Eight-dimensional Octonion-like but Associative Normed Division Algebra

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We present an eight-dimensional even sub-algebra of the $2^4 = 16$-dimensional associative Clifford algebra $Cl_{4,0}$ and show that its eight-dimensional elements denoted as $\mathbf{X}$ and $\mathbf{Y}$ respect the norm relation $||\mathbf{XY}|| = ||\mathbf{X}||\,||\mathbf{Y}||$, thus forming an octonion-like but associative normed division algebra.

Consider the following eight-dimensional vector space with graded Clifford-algebraic basis and orientation $\lambda = \pm 1$:

$$Cl^{\lambda}_{3,0} = \text{span}\{\, 1,\ \lambda\mathbf{e}_x,\ \lambda\mathbf{e}_y,\ \lambda\mathbf{e}_z,\ \lambda\mathbf{e}_x\mathbf{e}_y,\ \lambda\mathbf{e}_z\mathbf{e}_x,\ \lambda\mathbf{e}_y\mathbf{e}_z,\ \lambda\mathbf{e}_x\mathbf{e}_y\mathbf{e}_z\,\}. \tag{1}$$

Here $\{\,\mathbf{e}_x,\,\mathbf{e}_y,\,\mathbf{e}_z\}$ is a set of anti-commuting orthonormal vectors in $\mathrm{I\!R}^3$ such that $\mathbf{e}_j\mathbf{e}_i = -\,\mathbf{e}_i\mathbf{e}_j$ for any $i, j = x,\, y$, or z. In general the vectors $\mathbf{e}_i$ satisfy the following geometric product in this *associative* but non-commutative algebra [1]:

$$\mathbf{e}_i\,\mathbf{e}_j \;=\; \mathbf{e}_i\cdot\mathbf{e}_j + \mathbf{e}_i\wedge\,\mathbf{e}_j\,, \tag{2}$$

with

$$\mathbf{e}_i\cdot\mathbf{e}_j := \,\frac{1}{2}\,\{\mathbf{e}_i\mathbf{e}_j + \mathbf{e}_j\mathbf{e}_i\} \tag{3}$$

being the symmetric inner product and

$$\mathbf{e}_i\wedge\mathbf{e}_j := \,\frac{1}{2}\,\{\mathbf{e}_i\mathbf{e}_j - \mathbf{e}_j\mathbf{e}_i\} \tag{4}$$

being the anti-symmetric outer product, giving $(\mathbf{e}_i\wedge\mathbf{e}_j)^2 = -1$. There are thus basis elements of four different grades in this algebra: An identity element $\mathbf{e}_i^2 = 1$ of grade-0, three orthonormal vectors $\mathbf{e}_i$ of grade-1, three orthonormal bivectors $\mathbf{e}_j\mathbf{e}_k$ of grade-2, and a trivector $I_3 = \mathbf{e}_i\mathbf{e}_j\mathbf{e}_k$ of grade-3 representing a volume element in $\mathrm{I\!R}^3$. Since in $\mathrm{I\!R}^3$ there are $2^3 = 8$ ways to combine the vectors $\mathbf{e}_i$ using the geometric product (2) such that no two products are linearly dependent, the resulting algebra, $Cl^{\lambda}_{3,0}$, is a linear vector space of eight dimensions, spanned by these graded bases.

In this paper we are interested in a certain conformal completion[1] of this algebra, originally presented in Ref. [2]. This is accomplished by using an additional vector, $\mathbf{e}_\infty$, to close the lines and volumes of the Euclidean space, giving

$$\mathcal{K}^{\lambda} =\, \text{span}\{\,1,\,\lambda\mathbf{e}_x\mathbf{e}_y,\,\lambda\mathbf{e}_z\mathbf{e}_x,\,\lambda\mathbf{e}_y\mathbf{e}_z,\,\lambda\mathbf{e}_x\mathbf{e}_\infty,\,\lambda\mathbf{e}_y\mathbf{e}_\infty,\,\lambda\mathbf{e}_z\mathbf{e}_\infty,\,\lambda I_3\mathbf{e}_\infty\,\}. \tag{5}$$

With unit vector $\mathbf{e}_\infty$, this is an eight-dimensional *even* sub-algebra of the $2^4 = 16$-dimensional Clifford algebra $Cl_{4,0}$. As an eight-dimensional linear vector space, $\mathcal{K}^{\lambda}$ has some remarkable properties [2]. To begin with, it is *closed* under multiplication. Suppose $\mathbf{X}$ and $\mathbf{Y}$ are two vectors in $\mathcal{K}^{\lambda}$. Then $\mathbf{X}$ and $\mathbf{Y}$ can be expanded in the graded basis of $\mathcal{K}^{\lambda}$:

$$\mathbf{X} =\, X_0 + X_1\,\lambda\mathbf{e}_x\mathbf{e}_y + X_2\,\lambda\mathbf{e}_z\mathbf{e}_x + X_3\,\lambda\mathbf{e}_y\mathbf{e}_z + X_4\,\lambda\mathbf{e}_x\mathbf{e}_\infty + X_5\,\lambda\mathbf{e}_y\mathbf{e}_\infty + X_6\,\lambda\mathbf{e}_z\mathbf{e}_\infty + X_7\,\lambda I_3\mathbf{e}_\infty \tag{6}$$

and

$$\mathbf{Y} =\, Y_0 + Y_1\,\lambda\mathbf{e}_x\mathbf{e}_y + Y_2\,\lambda\mathbf{e}_z\mathbf{e}_x + Y_3\,\lambda\mathbf{e}_y\mathbf{e}_z + Y_4\,\lambda\mathbf{e}_x\mathbf{e}_\infty + Y_5\,\lambda\mathbf{e}_y\mathbf{e}_\infty + Y_6\,\lambda\mathbf{e}_z\mathbf{e}_\infty + Y_7\,\lambda I_3\mathbf{e}_\infty\,. \tag{7}$$

