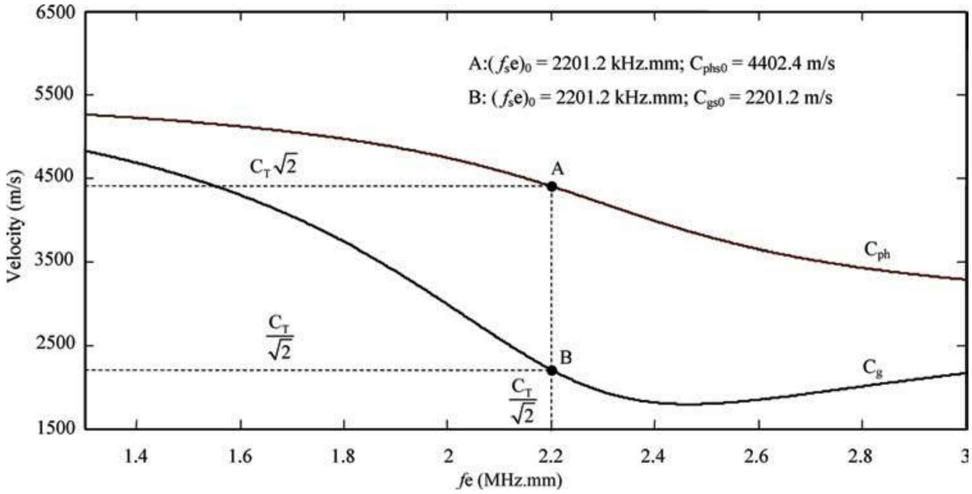

FIG. 3. (Color online) Dispersion curves of the group and phase velocities of S_0 mode in a duraluminum plate of thickness 5 mm.

velocity between C_{gs0} (group velocity of S_0) and $C_T/\sqrt{2}$ is less than $7 \times 10^{-5}\%$ and the error between C_{ga1}(group velocity of A_1) and $C_T/\sqrt{2}$ is less than $9 \times 10^{-6}\%$. We deduce therefrom that the group velocity becomes practically independent of the mode order. The phase-velocity $\times$ group-velocity product is then close to the square of the transverse velocity. Moreover, as it is pointed out by Pagneux[13] and Zernov *et al.*,[14] the S_0 Lamé mode is a necessary condition for the existence of the edge resonance. Indeed these authors have shown that this trapped mode (edge resonance) is due to a decoupling between the S_0 Lamb mode and the higher order evanescent modes.

In order to illustrate the third and last feature, the calculations of the normal displacement component U_n and the transverse displacement component U_T have been realized for a duraluminum plate in air. Both U_n and U_T are normalized by the local maximum of U_n in the considered frequency window. Figures 5(a) and 5(b) are, respectively, the plots of U_n and U_T for the S_0 Lamb mode in the surface of the plate. These plots clearly show that component U_n has a local maximum and the component U_T has a local minimum in the vicinity of a particular frequency-thickness product $(f_se)_0$ (i.e., when $(f_se)_0 = C_T/\sqrt{2}$). The same phenomena are observed in Figs. 6(a) and 6(b) for the A_1 Lamb mode but, this time, in the vicinity of the frequency-thickness product $(f_ae)_1$ (i.e., when $(f_ae)_1 = C_T\sqrt{2}$). We also note that transverse displacement components of modes S_0 and A_1 are very weak though not nil at, respectively, $(f_se)_0$ and $(f_ae)_1$ because the plate is air loaded. Therefore, the search for these minimum and maximum values could have a significant practical meaning, making it possible to obtain an estimation of the transverse velocity of a material. Indeed, the associated frequency-thickness products being dependent only on the transverse velocity C_T, the knowledge of these values enables us to estimate the value of C_T.

The measurement method described in details in Sec. III is used to obtain the normal displacement in the surface of the plate. Measured time signals allow us to obtain a representation of the wavenumber k as a function of frequency f(dispersion curve), after performing a 2D Fourier transform[15,16] on the signal. Figure 7 shows such presentation for the S_0 mode. This figure provides amplitude shading presentation in which the lighter the shading the higher the amplitude. The S_0 mode at the frequency $(f_s)_0 = 454.7$ kHz and wavenumber $k = 667.3$ m^{-1} ($C_{ph} = 4281.4$ m/s) dominates the normal displacement in the frequency range of 300–700 kHz. From the expression in Eq. (5), we deduce the value of transverse velocity of the bulk wave, which is equal to 3028

