On three domination numbers in block graphs

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Abstract

The problems of determining minimum identifying, locating-dominating or open locating-dominating codes are special search problems that are challenging both from a theoretical and a computational point of view. Hence, a typical line of attack for these problems is to determine lower and upper bounds for minimum codes in special graphs. In this work we study the problem of determining the cardinality of minimum codes in block graphs (that are diamond-free chordal graphs). We present for all three codes lower and upper bounds as well as block graphs where these bounds are attained.

Keywords: identifying, locating-dominating and open locating-dominating codes, block graphs

1 Introduction

For a graph G that models a facility or a multiprocessor network, detection devices can be placed at its vertices to locate an intruder (like a fire, a thief or a saboteur) or a faulty processor. Depending on the features of the detection devices, different dominating sets can be used to determine the optimal distribution of the detection devices in G. In the following, we study three problems arising in this context which all have been actively studied during the last decade, see e.g. the bibliography maintained by Lobstein [20].

Let $G = (V, E)$ be a graph. The (open) neighborhood of a vertex u is the set $N(u)$ of all vertices of G adjacent to u, and $N[u] = \{u\} \cup N(u)$ is the closed neighborhood of u.

A subset $C \subseteq V$ is an *identifying code* (for short: ID-code) of G if

- $N[u] \cap C \neq \emptyset$ for all $u \in V$ (domination),
- $N[u] \cap C \neq N[v] \cap C$ for all $u, v \in V$ (separation),

see Figure 1(a) for an example. Identifying codes were introduced in [19]. Not every graph G admits an identifying code, i.e. is *identifiable*: this holds if and only if there are no true twins in G (i.e., there is no pair of distinct vertices $u, v \in V$ with $N[u] = N[v]$) [19]. On the other hand, for every identifiable graph, its whole vertex set trivially forms an identifying code. The *identifying code number* $\gamma^{ID}(G)$ of a graph G is the minimum cardinality of an identifying code of G.

A subset $C \subseteq V$ is a *locating-dominating code* (for short: LD-code) if

- $N[u] \cap C \neq \emptyset$ for all $u \in V$ (domination),
- $N(u) \cap C \neq N(v) \cap C$ for all $u, v \in V - C$ (open-separation),

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see Figure 1(b) for an example. Locating-dominating codes were introduced in [22,23]. By definition, every graph has a locating-dominating code (as its whole vertex set trivially forms an LD-code). The *locating-dominating number* $\gamma^{LD}(G)$ of a graph G is the minimum cardinality of a locating-dominating code of G.

A subset $C \subseteq V$ is an *open locating-dominating code* (for short: OLD-code) of G if

- $N(u) \cap C \neq \emptyset$ for all $u \in V$ (open-domination),
- $N(u) \cap C \neq N(v) \cap C$ for all $u, v \in V$ (open-separation),

see Figure 1(c) for an example. Open locating-dominating codes were introduced in [24]. Not every graph G admits an OLD-code: this holds if and only if there are neither isolated vertices nor false twins in G (i.e., no pair of distinct vertices $u, v \in V$ with $N(u) = N(v)$) [24]. On the other hand, the whole vertex set forms an OLD-code of any connected false twin-free graph. The *open locating-dominating number* $\gamma^{OLD}(G)$ of a graph G is the minimum cardinality of an OLD-code of G.

Determining $\gamma^{ID}(G)$ or $\gamma^{LD}(G)$ is in general NP-hard [11] and remains hard for several graph classes where other in general hard problems are easy to solve, including bipartite graphs [11] and two classes of chordal graphs, namely split graphs and interval graphs [12,16]. Determining $\gamma^{OLD}(G)$ is also in general NP-hard [24] and remains NP-hard for perfect elimination bipartite graphs and interval graphs [16], and APX-complete for chordal graphs with maximum degree 4 [21]. Hence, a typical line of attack for these problems is to determine minimum identifying or (open) locating-dominating codes of special graphs. Closed formulas for these parameters have been found so far only for restricted graph families (e.g. for paths and cycles [9,23,24], for stars [17], for complete multipartite graphs [3,5] and for some subclasses of split graphs, including thin headless spiders [4]). In parallel, lower and upper bounds using the number of vertices and the VC-dimension of graphs have been determined [10,15]. More details will be given in Sections 2 and 3. More results on all three problems are listed in [20].