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[1] The conformal space we are considering is an *in*-homogeneous version of the space usually studied in Conformal Geometric Algebra [3]. It can be viewed as an 8-dimensional subspace of the 32-dimensional representation space postulated in Conformal Geometric Algebra. The larger representation space results from a homogeneous freedom of the origin within $\mathbb{E}^3$, which does not concern us in this paper.

$*$	1	$\lambda\,\mathbf{e}_x\mathbf{e}_y$	$\lambda\,\mathbf{e}_z\mathbf{e}_x$	$\lambda\,\mathbf{e}_y\mathbf{e}_z$	$\lambda\,\mathbf{e}_x\mathbf{e}_\infty$	$\lambda\,\mathbf{e}_y\mathbf{e}_\infty$	$\lambda\,\mathbf{e}_z\mathbf{e}_\infty$	$\lambda\,I_3\mathbf{e}_\infty$
1	1	$\lambda\,\mathbf{e}_x\mathbf{e}_y$	$\lambda\,\mathbf{e}_z\mathbf{e}_x$	$\lambda\,\mathbf{e}_y\mathbf{e}_z$	$\lambda\,\mathbf{e}_x\mathbf{e}_\infty$	$\lambda\,\mathbf{e}_y\mathbf{e}_\infty$	$\lambda\,\mathbf{e}_z\mathbf{e}_\infty$	$\lambda\,I_3\mathbf{e}_\infty$
$\lambda\,\mathbf{e}_x\mathbf{e}_y$	$\lambda\,\mathbf{e}_x\mathbf{e}_y$	-1	$\mathbf{e}_y\mathbf{e}_z$	$-\mathbf{e}_z\mathbf{e}_x$	$-\mathbf{e}_y\mathbf{e}_\infty$	$\mathbf{e}_x\mathbf{e}_\infty$	$I_3\mathbf{e}_\infty$	$-\mathbf{e}_z\mathbf{e}_\infty$
$\lambda\,\mathbf{e}_z\mathbf{e}_x$	$\lambda\,\mathbf{e}_z\mathbf{e}_x$	$-\mathbf{e}_y\mathbf{e}_z$	-1	$\mathbf{e}_x\mathbf{e}_y$	$\mathbf{e}_z\mathbf{e}_\infty$	$I_3\mathbf{e}_\infty$	$-\mathbf{e}_x\mathbf{e}_\infty$	$-\mathbf{e}_y\mathbf{e}_\infty$
$\lambda\,\mathbf{e}_y\mathbf{e}_z$	$\lambda\,\mathbf{e}_y\mathbf{e}_z$	$\mathbf{e}_z\mathbf{e}_x$	$-\mathbf{e}_x\mathbf{e}_y$	-1	$I_3\mathbf{e}_\infty$	$-\mathbf{e}_z\mathbf{e}_\infty$	$\mathbf{e}_y\mathbf{e}_\infty$	$-\mathbf{e}_x\mathbf{e}_\infty$
$\lambda\,\mathbf{e}_x\mathbf{e}_\infty$	$\lambda\,\mathbf{e}_x\mathbf{e}_\infty$	$\mathbf{e}_y\mathbf{e}_\infty$	$-\mathbf{e}_z\mathbf{e}_\infty$	$I_3\mathbf{e}_\infty$	-1	$-\mathbf{e}_x\mathbf{e}_y$	$\mathbf{e}_z\mathbf{e}_x$	$-\mathbf{e}_y\mathbf{e}_z$
$\lambda\,\mathbf{e}_y\mathbf{e}_\infty$	$\lambda\,\mathbf{e}_y\mathbf{e}_\infty$	$-\mathbf{e}_x\mathbf{e}_\infty$	$I_3\mathbf{e}_\infty$	$\mathbf{e}_z\mathbf{e}_\infty$	$\mathbf{e}_x\mathbf{e}_y$	-1	$-\mathbf{e}_y\mathbf{e}_z$	$-\mathbf{e}_z\mathbf{e}_x$
$\lambda\,\mathbf{e}_z\mathbf{e}_\infty$	$\lambda\,\mathbf{e}_z\mathbf{e}_\infty$	$I_3\mathbf{e}_\infty$	$\mathbf{e}_x\mathbf{e}_\infty$	$-\mathbf{e}_y\mathbf{e}_\infty$	$-\mathbf{e}_z\mathbf{e}_x$	$\mathbf{e}_y\mathbf{e}_z$	-1	$-\mathbf{e}_x\mathbf{e}_y$
$\lambda\,I_3\mathbf{e}_\infty$	$\lambda\,I_3\mathbf{e}_\infty$	$-\mathbf{e}_z\mathbf{e}_\infty$	$-\mathbf{e}_y\mathbf{e}_\infty$	$-\mathbf{e}_x\mathbf{e}_\infty$	$-\mathbf{e}_y\mathbf{e}_z$	$-\mathbf{e}_z\mathbf{e}_x$	$-\mathbf{e}_x\mathbf{e}_y$	1

