Asymptotic Analysis of a Thin Elastic Plate–Viscoelastic Layer Interaction

Frédéric Chardard, Alexander Elbert, Grigory Panasenko

Multiscale Model. Simul.
Vol. 16, No. 3, pp. 1258–1282

ASYMPTOTIC ANALYSIS OF A THIN ELASTIC PLATE–VISCOELASTIC LAYER INTERACTION*

FRÉDÉRIC CHARDARD†, ALEXANDER ELBERT‡, AND GRIGORY PANASENKO†

Abstract. The paper is devoted to an asymptotic analysis of a problem on interaction between a thin purely elastic plate and a thick viscoelastic layer described by the Kelvin–Voigt model. Such a problem appears in modeling of the earth crust–magma interaction. The small parameter is the ratio of the thicknesses of the elastic part and the viscoelastic one. At the same time the plate has a high Young's modulus, that is, an inverse to the third power of the small parameter. The complete asymptotic expansion of the solution is constructed. The error estimate is proved for the difference of the exact solution and a truncated expansion. The limit problem is the Kelvin–Voigt equations with a special boundary condition. This limit problem is solved numerically by a finite element scheme. The difference between the initial and limit problems is studied theoretically and by numerical computations.

Key words. elasticity, viscoelasticity, thin rigid layer, asymptotic expansion, numerical finite element scheme

AMS subject classifications. 35B27, 35Q53, 35C20

DOI. 10.1137/17M1138662

1. Introduction, formulation of the problem, and main results. We consider a coupled system "viscoelastic material–thin elastic plate" where a thin elastic plate "lies" on a thick layer of a viscoelastic medium. The ratio of the thickness of the plate and the thickness of the viscoelastic layer, ε, is a small positive parameter, and the Young's modulus of the plate material is of order ε^{-3}, while the moduli of the viscoelastic layer are all of order of 1. The mechanical properties of the thick layer are described by the Kelvin–Voigt model [8], [1], [10]. The elastic plate is described by the two-dimensional elasticity equation with a high (in comparison with the viscoelastic layer) Young's modulus. This formulation is motivated by the modeling of a geophysical problem: the earth crust–viscoelastic magma system. Namely, the crust has a stratified structure, and it is rigid and very thin with respect to the magma layer. Indeed, the thickness of the crust varies from 5 to 75 km, and its rigidity is close to the upper mantle, which is about 200 km thick. So, the crust with the upper mantle may be considered as a stratified plate, while the lower mantle is about 2500–2600 km thick, is viscoelastic, and has the elasticity moduli about three orders smaller than that of the crust. For example, the bulk modulus of the lower magma is 2.12–2.23 kbar, and its shear modulus is 1.30–1.35 kbar, while these constants in the crust vary from 100 to 300 GPa for the bulk modulus and from 60 to 200 GPa for the sear modulus [2, Chapter 6]. Locally, the layers of the earth crust are supposed to be isotropic, but after the homogenization the macroscopic description of the crust may

*Received by the editors July 14, 2017; accepted for publication (in revised form) May 29, 2018; published electronically August 9, 2018.
http://www.siam.org/journals/mms/16-3/M113866.html
Funding: The third author's research was supported by grant 14-11-00306 of the Russian Science Foundation executed by National Research University, Moscow Power Engineering Institute.
†Institute Camille Jordan UMR CNRS 5208 and SFR MODMAD FED 4169, University of Lyon, 42023 Saint-Etienne, France (frederic.chardard@univ-st-etienne.fr, Grigory.Panasenko@univ-st-etienne.fr).
‡Institute of Mechanics and Mathematics, Ural Branch of the Russian Academy of Sciences, Ekaterinburg 620990, Russia (aee@imm.uran.ru).

become anisotropic. The idea is to reduce the dimension of the crust and to transform it into some special boundary condition for the magma layer. This reduction economizes computational resources. On the other hand, this asymptotic reduction should be multiscale: if necessary, it should be possible to scale back the strain-stress field in the crust and to restore the solution at the microscopic level. That is why below the complete asymptotic expansion is constructed when ε tends to zero. It allows one to reconstruct the detailed information on the microscopic behavior of the solution. Earlier the problem of the elastic rigid plate–Newtonian fluid interaction was studied in [9] in the two-dimensional setting and in [7] in the three-dimensional setting.

Let us describe now the mathematical setting of the problem and formulate the main theoretical results.

Consider a strip $L_\varepsilon = \mathbb{R} \times (-1, \varepsilon)$ consisting of the elastic part $L_\varepsilon^+ = \mathbb{R} \times (0, \varepsilon)$ and the viscoelastic part $L^- = \mathbb{R} \times (-1, 0)$. Denote the boundaries of the layers by $\Gamma^- = \{(x_1, -1), x_1 \in \mathbb{R}\}$, $\tilde{\Gamma}^0 = \{(x_1, 0), x_1 \in \mathbb{R}\}$, $\Gamma^+ = \{(x_1, \varepsilon), x_1 \in \mathbb{R}\}$. Let T be a positive number, independent of ε.

Here the small parameter ε is the ratio between the thicknesses of the elastic and viscoelastic parts. The elastic part is much more rigid than the viscoelastic part: its Young's modulus is ε^{-3} times greater than the Young's modulus of the viscoelastic part. Denote the displacement function in the elastic part $\mathbf{u}_+$ and the displacement function in the viscoelastic part $\mathbf{u}_-$. At the interface Γ^0 between the elastic and viscoelastic parts the continuity condition is satisfied for the displacements and for the normal stresses. So, we get the following model for the interaction of the elastic and viscoelastic parts:

(1.1)

$$\begin{cases} \rho_+\left(\frac{x_2}{\varepsilon}\right)\frac{\partial^2 \mathbf{u}_+}{\partial t^2} - \varepsilon^{-3}\sum_{i,j=1}^{2}\frac{\partial}{\partial x_i}\left(A_{ij}^+\left(\frac{x_2}{\varepsilon}\right)\frac{\partial \mathbf{u}_+}{\partial x_j}\right) = \varepsilon^{-1}\mathbf{f}^+(x_1,t) & \text{in } L_\varepsilon^+ \times (0,T), \\ \rho_-(x_2)\frac{\partial^2 \mathbf{u}_-}{\partial t^2} - \sum_{i,j=1}^{2}\frac{\partial^2}{\partial t \partial x_i}\left(B_{ij}^-(x_2)\frac{\partial \mathbf{u}_-}{\partial x_j}\right) \\ \quad - \sum_{i,j=1}^{2}\frac{\partial}{\partial x_i}\left(A_{ij}^-(x_2)\frac{\partial \mathbf{u}_-}{\partial x_j}\right) = \mathbf{f}^-(\mathbf{x},t) & \text{in } L^- \times (0,T), \\ \sum_{j=1}^{2} A_{2j}^+\frac{\partial \mathbf{u}_+}{\partial x_j} = \mathbf{0} & \text{at } \Gamma_\varepsilon^+ \times (0,T), \\ \mathbf{u}_- = \mathbf{0} & \text{at } \Gamma^- \times (0,T), \\ \mathbf{u}_+ = \mathbf{u}_- & \text{at } \tilde{\Gamma}^0 \times (0,T), \\ \varepsilon^{-3}\left(\sum_{j=1}^{2} A_{2j}^+\frac{\partial \mathbf{u}_+}{\partial x_j}\right) = \sum_{j=1}^{2} A_{2j}^-\frac{\partial \mathbf{u}_-}{\partial x_j} + \sum_{j=1}^{2} B_{2j}^-\frac{\partial^2 \mathbf{u}_-}{\partial t \partial x_j} & \text{at } \tilde{\Gamma}^0 \times (0,T), \\ \mathbf{u}_+|_{t=0} = \left.\frac{\partial \mathbf{u}_+}{\partial t}\right|_{t=0} = \mathbf{0} & \text{in } L_\varepsilon^+, \\ \mathbf{u}_-|_{t=0} = \left.\frac{\partial \mathbf{u}_-}{\partial t}\right|_{t=0} = \mathbf{0} & \text{in } L^-, \\ \mathbf{u}_+,\ \mathbf{u}_- \text{ are 1-periodic in } x_1. \end{cases}$$

Here $\mathbf{x} = (x_1, x_2)$ and A_{ij}^+, A_{ij}^-, B_{ij}^- are the $2 \times 2-$matrix-valued coefficients defined below, and ρ_+ and ρ_- are the scalar coefficients. The right-hand sides are $\mathbf{f}^+$, 1-periodic in x_1, C^∞-smooth on an $\mathbb{R} \times [0, +\infty)$ vector-valued function; and $\mathbf{f}^-$, 1-periodic in x_1, C^∞-smooth on an $\bar{L}^- \times [0, +\infty)$ vector-valued function. Both right-hand-side functions $\mathbf{f}^+, \mathbf{f}^-$ are equal to zero for small values of t: there exists a positive number μ, such that $\mathbf{f}^+ = \mathbf{f}^- = 0$ for all $t \in [0, \mu)$.

Here the elastic stratified layer is described by the variable density ρ_+ and by matrix-valued coefficients A_{ij}^+ which depend on the Young's modulus E and on the Poisson's ratio $\hat{\nu}$. The viscoelastic medium is described by the density ρ_- and matrix-valued coefficients A^-_{ij}, B^-_{ij} being of order of 1. They correspond to the Kelvin–Voigt viscoelasticity model. Matrices B^-_{ij} characterize the linear law relating the viscous stress and the strain rate (see [8], [1], [10]), and the coefficients A^+_{ij}, A^-_{ij}, B^-_{ij} have the following structure:

$$\begin{cases} A^+_{11} = \begin{pmatrix} a_1 & 0 \\ 0 & a_2 \end{pmatrix}, \quad A^+_{12} = \begin{pmatrix} 0 & a_3 \\ a_2 & 0 \end{pmatrix}, \quad a_1 = \dfrac{E(1-\hat\nu)}{(1+\hat\nu)(1-2\hat\nu)}, \quad a_2 = \dfrac{E}{2(1+\hat\nu)}, \\ A^+_{21} = \begin{pmatrix} 0 & a_2 \\ a_3 & 0 \end{pmatrix}, \quad A^+_{22} = \begin{pmatrix} a_2 & 0 \\ 0 & a_1 \end{pmatrix}, \quad a_3 = \dfrac{E\hat\nu}{(1+\hat\nu)(1-2\hat\nu)}, \quad \dfrac{a_3}{a_1} = \dfrac{\hat\nu}{1-\hat\nu}, \end{cases}$$

$$\begin{cases} A^-_{11} = \begin{pmatrix} a^-_1 & 0 \\ 0 & a^-_2 \end{pmatrix}, \quad A^-_{12} = \begin{pmatrix} 0 & a^-_3 \\ a^-_2 & 0 \end{pmatrix}, \\ A^-_{21} = \begin{pmatrix} 0 & a^-_2 \\ a^-_3 & 0 \end{pmatrix}, \quad A^-_{22} = \begin{pmatrix} a^-_2 & 0 \\ 0 & a^-_1 \end{pmatrix}, \end{cases}$$

$$\begin{cases} B^-_{11} = \begin{pmatrix} b^-_1 & 0 \\ 0 & b^-_2 \end{pmatrix}, \quad B^-_{12} = \begin{pmatrix} 0 & b^-_3 \\ b^-_2 & 0 \end{pmatrix}, \\ B^-_{21} = \begin{pmatrix} 0 & b^-_2 \\ b^-_3 & 0 \end{pmatrix}, \quad B^-_{22} = \begin{pmatrix} b^-_2 & 0 \\ 0 & b^-_1 \end{pmatrix}. \end{cases}$$

Here $a^-_1, a^-_2, a^-_3, b^-_1, b^-_2, b^-_3$, and ρ^- are positive functions belonging to the space $C^\infty([-1,0])$. We assume that there exists a positive constant κ such that for any 2×2 symmetric matrices $(\xi_{ij})_{1\le i,j,\le 2}$ the following quadratic forms satisfy inequalities

$$a^-_1(\xi^2_{11}+\xi^2_{22}) + 4a^-_2\xi^2_{12} + 2a^-_3\xi_{11}\xi_{22} \ge \kappa(\xi^2_{11}+2\xi^2_{12}+\xi^2_{22}), \tag{1.2}$$

$$b^-_1(\xi^2_{11}+\xi^2_{22}) + 4b^-_2\xi^2_{12} + 2b^-_3\xi_{11}\xi_{22} \ge \kappa(\xi^2_{11}+2\xi^2_{12}+\xi^2_{22}), \tag{1.3}$$

and

$$\forall x_2 \in [-1,0], \quad \rho^-(x_2) \ge \kappa. \tag{1.4}$$

$E, \hat\nu, \rho_+$ are piecewise-smooth functions of the fast variable $\xi_2 = x_2/\varepsilon$ defined for $\xi_2 \in [0,1]$; namely, there exist real numbers $\theta_0 = 0 < \theta_1 < \cdots < \theta_{N-1} < 1 = \theta_N$ such that $E, \hat\nu, \rho_+ \in C^1([\theta_j, \theta_{j+1}])$, $j = 0,\ldots,N-1$; assume that for all $\xi \in [0,1]$, $E(\xi) \ge \kappa, \rho_+(\xi) \ge \kappa, -1+\kappa \le \hat\nu \le 1/2 - \kappa$.