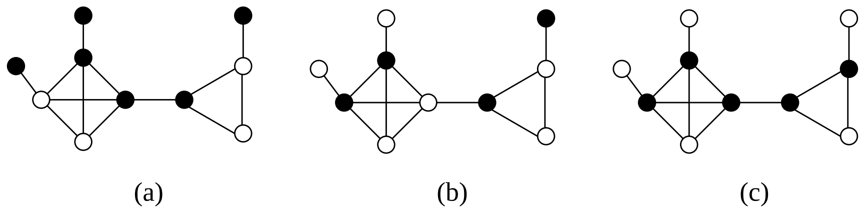

Fig. 1. A block graph where the black vertices form a minimum (a) ID-code, (b) LD-code, (c) OLD-code.

In this paper, we consider the family of block graphs. A *block graph* is a graph in which every maximal 2-connected subgraph (block) is a clique. Block graphs are precisely the diamond-free chordal graphs [7] resp. those chordal graphs in which every two maximal cliques have at most one vertex in common [18]. In particular, any tree is a block graph. Linear-time algorithms to compute all three domination numbers in block graphs are presented in [1], thus block graphs are chordal graphs for which the three problems can be solved in linear time.

In this paper, we complement this result by determining lower and upper bounds for all three domination numbers in block graphs. We give bounds using both the number of vertices – as it has been done for several classes of graphs – but also using the number $n_Q(G)$ of maximal cliques of G, that is more pertinent for block graphs. Note that for a connected block graph G that is not a single vertex, the two numbers $|V(G)|$ and $n_Q(G)$ are related: indeed, in a block graph, the number of vertices is always larger than the number of maximal cliques such that

$$n_Q(G) < |V(G)|$$

holds (which is not true in general as e.g. complete bipartite graphs $K_{n,m}$ with $2 \leq n < m$ show). There is no relation of the two numbers in the other direction (since any clique is a block graph with a single maximal clique), however, if the graph is identifiable, we can prove that $|V(G)| \leq 2n_Q(G) - 1$.

In [2] it was conjectured that $\gamma^{ID}(G)$ of any block graph G can be bounded from above by $n_Q(G)$; it was observed that this bound is true for trees and it was verified for some families of block graphs, including thin headless spiders and critical identifiable block graphs. Here, we prove the conjecture for all block graphs and address similar questions for LD- and OLD-codes.

Concerning the relation of the three studied domination numbers, it is immediate from the definitions that

$$\gamma^{LD}(G) \leq \min\{\gamma^{ID}(G), \gamma^{OLD}(G)\}$$

as every ID-code and every OLD-code satisfy the conditions for locating-domination. On the other hand, one can prove that both $\gamma^{ID}(G)$ and $\gamma^{OLD}(G)$ are smaller than $2\gamma^{LD}(G)$.

Note that $\gamma^{ID}(G)$ and $\gamma^{OLD}(G)$ are incomparable in general, which remains true on block graphs:

- we have from results in [9,24] that $\gamma^{ID}(G) < \gamma^{OLD}(G)$ holds for all paths P_n with $n \geq 7$, but
- we have $\gamma^{ID}(S_n) = n+1$ by [5] and we can show that $\gamma^{OLD}(S_n) = n$, thus $\gamma^{ID}(G) > \gamma^{OLD}(G)$ holds for all thin headless spiders S_n with $n \geq 3$.

2 Upper bounds

2.1 Identifying codes

Gravier and Moncel [17] proved that any identifiable graph with at least one edge satisfies $\gamma^{ID}(G) \leq |V(G)|-1$. Extremal graphs reaching this bound have been characterized in [14]. In particular, stars, that are trees and thus block graphs, reach this bound, hence this bound cannot be improved for block graphs. On the other hand, stars have many maximal cliques, namely $|V(G)|-1$, so we wonder whether we can improve the bound using $n_Q(G)$ as conjectured in [2]. The answer is affirmative, as proven in the next theorem.

Theorem 2.1 *Let G be an identifiable block graph. Then $\gamma^{ID}(G) \leq n_Q(G)$.*

Proof. In this proof, we will use the word *twin* for *true twin* (i.e. vertices with the same closed neighborhood). We recall that a graph is identifiable if and only if it is twin-free. The elements of an identifying code are called *code vertices*. Two vertices u, v are *separated* by a vertex c if c is in the closed neighborhood of exactly one vertex among $\{u, v\}$.