TABLE I: Multiplication Table for a "Conformal Geometric Algebra[1]" of $\mathbb{E}^3$. Here $I_3 = \mathbf{e}_x\mathbf{e}_y\mathbf{e}_z$, $\mathbf{e}_\infty^2 = +1$, and $\lambda = \pm 1$.

And using the definition $||\mathbf{X}|| := \sqrt{\mathbf{X}\cdot\mathbf{X}^\dagger}$ (where $\dagger$ represents the *reverse* operation [1]) they can be normalized as

$$||\mathbf{X}||^2 = \sum_{\mu\,=\,0}^{7} X_\mu^2 \,=\, 1 \quad\text{and}\quad ||\mathbf{Y}||^2 = \sum_{\nu\,=\,0}^{7} Y_\nu^2 \,=\, 1\,. \tag{8}$$

Now it is evident from the multiplication table above (Table I) that if $\mathbf{X}, \mathbf{Y} \in \mathcal{K}^\lambda$, then so is their product $\mathbf{Z} = \mathbf{XY}$:

$$\mathbf{Z} = \,Z_0 + Z_1\,\lambda\mathbf{e}_x\mathbf{e}_y + Z_2\,\lambda\mathbf{e}_z\mathbf{e}_x + Z_3\,\lambda\mathbf{e}_y\mathbf{e}_z + Z_4\,\lambda\mathbf{e}_x\mathbf{e}_\infty + Z_5\,\lambda\mathbf{e}_y\mathbf{e}_\infty + Z_6\,\lambda\mathbf{e}_z\mathbf{e}_\infty + Z_7\,\lambda I_3\mathbf{e}_\infty = \mathbf{XY}. \tag{9}$$

Thus $\mathcal{K}^\lambda$ remains closed under arbitrary number of multiplications of its elements. This is a powerful property. More importantly, we shall soon see that for vectors $\mathbf{X}$ and $\mathbf{Y}$ in $\mathcal{K}^\lambda$ (not necessarily unit) the following norm relation holds:

$$||\mathbf{XY}|| \,=\, ||\mathbf{X}||\;||\mathbf{Y}||\,. \tag{10}$$

In particular, this means that for any two unit vectors $\mathbf{X}$ and $\mathbf{Y}$ in $\mathcal{K}^\lambda$ with the geometric product $\mathbf{Z} = \mathbf{XY}$ we have

$$||\,\mathbf{Z}\,||^2 = \sum_{\rho\,=\,0}^{7} Z_\rho^2 \,=\, 1\,. \tag{11}$$

Now, in order to prove the norm relation (10), it is convenient to express the elements of $\mathcal{K}^\lambda$ as dual quaternions. The idea of dual numbers, z, analogous to complex numbers, was introduced by Clifford in his seminal work as follows:

$$z = r + d\,\varepsilon, \;\;\text{where}\;\; \varepsilon \neq 0 \;\;\text{but}\;\; \varepsilon^2 = 0\,. \tag{12}$$

Here ε is the dual operator, r is the real part, and d is the dual part [3][4]. Similar to how the "imaginary" operator i is introduced in the complex number theory to distinguish the "real" and "imaginary" parts of a complex number,

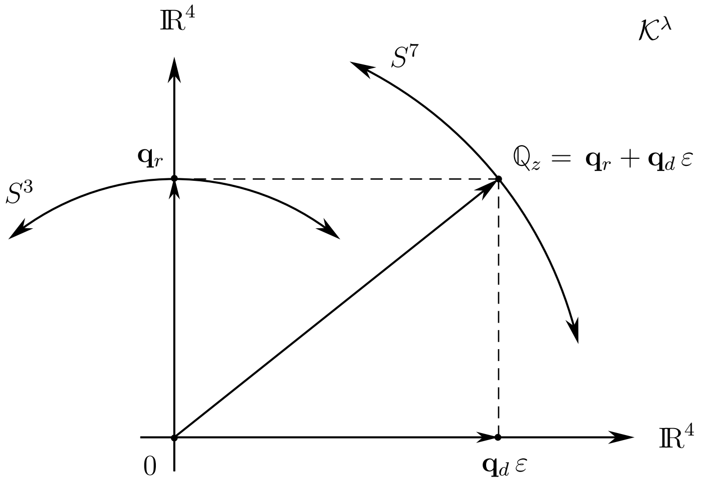

FIG. 1: An illustration of the 8D plane of $\mathcal{K}^\lambda$, which may be interpreted as an Argand diagram for a pair of quaternions.

