The use of heuristic strategies in modelling activities

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For students working on realistic modelling problems as autonomously as possible the support by a tutor is indispensable. However, how this support can be realised is still not a sufficiently answered question. In the paper we describe a study in which students worked on complex, realistic, authentic modelling problems over three days supported by tutors. The tutors participated in a teacher training prior to the modelling activity. The focus of the study is the usage of heuristic strategies by students within modelling activities and the promotion of strategic help provided by the tutors. Based on videotaping of ten groups while they were working on the modelling problem of the optimal placement of a bus stop, the study could show that heuristic strategies are an indispensable basis for adequate decisions in the modelling process. Their promotion by the tutors seems to be highly adequate in order to foster modelling competencies under a broader perspective.

Keywords: Mathematical modelling activities, heuristic strategies, modelling example.

Introduction

Modelling and applications are receiving increasing attention all over the world, modelling competencies are required internationally by many curricula. However, the complexity of real world examples and the according modelling process to tackle the problem leads to a strong discrepancy between the high relevance of these kinds of activities in curricula and their factual relevance in school. In the following we will present a study, which examines how tutors can foster the tackling of complex and authentic modelling problems by students in special learning environments, the so-called modelling days. We will in particular focus on heuristic strategies from the problem solving discussion and their possible usage in modelling classrooms.

Theoretical framework

Mathematical modelling and modelling cycle

In our research modelling is understood as a process where a ‘real situation’ from the ‘Rest of the World’ (Pollak, 1979) comes up and needs to be understood and simplified and transferred into a realworld model. The real world model is transformed into the world of mathematics, i.e. the formulated mathematical problem is (partly) solved and the solution is validated according to the real world situation. Often the first results do not answer the primary problem adequately, so the modelling cycle is run through again with an adjusted real world model. This process is repeated until a solution is produced which is adequate for the real situation from the standpoint of the modeller. The according process can be visualized with the following modelling cycle (figure 1).

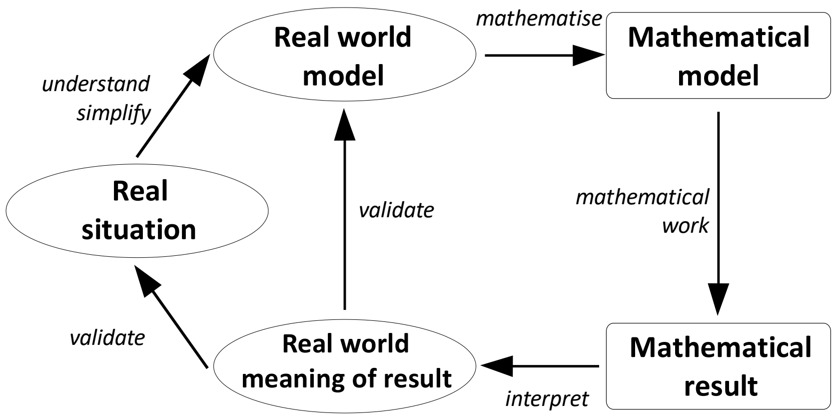

Figure 1: Modelling cycle (Kaiser & Stender, 2013, p. 279)

Fostering student's independent modelling activities in cooperative learning environments

If it is intended that students work as independent as possible on the modelling task in cooperative learning environments self-directed learning environments as required by many scholars in the modelling discussion (e.g. Kaiser and Stender, 2013, for an overview see Blum et al., 2007), the support by a tutor has to be adaptive. We use the definition given by Leiss (2007) of an *adaptive intervention*:

> Adaptive teacher interventions are defined as those kinds of assistance by the teacher to the student, which supports the individual learning and problem solving process of students minimally, so that students can continue to work at a maximal independent level. (Leiss, 2007, p. 65, own translation)

As guidelines for teachers supporting students who need help in their work, we refer to a framework developed Zech (1996), who suggests a five step approach to realise adaptivity: (1) *motivate*, e.g. 'You will make it'; (2) *feedback*, e.g.: 'Go on like this!'; (3) *strategic help* based on strategy what to do next, e.g. 'Simplify the situation by making it as symmetric as possible!"; (4) *content-related strategic help* gives a strategic help with additional information to the problem, e.g. which aspect should be described as symmetric; (5) *content-related help* shows the students aspects of concrete steps to work on.

If motivational and feedback support are not sufficient to enable the students to continue their work, strategic help is according to this framework the next step to support the students, as the students only get a possible way how to go on, but the students themselves still have to realise the work by their own. Important strategic helps are based on references to the steps of the modelling cycle: 'Simplify the situation!', 'Try to transfer this into a formula!', 'What does the mathematical result mean in the real world?', 'Does the result answers the real world situation meaningfully?' More specific support is offered by the use of heuristic strategies.