Since the functions $E, \hat\nu, \rho_+$ are piecewise-smooth, we add the interface conditions for the lines of discontinuity of coefficients $\tilde\Gamma^j = \left\{\left(x_1, \frac{\theta_j}{\varepsilon}\right),\ x_1 \in \mathbb{R}\right\}$, $j = 1,2,\ldots,N-1$:

$$\mathbf{u}_+|_{x_2=\frac{\theta_j}{\varepsilon}-0} = \mathbf{u}_+|_{x_2=\frac{\theta_j}{\varepsilon}+0} \text{ at } \tilde\Gamma^j \times (0,T), \tag*{$(1.1)_{10}$}$$

$$\left.\left(\sum_{j=1}^{2} A^+_{2j}\frac{\partial \mathbf{u}_+}{\partial x_j}\right)\right|_{x_2=\frac{\theta_j}{\varepsilon}-0} = \left.\left(\sum_{j=1}^{2} A^+_{2j}\frac{\partial \mathbf{u}_+}{\partial x_j}\right)\right|_{x_2=\frac{\theta_j}{\varepsilon}+0} \text{ at } \tilde\Gamma^j \times (0,T). \tag*{$(1.1)_{11}$}$$

However, these conditions are "automatically" satisfied for the weak solution in the variational formulation. That is why we will not write them below.

We introduce the following notations:

$$D^- = (0,1)\times(-1,0)\ ,\ \ D^+_\varepsilon = (0,1)\times(0,\varepsilon)\ ,\ \ D_\varepsilon = (0,1)\times(-1,\varepsilon),$$
$$\Omega = (0,1)\times(-1,\varepsilon)\times(0,T)\ ,\ \ \Omega^- = D^-\times(0,T)\ ,\ \ \Omega^+ = D^+_\varepsilon\times(0,T).$$

Let $\tilde{D}$ be a bounded domain in $\mathbb{R}$ or $\mathbb{R}^2$, and let D be the Cartesian product $(0,1) \times \tilde{D}$. Define $H^N_{per}(D)$ as a space of 1-periodic in x_1 functions f such that, for all $a, b \in \mathbb{R}$, $f \in H^N((a,b) \times \tilde{D})$, supplied with the norm $H^N(D)$ (see [4]): $\|u\|_{H^N(D)} = \sqrt{\int_D \sum_{|\alpha| \le N} (D^\alpha u)^2}$, where D^α are all partial derivatives of order N and smaller.

Introduce the following notations for the Sobolev spaces: $H^0 = L^2$, $H^N_T = (H^N_{per}(\Omega))^2$, $H^{N\pm}_T = (H^N_{per}(\Omega^\pm))^2$; the norms for the vector-valued functions are the euclidean norms $\sqrt{(\cdot)_1^2 + (\cdot)_2^2}$ of the corresponding norms for the entries (components) of a vector-valued function.

Denote

$$\mathcal{I}_{A^-}(\mathbf{v}, \omega)_{D^-} = \int_{D^-} \sum_{i,j=1}^{2} A^-_{ij}(x_2) \frac{\partial \mathbf{v}}{\partial x_j} \cdot \frac{\partial \omega}{\partial x_i}, \quad \mathcal{I}_{B^-}(\mathbf{v}, \omega)_{D^-} = \int_{D^-} \sum_{i,j=1}^{2} B^-_{ij}(x_2) \frac{\partial \mathbf{v}}{\partial x_j} \cdot \frac{\partial \omega}{\partial x_i},$$
$$\mathcal{I}_{A^+}(\mathbf{v}, \omega)_{D^+} = \int_{D^+_\varepsilon} \sum_{i,j=1}^{2} A^+_{ij}\left(\frac{x_2}{\varepsilon}\right) \frac{\partial \mathbf{v}}{\partial x_j} \cdot \frac{\partial \omega}{\partial x_i}.$$

For any function ω, defined in $\mathbb{R} \times (-1, \varepsilon)$, denote ω^- and ω^+ its restrictions on $\mathbb{R} \times (-1, 0)$ and $\mathbb{R} \times (0, \varepsilon)$, respectively. Define the following spaces and norms:

$$\begin{aligned} &\tilde{V} = \left\{\omega \in (H^1_{per}(D_\varepsilon))^2 : \omega|_{\Gamma^-} = 0, \right\}, \\ &\tilde{U} = \left\{\mathbf{u} \in H^1_T : \ \ \ddot{\mathbf{u}}^+ \in H^{0+}_T, \ \ \dot{\mathbf{u}}^- \in H^{1-}_T, \ \ \mathbf{u}^-|_{\Gamma^-} = 0, \right\}, \\ &\|\mathbf{u}\|^2_{\tilde{U}} = \|\mathbf{u}^+\|^2_{H^{1+}_T} + \|\ddot{\mathbf{u}}^+\|^2_{H^{0+}_T} + \|\mathbf{u}^-\|^2_{H^{1-}_T} + \|\dot{\mathbf{u}}^-\|^2_{H^{1-}_T}. \end{aligned} \tag{1.5}$$

Here and below we sometimes use the shortened dot-notation for the time derivative: $\dot{\mathbf{u}} = \frac{\partial \mathbf{u}}{\partial t}$.

Let us define a weak solution to problem (1.1) as a function $\mathbf{u}_\varepsilon \in \tilde{U}$ such that, for all $\omega \in \tilde{V}$,

(1.6)
$$\begin{cases} \int\limits_{D^+_\varepsilon} \rho_+ \left(\frac{x_2}{\varepsilon}\right) \ddot{\mathbf{u}}^+_\varepsilon \cdot \omega^+ + \varepsilon^{-3} \mathcal{I}_{A^+}(\mathbf{u}^+_\varepsilon, \omega^+)_{D^+_\varepsilon} + \int\limits_{D^-} \rho_-(x_2) \ddot{\mathbf{u}}^-_\varepsilon \cdot \omega^- + \mathcal{I}_{A^-}(\mathbf{u}^-_\varepsilon, \omega^-)_{D^-} \\ +\mathcal{I}_{B^-}(\dot{\mathbf{u}}^-_\varepsilon, \omega^-)_{D^-} = \int\limits_{D^+_\varepsilon} \varepsilon^{-1} \mathbf{f}^+ \cdot \omega^+ + \int\limits_{D^-} \mathbf{f}^- \cdot \omega^-, \\ \mathbf{u}_\varepsilon|_{t=0} = 0, \\ \dot{\mathbf{u}}_\varepsilon|_{t=0} = 0. \end{cases}$$

The following theorem ensures the existence and uniqueness of a weak solution.

THEOREM 1.1. *Problem* (1.6) *admits a unique solution.*

This theorem and an a priori estimate of the solution will be proved in Appendix A.

The main theoretical result is the construction of an asymptotic expansion of the solution to problem (1.1) as $\varepsilon \to 0$. This construction is described in the next section. The leading term of this expansion is a solution to the limit problem. This limit problem is the viscoelasticity equations stated in L^- with the nonstandard boundary condition on the upper boundary; namely, the solution has a form

$$\mathbf{v_0} = \hat{\mathbf{v}} + \begin{pmatrix} V \\ 0 \end{pmatrix},$$

where $\hat{\mathbf{v}}$ is a solution to the following problem:

(1.7)
$$\begin{cases} \rho_-(x_2)\frac{\partial^2 \hat{\mathbf{v}}}{\partial t^2} - \sum\limits_{i,j=1}^{2} \frac{\partial^2}{\partial t \partial x_i}\left(B^-_{ij}(x_2)\frac{\partial \hat{\mathbf{v}}}{\partial x_j}\right) - \sum\limits_{i,j=1}^{2} \frac{\partial}{\partial x_i}\left(A^-_{ij}(x_2)\frac{\partial \hat{\mathbf{v}}}{\partial x_j}\right) = \mathbf{f}^-(\mathbf{x},t) & \text{in } L^- \times (0,T), \\ \hat{\mathbf{v}} = \mathbf{0} & \text{at } \Gamma^- \times (0,T), \\ (\hat{\mathbf{v}})_1 = 0 & \text{at } \tilde{\Gamma}^0 \times (0,T), \\ \frac{\partial^4 (\hat{\mathbf{v}})_2}{\partial x_1^4} + h\left(\sum\limits_{j=1}^{2} A^-_{2j}\frac{\partial \hat{\mathbf{v}}}{\partial x_j} + \sum\limits_{j=1}^{2} B^-_{2j}\frac{\partial^2 \hat{\mathbf{v}}}{\partial t \partial x_j}\right)_2 = h(\mathbf{f}^+)_2 & \text{at } \tilde{\Gamma}^0 \times (0,T), \\ \hat{\mathbf{v}}|_{t=0} = \frac{\partial \hat{\mathbf{v}}}{\partial t}\Big|_{t=0} = \mathbf{0} & \text{in } L^-, \\ \mathbf{V}_1 \text{ is 1-periodic in } x_1, \end{cases}$$

$$h = \frac{\left\langle \frac{E}{1-\hat{\nu}^2}\right\rangle_{\xi_2}}{\left\langle \frac{E}{1-\hat{\nu}^2}\right\rangle_{\xi_2}\left\langle \mathcal{A}\left(\frac{E}{1-\hat{\nu}^2}\left(\frac{1}{2}-\xi_2\right)\right)\right\rangle_{\xi_2} - \left\langle \frac{E}{1-\hat{\nu}^2}\left(\frac{1}{2}-\xi_2\right)\right\rangle_{\xi_2}\left\langle \mathcal{A}\left(\frac{E}{1-\hat{\nu}^2}\right)\right\rangle_{\xi_2}},$$

where the denominator is different from zero (see [10]), and the function V is a solution to the problem

$$\text{(1.8)} \quad \begin{cases} \rho_-(x_2)\frac{\partial^2 V}{\partial t^2} - \frac{\partial}{\partial x_2}\left(b_2^-\frac{\partial^2 V}{\partial t \partial x_2} + a_2^-\frac{\partial V}{\partial x_2}\right) = 0, \quad (x_2,t) \in (-1,0)\times(0,T), \\ V|_{x_2=-1} = 0, & t \in (0,T), \\ \left(b_2^-\frac{\partial^2 V}{\partial t \partial x_2} + a_2^-\frac{\partial V}{\partial x_2}\right)|_{x_2=0} = \theta(t), & t \in (0,T), \\ V|_{t=0} = \frac{\partial V}{\partial t}\Big|_{t=0} = 0, & x_2 \in (-1,0), \end{cases}$$

where

$$\theta(t) = \left(\langle(\mathbf{f}^+)_1\rangle_{x_1} - \left\langle b_2^-\frac{\partial^2(\hat{\mathbf{v}})_1}{\partial t \partial x_2} + a_2^-\frac{\partial(\hat{\mathbf{v}})_1}{\partial x_2}\right\rangle_{x_1}\right)\Bigg|_{x_2=0}$$

and for any vector $(\mathbf{u})$, $(\mathbf{u})_1$ is the first component of this vector $(\mathbf{u})$; for example, $(\hat{\mathbf{v}})_1$ is the first component of the vector $\hat{\mathbf{v}}$. Here we used the following notations:

$$\langle F\rangle_x = \int_0^1 F(x)dx,$$

$$\text{(1.9)} \qquad \mathcal{A}F(x) = x\langle F\rangle_x - \int_0^x F(x)dx.$$

Sometimes we will omit the subscript x in the notation of a mean value and write $\langle F\rangle$ instead of $\langle F\rangle_x$.

This limit problem has a nonstandard boundary condition, and its mathematical analysis is given in Appendix A. In particular, we prove the following theorem.

THEOREM 1.2. *Problem* (1.7) *admits a unique solution. This solution belongs to the space* $C^\infty(\overline{L^-}\times[0,T])$. *Problem* (1.8) *admits a unique solution belonging to the space* $C^\infty([-1,0]\times[0,T])$.

The main result on the justification of the asymptotic analysis of problem (1.1) claims that the solutions to the initial and limit problems are close in the norm $L^2(D^-\times(0,T))$; namely, the following theorem holds.

THEOREM 1.3. *The following estimates hold:*

$$\|\mathbf{u}_+ - \mathbf{v_0}|_{x_2=0}\|_{L^2(D_\varepsilon^+\times(0,T))} = O(\varepsilon\sqrt{\varepsilon}); \quad \|\mathbf{u}_- - \mathbf{v_0}\|_{H^1(D^-\times(0,T))} = O(\varepsilon).$$

Moreover, for the partial sums of any order J of an asymptotic expansion an estimate of order $O(\varepsilon^J)$ will be proved in the norm $H^1(D^- \times (0,T))$.

The structure of the paper is as follows. In the next section we construct an asymptotic expansion of the solution to problem (1.1). The algorithm is recursive, and the coefficients of the ansatz will be constructed successfully by induction. For the leading term of the expansion we will obtain the limit problem (1.7), (1.8). In section 3 we provide the numerical comparison of the solutions to the initial and limit problems, describing the numerical method used for this comparison. Theoretical results on mathematical analysis of the initial and limit problems are given in Appendix A. The existence and uniqueness of solutions to (1.1) are proved by means of Galerkin's method. In Appendix B the residual estimates are proved, and in particular, for the partial sums of any order J of an asymptotic expansion an estimate of order $O(\varepsilon^J)$ is proved in the norm $H^1(D^- \times (0,T))$. Theorem 1.3 is a direct consequence of this estimate.