Clifford introduced the dual operator ε to distinguish the "real" and "dual" parts of a dual number. The dual number theory can be extended to numbers of higher grades, including to numbers of composite grades, such as quaternions:

$$\mathbb{Q}_z = \mathbf{q}_r + \mathbf{q}_d\,\varepsilon\,, \tag{13}$$

where $\mathbf{q}_r$ and $\mathbf{q}_d$ are quaternions and $\mathbb{Q}_z$ is a dual-quaternion (or in Clifford's terminology, $\mathbb{Q}_z$ is a bi-quaternion). Recall that the set of unit quaternions is a 3-sphere, which can be normalized to a radius ϱ_r and written as the set

$$S^3 = \left\{ \mathbf{q}_r := q_0 + q_1\,\lambda\,\mathbf{e}_x\mathbf{e}_y + q_2\,\lambda\,\mathbf{e}_z\mathbf{e}_x + q_3\,\lambda\,\mathbf{e}_y\mathbf{e}_z \;\middle|\; ||\mathbf{q}_r|| = \sqrt{\mathbf{q}_r\,\mathbf{q}_r^\dagger} = \varrho_r \right\}. \tag{14}$$

Consider now a second, dual copy of the set of quaternions within $\mathcal{K}^\lambda$, corresponding to the fixed orientation $\lambda = +1$:

$$S^3 = \left\{ \mathbf{q}_d := -q_7 + q_6\,\mathbf{e}_x\mathbf{e}_y + q_5\,\mathbf{e}_z\mathbf{e}_x + q_4\,\mathbf{e}_y\mathbf{e}_z \;\middle|\; ||\mathbf{q}_d|| = \sqrt{\mathbf{q}_d\,\mathbf{q}_d^\dagger} = \varrho_d \right\}. \tag{15}$$

If we now identify $\lambda\,I_3\mathbf{e}_\infty$ appearing in (5) as the duality operator $-\,\varepsilon$, then (in the reverse additive order) we obtain

$$\varepsilon \equiv -\lambda\,I_3\mathbf{e}_\infty \;\text{ with }\; \varepsilon^\dagger = \varepsilon \;\text{ and }\; \varepsilon^2 = +1 \;\text{ (since } \mathbf{e}_\infty \text{ is a unit vector in } \mathcal{K}^\lambda) \tag{16}$$

$$\text{and }\; \mathbf{q}_d\,\varepsilon \equiv -\mathbf{q}_d\,\lambda\,I_3\mathbf{e}_\infty = q_4\,\lambda\,\mathbf{e}_x\mathbf{e}_\infty + q_5\,\lambda\,\mathbf{e}_y\mathbf{e}_\infty + q_6\,\lambda\,\mathbf{e}_z\mathbf{e}_\infty + q_7\,\lambda\,I_3\mathbf{e}_\infty\,, \tag{17}$$

which is a multi-vector "dual" to the quaternion $\mathbf{q}_d$. Note that we write ε as if it were a scalar because it commutes with each element of $\mathcal{K}^\lambda$ in (5). Comparing (14) and (17) with (5) we can now rewrite $\mathcal{K}^\lambda$ as a set of paired quaternions:

$$\mathcal{K}^\lambda = \left\{ \mathbb{Q}_z := \mathbf{q}_r + \mathbf{q}_d\,\varepsilon \;\middle|\; ||\mathbb{Q}_z|| = \sqrt{\mathbb{Q}_z\,\mathbb{Q}_z^\dagger} = \sqrt{\varrho_r^2 + \varrho_d^2}\,,\; 0 < \varrho_r < \infty,\; 0 < \varrho_d < \infty \right\}. \tag{18}$$

Now the normalization of $\mathbb{Q}_z$ necessitates that for that condition to hold every $\mathbf{q}_r$ must be orthogonal to its dual $\mathbf{q}_d$:

$$||\mathbb{Q}_z|| = \sqrt{\mathbb{Q}_z\,\mathbb{Q}_z^\dagger} = \sqrt{\varrho_r^2 + \varrho_d^2} \iff \mathbf{q}_r\,\mathbf{q}_d^\dagger + \mathbf{q}_d\,\mathbf{q}_r^\dagger = 0\,, \tag{19}$$

or equivalently, $(\mathbf{q}_r\,\mathbf{q}_d^\dagger)_s = 0$; *i.e.*, $\mathbf{q}_r\,\mathbf{q}_d^\dagger$ must be a pure quaternion (for a pedagogical discussion of (19) see section 7.1 of Ref. [4]). We can see this by working out the geometric product of $\mathbb{Q}_z$ with $\mathbb{Q}_z^\dagger$ while using $\varepsilon^2 = +1$, which gives

$$\mathbb{Q}_z\,\mathbb{Q}_z^\dagger = \left(\mathbf{q}_r\,\mathbf{q}_r^\dagger + \mathbf{q}_d\,\mathbf{q}_d^\dagger\right) + \left(\mathbf{q}_r\,\mathbf{q}_d^\dagger + \mathbf{q}_d\,\mathbf{q}_r^\dagger\right)\varepsilon\,. \tag{20}$$

Now, using definitions (14) and (15), it is easy to see that $\mathbf{q}_r\,\mathbf{q}_r^\dagger = \varrho_r^2$ and $\mathbf{q}_d\,\mathbf{q}_d^\dagger = \varrho_d^2$, reducing the above product to

$$\mathbb{Q}_z\,\mathbb{Q}_z^\dagger \;=\; \varrho_r^2 + \varrho_d^2 \,+\, \Big(\mathbf{q}_r\,\mathbf{q}_d^\dagger \,+\, \mathbf{q}_d\,\mathbf{q}_r^\dagger\Big)\;\varepsilon\,. \tag{21}$$

It is thus clear that for $\mathbb{Q}_z\,\mathbb{Q}_z^\dagger$ to be a scalar $\mathbf{q}_r\,\mathbf{q}_d^\dagger + \mathbf{q}_d\,\mathbf{q}_r^\dagger$ must vanish, or equivalently $\mathbf{q}_r$ must be orthogonal to $\mathbf{q}_d$.