Heuristic strategies

The usage of heuristic strategies is a well-known approach in mathematical problem solving (e.g. Pólya, 1973), that can be used while tackling modelling problems as well. Based on the work of Pólya and others we distinguish the following heuristic strategies, which were formulated in the frame of ongoing empirical research in mathematical modelling (for details see Stender and Kaiser, 2015):

- *organise your material / understand the problem*: change the representation of the situation if useful, try out systematically, (Pólya, 1973) use simulations with or without computers, discretize situations,
- *use the working memory effective*: combine complex items to supersigns, which represent the concept of 'chunks' (Miller, 1956), use symmetry, break down your problem into sub-problems),
- *think big*: do not think inside dispensable borders, generalise the situation (Pólya, 1973),
- *use what you know*: use analogies from other problems, trace back new problems to familiar ones, combine partial solutions to get a global solution, use algorithms where possible (Pólya, 1973),
- *functional aspects*: analyse special cases or borderline cases (Pólya, 1973), in order to optimise you have to vary the input quantity,
- *organise the work*: work backwards and forwards, keep your approach – change your approach – both at the right moment (Pólya, 1973).

Although these heuristic strategies are well-known in the problem solving discussion, there is only little empirical research known, how these heuristic strategies can be implemented in classroom teaching and how far these heuristic strategies can be transferred to other mathematical activities such as mathematical modelling.

Research Question

In our research study we aim to evaluate, how far heuristic strategies developed in the problem solving discussion can be transferred into the teaching and learning of mathematical modelling. Furthermore, we examine how far these heuristic strategies are appropriate strategic interventions for the tutoring of students who are working on complex, realistic and authentic modelling problems.

Design of the Empirical Study

Modelling Days as learning environment

As this kind of work is not very common in usual classes we established modelling projects in schools as learning environment, called modelling days and offer these to schools as special, project-oriented activity organised in their school. The problems, on which the students work, are developed by the university research group. The tutoring of the students, who work in groups of four to six, is organised either by the teachers or future teachers within their master studies. Both groups receive a special training, in which they become acquainted with complex modelling examples and how to support students during their modelling activities. During the modelling days students of grade 9 (15 years old at the end of lower secondary level) work on one modelling problem for three full days in school. The students can choose the problem out of three problems presented by the research group.

In the following we describe one example and describe exemplarily one possible solution.

Modelling problem: The Bus Stop Problem – a possible solution

We used the bus stop problem in two different versions within various modelling days: the more complex one asks for the best positions of the bus stops for the entire public transport system of the city of Hamburg. The simplified (but still complex) version only asks for the bus stops of one single bus line.

A solution for the more complex version is based on the idea of covering the city with circles of the same diameter in a regular pattern where the centres of the circles are the bus stops. In a second step a rule is developed based on the adjustment of the bus stops to the requirements of the city-map. The diameter of the circles has to be calculated, which leads to the distance of two bus stops by a certain bus line. For this problem there are a lot of possible aspects that can be considered. One possible approach is to reduce the bus line to a straight line, where the bus stops all have the same distance to the next stop.

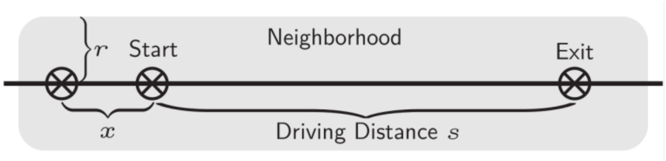

Figure 2: Bus line

In this solution the following aspects were taken into consideration: the average walking time from and to the bus station (velocity v_F), the time the bus drives (velocity v_B) over the distance s and the extra time (T_H) each stop causes in between. An optimal bus stop distance shall minimize the total travel time $T(x)$.

This leads to the function $T(x) = \frac{2r+x}{2v_F} + \frac{s}{v_B} + \frac{s}{x} \cdot T_H$ and setting the derivation as zero yields the following solution $x = \sqrt{2v_F \cdot s \cdot T_H}$. These formulae now can be interpreted according to the situation, e.g. in respect of the influence of the distance s or the walking velocity. Students usually will not receive this general result using variables, but with set numbers and they often produce a graph like figure 3. For a more detailed version of this solution see Stender (2016).