In what follows, we will use the following notations:

$$\mathcal{I}F(x) = \int_0^x F(s)ds, \quad \mathcal{B}F(x) = \left\langle \int_0^x F(\theta)d\theta \right\rangle_x - \int_0^x F(x)dx. \tag{1.10}$$

2. Asymptotic analysis. An asymptotic solution of order J is sought in the form of truncated series in powers of ε with 1-periodic in x_1 coefficients.

Let us set

$$\begin{cases} \mathbf{u}_+^{(J)}(x_1,x_2,t) = \sum\limits_{i=1}^{2} \sum\limits_{k,j\geqslant 0}^{5k+2j\leqslant J} \varepsilon^{5k+2j} \mathcal{N}_{2k,2j}^{(i)} \mathbf{w}^{(i)(J-2)}, \\ \mathbf{u}_-^{(J)}(x_1,x_2,t) = \sum\limits_{k=0}^{J} \varepsilon^k \mathbf{v}_k(x_1,x_2,t), \\ w_1^{(1)(J)}(x_1,t) = \sum\limits_{k=0}^{J} \varepsilon^k y_k^{(1)}(x_1,t), \quad w_2^{(1)(J)}(x_1,t) = \sum\limits_{k=0}^{J} \varepsilon^k z_k^{(1)}(x_1,t), \\ w_1^{(2)(J)}(x_1,t) = \sum\limits_{k=4}^{J+4} \varepsilon^k y_k^{(2)}(x_1,t), \quad w_2^{(2)(J)}(x_1,t) = \sum\limits_{k=4}^{J+4} \varepsilon^k z_k^{(2)}(x_1,t). \end{cases}$$

Here vectors $\mathbf{w}^{(i)}$ are the truncated series with coefficients $\mathbf{w}_k^{(i)}$ having the components $y_k^{(i)}, z_k^{(i)}$. So, we seek

$$\mathbf{w}^{(1)(J)} = \sum_{k=0}^{J} \varepsilon^k \mathbf{w}_k^{(1)} = \sum_{k=0}^{J} \varepsilon^k \begin{pmatrix} y_k^{(1)} \\ z_k^{(1)} \end{pmatrix},$$

$$\mathbf{w}^{(2)(J)} = \varepsilon^4 \left(\sum_{j=1}^{2} A_{2j}^- \frac{\partial \mathbf{u}_-^{(J)}}{\partial x_j} + \sum_{j=1}^{2} B_{2j}^- \frac{\partial^2 \mathbf{u}_-^{(J)}}{\partial t \partial x_j} \right)\Bigg|_{x_2=0} = \begin{pmatrix} \varepsilon^4 y_4^{(2)} + \varepsilon^5 y_5^{(2)} + \cdots \\ \varepsilon^4 z_4^{(2)} + \varepsilon^5 z_5^{(2)} + \cdots \end{pmatrix}.$$

For any k, i functions $y_k^{(i)}, z_k^{(i)}, \mathbf{v}_k$ are C^∞-smooth and 1-periodic in x_1.

The operator-valued matrices $\mathcal{N}_{kl}^{(i)}$ contain differential operators $D_{kj} = \frac{\partial^{k+j}}{\partial t^k \partial x_1^j}$:

$$\mathcal{N}_{2k,2j}^{(i)} = \begin{pmatrix} b_{2k,2j}^{(i)}(\xi_2) D_{2k,2j} & \varepsilon b_{2k,2j+1}^{(i)}(\xi_2) D_{2k,2j+1} \\ \varepsilon c_{2k,2j+1}^{(i)}(\xi_2) D_{2k,2j+1} & c_{2k,2j}^{(i)}(\xi_2) D_{2k,2j} \end{pmatrix},$$

where $b_{\alpha,\beta}^{(i)}$ and $c_{\alpha,\beta}^{(i)}$ are some piecewise-smooth functions.

Further, we plug this ansatz in (1.1) and determine successively functions $\mathbf{v}_k$, $\mathbf{w}_k^{(i)}$ and the operator-valued matrices $\mathcal{N}_{kl}^{(i)}$.

Constructing coefficients of $\mathcal{N}^{(i)}_{2k,2j}$. After substitution of the asymptotic expansion into the stresses $\sigma_i^+ = \sum_{j=1}^2 A^+_{ij}\frac{\partial \mathbf{u}_+}{\partial x_j}$ we get the following formulas, where functions $\alpha^{(i)}_{kj}, \beta^{(i)}_{kj}$, and $\gamma^{(i)}_{kj}$ depend on $\xi_2 = \frac{x_2}{\varepsilon}$:

$$\sum_{k,j\geqslant 0}^{5k+2j\leqslant J} \varepsilon^{5k+2j}\left(A^+_{11}\frac{\partial \mathcal{N}^{(i)}_{2k,2j}}{\partial x_1} + A^+_{12}\frac{\partial \mathcal{N}^{(i)}_{2k,2j}}{\partial x_2}\right)$$

$$= \begin{pmatrix} \sum_{k,j\geqslant 0}^{5k+2j\leqslant J} \varepsilon^{5k+2j}\alpha^{(i)}_{2k,2j+1}D_{2k,2j+1} & \sum_{k,j\geqslant 0}^{5k+2j\leqslant J+2} \varepsilon^{5k+2j}\cdot\varepsilon^{-1}\alpha^{(i)}_{2k,2j}D_{2k,2j} \\ \sum_{k,j\geqslant 0}^{5k+2j\leqslant J+2} \varepsilon^{5k+2j}\cdot\varepsilon^{-1}\beta^{(i)}_{2k,2j}D_{2k,2j} & \sum_{k,j\geqslant 0}^{5k+2j\leqslant J} \varepsilon^{5k+2j}\beta^{(i)}_{2k,2j+1}D_{2k,2j+1} \end{pmatrix},$$

(2.1)

$$\sum_{k,j\geqslant 0}^{5k+2j\leqslant J} \varepsilon^{5k+2j}\left(A^+_{21}\frac{\partial \mathcal{N}^{(i)}_{2k,2j}}{\partial x_1} + A^+_{22}\frac{\partial \mathcal{N}^{(i)}_{2k,2j}}{\partial x_2}\right)$$

$$= \begin{pmatrix} \sum_{k,j\geqslant 0}^{5k+2j\leqslant J+2} \varepsilon^{5k+2j}\cdot\varepsilon^{-1}\beta^{(i)}_{2k,2j}D_{2k,2j} & \sum_{k,j\geqslant 0}^{5k+2j\leqslant J} \varepsilon^{5k+2j}\beta^{(i)}_{2k,2j+1}D_{2k,2j+1} \\ \sum_{k,j\geqslant 0}^{5k+2j\leqslant J} \varepsilon^{5k+2j}\gamma^{(i)}_{2k,2j+1}D_{2k,2j+1} & \sum_{k,j\geqslant 0}^{5k+2j\leqslant J+2} \varepsilon^{5k+2j}\cdot\varepsilon^{-1}\gamma^{(i)}_{2k,2j}D_{2k,2j} \end{pmatrix}.$$

In the same way,

$$\sum_{i,l=1}^{2}\frac{\partial}{\partial x_i}\left(A^+_{il}\left(\frac{x_2}{\varepsilon}\right)\frac{\partial \mathbf{u}^{(J)}_+}{\partial x_l}\right)$$

$$= \sum_{i=1}^{2}\sum_{k,j\geqslant 0}^{5k+2j\leqslant J+2} \varepsilon^{5k+2j}\begin{pmatrix} \varepsilon^{-2}p^{(i)}_{2k,2j}D_{2k,2j} & \varepsilon^{-1}p^{(i)}_{2k,2j+1}D_{2k,2j+1} \\ \varepsilon^{-1}q^{(i)}_{2k,2j+1}D_{2k,2j+1} & \varepsilon^{-2}q^{(i)}_{2k,2j}D_{2k,2j} \end{pmatrix}\mathbf{w}^{(i)(J-2)}$$

and

$$\frac{\partial^2 \mathbf{u}^{(J)}_+}{\partial t^2} = \sum_{i=1}^{2}\sum_{k\geqslant 1, j\geqslant 0}^{5k+2j\leqslant J+5} \varepsilon^{5k+2j-5}\begin{pmatrix} b^{(i)}_{2k-2,2j}D_{2k,2j} & \varepsilon b^{(i)}_{2k-2,2j+1}D_{2k,2j+1} \\ \varepsilon c^{(i)}_{2k-2,2j+1}D_{2k,2j+1} & c^{(i)}_{2k-2,2j}D_{2k,2j} \end{pmatrix}\mathbf{w}^{(i)(J-2)}.$$

Denote

$$\mathcal{P}_\varepsilon \mathbf{u}_+ = \rho_+\left(\frac{x_2}{\varepsilon}\right)\frac{\partial^2 \mathbf{u}_+}{\partial t^2} - \varepsilon^{-3}\sum_{i,j=1}^{2}\frac{\partial}{\partial x_i}\left(A^+_{ij}\left(\frac{x_2}{\varepsilon}\right)\frac{\partial \mathbf{u}_+}{\partial x_j}\right).$$

Combining the previous expansions we get the following formula with currently undefined coefficients $h^{(i)}_{kj}$ and $m^{(i)}_{kj}$:

(2.2)

$$\mathcal{P}_\varepsilon \mathbf{u}^{(J)}_+ = \varepsilon^{-5}\sum_{i=1}^{2}\sum_{k,j\geqslant 0}^{5k+2j\leqslant J+2} \varepsilon^{5k+2j}\begin{pmatrix} h^{(i)}_{2k,2j}D_{2k,2j} & \varepsilon h^{(i)}_{2k,2j+1}D_{2k,2j+1} \\ \varepsilon m^{(i)}_{2k,2j+1}D_{2k,2j+1} & m^{(i)}_{2k,2j}D_{2k,2j} \end{pmatrix}\mathbf{w}^{(i)(J-2)} + \mathcal{R}^{(J)},$$

where

(2.3)

$$\mathcal{R}^{(J)} = \varepsilon^{-5}\sum_{i=1}^{2}\sum_{5k+2j\leqslant J+3}^{5k+2j\leqslant J+5} \varepsilon^{5k+2j}\begin{pmatrix} b^{(i)}_{2k-2,2j}D_{2k,2j} & \varepsilon b^{(i)}_{2k-2,2j+1}D_{2k,2j+1} \\ \varepsilon c^{(i)}_{2k-2,2j+1}D_{2k,2j+1} & c^{(i)}_{2k-2,2j}D_{2k,2j} \end{pmatrix}\mathbf{w}^{(i)(J-2)} = O(\varepsilon^{J-2}).$$

Comparing coefficients in four previous expansions, we conclude that currently undefined functions $b, c, \alpha, \beta, \gamma, p, q, h, m$ satisfy the following relations:

$$\alpha_{k,l}^{(i)} = a_1 b_{k,l-1}^{(i)} + a_3 (c_{k,l}^{(i)})', \quad p_{k,l}^{(i)} = \alpha_{k,l-1}^{(i)} + (\beta_{k,l}^{(i)})', \quad h_{k,l}^{(i)} = b_{k-2,l}^{(i)} \rho_+ - p_{k,l}^{(i)},$$
$$\beta_{k,l}^{(i)} = a_2 c_{k,l-1}^{(i)} + a_2 (b_{k,l}^{(i)})', \quad q_{k,l}^{(i)} = \beta_{k,l-1}^{(i)} + (\gamma_{k,l}^{(i)})', \quad m_{k,l}^{(i)} = c_{k-2,l}^{(i)} \rho_+ - q_{k,l}^{(i)},$$
$$\gamma_{k,l}^{(i)} = a_3 b_{k,l-1}^{(i)} + a_1 (c_{k,l}^{(i)})'.$$

As usual in the homogenization [3] we require that functions $h_{k,l}^{(i)}$, $m_{k,l}^{(i)}$ be constants. The boundary conditions $(1.1)_{3,5}$ generate the corresponding boundary conditions for these coefficients:

$$\beta_{0,0}^{(2)}(0) = \gamma_{0,0}^{(2)}(0) = 1, \quad \beta_{k,l}^{(i)}(0) = \gamma_{k,l}^{(i)}(0) = 0, \quad \beta_{k,l}^{(i)}(1) = \gamma_{k,l}^{(i)}(1) = 0.$$

Let

$$b_{0,0}^{(1)} = c_{0,0}^{(1)} = 1, \quad \alpha_{0,0}^{(1)} = \beta_{0,0}^{(1)} = \gamma_{0,0}^{(1)} = p_{0,0}^{(1)} = q_{0,0}^{(1)} = h_{0,0}^{(1)} = m_{0,0}^{(1)} = 0,$$
$$\beta_{0,0}^{(2)} = \gamma_{0,0}^{(2)} = 1 - \xi_2, \quad \alpha_{0,0}^{(2)} = \frac{\hat{\nu}}{1-\hat{\nu}}(1-\xi_2), \quad h_{0,0}^{(2)} = m_{0,0}^{(2)} = 1, \quad p_{0,0}^{(2)} = q_{0,0}^{(2)} = -1,$$
$$b_{0,0}^{(2)} = \int_0^{\xi_2} \frac{1-s}{a_2(s)} ds, \quad c_{0,0}^{(2)} = \int_0^{\xi_2} \frac{1-s}{a_1(s)} ds.$$