But there is more to the normalization condition $\mathbf{q}_r\,\mathbf{q}_d^\dagger + \mathbf{q}_d\,\mathbf{q}_r^\dagger = 0$ than meets the eye. It also leads to the crucial norm relation (10), which is at the heart of the only possible four normed division algebras $\mathbb{R}$, $\mathbb{C}$, $\mathbb{H}$ and $\mathbb{O}$ associated with the four parallelizable spheres S^0, S^1, S^3 and S^7, with octonions forming a non-associative algebra in addition to forming a non-commutative algebra [5]. To verify this, consider a product of two distinct members of the set $\mathcal{K}^\lambda$,

$$\mathbb{Q}_{z1}\,\mathbb{Q}_{z2} \;=\; (\mathbf{q}_{r1}\,\mathbf{q}_{r2} \,+\, \mathbf{q}_{d1}\,\mathbf{q}_{d2}) \,+\, (\mathbf{q}_{r1}\,\mathbf{q}_{d2} \,+\, \mathbf{q}_{d1}\,\mathbf{q}_{r2})\;\varepsilon\,, \tag{22}$$

together with their individual definitions

$$\mathbb{Q}_{z1} \,=\, \mathbf{q}_{r1} + \mathbf{q}_{d1}\,\varepsilon \quad\text{and}\quad \mathbb{Q}_{z2} \,=\, \mathbf{q}_{r2} + \mathbf{q}_{d2}\,\varepsilon\,. \tag{23}$$

If we now use the fact that ε, along with $\varepsilon^\dagger = \varepsilon$ and $\varepsilon^2 = 1$, commutes with every element of $\mathcal{K}^\lambda$ defined in (5) and consequently with all $\mathbf{q}_r$, $\mathbf{q}_r^\dagger$, $\mathbf{q}_d$ and $\mathbf{q}_d^\dagger$, and work out $\mathbb{Q}_{z1}^\dagger$, $\mathbb{Q}_{z2}^\dagger$ and the products $\mathbb{Q}_{z1}\mathbb{Q}_{z1}^\dagger$, $\mathbb{Q}_{z2}\mathbb{Q}_{z2}^\dagger$ and $(\mathbb{Q}_{z1}\,\mathbb{Q}_{z2})^\dagger$ as

$$\mathbb{Q}_{z1}^\dagger \,=\, \mathbf{q}_{r1}^\dagger + \mathbf{q}_{d1}^\dagger\,\varepsilon\,, \tag{24}$$

$$\mathbb{Q}_{z2}^\dagger \,=\, \mathbf{q}_{r2}^\dagger + \mathbf{q}_{d2}^\dagger\,\varepsilon\,, \tag{25}$$

$$\mathbb{Q}_{z1}\,\mathbb{Q}_{z1}^\dagger \,=\, \Big(\mathbf{q}_{r1}\,\mathbf{q}_{r1}^\dagger \,+\, \mathbf{q}_{d1}\,\mathbf{q}_{d1}^\dagger\Big) \,+\, \Big(\mathbf{q}_{r1}\,\mathbf{q}_{d1}^\dagger \,+\, \mathbf{q}_{d1}\,\mathbf{q}_{r1}^\dagger\Big)\;\varepsilon\,, \tag{26}$$

$$\mathbb{Q}_{z2}\,\mathbb{Q}_{z2}^\dagger \,=\, \Big(\mathbf{q}_{r2}\,\mathbf{q}_{r2}^\dagger \,+\, \mathbf{q}_{d2}\,\mathbf{q}_{d2}^\dagger\Big) \,+\, \Big(\mathbf{q}_{r2}\,\mathbf{q}_{d2}^\dagger \,+\, \mathbf{q}_{d2}\,\mathbf{q}_{r2}^\dagger\Big)\;\varepsilon\,, \tag{27}$$

$$\text{and}\quad (\mathbb{Q}_{z1}\,\mathbb{Q}_{z2})^\dagger \,=\, \Big(\mathbf{q}_{r2}^\dagger\,\mathbf{q}_{r1}^\dagger \,+\, \mathbf{q}_{d2}^\dagger\,\mathbf{q}_{d1}^\dagger\Big) \,+\, \Big(\mathbf{q}_{d2}^\dagger\,\mathbf{q}_{r1}^\dagger \,+\, \mathbf{q}_{r2}^\dagger\,\mathbf{q}_{d1}^\dagger\Big)\;\varepsilon\,, \tag{28}$$