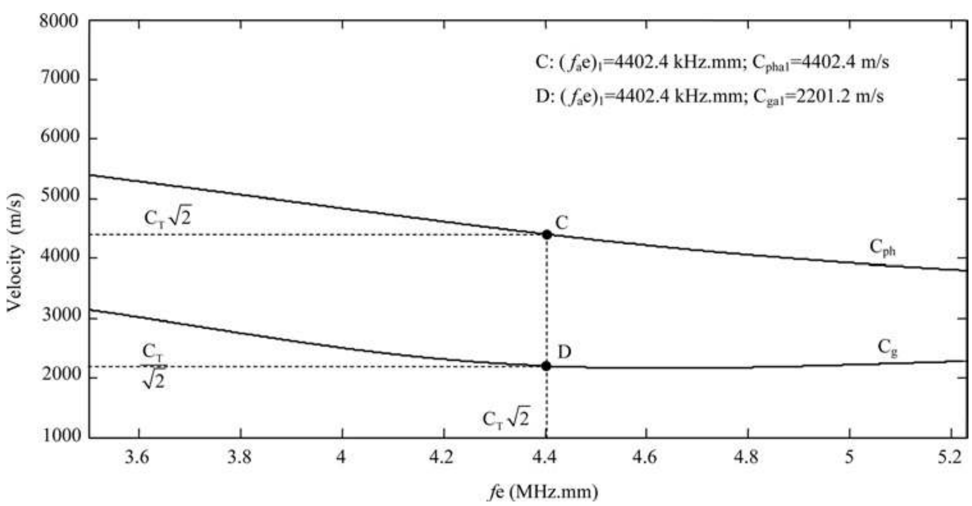

FIG. 4. Dispersion curves of the group and phase velocities of A_1 mode in a duraluminum plate of thickness 5 mm.

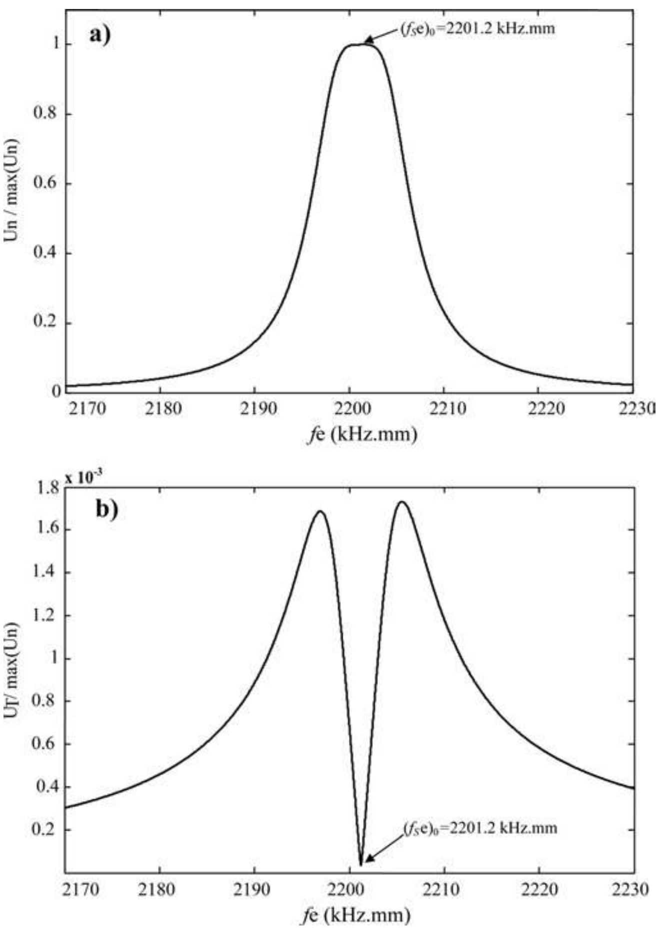

FIG. 5. (a) Normal displacement component of the S_0 mode in the surface of a duraluminum plate of thickness 5 mm. (b) Transverse displacement component of the S_0 mode in the surface of a duraluminum plate of thickness 5 mm.