The coefficients are found by induction in k, j. We require that $h_{kl}^{(i)}$, $m_{kl}^{(i)}$ be independent of ξ_2:

$$h_{k,l}^{(i)} = \langle b_{k-2,l}^{(i)} \rho_+ - \alpha_{k,l-1}^{(i)} \rangle_{\xi_2}, \quad m_{k,l}^{(i)} = \langle c_{k-2,l}^{(i)} \rho_+ - \beta_{k,l-1}^{(i)} \rangle_{\xi_2}$$

and

$$p_{k,l}^{(i)} = b_{k-2,l}^{(i)} \rho_+ - h_{k,l}^{(i)}, \quad q_{k,l}^{(i)} = c_{k-2,l}^{(i)} \rho_+ - m_{k,l}^{(i)};$$

below we use the integral operators $\mathcal{A}$ and $\mathcal{B}$ (see (1.9), (1.10)) (operator $\mathcal{B}$ appears after double integration in ξ_2):

$$\beta_{k,l}^{(i)} = \mathcal{A}\left(\alpha_{k,l-1}^{(i)} - b_{k-2,l}^{(i)} \rho_+\right), \quad \gamma_{k,l}^{(i)} = \mathcal{A}\left(\beta_{k,l-1}^{(i)} - c_{k-2,l}^{(i)} \rho_+\right), \quad \alpha_{k,l}^{(i)} = \frac{\hat{\nu}}{1-\hat{\nu}} \gamma_{k,l}^{(i)} + \frac{E}{1-\hat{\nu}^2} b_{k,l-1}^{(i)},$$
$$b_{k,l}^{(1)} = \mathcal{B}\left(c_{k,l-1}^{(1)}(\xi_2) - \frac{\beta_{k,l}^{(1)}(\xi_2)}{a_2(\xi_2)}\right), \quad c_{k,l}^{(1)} = \mathcal{B}\left(\frac{\hat{\nu}}{1-\hat{\nu}} b_{k,l-1}^{(1)} - \frac{\gamma_{k,l}^{(1)}(\xi_2)}{a_1(\xi_2)}\right),$$
$$b_{k,l}^{(2)} = -\int_0^{\xi_2} \left(c_{k,l-1}^{(2)}(s) - \frac{\beta_{k,l}^{(2)}(s)}{a_2(s)}\right) ds, \quad c_{k,l}^{(2)} = -\int_0^{\xi_2} \left(\frac{\hat{\nu}}{1-\hat{\nu}} b_{k,l-1}^{(2)} - \frac{\gamma_{k,l}^{(2)}(s)}{a_1(s)}\right) ds\,.$$

For any function F we have $\mathcal{A}F(x)|_{x=1} = 0$, so for all i, k, l,

$$\beta_{k,l}^{(i)}(1) = \gamma_{k,l}^{(i)}(1) = 0,$$

and therefore condition $(1.1)_3$ is satisfied.

Let us calculate first values of coefficients:

$$h_{0,1}^{(1)} = m_{0,1}^{(1)} = p_{0,1}^{(1)} = q_{0,1}^{(1)} = \beta_{0,1}^{(1)} = \gamma_{0,1}^{(1)} = 0, \quad \alpha_{0,1}^{(1)} = \frac{E}{1-\hat{\nu}^2}\,,$$
$$b_{0,1}^{(1)} = \mathcal{B}(1) = \frac{1}{2} - \xi_2, \quad c_{0,1}^{(1)} = \mathcal{B}\left(\frac{\hat{\nu}}{1-\hat{\nu}}\right),$$

$$h_{0,0}^{(1)} = h_{0,1}^{(1)} = m_{0,0}^{(1)} = m_{0,1}^{(1)} = 0, \quad h_{0,2}^{(1)} \neq 0, \quad m_{0,2}^{(1)} = 0, \quad m_{2,0}^{(1)} = 1, \quad m_{0,0}^{(2)} = 1.$$

Note that the functions $\beta_{k,l}^{(i)}$, $\gamma_{k,l}^{(i)}$, $b_{k,l}^{(i)}$, $c_{k,l}^{(i)}$ are continuous, and therefore the conditions $(1.1)_{10}$, $(1.1)_{11}$ are satisfied (due to (2.1)).

Now, all coefficients $\mathcal{N}_{2k,2j}^{(i)}$ are defined and we pass to the equations for the terms of expansions of $\mathbf{w}^{(i)(J)}$ in powers of ε, i.e., for their components $y_k^{(i)}$ and $z_k^{(i)}$.

Equations for $y_k^{(i)}$ and $z_k^{(i)}$. In the same way as above, substituting expansion $\mathbf{w}^{(i)(J-2)} = \sum_{k=0}^{J-2} \varepsilon^k \mathbf{w}_k^{(1)}$ into (2.2), we get

$$\mathcal{P}_\varepsilon \mathbf{u}_+^{(J)} = \varepsilon^{-5} \sum_{i=1}^{2} \sum_{k,j\geqslant 0}^{5k+2j\leqslant J+2} \sum_{l\geqslant 0:\ 5k+2j+l-5<J-4}^{J-2+\delta_{i2}\cdot 4} \varepsilon^{5k+2j+l}$$
$$\times \begin{pmatrix} h_{2k,2j}^{(i)} D_{2k,2j} & \varepsilon h_{2k,2j+1}^{(i)} D_{2k,2j+1} \\ \varepsilon m_{2k,2j+1}^{(i)} D_{2k,2j+1} & m_{2k,2j}^{(i)} D_{2k,2j} \end{pmatrix} \mathbf{w}_l^{(i)} + \mathcal{S}^{(J)},$$

$$\mathcal{S}^{(J)} = \mathcal{R}^{(J)} + \sum_{i=1}^{2} \sum_{k,j\geqslant 0}^{5k+2j\leqslant J+2} \sum_{l\geqslant 0:\ 5k+2j+l-5\geqslant J-4}^{J-2+\delta_{i2}\cdot 4} \varepsilon^{5k+2j+l-5}$$
$$\times \begin{pmatrix} h_{2k,2j}^{(i)} D_{2k,2j} & \varepsilon h_{2k,2j+1}^{(i)} D_{2k,2j+1} \\ \varepsilon m_{2k,2j+1}^{(i)} D_{2k,2j+1} & m_{2k,2j}^{(i)} D_{2k,2j} \end{pmatrix} \mathbf{w}_l^{(i)}, \tag{2.4}$$

where $\mathcal{R}^{(J)}$ is defined in (2.3).

Equating $\mathcal{P}_\varepsilon \mathbf{u}_+^{(J)}$ to the right-hand side of $(1.1)_1$ and collecting together the terms of order ε^{l-2} for the first component of $\mathcal{P}_\varepsilon \mathbf{u}_+^{(J)}$ and of order ε^{l-1} for the second component, we get a recurrent relation for the functions $y_k^{(i)}$ and $z_k^{(i)}$ depending on the slow variable x_1:

$$\begin{cases} h_{0,2}^{(1)} D_{0,2} y_{l+1}^{(1)} + h_{0,3}^{(1)} D_{0,3} z_l^{(1)} + R_{l,1} = (\mathbf{f}^+)_1 \delta_{l-2,-1}, \\ m_{0,3}^{(1)} D_{0,3} y_{l+1}^{(1)} + m_{0,4}^{(1)} D_{0,4} z_l^{(1)} + z_{l+4}^{(2)} + R_{l,2} = (\mathbf{f}^+)_2 \delta_{l-1,-1}, \end{cases} \quad l = 0, \dots, J-2. \tag{2.5}$$

Here $R_{l,j}$ are some functions determined by the values of $(y_{i_1}^{(1)}, z_{i_2}^{(1)}, y_{i_3}^{(2)}, z_{i_4}^{(2)})$, $i_1 < l+1,\ i_2 < l,\ i_3, i_4 < l+4,\ j = 1, 2$, and their derivatives; in particular,

$$R_{0,1} = R_{0,2} = 0, \quad R_{1,1} = y_4^{(2)}, \quad R_{1,2} = D_{0,1} y_4^{(2)} + m_{2,0} D_{2,0} z_0^{(1)},$$

$$R_{l+1,1} = y_{l+4}^{(2)} + \varphi_l(y_{i_1}^{(1)}, z_{i_2}^{(1)}, y_{i_3}^{(2)}, z_{i_4}^{(2)}), \quad i_1 < l+2,\ i_2 < l+1,\ i_3, i_4 < l+4, \tag{2.6}$$

where $\varphi_l(y_{i_1}^{(1)}, z_{i_2}^{(1)}, y_{i_3}^{(2)}, z_{i_4}^{(2)})$ is an expression depending on $(y_{i_1}^{(1)}, z_{i_2}^{(1)}, y_{i_3}^{(2)}, z_{i_4}^{(2)})$ with $i_1 < l+2,\ i_2 < l+1,\ i_3, i_4 < l+4$. Here $w_{n1}^{(i)}$ and $w_{n2}^{(i)}$ are the components of vector $\mathbf{w}_n^{(i)}$.

Now let us define a constant Δ as follows:

$$\Delta = h_{0,2}^{(1)} m_{0,4}^{(1)} - m_{0,3}^{(1)} h_{0,3}^{(1)}.$$

A similar expression appeared in [7], denoted as follows:

$$h_{0,2}^{(1)} = -\hat{E}, \quad m_{0,4}^{(1)} = -\hat{J}, \quad h_{0,3}^{(1)} = -\hat{\hat{E}}, \quad m_{0,3}^{(1)} = -\hat{\hat{\hat{E}}}, \quad \frac{\Delta}{h_{0,2}^{(1)}} > 0.$$

According to [7], $\Delta \neq 0$.

Differentiating the first equation in (2.5) with respect to x_1 and subtracting the second equation multiplied by an appropriate factor, we get an equation for $z_l^{(1)}$, keeping for $y_{l+1}^{(1)}$ equation $(2.5)_1$:

(2.7)

$$\begin{cases} \Delta D_{0,4} z_l^{(1)} + h_{0,2}^{(1)}(z_{4+l}^{(2)} + R_{l,2} - (\mathbf{f}^+)_2 \delta_{l-1,-1}) - m_{0,3}^{(1)} \dfrac{d}{dx_1}(R_{l,1} - (\mathbf{f}^+)_1 \delta_{l-2,-1}) = 0 & \text{in } L_\varepsilon^+ \times (0,T), \\ h_{0,2}^{(1)} D_{0,2} y_{l+1}^{(1)} + h_{0,3}^{(1)} D_{0,3} z_l^{(1)} + R_{l,1} = (\mathbf{f}^+)_1 \delta_{l-2,-1} & \text{in } L_\varepsilon^+ \times (0,T), \\ \rho_-(x_2) \dfrac{\partial^2 \mathbf{v}_l}{\partial t^2} - \sum\limits_{i,j=1}^{2} \dfrac{\partial^2}{\partial t \partial x_i}\left(B_{ij}^-(x_2) \dfrac{\partial \mathbf{v}_l}{\partial x_j}\right) - \sum\limits_{i,j=1}^{2} \dfrac{\partial}{\partial x_i}\left(A_{ij}^-(x_2)\dfrac{\partial \mathbf{v}_l}{\partial x_j}\right) = \mathbf{f}^-(\mathbf{x},t)\delta_{l,0} & \text{in } L^- \times (0,T), \\ \mathbf{v}_l = \mathbf{0} & \text{at } \tilde{\Gamma}^- \times (0,T), \\ \mathbf{v}_l = \begin{pmatrix} y_l^{(1)} \\ z_l^{(1)} \end{pmatrix} + \sum\limits_{k,j,r:5k+2j+r=l;(k,j)\neq(0,0)} \mathcal{N}_{2k,2j}^{(1)}(0) \begin{pmatrix} y_r^{(1)} \\ z_r^{(1)} \end{pmatrix} & \text{at } \Gamma^0 \times (0,T), \\ \begin{pmatrix} y_{4+l}^{(2)} \\ z_{4+l}^{(2)} \end{pmatrix} = \sum\limits_{j=1}^{2} A_{2j}^- \dfrac{\partial \mathbf{v}_l}{\partial x_j} + \sum\limits_{j=1}^{2} B_{2j}^- \dfrac{\partial^2 \mathbf{v}_l}{\partial t \partial x_j} & \text{at } \tilde{\Gamma}^0 \times (0,T), \\ \mathbf{v}_l|_{t=0} = \dfrac{\partial \mathbf{v}_l}{\partial t}\Big|_{t=0} = \mathbf{0} & \text{in } L^-, \\ y_l^{(1)}, \ z_l^{(1)}, \ \mathbf{v}_l \text{ are 1-periodic in } x_1. \end{cases}$$

A necessary and sufficient condition of the existence of a periodic solution $y_{l+1}^{(1)}$ for the second equation in (2.7) is

$$\langle R_{l,1} - (\mathbf{f}^+)_1 \delta_{l-2,-1} \rangle_{x_1} = 0, \tag{2.8}$$

while the first equation together with the fifth will give a boundary condition for $(2.7)_3$.