then, thanks to the normalization condition $\mathbf{q}_r\,\mathbf{q}_d^\dagger + \mathbf{q}_d\,\mathbf{q}_r^\dagger = 0$ of (19), the norm relation (10) is not difficult to verify. To that end, we first work out the geometric product $(\mathbb{Q}_{z1}\,\mathbb{Q}_{z2})(\mathbb{Q}_{z1}\,\mathbb{Q}_{z2})^\dagger$ using expressions (22) and (28), which gives

$$\begin{aligned}(\mathbb{Q}_{z1}\,\mathbb{Q}_{z2})(\mathbb{Q}_{z1}\,\mathbb{Q}_{z2})^\dagger = &\Big\{(\mathbf{q}_{r1}\,\mathbf{q}_{r2} \,+\, \mathbf{q}_{d1}\,\mathbf{q}_{d2})\Big(\mathbf{q}_{r2}^\dagger\,\mathbf{q}_{r1}^\dagger \,+\, \mathbf{q}_{d2}^\dagger\,\mathbf{q}_{d1}^\dagger\Big) + (\mathbf{q}_{r1}\,\mathbf{q}_{d2} \,+\, \mathbf{q}_{d1}\,\mathbf{q}_{r2})\Big(\mathbf{q}_{d2}^\dagger\,\mathbf{q}_{r1}^\dagger \,+\, \mathbf{q}_{r2}^\dagger\,\mathbf{q}_{d1}^\dagger\Big)\Big\}\\ &+\Big\{(\mathbf{q}_{r1}\,\mathbf{q}_{d2} \,+\, \mathbf{q}_{d1}\,\mathbf{q}_{r2})\Big(\mathbf{q}_{r2}^\dagger\,\mathbf{q}_{r1}^\dagger \,+\, \mathbf{q}_{d2}^\dagger\,\mathbf{q}_{d1}^\dagger\Big) + (\mathbf{q}_{r1}\,\mathbf{q}_{r2} \,+\, \mathbf{q}_{d1}\,\mathbf{q}_{d2})\Big(\mathbf{q}_{d2}^\dagger\,\mathbf{q}_{r1}^\dagger \,+\, \mathbf{q}_{r2}^\dagger\,\mathbf{q}_{d1}^\dagger\Big)\Big\}\,\varepsilon\,.\end{aligned} \tag{29}$$

Now the "real" part of the above product simplifies to (32) as follows:

$$\begin{aligned}\big\{(\mathbb{Q}_{z1}\,\mathbb{Q}_{z2})(\mathbb{Q}_{z1}\,\mathbb{Q}_{z2})^\dagger\big\}_{real} &= \mathbf{q}_{r1}\,\mathbf{q}_{r2}\,\mathbf{q}_{r2}^\dagger\,\mathbf{q}_{r1}^\dagger \,+\, \mathbf{q}_{d1}\,\mathbf{q}_{d2}\,\mathbf{q}_{r2}^\dagger\,\mathbf{q}_{r1}^\dagger \,+\, \mathbf{q}_{r1}\,\mathbf{q}_{r2}\,\mathbf{q}_{d2}^\dagger\,\mathbf{q}_{d1}^\dagger \,+\, \mathbf{q}_{d1}\,\mathbf{q}_{d2}\,\mathbf{q}_{d2}^\dagger\,\mathbf{q}_{d1}^\dagger\\ &\quad+\, \mathbf{q}_{r1}\,\mathbf{q}_{d2}\,\mathbf{q}_{d2}^\dagger\,\mathbf{q}_{r1}^\dagger \,+\, \mathbf{q}_{d1}\,\mathbf{q}_{r2}\,\mathbf{q}_{d2}^\dagger\,\mathbf{q}_{r1}^\dagger \,+\, \mathbf{q}_{r1}\,\mathbf{q}_{d2}\,\mathbf{q}_{r2}^\dagger\,\mathbf{q}_{d1}^\dagger \,+\, \mathbf{q}_{d1}\,\mathbf{q}_{r2}\,\mathbf{q}_{r2}^\dagger\,\mathbf{q}_{d1}^\dagger \qquad (30)\\ &= \mathbf{q}_{r1}\,\mathbf{q}_{r2}\,\mathbf{q}_{r2}^\dagger\,\mathbf{q}_{r1}^\dagger \,+\, \mathbf{q}_{d1}\,\mathbf{q}_{d2}\,\mathbf{q}_{d2}^\dagger\,\mathbf{q}_{d1}^\dagger \,+\, \mathbf{q}_{r1}\,\mathbf{q}_{d2}\,\mathbf{q}_{d2}^\dagger\,\mathbf{q}_{r1}^\dagger \,+\, \mathbf{q}_{d1}\,\mathbf{q}_{r2}\,\mathbf{q}_{r2}^\dagger\,\mathbf{q}_{d1}^\dagger \qquad (31)\\ &= \varrho_{r1}^2\,\varrho_{r2}^2 + \varrho_{d1}^2\,\varrho_{d2}^2 + \varrho_{r1}^2\,\varrho_{d2}^2 + \varrho_{d1}^2\,\varrho_{r2}^2\,. \qquad (32)\end{aligned}$$