Assume by contradiction that there is a twin-free block graph G with $\gamma^{ID}(G) > n_Q(G)$ and let G be such a counterexample with the minimum number of vertices. Note that G has at least four vertices since it can be easily checked that the theorem is true for graphs with three vertices. First note that G has a vertex x of degree 1 (take any vertex of a leaf-clique which is not the articulation vertex). Let y be the unique neighbor of x and let $G' = G - x$. Then G' is a block graph with $n_Q(G') < n_Q(G)$ (since G' is not a single vertex).

Case 1 : G' is twin-free. By minimality of G, there is an identifying code C' of G' of size at most $n_Q(G)-1$. If $y \notin C'$, then $C = C' \cup \{x\}$ is an identifying code of G. Indeed, x is separated from all the vertices except y by itself and from y by any vertex dominating y in C'. Any other pair of vertices was separated in C' and is still separated in C. Then we have a contradiction since C is an identifying code of size at most $n_Q(G)$ of G. So we can assume that $y \in C'$. If y has a neighbor z in C', then again $C' \cup \{x\}$ is an identifying code of G. Otherwise, it means that $N[y] \cap C' = \{y\}$ and by definition of an identifying code, y is the only vertex with only y as code vertex in its closed neighborhood. Then $C' \cup \{z\}$ where z is any neighbor of y but not x is an identifying code. Indeed, x is separated from all the other vertices since it is the only one to have precisely code vertex y in its closed neighborhood, and all the other pairs of vertices where separated in C' so they are still separated.

Case 2 : G' has twins. Let u, v be a pair of twins of G'. They were not twins in G, thus x is adjacent to exactly one of them, without loss of generality, let x be adjacent to u but not to v. Thus $u = y$. Note that v is then unique since if u, v' were twins in G' then v and v' would be twins in G. Then $G'' = G' - v$ is twin-free since v and u have the same closed neighborhood in G', if one of them separates a pair of vertices, so does the other. By minimality of G, there is an identifying code C'' of G'' of size at most $n_Q(G'') < n_Q(G)$. If $y \notin C''$, then $C'' \cup \{x\}$ is an identifying code of G. Indeed x is the unique vertex having exactly $\{x\}$ has identifying set. Thus y and v are separated by x. Vertex y is still separated from all the other vertices and so is v since they have the same closed neighborhood except x. If $y \in C''$, then $C = C'' - y \cup \{x, v\}$ is an identifying code. Indeed, as before, x is the unique vertex having exactly $\{x\}$ has identifying set. Vertices y and v are separated by x. Vertex y is the only vertex having x and v in its closed neighborhood. Finally v is separated from any vertex $t \neq x, y$ by any vertex c that was separating t and y in C''.

In all cases, there is an identifying code of size at most $n_Q(G)$ in G, a contradiction. □

Note that, besides for stars [17], this bound is attained for e.g. thin headless spiders [4]. Note further that this bound does not hold for general graphs and even not for chordal graphs since the chordal graph P_{2k}^{k-1} is identifiable, has only two maximal cliques, but needs $2k-1$ vertices in any identifying code [14].

2.2 (Open) locating dominating codes

There are false twin-free graphs having $\gamma^{OLD}(G) = |V(G)|$ which even remains true if we restrict our considerations to block graphs, as $\gamma^{OLD}(P_2) = 2$ and $\gamma^{OLD}(P_4) = 4$ holds. However, we can show that the only block graphs that reach the upper bound $|V(G)|$ are P_2 and P_4.

Theorem 2.2 *Let $G = (V, E)$ be a connected false twin-free block graph different from P_2 and P_4. Then $\gamma^{OLD}(G) \le |V(G)| - 1$.*

Proof. Let G be a connected false twin-free block graph that satisfies $\gamma^{OLD}(G) = |V(G)|$. Assume that G is neither P_2 nor P_4.

Using Bondy's theorem [6], there is a vertex x such that $G - x$ does not have false twins. Then $S_1 = V(G) - \{x\}$ is not an OLD-code, because one vertex, say y, is not open-dominated. This means that y has degree 1 with unique neighbor x.