Condition (2.8) can be presented in the form

$$\langle y_{l+4}^{(2)} + \varphi_l(y_{i_1}^{(1)}, z_{i_2}^{(1)}, y_{i_3}^{(2)}, z_{i_4}^{(2)}) - (\mathbf{f}^+)_1 \delta_{l,0} \rangle_{x_1} = 0. \tag{2.9}$$

Denote

$$\begin{pmatrix} \hat{y}_l^{(1)} \\ \hat{z}_l^{(1)} \end{pmatrix} = \sum_{k,j,r:5k+2j+r=l;(k,j)\neq(0,0)} \mathcal{N}_{2k,2j}^{(1)}(0) \begin{pmatrix} y_r^{(1)} \\ z_r^{(1)} \end{pmatrix}. \tag{2.10}$$

Then we obtain a problem for $\mathbf{v}_l$ coupled with the problem for $y_{l+1}^{(1)}$

(2.11)

$$\begin{cases} \rho_-(x_2) \dfrac{\partial^2 \mathbf{v}_l}{\partial t^2} - \sum\limits_{i,j=1}^{2} \dfrac{\partial^2}{\partial t \partial x_i}\left(B_{ij}^-(x_2) \dfrac{\partial \mathbf{v}_l}{\partial x_j}\right) - \sum\limits_{i,j=1}^{2} \dfrac{\partial}{\partial x_i}\left(A_{ij}^-(x_2)\dfrac{\partial \mathbf{v}_l}{\partial x_j}\right) = \mathbf{f}^-(\mathbf{x},t)\delta_{l,0} & \text{in } L^- \times (0,T), \\ \mathbf{v}_l = \mathbf{0} & \text{at } \Gamma^- \times (0,T), \\ (\mathbf{v}_l)_1 = y_l^{(1)}(x_1,t) + \hat{y}_l^{(1)}(x_1,t) & \text{at } \tilde{\Gamma}^0 \times (0,T), \\ \quad \Delta D_{0,4}(\mathbf{v}_l)_2 + h_{0,2}^{(1)} \left(\sum\limits_{j=1}^{2} A_{2j}^- \dfrac{\partial \mathbf{v}_l}{\partial x_j} + \sum\limits_{j=1}^{2} B_{2j}^- \dfrac{\partial^2 \mathbf{v}_l}{\partial t \partial x_j}\right)_2 & \text{at } \tilde{\Gamma}^0 \times (0,T), \\ \quad = \Delta D_{0,4} \hat{z}_l^{(1)} - h_{0,2}^{(1)}(R_{l,2} - (\mathbf{f}^+)_2 \delta_{l-1,-1}) + m_{0,3}^{(1)} \dfrac{d}{dx_1}(R_{l,1} - (\mathbf{f}^+)_1 \delta_{l-2,-1}) & \\ \begin{cases} \langle R_{l+1,1} - (\mathbf{f}^+)_1 \delta_{l-1,-1} \rangle_{x_1} = 0, \\ y_{4+l}^{(2)} = \left(\sum\limits_{j=1}^{2} A_{2j}^- \dfrac{\partial \mathbf{v}_l}{\partial x_j} + \sum\limits_{j=1}^{2} B_{2j}^- \dfrac{\partial^2 \mathbf{v}_l}{\partial t \partial x_j}\right)_1 \end{cases} & \text{at } \tilde{\Gamma}^0 \times (0,T), \\ \mathbf{v}_l|_{t=0} = \dfrac{\partial \mathbf{v}_l}{\partial t}\Big|_{t=0} = \mathbf{0} & \text{in } L^-, \\ \mathbf{v}_l \text{ is 1-periodic in } x_1 \end{cases}$$

and a problem for $y_{l+1}^{(1)}$

$$\begin{cases} h_{0,2}^{(1)} D_{0,2} y_{l+1}^{(1)} + h_{0,3}^{(1)} D_{0,3} z_l^{(1)} + R_{l,1} = (\mathbf{f}^+)_1 \delta_{l-2,-1}, \quad x_1 \in \mathbb{R}, t > 0, \\ y_{l+1}^{(1)} \text{ is 1-periodic in } x_1. \end{cases} \tag{2.12}$$

As we have noted, this problem admits a solution if and only if condition (2.9) is satisfied, and its solution is defined up to an additive function of t and can be decomposed as follows:

$$y_{l+1}^{(1)} = \tilde{y}_{l+1}^{(1)} + \langle y_{l+1}^{(1)} \rangle, \tag{2.13}$$

where

$$\langle \tilde{y}_{l+1}^{(1)} \rangle = 0. \tag{2.14}$$

In turn, applying the superposition principle, we can reduce problem (2.11) to two problems. The first one is

$$\begin{cases} \rho_-(x_2) \dfrac{\partial^2 \hat{\mathbf{v}}_l}{\partial t^2} - \sum\limits_{i,j=1}^{2} \dfrac{\partial^2}{\partial t \partial x_i} \left(B_{ij}^-(x_2) \dfrac{\partial \hat{\mathbf{v}}_l}{\partial x_j} \right) - \sum\limits_{i,j=1}^{2} \dfrac{\partial}{\partial x_i} \left(A_{ij}^-(x_2) \dfrac{\partial \hat{\mathbf{v}}_l}{\partial x_j} \right) = \mathbf{f}^-(\mathbf{x}, t) \delta_{l,0} & \text{in } L^- \times (0,T), \\ \hat{\mathbf{v}}_l = \mathbf{0} & \text{at } \Gamma^- \times (0,T), \\ (\hat{\mathbf{v}}_l)_1 = \tilde{y}_l^{(1)}(x_1, t) + \hat{y}_l^{(1)}(x_1, t) & \text{at } \tilde{\Gamma}^0 \times (0,T), \\ \quad \Delta D_{0,4} (\hat{\mathbf{v}}_l)_2 + h_{0,2}^{(1)} \left(\sum\limits_{j=1}^{2} A_{2j}^- \dfrac{\partial \hat{\mathbf{v}}_l}{\partial x_j} + \sum\limits_{j=1}^{2} B_{2j}^- \dfrac{\partial^2 \hat{\mathbf{v}}_l}{\partial t \partial x_j} \right)_2 & \text{at } \tilde{\Gamma}^0 \times (0,T), \\ \quad = \Delta D_{0,4} \hat{z}_l^{(1)} - h_{0,2}^{(1)} (R_{l,2} - (\mathbf{f}^+)_2 \delta_{l-1,-1}) + m_{0,3}^{(1)} \dfrac{d}{dx_1} (R_{l,1} - (\mathbf{f}^+)_1 \delta_{l-2,-1}) & \\ \hat{\mathbf{v}}_l|_{t=0} = \left.\dfrac{\partial \hat{\mathbf{v}}_l}{\partial t}\right|_{t=0} = \mathbf{0} & \text{in } L^-, \\ \hat{\mathbf{v}}_l \text{ is 1-periodic in } x_1, & \end{cases} \tag{2.15}$$

and the second is

$$\begin{cases} \rho_-(x_2) \frac{\partial^2 V_l}{\partial t^2} - \frac{\partial}{\partial x_2} \left(b_2^- \frac{\partial^2 V_l}{\partial t \partial x_2} + a_2^- \frac{\partial V_l}{\partial x_2} \right) = 0, & (x_2, t) \in (-1,0) \times (0,T), \\ V_l|_{x_2=-1} = 0, & t \in (0,T), \\ \left(b_2^- \frac{\partial^2 V_l}{\partial t \partial x_2} + a_2^- \frac{\partial V_l}{\partial x_2} \right)|_{x_2=0} = \theta_l(t), & t \in (0,T), \\ V_l|_{t=0} = \left.\frac{\partial V_l}{\partial t}\right|_{t=0} = 0, & x_2 \in (-1,0), \end{cases} \tag{2.16}$$

where

$$\theta_l(t) = \left.\left(\langle -\varphi_l(y_{i_1}^{(1)}, z_{i_2}^{(1)}, y_{i_3}^{(2)}, z_{i_4}^{(2)}) + (\mathbf{f}^+)_1 \delta_{l,0} \rangle_{x_1} - \left\langle b_2^- \frac{\partial^2 (\hat{\mathbf{v}}_l)_1}{\partial t \partial x_2} + a_2^- \frac{\partial (\hat{\mathbf{v}}_l)_1}{\partial x_2} \right\rangle_{x_1} \right)\right|_{x_2=0}.$$

The sum $\hat{\mathbf{v}}_l + \mathbf{V_l}$ with $\mathbf{V_l} = (V_l, 0)^T$ satisfies problem (2.11) if $\langle y_l^{(1)} \rangle$ is defined as $V_l(0,t)$.

Let us describe the algorithm of construction of an asymptotic expansion by recurrent determining of functions $\mathbf{v}_l$ and $(y_l^{(i)}), (z_l^{(i)})$. Initiating the induction by problem for $l = 0$ we solve first the couple of problems (2.15) and (2.16); for $l = 0$ these problems are (1.7) and (1.8). Define $\mathbf{v}_0$ as a sum of solutions of these two problems. Define $y_0^{(1)} = V(0,t)$, $z_0^{(1)} = \hat{\mathbf{v}}_2$, $(y_4^{(2)}, z_4^{(2)})^T = \left(\sum_{j=1}^2 A_{2j}^- \frac{\partial \mathbf{v}_0}{\partial x_j} + \sum_{j=1}^2 B_{2j}^- \frac{\partial^2 \mathbf{v}_0}{\partial t \partial x_j} \right)$.

Assume now that we have found all $\mathbf{v}_r, y_r^{(1)}, z_r^{(1)}, y_{r+4}^{(2)}, z_{r+4}^{(2)}$, $r < l$. Describe the step $r = l$. Solve problem

$$\begin{cases} h_{0,2}^{(1)} D_{0,2} \tilde{y}_l^{(1)} + h_{0,3}^{(1)} D_{0,3} z_{l-1}^{(1)} + R_{l-1,1} = (\mathbf{f}^+)_1 \delta_{l,1}, \quad x_1 \in \mathbb{R}, \, t > 0, \\ \tilde{y}_l^{(1)} \text{ is 1-periodic in } x_1, \langle \tilde{y}_l^{(1)} \rangle = 0. \end{cases} \tag{2.17}$$

Then we solve problems (2.15) and (2.16), define $\mathbf{v}_l$, and then define $\langle y_l^{(1)}\rangle = V_l(0,t)$, and so $y_l^{(1)} = \tilde{y}_l^{(1)} + V_l(0,t)$, $z_l^{(1)} = \hat{\mathbf{v}}_l - \hat{z}_l^{(1)}$, $(y_{l+4}^{(2)}, z_{l+4}^{(2)})^T = \big(\sum_{j=1}^2 A_{2j}^- \frac{\partial \mathbf{v}_l}{\partial x_j} + \sum_{j=1}^2 B_{2j}^- \frac{\partial^2 \mathbf{v}_l}{\partial t \partial x_j}\big)$. The step is finished.

Note that problem (1.8) can be solved by means of Fourier series after the change

$$V = \frac{x_2+1}{b_2^-}\int_0^t \theta(s)ds + U(t,x_2)e^{-\kappa t}, \quad \kappa = \frac{a_2^-}{b_2^-}.$$

In Appendix B we prove the following estimate for the difference of the exact solution and the asymptotic approximation:

$$\|\mathbf{u}^{(J)} - \mathbf{u}_\varepsilon\|_{H_T^1} = O(\varepsilon^{J-2}).$$

Theorem 1.3 is a corollary of this estimate.

3. Numerical evaluation of the error of the asymptotic approximation.

The goal of the section is to provide the numerical simulations for the original problem (1.1) and the asymptotic approximation (2.11) for $l = 0$ (which is equivalent to solving problem (1.7), (1.8)) and evaluate numerically the error between the exact solution and the asymptotic approximation. The finite element schemes use the weak formulation (A1.13) and a modified version of (A1.4), respectively. They use the $\mathbb{P}^3$-type conform finite elements for the space discretization and the Newmark method for the time discretization, generating an unconditionally stable second order implicit scheme, as explained in [11, 5]. These schemes are implemented with Freefem++ software [6].

3.1. Description of the numerical scheme. Let k be the time step, η be the space step, $\mathcal{T}_\eta^+$ be a triangulation in the upper domain D_ε^+, and $\mathcal{T}_\eta^-$ be a triangulation of the lower domain D^-. Let us denote $\mathbb{P}^3(\mathcal{T}_\eta^\pm)$ the continuous functions which are equal to polynomials of degree at most 3 on each triangle of the triangulation.