Here (31) follows from (30) upon inserting the normalization condition (19) in the form $\mathbf{q}_r\,\mathbf{q}_d^\dagger = -\,\mathbf{q}_d\,\mathbf{q}_r^\dagger$ into the second and third terms of (30), which then cancel out with the sixth and seventh terms of (30), respectively; and (32) follows from (31) upon inserting the normalization conditions $||\mathbf{q}||^2 = \mathbf{q}\mathbf{q}^\dagger = \varrho^2$ for the real and dual quaternions specified in (14) and (15), for each of the four terms of (31). Similarly, the "dual" part of the product (29) simplifies to

$$\begin{aligned}\big\{(\mathbb{Q}_{z1}\,\mathbb{Q}_{z2})(\mathbb{Q}_{z1}\,\mathbb{Q}_{z2})^\dagger\big\}_{dual} &= \Big\{\mathbf{q}_{r1}\,\mathbf{q}_{d2}\,\mathbf{q}_{r2}^\dagger\,\mathbf{q}_{r1}^\dagger \,+\, \mathbf{q}_{d1}\,\mathbf{q}_{r2}\,\mathbf{q}_{r2}^\dagger\,\mathbf{q}_{r1}^\dagger \,+\, \mathbf{q}_{r1}\,\mathbf{q}_{d2}\,\mathbf{q}_{d2}^\dagger\,\mathbf{q}_{d1}^\dagger \,+\, \mathbf{q}_{d1}\,\mathbf{q}_{r2}\,\mathbf{q}_{d2}^\dagger\,\mathbf{q}_{d1}^\dagger\\ &\quad+\, \mathbf{q}_{r1}\,\mathbf{q}_{r2}\,\mathbf{q}_{d2}^\dagger\,\mathbf{q}_{r1}^\dagger \,+\, \mathbf{q}_{d1}\,\mathbf{q}_{d2}\,\mathbf{q}_{d2}^\dagger\,\mathbf{q}_{r1}^\dagger \,+\, \mathbf{q}_{r1}\,\mathbf{q}_{r2}\,\mathbf{q}_{r2}^\dagger\,\mathbf{q}_{d1}^\dagger \,+\, \mathbf{q}_{d1}\,\mathbf{q}_{d2}\,\mathbf{q}_{r2}^\dagger\,\mathbf{q}_{d1}^\dagger\Big\}\,\varepsilon \qquad (33)\\ &= 0\,. \qquad (34)\end{aligned}$$

We can see this again by inserting into (33) the normalization condition (19) in the form $\mathbf{q}_r\,\mathbf{q}_d^\dagger = -\,\mathbf{q}_d\,\mathbf{q}_r^\dagger$ and the normalization conditions $||\mathbf{q}||^2 = \mathbf{q}\mathbf{q}^\dagger = \varrho^2$ for the quaternions in (14) and (15), which cancels out the first four terms of (33) with the last four. Consequently, combining the results of (32) and (34), for the left-hand side of (10) we have

$$||\mathbb{Q}_{z1}\,\mathbb{Q}_{z2}|| \,=\, \sqrt{\varrho_{r1}^2\,\varrho_{r2}^2 + \varrho_{r1}^2\,\varrho_{d2}^2 + \varrho_{d1}^2\,\varrho_{r2}^2 + \varrho_{d1}^2\,\varrho_{d2}^2}\,. \tag{35}$$

On the other hand, once again using *the same* normalization condition (19), for the right-hand side of (10) we have

$$||\mathbb{Q}_{z1}||\,||\mathbb{Q}_{z2}|| \,=\, \left(\sqrt{\varrho_{r1}^2 + \varrho_{d1}^2}\,\right)\left(\sqrt{\varrho_{r2}^2 + \varrho_{d2}^2}\,\right) \,=\, \sqrt{\varrho_{r1}^2\,\varrho_{r2}^2 + \varrho_{r1}^2\,\varrho_{d2}^2 + \varrho_{d1}^2\,\varrho_{r2}^2 + \varrho_{d1}^2\,\varrho_{d2}^2}\,. \tag{36}$$

Thus, comparing the results in (35) and (36), we finally arrive at the relation

$$||\mathbb{Q}_{z1}\,\mathbb{Q}_{z2}|| \,=\, ||\mathbb{Q}_{z1}||\,||\mathbb{Q}_{z2}||\,, \tag{37}$$

which is evidently the same as the norm relation (10) in every respect apart from the appropriate change in notation.

Without loss of generality we can now restrict space $\mathcal{K}^\lambda$ to a unit 7-sphere by setting the radii ϱ_r and ϱ_d to $\frac{1}{\sqrt{2}}$:

$$\mathcal{K}^\lambda \supset S^7 := \left\{\,\mathbb{Q}_z := \mathbf{q}_r + \mathbf{q}_d\,\varepsilon \;\middle|\; ||\mathbb{Q}_z|| = 1 \text{ and } \mathbf{q}_r\,\mathbf{q}_d^\dagger + \mathbf{q}_d\,\mathbf{q}_r^\dagger = 0\,\right\}, \tag{38}$$

where $\varepsilon = -\lambda\,I_3\mathbf{e}_\infty\,,\ \varepsilon^\dagger = \varepsilon,\ \varepsilon^2 = \mathbf{e}_\infty^2 = +1\,,$