Now, consider the set $S_2 = V(G) - \{y\}$. Since G is not P_2, S_2 is an open-dominating set. But S_2 is not an OLD-code. It means that there are two vertices that are not separated by S_2. Since G has no false twins, these two vertices are separated by y and necessarily, one of them is actually the vertex x. Let z be the other vertex. We must have $N(x) = N(z) \cup \{y\}$.

Consider now $S_3 = V(G) - \{z\}$. Since $N(z) \subset N(x)$, S_3 is an open-dominating set. As before, it is not an OLD-code of G and there are two vertices in G that are only separated by z, and necessarily one of them is y. Let u be the other vertex, it has exactly two neighbors that are x and z. Note that the four vertices x, y, z, u induce P_4. Since G is not P_4, there must be at least another vertex v adjacent to these vertices. Since y and u have degrees 1 and 2, v must be adjacent to x and z (since they have the same neighborhood except y). But then there is a C_4 induced in G, a contradiction since G is a block graph. □

Note that a characterization of all graphs satisfying $\gamma^{OLD}(G) = |V(G)|$ can be obtained by extending the argumentation of the previous proof.

There is no hope for a general upper bound on $\gamma^{OLD}(G)$ using uniquely $n_Q(G)$. Indeed, this parameter is well defined for cliques and equal to $|V(G)| - 1$ whereas there is a unique maximal clique.

For locating-dominating codes, from $\gamma^{LD}(G) \le \min\{\gamma^{ID}(G), \gamma^{OLD}(G)\}$, we conclude that the upper bound for $\gamma^{ID}(G)$ carries over to $\gamma^{LD}(G)$ whenever the block graph G is identifiable, but we cannot improve the general upper bound in the other case:

Theorem 2.3 *Let $G = (V, E)$ be a block graph. Then*

$$\gamma^{LD}(G) \le \begin{cases} n_Q(G) & \text{if } G \text{ is identifiable,} \\ |V| - 1 & \text{otherwise.} \end{cases}$$

Note that these bounds are attained for stars (which are identifiable) and cliques (which are not identifiable) by [5].

3 Lower bounds

The general lower bound for the size of an identifying code using the number of vertices is $\gamma^{ID}(G) \ge \lceil \log_2(|V(G)| + 1) \rceil$ [19]. However, to reach this bound, a graph needs to have a large *VC-dimension* [10]. Indeed, if a graph has VC-dimension c then any identifying code has size at least $O(|V(G)|^{1/c})$. The value $1/c$ is not always tight, see for example the case of line graphs, which have VC-dimension at most 4 but for which the tight order for the lower bound is $O(|V(G)|^{1/2})$ [13]. Similar results hold for LD- and OLD-codes.

Block graphs have VC-dimension at most 2, and thus, using the previous result, their (identifying) codes are lower bounded by $O(|V(G)|^{1/2})$. In the following theorem, we improve this lower bound and give a tight result.

Theorem 3.1 *Let G be a block graph. Then we have:*

- $\gamma^{ID}(G), \gamma^{OLD}(G) \ge \frac{|V(G)|}{3} + 1$,
- $\gamma^{LD}(G) \ge \frac{|V(G)|+1}{3}$.

Proof. Let C be a minimum code of G. Let $C_1, C_2, ..., C_k$ be the connected components in the subgraph $G[C]$ induced by C.

For each vertex x, the subset $N(x) \cap C$ (resp. $N[x] \cap C$) is called its signature if C is a LD-code or an OLD-code (resp. an ID-code).

The vertices of the graph are partitioned into the four following parts:

- $V_1 = C$ contains the vertices of the code,
- V_2 contains the vertices that have exactly one vertex of the code in their neighborhood,

- V_3 contains the vertices that have neighbors in at least two different connected components of C,
- V_4 contains the other vertices. Note that all their code neighbors are in the same connected component and that there are at least two such neighbors.

Let $n_i(G)$ denote the number of vertices of degree i in G.

Claim 3.2 *We have:*

- $|V_2| \leq |C|$ *if C is an LD-code.*
- $|V_2| \leq |C| - n_0(G[C])$ *if C is an ID-code.*
- $|V_2| \leq |C| - n_1(G[C])$ *if C is an OLD-code.*

By definition of V_2, each vertex $x \in V_2$ has its unique neighbor $u \in C$ as signature: $N(x) \cap C = \{u\}$ if C is an LD-code or an OLD-code, and $N[x] \cap C = \{u\}$ if C is an ID-code. Hence, there can be at most $|C|$ vertices in V_2.