First, let us discretize the full problem (1.1). We consider the following finite element space:

$$\text{(3.1)} \quad \overline{\mathbb{V}}_\eta = \tilde{V} \cap \mathbb{P}^3(\mathcal{T}_\eta^+)^2 \times \mathbb{P}^3(\mathcal{T}_\eta^-)^2$$
$$= \{(\mathbf{u}_\eta^+, \mathbf{u}_\eta^-) \in \mathbb{P}^3(\mathcal{T}_\eta^+)^2 \times \mathbb{P}^3(\mathcal{T}_\eta^-)^2 | \quad \mathbf{u}_\eta^\pm \ x_1\text{-periodic}, \ u_\eta^+ = \mathbf{u}_\eta^- \text{ on } \Gamma^0, \ \mathbf{u}_\eta^- = 0 \text{ on } \Gamma^-\}.$$

We look for $(\mathbf{u}_\eta^{n+}, \mathbf{u}_\eta^{n-})_n \in \overline{\mathbb{V}}_\eta^{\mathbb{N}}$ such that, for all $(\omega_\eta^+, \omega_\eta^-) \in \mathbb{V}_\eta$,

$$\int_{D^+} \rho^+\Big(\frac{x_2}{\varepsilon}\Big)\frac{\mathbf{u}_\eta^{(n+1)+} - 2\mathbf{u}_\eta^{n+} + \mathbf{u}_\eta^{(n-1)+}}{k^2} \cdot \omega_\eta^+ + \varepsilon^{-3} I_{A^+}\Big(\frac{\mathbf{u}_\eta^{(n+1)+} + 2\mathbf{u}_\eta^{n+} + \mathbf{u}_\eta^{(n-1)+}}{4}, \omega_\eta^+\Big)_{D_\varepsilon^+}$$
$$+ \int_{D^-} \rho^-(x_2)\frac{\mathbf{u}_\eta^{(n+1)-} - 2\mathbf{u}_\eta^{n-} + \mathbf{u}_\eta^{(n-1)-}}{k^2} \cdot \omega_\eta^- + \mathcal{I}_{B^-}\Big(\frac{\mathbf{u}_\eta^{(n+1)-} - \mathbf{u}_\eta^{(n-1)-}}{2k}, \omega_\eta^-\Big)_{D^-}$$
$$+ \mathcal{I}_{A^-}\Big(\frac{\mathbf{u}_\eta^{(n+1)-} + 2\mathbf{u}_\eta^{n-} + \mathbf{u}_\eta^{(n-1)-}}{4}, \omega_\eta^-\Big)_{D^-}$$
$$= \varepsilon^{-1}\int_{D_\varepsilon^+} \mathbf{f}^+ \cdot \omega_\eta^+ + \int_{D^-} \mathbf{f}^- \cdot \omega_\eta^- + \int_{\Gamma^0} g^0 \cdot \omega^\pm + \int_{\Gamma_\varepsilon^+} g^+ \cdot \omega^+,$$
$$\mathbf{u}_\eta^{(0)\pm} = \mathbf{u}_\eta^{(-1)\pm} = 0.$$

Here, g^0, g^+ represent lineic forces on Γ^0 and Γ_ε^+ that are not present in problem (1.1) and the corresponding weak form (A1.13). However, we have added them in order to be able to build an exact test case for this system.

For the asymptotic system (1.7)–(1.8), we use the following finite element space:

(3.2)
$$\mathbb{V}_\eta = \mathbb{P}^3(\mathcal{T}_\eta^-)^2 \cap V + \mathbb{R}\begin{pmatrix} x_2+1 \\ 0 \end{pmatrix}$$
$$= \left\{ \omega_\eta \in \mathbb{P}^3(\mathcal{T}_\eta^-)^2 | \left(\frac{\partial \omega_\eta}{\partial x_1}\right)_1 = 0 \text{ on } \Gamma^0, \quad (\omega_\eta)_2(.,0) \in H^2_{per}(\Gamma^0), \quad \omega_\eta = 0 \text{ on } \Gamma^- \right\}.$$

We look for solutions $(\mathbf{v}_\eta^n)_n \in (\mathbb{V}_\eta)^{\mathbb{N}}$ such that, for all $\omega_\eta \in \mathbb{V}_\eta$,

$$\int_{D^-} \rho^- \frac{\mathbf{v}_\eta^{n+1} - 2\mathbf{v}_\eta^n + \mathbf{v}_\eta^{(n-1)}}{k^2} \cdot \omega_\eta + \mathcal{I}_{B^-}\left(\frac{\mathbf{v}_\eta^{n+1} - \mathbf{v}_\eta^{n-1}}{2k}, \omega_\eta\right)_{D^-}$$
$$+ \mathcal{I}_{A^-}\left(\frac{\mathbf{v}_\eta^{n+1} + 2\mathbf{v}_\eta^n + \mathbf{v}_\eta^{n-1}}{4}, \omega_\eta\right)_{D^-} + \mathcal{I}_0\left(\frac{\mathbf{v}_\eta^{n+1} + 2\mathbf{v}_\eta^n + \mathbf{v}_\eta^{n-1}}{4}, \omega_\eta\right)$$
$$= \int_{\Gamma^0} \mathbf{f}^+ \cdot \omega_\eta + \int_{D^-} \mathbf{f}^- \cdot \omega_\eta,$$
$$\mathbf{v}_\eta^0 = \mathbf{v}_\eta^{-1} = 0.$$

For the notation $\mathcal{I}_0$, see Appendix A. It is a discretized version of the following weak form satisfied by $\mathbf{v}_0 \in V + \mathbb{R}\left(\begin{smallmatrix} x_2+1 \\ 0 \end{smallmatrix}\right)$: for all $\omega \in V + \mathbb{R}\left(\begin{smallmatrix} x_2+1 \\ 0 \end{smallmatrix}\right)$,

(3.3)
$$\int_{D^-} \rho_-(x_2)\ddot{\mathbf{v}}_0 \cdot \omega + \mathcal{I}_{A^-}(\mathbf{v}_0, \omega)_{D^-} + \mathcal{I}_{B^-}(\dot{\mathbf{v}}_0, \omega)_{D^-} + \mathcal{I}_0(\mathbf{v}_0, \omega) = \int_{D^-} \mathbf{f}^- \cdot \omega + \int_{\Gamma^0} \mathbf{f}^+ \cdot \omega.$$

For this problem, a Lagrange multiplier method is used to implement the boundary conditions on Γ^0.

3.2. Numerical order of the method. The numerical accuracy of the method is tested by running a test case for various values of the time and space steps. More precisely, we take the following set of parameters:

	Thickness	Density	Elasticity	Viscosity
Upper layer	$\varepsilon = 0.1$	$\rho^+(\frac{x_2}{\varepsilon}) = 1 + x_2^2$	$\varepsilon^{-3}E^+(\frac{x_2}{\varepsilon}) = \frac{(8\frac{x_2}{\varepsilon}+35)(\frac{x_2}{\varepsilon}+10)}{3(\frac{x_2}{\varepsilon}+5)}$ $\nu^+(\frac{x_2}{\varepsilon}) = \frac{2\frac{x_2}{\varepsilon}+5}{6(\frac{x_2}{\varepsilon}+5)}$	None
Lower layer	1	$\rho^- = 1 + x_2^2$	$E^- = \frac{(4x_2+5)(x_2+2)}{3x_2+4}$ $\nu^- = \frac{x_2+1}{3x_2+4}$	$\kappa = 0.3$ $B_{ij}^- = \kappa A_{ij}^-$

We choose[1] the force $\mathbf{f}^\pm$ and lineic forces g^0, g^+ on Γ^0 and Γ_ε^+ such that $\mathbf{u}^\pm = \Xi$ with

$$\Xi = \begin{cases} \begin{pmatrix} \left(e^{-t^{-1}}\cos(2\pi x_1) + x_2 e^{-t^{-2}}\right)(1+x_2)e^{-2x_2^2} \\ \left(e^{-t^{-2}}\sin(2\pi x_1)^5 + \sin(2\pi x_1)e^{-3t^{-1}}x_2^2\right)(1+x_2)e^{-x_2^2} \end{pmatrix} & \text{if } t > 0, \\ \begin{pmatrix} 0 \\ 0 \end{pmatrix} & \text{if } t \le 0. \end{cases}$$

[1] For this particular test case, we allow $\mathbf{f}^+$ to be dependent on x_2.

TABLE 3.1

		Full				Asymptotic			
η	k	L^2-error	Order	H^1-error	Order	L^2-error	Order	H^1-error	Order
$\frac{1}{10}$	0.001	0.000510703		0.0122747		0.000691715		0.0240125	
$\frac{1}{20}$	0.001	2.74389e-05	4.22	0.00132727	3.21	3.6356e-05	4.25	0.00185208	3.70
$\frac{1}{30}$	0.001	5.45203e-06	3.99	0.000372288	3.14	7.24062e-06	3.98	0.000462021	3.42
$\frac{1}{40}$	0.001	1.72681e-06	4.00	0.000153048	3.09	2.29106e-06	4.00	0.000175482	3.37
$\frac{1}{60}$	0.001	3.44016e-07	3.98	4.4198e-05	3.06	4.4673e-07	4.03	4.61036e-05	3.30
$\frac{1}{80}$	0.001	1.17389e-07	3.74	1.84064e-05	3.04	1.40427e-07	4.02	1.81944e-05	3.23

TABLE 3.2

		Full				Asymptotic			
η	k	L^2-error	Order	H^1-error	Order	L^2-error	Order	H^1-error	Order
$\frac{1}{80}$	0.2	0.0049553		0.0863031		0.0283608		0.0586775	
$\frac{1}{80}$	0.1	0.000497457	3.32	0.00900011	3.26	0.00315082	3.17	0.00811027	2.85
$\frac{1}{80}$	0.05	0.00011747	2.08	0.00126194	2.83	0.000100676	4.97	0.000952858	3.09
$\frac{1}{80}$	0.04	7.47785e-05	2.02	0.00079248	2.08	7.53285e-05	1.30	0.000612418	1.98
$\frac{1}{80}$	0.02	1.86234e-05	2.01	0.000197224	2.01	1.69191e-05	2.15	0.00015023	2.03
$\frac{1}{80}$	0.01	4.65172e-06	2.00	5.1934e-05	1.93	4.19908e-06	2.01	4.01e-05	1.91
$\frac{1}{80}$	0.008	2.97781e-06	2.00	3.59255e-05	1.65	2.67665e-06	2.02	2.86519e-05	1.51
$\frac{1}{80}$	0.004	7.51514e-07	1.99	1.97811e-05	0.86	6.56719e-07	2.03	1.85818e-05	0.62
$\frac{1}{80}$	0.002	2.14857e-07	1.81	1.84552e-05	0.10	1.93902e-07	1.76	1.81321e-05	0.04
$\frac{1}{80}$	0.001	1.17389e-07	0.87	1.84064e-05	0.00	1.40427e-07	0.47	1.81944e-05	0.00

The same is done for the asymptotic problem. We take the same parameters, $\frac{1}{h} = \frac{\Delta}{h_{0,2}^{(1)}} = 4$, and choose $\mathbf{f}^+, \mathbf{f}^-$ such that $\mathbf{v}_0 = \Xi$.

We then compute the distance between the exact solution and the numerical solution for L^2- and H^1-norms at $t = 1$. We obtained the data presented below.

In Table 3.1, we give a numerical estimate of the space order of the method by taking the rate of increase of the logarithm of the errors with respect to the logarithm of the step. For both schemes, the order seems to be 3 for the H^1-norm and 4 for the L^2-norm.

In Table 3.2, we do the same for the time order. For both problems and for both norms, the estimated order is 2.

3.3. Comparison of the asymptotic model with the exact model. Several tests are used to check the accuracy of the asymptotic model. We take $\eta = \frac{1}{100}$ for the spatial resolution and the following parameters:

	Thickness	Density	Elasticity	Viscosity
Upper layer	$\varepsilon \in \{0.01, 0.02, 0.05, 0.07, 0.1, 0.2, 0.3\}$	$\rho^+ = 0.1$	$\varepsilon^{-3}E^+(\frac{x_2}{\varepsilon}) = \varepsilon^{-3}(1+\frac{x_2}{\varepsilon})$ $\nu^+(\frac{x_2}{\varepsilon}) = -0.3 + 0.2\frac{x_2}{\varepsilon}$	None
Lower layer	1	$\rho^- = 0.1$	$E^- = \frac{11}{40}$ $\nu^- = \frac{3}{8}$	$\kappa = 0.4$ $B^-_{ij} = \kappa A^-_{ij}$

Here we report two tests:

- one for the nonstationary case and
- one for the stationary case when the volumic forces are only present on the upper layer ($F^- = 0$), are vertical ($F_1^+ = 0$), and have zero mean.

For the first one, we take $k = \frac{1}{1000}$ for the time step,

$$\varepsilon^{-1}\mathbf{f}^+ = \varepsilon^{-1}\begin{pmatrix} \sin(2\pi x_1)g(t) \\ 20\sin^3(2\pi x_1)g(t) \end{pmatrix},$$

and

$$\mathbf{f}^- = \begin{pmatrix} 0.1\Big(x_2 + \Big(\frac{1+\cos(2\pi x_1)}{2}\Big)^4\Big)g(t) \\ 0.1\Big(x_2 + \Big(\frac{1+\cos(2\pi x_1)}{2}\Big)^4\Big)g(t) \end{pmatrix}$$

for the volumic forces, where

$$g(t) = \begin{cases} e^{-\frac{1}{t}} & \text{if } t > 0, \\ 0 & \text{if } t \leq 0. \end{cases}$$

We then compute the distance between the numerical solutions of the full problem and of the asymptotic approximation for the L^2- and H^1-norms at $t = 1$. The results are summarized in the table below.