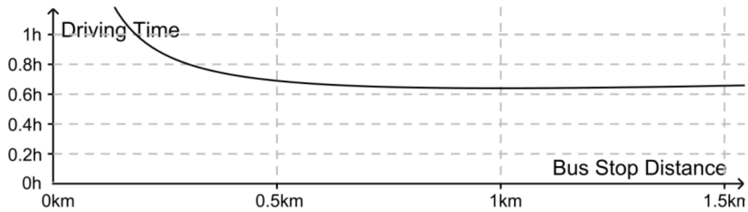

Figure 3: $T(x)$: Travel time depending on the bus stop distance

Data collection and data evaluation

Within our study we videotaped ten groups of students who were working in five rooms at higher track school in Hamburg (so-called Gymnasium), overall about 40 students participated in the study. Over three days the students worked around 15 hours on this modelling problem. We transcribed the phases during which the tutor communicated with the students, including a short time before, so that we could identify the causes leading to the contact and a few minutes after the communication so it was possible to analyse the effect of the tutors' intervention. In total 238 contacts between tutors and individual groups were transcribed and coded using qualitative content analysis (Mayring, 2015). Based on the analysis of the codes for the phases before, during and after the intervention the success of the interventions could be determined. Detailed findings were presented in Stender & Kaiser

(2015). Interventions that were not successful or gave too strong content-related help were subject to a more detailed examination. In these cases we tried to formulate alternative strategic interventions for use in further teacher training. The solution processes of the students were reconstructed based on the work of different groups and hereby an idealised modelling process could be reconstructed (for more details see Stender, 2016).

Results of the study

The reconstructed and idealised modelling process is in the first part of the results section used in order to identify, which heuristic strategies students used either intuitively or by referring explicitly to the modelling cycle, to which they had been introduced explicitly using the example of the length of traffic light phases. All students had worked on this example as introduction. The second part of the results section identifies possible interventions by tutors, introducing the students to the usage of heuristic strategies or by using these heuristic strategies by themselves in order to support the students.

Reconstruction of heuristic strategies in the solution process

In the following we analyse this reconstructed idealised solution regarding the use of heuristic strategies, which are highlighted in *italic*.

The first step of every modelling process is the exploration of the situation, that means as heuristic strategy *organise your material / understand the problem*. The students explored public transport maps and collected important places like schools or hospitals. It took a longer time to *change* this point of view to a more abstract *representation* of the situation, where the bus line is a straight line and special places do not matter. In this situation the more abstract representation is less complex as a lot of details from reality (traffic lights, curves, crossings, hills, …) are missing. So, here a heuristic strategy derived from the modelling cycle is applicable: *simplify* the situation as much as possible at the beginning! Describing the representation of a bus line as a straight line needs another heuristic strategy, namely to construct the situation *symmetrically*. Using this strategy leads to the assumption that the distance between two contiguous bus stops should be all the same. Figure 2 shows that even more aspects are symmetrical in this model. The transfer from the complete public transport system to one single bus line, that is used later to reconstruct the whole transport-system, uses as heuristic strategy to *break down your problem into sub-problems*! This is another way to *simplify* the situation.

To understand the problem of the simple straight bus line *two extreme cases* should be *analysed*, a powerful heuristic strategy: If there are few bus stops, the bus can drive fast without being interrupted by time consuming boarding, but the walk to the next bus stop will be very long for many passengers, which leads to a high total travel time. The other extreme situation has many bus stops, e.g. every 50 m. Now the walking time to the bus stop will be short for all passengers, but the bus will need a long time for a certain distance, because it is stopping every 50 m. So it becomes clear that between these two extremes there is an optimal distance between two bus stops that minimises the total travel time.

For the students the situation was still too complex and they were not able to formulate a functional based approach as they did not have enough experience with these kinds of problems. Now several heuristic strategies were employed: *use analogies, break down your problem into sub-problems, try out systematically or work on special cases.* As already mentioned all groups had worked on the length of traffic light phases as introductory example, they therefore knew the formulae $s(t) = \frac{1}{2}at^2$

and $v(t) = at$ and how to calculate acceleration processes. They reduced complexity again by the heuristic strategy *simplify* in the modelling cycle setting the distance between two bus stops a x = 500 m. This heuristic strategy is related to the heuristic strategy of *working backwards* as x should be the result of the calculation but is used here as if the result is already achieved. Then the students calculated the driving time between the two bus stops using certain values for the acceleration and the velocity of the bus and using *analogies* from the traffic light problem as heuristic strategy. This can be described again as using the heuristic strategy of *break down into sub-problems* as the calculation was not done for the whole bus line, but only for the part between two bus stops. As the choice of x = 500 m was made as ad hoc decision it can be described as heuristic strategy of *trying out* a *special case*. The calculation itself that led to a certain driving time includes several steps and is mathematically challenging, but was achieved due to the use of the *analogy* from the traffic light problem. The result of this approach was the calculation of a certain traveling time that unfortunately was not the answer to the question of the best distance between two bus stops. So, in a second loop through the modelling cycle the students calculated the time for a longer distance of 12 km with a bus stop every 500m *combining the partial results from above*. Again this shows an interesting result but no answer to the initial question.