Moreover, if C is an ID-code, u cannot be isolated in $G[C]$ (otherwise it will not be separated from x) thus there are at most $|C| - n_0(G[C])$ vertices in V_2.

Finally, if C is an OLD-code, and if u has a neighbor t of degree 1 in $G[C]$, then t and x are not separated. Note that u has at most one such neighbor (if there were two such neighbors t_1 and t_2, these two vertices would not be separated). Thus, there are at most $|C| - n_1(G[C])$ vertices in V_2. $\diamond$

Claim 3.3 $|V_3| \leq k - 1$ *where k is the number of connected components of $G[C]$.*

Indeed, consider the graph H having as vertex set $V_3 \cup \{u_1, u_2, ..., u_k\}$ where u_1, u_2,... u_k are k new vertices. For each vertex x in V_3, add an edge between x and u_i if x is adjacent to a vertex of C_i. If there is a cycle in H, there would be a cycle in G involving two vertices of different connected components C_i and C_j. By definition of a block graph, this cycle has to induce a clique, but then C_i and C_j have to be in the same connected component in $G[C]$, a contradiction. Thus H is a (bipartite) tree. Since any vertex in V_3 connects at least two vertices u_i and u_j, there are at most $k - 1$ such vertices. $\diamond$

Claim 3.4 *We have in all cases that $|V_4| \leq |C| - k$ and in particular*

- $|V_4| \leq |C| - 3k + 2n_0(G[C])$ *if C is an ID-code;*
- $|V_4| \leq |C| - 3k + n_1(G[C])$ *if C is an OLD-code;*
- $|V_4| \leq |C| - 3k + n_1(G[C]) + 2n_0(G[C])$ *if C is an LD-code.*

Consider a component C_i of $G[C]$ and a vertex $x \in V_4$ connected to C_i only. Note that C_i must have at least two vertices. By definition of a block graph, $N(x) \cap C_i$ is a clique (since C_i is connected) and, moreover, a maximal clique of C_i of size at least 2. Thus, if x and x' are two elements of V_4 connected to C_i, we must have $N(x) \cap N(x') \cap C_i$ of size at most 1 (since the signatures $N(x) \cap C$ and $N(x') \cap C$ (resp. $N[x] \cap C$ and $N[x'] \cap C$) are distinct).

This shows that each vertex $x \in V_4$ corresponds to a *unique* maximal clique of $G[C]$ of size at least 2. Since there are at most $|C_i| - 1$ maximal cliques of size at most 2 in $G[C_i]$, we obtain $|V_4| \leq |C| - k$ in any case (i.e., regardless whether C is an ID-, LD- or OLD-code).

We now precise the bound further.

Consider a component C_i with at least two vertices. We know that there are at most $|C_i| - 1$ maximal cliques in $G[C_i]$. But if there are no vertices of degree 1, then there are less maximal cliques. Indeed, consider the tree with maximal cliques as vertex set and two maximal cliques are adjacent if they share one vertex. It follows that the number of maximal cliques is the number of edges plus one of this tree. The number of edges is at most the number of vertices that are in at least two cliques. Otherwise said, $n_Q(G[C_i]) \leq |C_i| - |V_{LQ}| + 1$ where V_{LQ} denotes the vertices that are in only one maximal clique. Let n_{LQ} be the number of leaf cliques of C_i. Since C_i has size 2, we have $n_{LQ} \geq 2$. There are $n_1(G[C_i])$ leaf cliques of size 2, and each one gives one vertex in V_{LQ}. The others give at least two vertices in V_{LQ}. Finally, $|V_{LQ}| \geq n_1(G[C_i]) + 2(n_{LQ} - n_1(G[C_i])) \geq 4 - n_1(G[C_i])$, and $n_Q(G[C_i]) \leq |C_i| - 3 + n_1(G[C_i])$.

We further have $|V_4| \leq \sum_{|C_i| \geq 2}(|C_i| - 3 + n_1(G[C_i])) = |C| - 3k' + n_1(G[C]) - n_0(G[C])$ where k' denotes the number of components of size at least 2 in C. We have $k = k' + n_0(G[C])$ and thus $|V_4| \leq |C| - 3k + n_1(G[C]) + 2n_0(G[C])$.