These distances are expected to be proportional to some power of ε. In order to determine the corresponding exponent, we compute the rate of increase of the logarithm of these distances with respect to $\log(\varepsilon)$. The results are shown in the columns labeled by L^2-order and H^1-order.

ε	L^2-error	L^2-order	H^1-error	H^1-order
0.3	0.00632305	-	0.0672804	-
0.2	0.00351438	1.44856	0.0410133	1.22075
0.1	0.00160106	1.13424	0.0198294	1.04845
0.07	0.00112146	0.99820	0.0139279	0.99046
0.05	8.07795e-04	0.97505	0.0100064	0.98275
0.03	4.91754e-04	0.97162	0.00605542	0.98325
0.02	3.30933e-04	0.97681	0.00405831	0.98698
0.01	1.67271e-04	0.98435	0.00204132	0.99138

In Figure 1, the displacement is drawn for the full problem when $\varepsilon = 0.2, 0.02$ and for the asymptotic approximation.

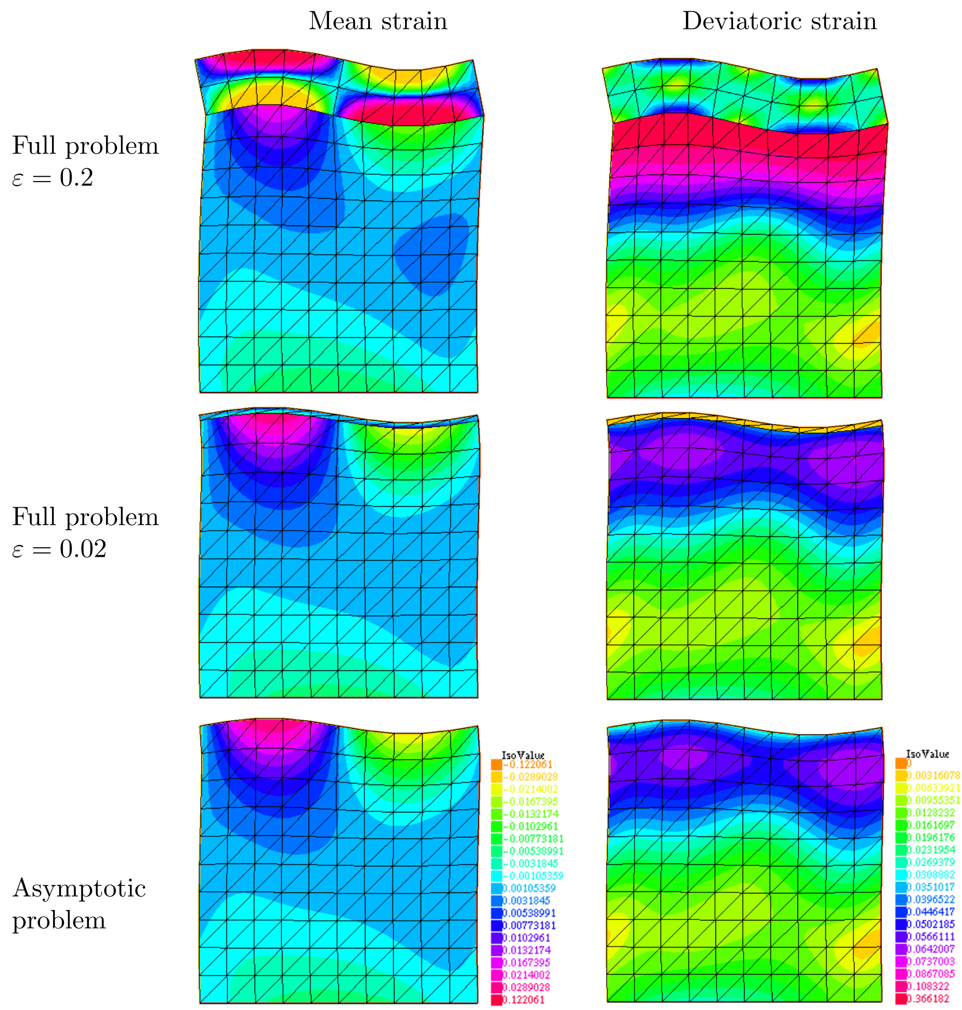

FIG. 1. *Results of the first test case of section* 3.3. *The plots above represent* (1) *a regular grid of D_ε shifted by the displacement found at $t = 1$ with the full model when $\varepsilon = 0.2$,* (2) *the same plot, but for $\varepsilon = 0.02$, and* (3) *a regular grid of D^- shifted by the displacement found at $t = 1$ with the asymptotic model. Each deformed grid is represented twice: the left one is colored (able to be seen online only) according to the mean strain and the right one according to the deviatoric strain. One notices that when ε is closer to zero, the solution is closer to the solution of the asymptotic problem.*

Since we find a numerical order equal to 1, we conclude that the distance between the full model and the asymptotic model decreases linearly with ε both in the L^2- and the H^1-norms.

For the second test, we test the relevance of the fourth order derivative in the fourth equation in problem (1.8). That is why we consider the stationary problem (i.e., we assume that the data and the solution are constant with respect to time) when no horizontal force is applied and when the average of vertical forces is zero.

We take

$$\varepsilon^{-1}\mathbf{f}^{+} = \varepsilon^{-1}\begin{pmatrix} 0 \\ 8\sin(2\pi x_1)^3 \end{pmatrix}$$

and

$$\mathbf{f}^{-} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

for the volumic forces.

We computed the distance between the numerical solution of the full problem and that of the asymptotic problem. The results are summarized below:

ε	L^2-error	L^2-order	H^1-error	H^1-order
0.3	0.00173782	-	0.0184646	-
0.2	0.000960346	1.46275	0.0112018	1.23261
0.1	0.000435691	1.14025	0.00540184	1.05221
0.07	0.000304926	1.00053	0.00379259	0.99163
0.05	0.000219549	0.97629	0.0027242	0.98336
0.03	0.000133611	0.97224	0.00164833	0.98353
0.02	8.99047e-05	0.97711	0.00110465	0.98710
0.01	4.54378e-05	0.98450	0.000555612	0.99144

Again, the distance between the solutions of the two problems decreases linearly with ε, both in the L^2- and the H^1-norms.

4. Conclusion. The two-dimensional model of the elastic plate–thick viscoelastic layer interaction can be reduced to the viscoelastic Kelvin–Voigt model with special boundary conditions replacing the plate. This dimension reduction gives a limit model, justified asymptotically and tested numerically. Theoretical analysis as well as numerical experiments show good approximation of the exact solution by the solution of the limit model.

Appendix A. Existence and uniqueness of a solution to the original problem and of a solution to the limit problem.

Denote the spaces

$$H^N = (H^N_{per}(D_\varepsilon))^2, \quad H^{N+} = (H^N_{per}(D^+_\varepsilon))^2, \quad H^{N-} = (H^N_{per}(D^-))^2,$$

$$H^N_{\Gamma^0} = (H^N_{per}((0,1)\times(0,T)))^2,$$

the norms

$$\|\nabla\mathbf{v}\|_{L^2(D^-)} = \sqrt{\left\|\frac{\partial(\mathbf{v})_1}{\partial x_1}\right\|^2_{L^2(D^-)} + \left\|\frac{\partial(\mathbf{v})_1}{\partial x_2}\right\|^2_{L^2(D^-)} + \left\|\frac{\partial(\mathbf{v})_2}{\partial x_1}\right\|^2_{L^2(D^-)} + \left\|\frac{\partial(\mathbf{v})_2}{\partial x_2}\right\|^2_{L^2(D^-)}},$$

$$\|D(\mathbf{v})\|_{L^2(D^-)} = \sqrt{\left\|\frac{\partial(\mathbf{v})_1}{\partial x_1}\right\|^2_{L^2(D^-)} + (1/2)\left\|\frac{\partial(\mathbf{v})_1}{\partial x_2} + \frac{\partial(\mathbf{v})_2}{\partial x_1}\right\|^2_{L^2(D^-)} + \left\|\frac{\partial(\mathbf{v})_2}{\partial x_2}\right\|^2_{L^2(D^-)}},$$

a bilinear form

$$\mathcal{I}_0(\mathbf{v},\omega) = \int_0^1 \frac{\Delta}{h_{0,2}^{(1)}} \frac{\partial^2(\mathbf{v})_2}{\partial x_1^2} \cdot \frac{\partial^2(\omega)_2}{\partial x_1^2}\bigg|_{x_2=0} dx_1,$$

and operators

$$l^+\mathbf{u}_+ = \sum_{j=1}^{2} A_{2j}^+\left(\frac{x_2}{\varepsilon}\right)\frac{\partial \mathbf{u}_+}{\partial x_j}, \quad l^-\mathbf{u}_- = \sum_{j=1}^{2} A_{2j}^-(x_2)\frac{\partial \mathbf{u}_-}{\partial x_j} + \sum_{j=1}^{2} B_{2j}^-(x_2)\frac{\partial^2 \mathbf{u}_-}{\partial t \partial x_j}.$$

Due to the coercivity conditions for the coefficients a_1^-, a_2^-, a_3^-, b_1^-, b_2^-, b_3^- and Korn's inequality, we get that there exists a positive constant ν independent of ε, such that for any $\mathbf{v} \in H^{1-}$, vanishing on Γ^-,

$$\text{(A1.1)} \qquad \mathcal{I}_{A^-}(\mathbf{v},\mathbf{v})_{D^-} \geqslant \nu\|\nabla \mathbf{v}\|_{L^2(D^-)}, \quad \mathcal{I}_{B^-}(\mathbf{v},\mathbf{v})_{D^-} \geqslant \nu\|\nabla \mathbf{v}\|_{L^2(D^-)}$$

and for any $\mathbf{v} \in H^1$, vanishing on Γ^-,

$$\text{(A1.2)} \qquad \mathcal{I}_{A^+}(\mathbf{v},\mathbf{v})_{D^+} + \mathcal{I}_{A^-}(\mathbf{v},\mathbf{v})_{D^-} \geqslant \nu\|\nabla \mathbf{v}\|_{L^2(D_\varepsilon)}.$$

Here ν is independent of ε.

Consider first the variational formulation for the limit problem.

Introduce the following spaces:

$$V = \left\{\omega \in H^{1-}: \ (\omega)_2(\cdot,0) \in H^2_{per}(0,1), \ \omega = \mathbf{0} \text{ at } \Gamma^-, \ (\omega)_1(\cdot,0) = 0\right\},$$

$$\|\omega\|_V = \|\omega\|_{H^{1-}} + \|(\omega)_2(\cdot,0)\|_{H^2(0,1)},$$

$$U = \left\{\mathbf{u} \in H_T^{1-}: \ \ \dot{\mathbf{u}} \in H_T^{1-}, \ \ddot{\mathbf{u}} \in H_T^{0-} \ \forall t \in [0,T], \ \mathbf{u}(\cdot,t) \in V\right\},$$

$$\|\mathbf{u}\|_U = \|\mathbf{u}\|_{H_T^{1-}} + \|\dot{\mathbf{u}}\|_{H_T^{1-}} + \|\ddot{\mathbf{u}}\|_{H_T^{0-}} + \text{ess sup}_{t\in(0,T)}\|(\mathbf{u})_2\|_{H^2(0,1)}.$$

Here $(\cdot)_2$ is the second component of a vector.