The next step was to vary the number of bus stops on the 12 km journey using a heuristic strategy *for optimisation you have to vary the input quantity*. The travel time was calculated in the same way as before so an *analogy* was used. The calculation was realised using a spreadsheet and used the heuristic strategy *trying out systematically* different distances. This approach yields the result that with increasing number of bus stops the traveling time increases proportionally. This was expectable as one of the crucial aspects – the walk to the bus stop – was not considered up to now.

Based on this insight an average walking time to the bus stop was included in the calculation while the rest of the calculation was *analogue* to the previous one. This led to a result similar to figure 3, but the students still used as variable the number of bus stops, not the distance between two bus stops.

With the heuristic strategy of *a change of representation* the students switched to the more meaningful variable, again the new calculation was *analogue* to the previous one. Still everything was calculated with a spreadsheet so it had the character of a *simulation* or just *trying* out special cases. Only few students were able to realise the next *change of representation* and combined the single steps of the calculation into one formula. They developed one function $T(x)$ that included several steps of the calculation in one single mathematical term and $T(x)$ works as a *supersign*, which is another, less discussed heuristic strategy. This approach opened the way to use the derivate and calculate a solution like it is shown above. This was done with concrete numbers instead of parameters (v_F, v_B, T_H, s, r) by one group of students, but in another group there were students able to use parameters instead of concrete numbers, which again means the use of *supersigns* as each character stands for an infinite amount of numbers.

The results from different traveling distances s were compared and it became clear as meaningful result that on shorter trips the bus stops may be closer together. This result was *validated*, a heuristic strategy from the modelling cycle, by analysing the map of the public transport in Hamburg. Near the city, where people often use the bus only for the short distance to the next metro station, the bus stops were much closer than in the outer parts of Hamburg. The calculated distances matched very well to the distances in the map.

The subsequent step was to go back to the public transport network and cover the city with circles of a certain diameter. Now the results from the single bus line were used, which uses the heuristic strategy to *combine partial solutions to get a global solution* in order to choose meaningful diameters. These diameters were not the same over the whole city according to the previous results and in opposition to the initial idea.

To summarise, the analyses of the modelling activities by the students showed an intensive usage of heuristic strategies, partly referring to the various phases of the modelling cycle and partly as intuitive usage.

Heuristic strategies as strategic interventions

The heuristic strategies that were used quite often intuitively by the students in the modelling process can be transferred into strategic interventions by tutors, if the students are not able to continue their work on their own. In the following examples for these activities are described, which were shortly included in the teacher training beforehand and which were used by the tutors, but not as intensively as wished, probably due to their low importance in teacher training.

In each situation, where an analogue acting to previous work occurs, the following hints are possible: "This work is analogue to something you have done bevor!" or "Calculate this in the same way you did in the traffic light problem".

While constructing the real model the simplification of the situation is essential. "In your first approach build the real model as simple as possible – for this, it's a good idea to describe the model as symmetric as possible!"

The idea to break down the problem into sub-problems can be initiated by "For this problem you have to work on several steps – try to solve only one simple part at the beginning and then try to use these result for the next steps!"

The idea of using special cases can be implemented as follows: "If you have no idea how to go on, select specific numbers and work with them! Just work on special cases in the beginning!" As shown above this strategic intervention is a powerful mean for modelling activities.

The heuristic strategy *For optimisation you have to vary the input quantity* can be helpful for students who are not familiar with functional thinking. The following hints can be given: "You calculated with 23 bus stops. What happens if you use more or less bus stops?" "Vary the number of bus stops!" "If you look for an optimal solution you have to make sure that a nearby situation is less good!"

To summarise, these examples show how a heuristic strategy can be used to create a strategic intervention. Depending on the work of the students, more or less information on the concrete modelling problem can be included in the intervention in order to give a less abstract input to the students if necessary.

Summary and conclusions

The empirical study displays a great variety of heuristic strategies used by the students within their modelling activities, a few were developed intuitively, a few derived from the description of the modelling cycle introduced beforehand.

An important result of the study is that strategic interventions often were successful when a tutor supported students working on complex modelling problems, because the usage of these heuristic strategies is not self-evident. Because the usage of adequate strategic interventions by tutors is very hard, it often only will be possible if prepared beforehand. In order to react on the students in class in an adaptive way the tutor needs a deep insight into the modelling process, the modelling problem, possible solutions as well as heuristic strategies and strategic interventions.

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