In the case of an OLD-code, there are no isolated vertices in $G[C]$ (they would not be open-dominated), and thus $n_0(G[C]) = 0$.

In the case of an ID-code, the leaf cliques cannot be the signature of a vertex in V_4 and there is exactly one vertex of degree 1 per leaf clique (since the graph $G[C]$ is identifiable). Thus one can remove the term $n_1(G[C])$. $\diamond$

In conclusion, we compute the total number of vertices $n = |C| + |V_2| + |V_3| + |V_4|$ and discuss when the maximum is attained.

If C is an ID-code, then

$$\begin{aligned} n &= |C| + |V_2| + |V_3| + |V_4| \\ &\leq |C| + |C| - n_0(G[C]) + k - 1 + |C| - 3k + 2n_0(G[C]) \\ &= 3|C| - 2k - 1 + n_0(G[C]) \end{aligned}$$

follows. Since $n_0(G[C]) \leq k$, the maximum is attained when $k = 1$ and then $n_0(G[C]) = 0$.

If C is an OLD-code, then

$$\begin{aligned} n &= |C| + |V_2| + |V_3| + |V_4| \\ &\leq |C| + |C| - n_1(G[C]) + k - 1 + |C| - 3k + n_1(G[C]) \\ &= 3|C| - 2k - 1 \end{aligned}$$

follows. The maximum is reached when $k = 1$.

If C is an LD-code, then

$$\begin{aligned} n &= |C| + |V_2| + |V_3| + |V_4| \\ &\leq |C| + |C| + k - 1 + |C| - k \\ &= 3|C| - 1 \end{aligned}$$

follows (and the value is independent from the number k of components of $G[C]$). □

Extremal cases where these bounds are attained can be constructed as follows (see Fig. 2): Consider the graph with one path $u_1, ..., u_k$ (the vertices in the code C) and attach further vertices:

- for an ID-code C: attach a single vertex to each u_i and vertices to the pairs u_i, u_{i+1} for $1 < i < k - 1$,
- for an OLD-code C: attach a single vertex to u_1, u_k and each u_i for $2 < i < k - 1$ and vertices to all the pairs u_i, u_{i+1},
- for an LD-code C: attach a single vertex to each u_i and vertices to all the pairs u_i, u_{i+1}.

Note that the here presented graphs are all extremal cases for ID-codes, whereas further extremal graphs exist for OLD-codes and for LD-codes. In fact, the reasoning from the proof of Theorem 3.1 can be used to characterize the extremal graphs for all three cases.

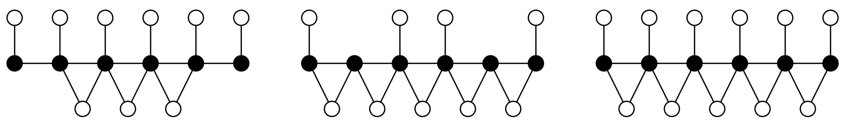

Fig. 2. Extremal cases where the lower bounds are attained, black vertices form a minimum (a) ID-code, (b) OLD-code, (c) LD-code.

If we now consider the parameter $n_Q(G)$, we can use the relation $|V(G)| \geq n_Q(G) + 1$ to obtain a similar lower bound. But this lower bound can be improved:

Theorem 3.5 *Let G be a block graph. Then we have:*

- $\gamma^{ID}(G), \gamma^{OLD}(G) \geq \frac{3(n_Q(G)+2)}{7}$,
- $\gamma^{LD}(G) \geq \frac{n_Q(G)+2}{3}$,
- $\gamma^{OLD}(G) \geq \frac{n_Q(G)+3}{2}$.

Proof. The proof uses the notations V_1, V_2, V_3, V_4, n_0, n_1 from the proof of Theorem 3.1. We will bound the number of maximal cliques. There are four types of maximal cliques:

(i) Maximal cliques that are maximal cliques of C (of size at least 2) or a maximal clique of C with one vertex of V_4. There are at most $n_Q(G[C]) - n_0(G[C]) \leq |C| - k$ of them ($|C_i| - 1$ for each of component).