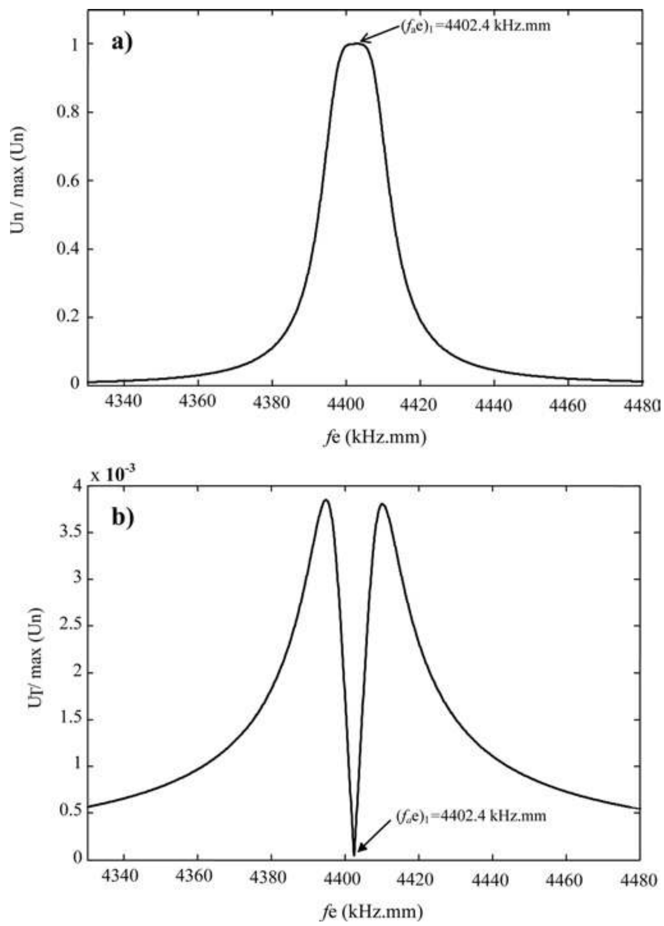

FIG. 6. (a) Normal displacement component of the A_1 mode in the surface of a duraluminum plate of thickness 5 mm. (b) Transverse displacement component of the A_1 mode in the surface of a duraluminum plate of thickness 5 mm.

m/s. The same experimental work has been achieved for a PMMA plate. The main results are presented in Table I.

In the case of duraluminum, the estimated value of C_T differs slightly from the standard parameter C_T (see Table1): the transverse velocity error is less than 3%. This can partly be explained by the non-uniformity of the plate thickness and partly by the type of duraluminum used experimentally which is different from the standard one. For PMMA, the estimated value of C_T is also close to theoretical results: The transverse velocity error is less than 3%.

V. CONCLUSION

The vanishing of the transverse displacement components of symmetric and antisymmetric Lamb waves in a free plate has been revisited. This occurs when the phase velocities are equal to $C_T\sqrt{2}$ (C_T is the velocity of bulk transverse waves). Thus, two analytical expressions of the frequency-thickness product are given: One for the symmetric waves $(f_se)_p$ and another for antisymmetric waves $(f_ae)_p$ (index p is the mode order). Then, two formulations of the group velocity have been established when the phase velocity tends toward $C_T\sqrt{2}$ and the frequency-thickness product toward $(f_se)_p$ or $(f_ae)_p$. From the results obtained, it was noted that two distinctive features for the symmetric or antisymmetric Lamb waves stand out clearly when the phase velocity is equal to $C_T\sqrt{2}$: (i) the interval between two successive frequency-thickness products is always equal to $C_T/\sqrt{2}$ and (ii) their group velocities tend toward the limit $C_T/\sqrt{2}$.

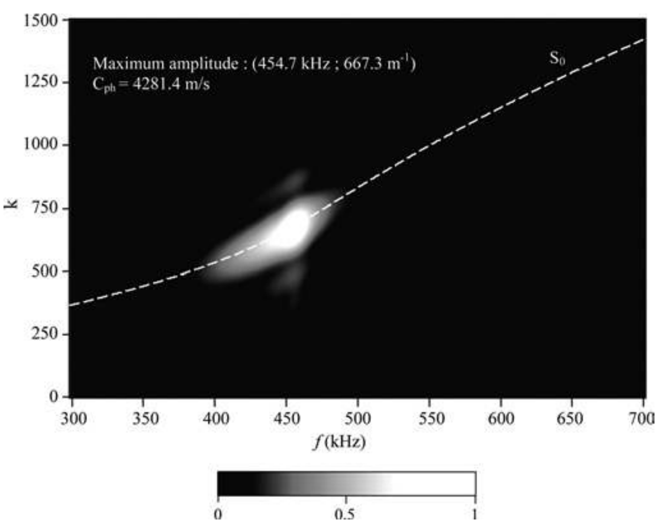

FIG. 7. Modulus of the space-time Fourier transform of experimental time signals of the S_0 mode in the surface of the duraluminum plate of thickness 5 mm. White dashed line: theoretical trajectory of the S_0 mode.

TABLE I. Estimation of the transverse velocity C_T of the two materials: duraluminum and PMMA.

Material plate	Duraluminum	PMMA
Experimental frequency $(f_S e)_0$ (kHz)	454.7	211.2
Estimated phase velocity C_{ph} (m/s)	4281.4	1875.4
Estimated transverse velocity C_T (m/s)	3028	1326
Standard transverse velocity C_T (m/s) (Refs. 9, 11)	3113	1340

Moreover, an additional theoretical study of absolute normal displacements of S_0 and A_1 waves in the surface of an air loaded duraluminum plate has been carried out. This shows that the normal displacements have a local maximum in the vicinity of their respective frequency-thickness products $(f_s e)_0 = C_T/\sqrt{2}$ and $(f_a e)_1 = C_T\sqrt{2}$. This happens when their absolute transverse displacements have a local minimum. An experimental investigation has allowed us to corroborate this third feature for the S_0 wave. This has been achieved for both duraluminum and PMMA plates. The velocity of the bulk waves C_T, of each of these materials, has been estimated with a good experimental accuracy.

ACKNOWLEDGMENTS

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