$$\mathbf{q}_r = q_0 + q_1\,\lambda\,\mathbf{e}_x\mathbf{e}_y + q_2\,\lambda\,\mathbf{e}_z\mathbf{e}_x + q_3\,\lambda\,\mathbf{e}_y\mathbf{e}_z\,, \quad \text{and} \quad \mathbf{q}_d = -q_7 + q_6\,\mathbf{e}_x\mathbf{e}_y + q_5\,\mathbf{e}_z\mathbf{e}_x + q_4\,\mathbf{e}_y\mathbf{e}_z\,, \tag{39}$$

so that

$$\mathbb{Q}_z = q_0 + q_1\,\lambda\mathbf{e}_x\mathbf{e}_y + q_2\,\lambda\mathbf{e}_z\mathbf{e}_x + q_3\,\lambda\mathbf{e}_y\mathbf{e}_z + q_4\,\lambda\mathbf{e}_x\mathbf{e}_\infty + q_5\,\lambda\mathbf{e}_y\mathbf{e}_\infty + q_6\,\lambda\mathbf{e}_z\mathbf{e}_\infty + q_7\,\lambda I_3\mathbf{e}_\infty\,. \tag{40}$$

Needless to say, since all Clifford algebras are associative division algebras by definition, unlike the non-associative octonionic algebra the 7-sphere we have constructed here corresponds to an *associative* (but non-commutative) algebra.

We may now view the four *associative* normed division algebras in the only possible dimensions 1, 2, 4 and 8, respectively, as even sub-algebras of the Clifford algebras

$$Cl_{1,0}^\lambda = \text{span}\{\,1,\ \lambda\mathbf{e}_x\,\}, \tag{41}$$

$$Cl_{2,0}^\lambda = \text{span}\{\,1,\ \lambda\mathbf{e}_x,\ \lambda\mathbf{e}_y,\ \lambda\mathbf{e}_x\mathbf{e}_y\,\}, \tag{42}$$

$$Cl_{3,0}^\lambda = \text{span}\{\,1,\ \lambda\mathbf{e}_x,\ \lambda\mathbf{e}_y,\ \lambda\mathbf{e}_z,\ \lambda\mathbf{e}_x\mathbf{e}_y,\ \lambda\mathbf{e}_z\mathbf{e}_x,\ \lambda\mathbf{e}_y\mathbf{e}_z,\ \lambda\mathbf{e}_x\mathbf{e}_y\mathbf{e}_z\,\}, \tag{43}$$

$$\text{and } Cl_{4,0}^\lambda = \text{span}\big\{\,1,\ \lambda\mathbf{e}_x,\ \lambda\mathbf{e}_y,\ \lambda\mathbf{e}_z,\ \lambda\mathbf{e}_\infty,\ \lambda\mathbf{e}_x\mathbf{e}_y,\ \lambda\mathbf{e}_z\mathbf{e}_x,\ \lambda\mathbf{e}_y\mathbf{e}_z,\ \lambda\mathbf{e}_x\mathbf{e}_\infty,\ \lambda\mathbf{e}_y\mathbf{e}_\infty,\ \lambda\mathbf{e}_z\mathbf{e}_\infty,$$
$$\lambda\mathbf{e}_x\mathbf{e}_y\mathbf{e}_z,\ \lambda\mathbf{e}_x\mathbf{e}_y\mathbf{e}_\infty,\ \lambda\mathbf{e}_z\mathbf{e}_x\mathbf{e}_\infty,\ \lambda\mathbf{e}_y\mathbf{e}_z\mathbf{e}_\infty,\ \lambda\mathbf{e}_x\mathbf{e}_y\mathbf{e}_z\mathbf{e}_\infty\,\big\}. \tag{44}$$

It is easy to verify that the even subalgebras of $Cl_{1,0}^\lambda$, $Cl_{2,0}^\lambda$ and $Cl_{3,0}^\lambda$ are indeed isomorphic to $\mathbb{R}$, $\mathbb{C}$ and $\mathbb{H}$, respectively.

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[1] C. Doran and A. Lasenby, *Geometric Algebra for Physicists* (Cambridge University Press, Cambridge, 2003).

[2] J. Christian, *Quantum correlations are weaved by the spinors of the Euclidean primitives*, R. Soc. Open Sci., **5**, 180526 (2018); `https://doi.org/10.1098/rsos.180526`; See also `https://arxiv.org/abs/1806.02392` (2018).

[3] L. Dorst, D. Fontijne, and S. Mann, *Geometric Algebra for Computer Science* (Elsevier, Amsterdam, 2007).

[4] B. Kenwright, *A beginners guide to dual-quaternions: what they are, how they work, and how to use them for 3D character hierarchies*, in Proceedings of the 20th International Conferences on Computer Graphics, Visualization and Computer Vision, 1-10 (2012).

[5] A. Hurwitz, *Über die Composition der quadratischen Formen von beliebig vielen Variabeln*, Nachr. Ges. Wiss. Göttingen, **1898**, 309-316 (1898); For a recent review, see also J. C. Baez, *The Octonions*, Bull. Am. Math. Soc., **39**, 145-205 (2002).