(ii) Maximal cliques of size 2 between V_2 and its unique neighbor in the code. There are at most $|V_2|$ such cliques.

(iii) Maximal cliques formed with a vertex of V_3 and some vertices of the code.

(iv) Maximal cliques that are included in $V(G) \setminus C$.

To count the cliques of the two last types, we consider the same auxiliary graph H as in the proof of Claim 3.3. Let ℓ be the number of components in this graph that is actually a forest. The number of cliques of type (iii) is at most the number of edges in this graph that is $|V_3| + k - \ell$. The number of cliques of type (iv) is at most $\ell - 1$ since any such clique has to connect two connected components of the graph obtained from $G[C \cup V_3]$ where edges inside V_3 are removed. Since each vertex corresponding to a vertex of V_3 in H has degree at least 2 in H, there are at least $2|V_3|$ edges in H which implies that $|V_3| \leq k - \ell$. Finally, the number of cliques of type (iii) and (iv) is at most $2(k - \ell) \leq 2k - 2$. This implies

$$n_Q(G) \leq n_Q(G[C]) - n_0(G[C]) + |V_2| + 2k - 2.$$

We now discuss this inequality using the different upper bounds for $|V_2|$ proved in Claim 3.2 and using the fact that $n_Q(G[C]) - n_0(G[C]) \leq |C| - k$.

For ID-codes, we obtain

$$n_Q(G) \leq 2|C| + k - n_0(G[C]) - 2$$

and this value is maximized when $k - n_0(G[C])$ is maximal which happens when the number of components of size at least 2 is maximal. In an identifying code, this component must have size at least 3, and thus we have $k - n_0(G[C]) \leq |C|/3$ and $n_Q(G) \leq 7|C|/3 - 2$.

For LD-codes, we obtain

$$n_Q(G) \leq 2|C| + k - 2$$

and this value is maximized when k is maximal, i.e. $k = |C|$ and then $n_Q(G) \leq 3|C| - 2$.

For OLD-codes, we obtain, using the bound $n_Q(G[C]) - n_0(G[C]) \leq |C| - 3k + n_1(G[C])$ proved for OLD-codes in Claim 3.4, that

$$n_Q(G) \leq |C| - 3k + n_1(G[C]) + |C| - n_1(G[C]) + 2k - 2 = 2|C| - k - 2$$

and this value is maximal when $k = 1$ and we finally have $n_Q(G) \leq 2|C| - 3$. □

Note that for trees, since we have $n_Q(G) = |E(G)| = |V(G)| - 1$, these bounds are equivalent to the known lower bounds using the number of vertices (see [8] for ID-codes, [22] for LD-codes and [24] for OLD-codes). In particular, there are infinite families of trees reaching the three bounds.

Moreover, there is no such lower bound for general graphs. Indeed, consider the following split graph G with vertex set $V = \{v_1, ..., v_k\} \cup \{u_X, X \subseteq \{1, ..., k\}, X \neq \emptyset\}$. Vertices $v_1, ...v_k$ induce a clique whereas vertices u_X induce an independent set. Moreover, there is an edge between v_i and u_X iff $i \in X$. This graph has an identifying code of size $2k$ (the clique with the vertices corresponding to the singletons) but the number of maximum cliques is 2^k.

4 Concluding remarks

The three here studied domination problems are challenging both from a theoretical and a computational point of view and even remain hard for several graph classes where other in general hard problems are easy to solve, including bipartite graphs and chordal graphs. Block graphs form a subclass of chordal graphs for which all three domination problems can be solved in linear time [1]. In this paper, we complement this result by presenting for all three codes lower and upper bounds. We give bounds using both the number of vertices – as it has been done for several classes of graphs – but also using the number $n_Q(G)$ of maximal cliques of G, that is more pertinent for block graphs. In particular, we verify a conjecture from [2] concerning an upper bound for $\gamma^{ID}(G)$. Moreover, we address the questions to find block graphs where the provided lower and upper bounds are attained.

It is interesting whether similar ideas work for graph classes with a similar structure, e.g. for cacti (graphs in which every maximal 2-connected subgraph is an edge or a cycle) or for block-cacti (graphs in which every maximal 2-connected subgraph is a clique or a cycle).

Acknowledgement. We would like to thank Florent Foucaud for interesting discussions on the topic, in particular concerning lower bounds for the studied domination numbers.

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