Let $\varphi \in H_T^1$, $\psi, \dot{\psi} \in H^0_{\Gamma^0}$. Consider the following problem:

$$\text{(A1.3)} \quad \begin{cases} \rho_- \dfrac{\partial^2 \mathbf{V}}{\partial t^2} - \sum\limits_{i,j=1}^{2} \dfrac{\partial^2}{\partial t \partial x_i}\left(B_{ij}^-(x_2)\dfrac{\partial \mathbf{V}}{\partial x_j}\right) - \sum\limits_{i,j=1}^{2} \dfrac{\partial}{\partial x_i}\left(A_{ij}^-(x_2)\dfrac{\partial \mathbf{V}}{\partial x_j}\right) = \varphi(\mathbf{x},t) & \text{in } L^- \times (0,T), \\ \mathbf{V} = \mathbf{0} & \text{at } \Gamma^- \times (0,T), \\ (\mathbf{V})_1 = 0 & \text{at } \Gamma^0 \times (0,T), \\ \dfrac{\partial^4 (\mathbf{V})_2}{\partial x_1^4} + \dfrac{h_{0,2}^{(1)}}{\Delta}\left(a_1^- \dfrac{\partial (\mathbf{V})_2}{\partial x_2} + b_1^- \dfrac{\partial^2 (\mathbf{V})_2}{\partial t \partial x_2}\right)\bigg|_{x_2=0} = \dfrac{h_{0,2}^{(1)}}{\Delta}\psi(x_1,t) & \text{at } \Gamma^0 \times (0,T), \\ \mathbf{V}|_{t=0} = \dfrac{\partial \mathbf{V}}{\partial t}\bigg|_{t=0} = \mathbf{0} & \text{in } L^-, \\ \mathbf{V} \text{ is 1-periodic in } x_1. \end{cases}$$

Define its weak solution as a function of the space U such that, for any $\omega \in V$,

(A1.4)

$$\int_{D^-} \rho_-(x_2)\ddot{\mathbf{v}} \cdot \omega + \mathcal{I}_{A^-}(\mathbf{v},\omega)_{D^-} + \mathcal{I}_{B^-}(\dot{\mathbf{v}},\omega)_{D^-} + \mathcal{I}_0(\mathbf{v},\omega) = \int_{D^-} \varphi \cdot \omega + \int_{\Gamma^0} \psi \cdot (\omega)_2.$$

THEOREM A.1. *Problem* (A1.3) *admits a unique weak solution* $\mathbf{v} \in U$, *and there exists a constant* c_5 *such that*

$$\text{(A1.5)} \qquad \|\mathbf{v}\|_U^2 \leqslant c_5\left(\|\varphi\|^2_{H_T^{0-}} + \|\dot{\varphi}\|^2_{H_T^{0-}} + \|\psi\|^2_{H^0_{\Gamma^0}} + \|\dot{\psi}\|^2_{H^0_{\Gamma^0}}\right).$$

Proof. Apply the Faedo–Galerkin method. Denote by $(\omega_j)_{j\in\mathbb{N}}$ an orthogonal basis of V, and for any fixed positive integer n introduce the Galerkin approximate problem which consists in finding a function $\mathbf{v}_n$ defined by

$$\mathbf{v}_n(\mathbf{x},t) = \sum_{l=1}^{n} a_l(t)\omega_l(\mathbf{x}),$$

where $(a_l(t))_{l\in\mathbb{N}}$ satisfy the following system of ordinary differential equations for $1 \leqslant j \leqslant n$:

(A1.6)

$$\begin{cases} \int\limits_{D^-} \rho_-(x_2)\ddot{\mathbf{v}}_n\cdot\omega_j + \mathcal{I}_{A^-}(\mathbf{v}_n,\omega_j)_{D^-} + \mathcal{I}_{B^-}(\dot{\mathbf{v}}_n,\omega_j)_{D^-} + \mathcal{I}_0(\mathbf{v}_n,\omega_j) = \int\limits_{D^-}\varphi\cdot\omega_j + \int\limits_{\Gamma^0}\psi\cdot(\omega_j)_2, \\ a_l(0) = \dot{a}_l(0) = 0. \end{cases}$$

Applying (1.4) we conclude the matrix with entries $(\int_{D^-}\rho_-(x_2)\,\omega_i\,\omega_j dx)_{1\leqslant i,j\leqslant n}$ is invertible, so the system (A1.6) admits a unique solution $\mathbf{v}_n$.

Derive next a priori estimates for $\mathbf{v}_n$. Multiply (A1.6) by $\dot{a}_j(t)$, and sum up all these problems for $j = 1,\dots,n$. We get

$$\int_{D^-}\rho_-(x_2)\ddot{\mathbf{v}}_n\cdot\dot{\mathbf{v}}_n + \mathcal{I}_{A^-}(\mathbf{v}_n,\dot{\mathbf{v}}_n)_{D^-} + \mathcal{I}_{B^-}(\dot{\mathbf{v}}_n,\dot{\mathbf{v}}_n)_{D^-} + \mathcal{I}_0(\mathbf{v}_n,\dot{\mathbf{v}}_n) = \int_{D^-}\varphi\cdot\dot{\mathbf{v}}_n + \int_{\Gamma^0}\psi\cdot(\dot{\mathbf{v}}_n)_2.$$

Since for any smooth function g defined in D^- the following relation holds,

$$g^2(x_1,0) = g^2(x_1,x_2) - \int_0^{x_2}\frac{\partial g^2}{\partial x_2}(x_1,s_2)ds_2,$$

we have

$$\|g\|^2_{L^2(\Gamma^0)} = \|g\|^2_{H^{0-}} - \int_0^1\int_0^1\int_0^{x_2}\frac{\partial g^2}{\partial x_2}(x_1,s_2)ds_2dx_1dx_2 \leqslant 2\|g\|^2_{H^{1-}},$$

$$\frac{1}{2}\frac{d}{dt}\left(\|\sqrt{\rho_-}\dot{\mathbf{v}}_n\|^2_{H^{0-}} + \mathcal{I}_{A^-}(\mathbf{v}_n,\mathbf{v}_n)_{D^-} + \mathcal{I}_0(\mathbf{v}_n,\mathbf{v}_n)\right) + \mathcal{I}_{B^-}(\dot{\mathbf{v}}_n,\dot{\mathbf{v}}_n)_{D^-} = \int_{D^-}\varphi\cdot\dot{\mathbf{v}}_n + \int_{\Gamma^0}\psi\cdot(\dot{\mathbf{v}}_n)_2$$

$$\leqslant \frac{\gamma_1}{2}\|\varphi\|^2_{H^{0-}} + \frac{1}{2\gamma_1}\|\dot{\mathbf{v}}_n\|^2_{H^{0-}} + \frac{\gamma_2}{2}\|\psi\|^2_{L^2(\Gamma^0)} + \frac{1}{\gamma_2}\|\dot{\mathbf{v}}_n\|^2_{H^{1-}}.$$

Integrating the last inequality from 0 to $t \leqslant T$, we get

$$\frac{1}{2}\|\sqrt{\rho_-}\dot{\mathbf{v}}_n\|^2_{H^{0-}} + \frac{1}{2}\mathcal{I}_{A^-}(\mathbf{v}_n,\mathbf{v}_n)_{D^-} + \frac{1}{2}\mathcal{I}_0(\mathbf{v}_n,\mathbf{v}_n) + \nu\|\nabla(\dot{\mathbf{v}}_n)\|^2_{H^{0-}_T}$$

$$\leqslant \frac{\gamma_1}{2}\|\varphi\|^2_{H^{0-}_T} + \frac{1}{2\gamma_1}\|\dot{\mathbf{v}}_n\|^2_{H^{0-}_T} + \frac{\gamma_2}{2}\|\psi\|^2_{L^2(\Gamma^0\times(0,T))} + \frac{1}{\gamma_2}\|\dot{\mathbf{v}}_n\|^2_{H^{1-}_T}.$$

Let $\gamma_1 = \frac{3T}{\kappa_-}$, $\gamma_2 = T\cdot\min\{\frac{6}{\kappa_-},\frac{2}{\nu}\}$, and integrate the last inequality from 0 to T, using (A1.1):

(A1.7)

$$\frac{\kappa_-}{6}\|\dot{\mathbf{v}}_n\|^2_{H^{0-}_T} + \frac{\nu}{2}\|\nabla(\mathbf{v}_n)\|^2_{H^{0-}_T} \leqslant \frac{1}{6}\|\sqrt{\rho_-}\dot{\mathbf{v}}_n\|^2_{H^{0-}_T} + \frac{\nu}{2}\|\nabla(\mathbf{v}_n)\|^2_{H^{0-}_T} \leqslant c_1(\|\varphi\|^2_{H^{0-}_T} + \|\psi\|^2_{H^0_{\Gamma^0}}),$$

(A1.8)
$$\int_0^T\mathcal{I}_0(\mathbf{v}_n,\mathbf{v}_n)dt + 2\nu\|\nabla(\dot{\mathbf{v}}_n)\|^2_{H^0_{\Gamma_0}} \leqslant c_2(\|\varphi\|^2_{H^{0-}_T} + \|\psi\|^2_{L^2(\Gamma^0\times(0,T))}).$$

Applying the Poincaré–Friedrichs inequality to $\mathbf{v}_n$, vanishing at $t = 0$, we evaluate its L^2-norm via $\|\dot{\mathbf{v}}_n\|^2_{H^{0-}_T}$ and then estimate (A1.7):

$$\|\mathbf{v}_n\|^2_{H^{0-}_T} \leqslant c_3(\|\varphi\|^2_{H^{0-}_T} + \|\psi\|^2_{H^0_{\Gamma^0}}). \tag{A1.9}$$

Consider now (A1.4) with $\dot{\varphi}$ instead of φ. Then (A1.7), (A1.8) give

$$\frac{\kappa_-}{2}\|\ddot{\mathbf{v}}_n\|^2_{H^{0-}_T} + \nu\|\nabla(\dot{\mathbf{v}}_n)\|^2_{H^{0-}_T} + \int_0^T \mathcal{I}_0(\mathbf{v}_n, \mathbf{v}_n)dt + 2\nu\|\nabla(\ddot{\mathbf{v}}_n)\|^2_{H^{0-}_T} \leqslant c_4(\|\dot{\varphi}\|^2_{H^{0-}_T} + \|\dot{\psi}\|^2_{H^0_{\Gamma^0}}). \tag{A1.10}$$

So

$$\|\mathbf{v}_n\|^2_U \leqslant c_5\left(\|\varphi\|^2_{H^{0-}_T} + \|\dot{\varphi}\|^2_{H^{0-}_T} + \|\psi\|^2_{H^0_{\Gamma^0}} + \|\dot{\psi}\|^2_{H^0_{\Gamma^0}}\right). \tag{A1.11}$$

We conclude that the sequence $(\mathbf{v}_n)_n$ is bounded in U. Therefore we can extract a subsequence $(\mathbf{v}_{n'})_{n'}$ such that it weakly converges in H^{1-}_T to some function $\mathbf{v} \in U$, so that for all functions $\mathbf{w} \in H^{1-}_T \cap H^2_{\Gamma^0}$,

$$\int_{D^-} \rho_-(x_2)\ddot{\mathbf{v}}_{n'} \cdot \mathbf{w} \to \int_{D^-} \rho_-(x_2)\ddot{\mathbf{v}} \cdot \mathbf{w},$$

$$\mathcal{I}_{A^-}(\mathbf{v}_{n'}, \mathbf{w})_{D^-} \to \mathcal{I}_{A^-}(\mathbf{v}, \mathbf{w})_{D^-},$$

$$\mathcal{I}_{B^-}(\dot{\mathbf{v}}_{n'}, \mathbf{w})_{D^-} \to \mathcal{I}_{B^-}(\dot{\mathbf{v}}, \mathbf{w})_{D^-},$$

$$\mathcal{I}_0(\mathbf{v}_{n'}, \mathbf{w}) \to \mathcal{I}_0(\mathbf{v}, \mathbf{w}),$$

and so $\mathbf{v}$ is a weak solution to (A1.3).

Thus we proved that there exists a function $\mathbf{v}$ belonging to U and satisfying the weak formulation (A1.4).

Now we prove the uniqueness of this solution. Let $\mathbf{v}$, $\tilde{\mathbf{v}}$ be two solutions of (A1.4). Then their difference $\mathbf{r} = \mathbf{v} - \tilde{\mathbf{v}}$ satisfies (A1.4) with zero right-hand side. Taking the test function $\omega = \mathbf{r}|_{t=t_0}$and integrating, then over t_0 we get that the following three terms are equal to zero:

$$\sup_{t\in(0,T)} \|r_2\|_{H^2} = 0, \quad \sup_{t\in(0,T)} \|\dot{\mathbf{r}}\|_{H^0} = 0, \quad \|\ddot{\mathbf{r}}\|_{H^0_T} = 0.$$

So, $\mathbf{r} = 0$ and the solution is unique. Now, letting n tend to infinity we can pass to the limit in (A1.11) and get the a priori estimate (A1.5) of the theorem. Theorem A.1 is proved. ∎

Consider now the initial problem (1.5) and prove the existence and uniqueness of its solution.

Recall the notations for the following spaces (1.5):

$$\tilde{V} = \left\{\omega \in (H^1_{per}(D_\varepsilon))^2 : \omega|_{\Gamma^-} = 0\right\},$$

$$\tilde{U} = \left\{\mathbf{u} \in H^1_T : \quad \ddot{\mathbf{u}}^+ \in H^{0+}_T, \quad \dot{\mathbf{u}}^- \in H^{1-}_T, \quad \mathbf{u}^-|_{\Gamma^-} = 0\right\},$$

$$\|\mathbf{u}\|_{\tilde{U}}^2 = \|\mathbf{u}^+\|_{H_T^{1+}}^2 + \|\ddot{\mathbf{u}}^+\|_{H_T^{0+}}^2 + \|\mathbf{u}^-\|_{H_T^{1-}}^2 + \|\dot{\mathbf{u}}^-\|_{H_T^{1-}}^2.$$

Let $\psi, \dot{\psi} \in H_T^{0+}$, $\varphi, \dot{\varphi} \in H_T^{0-}$. Consider for all $\varepsilon \in (0,1)$ the following problem:
(A1.12)

$$\begin{cases} \rho_+\left(\frac{x_2}{\varepsilon}\right)\frac{\partial^2 \mathbf{u}^+}{\partial t^2} - \varepsilon^{-3}\sum\limits_{i,j=1}^{2}\frac{\partial}{\partial x_i}\left(A_{ij}^+\left(\frac{x_2}{\varepsilon}\right)\frac{\partial \mathbf{u}^+}{\partial x_j}\right) = \psi(x_1,t,\varepsilon) & \text{in } L_\varepsilon^+ \times (0,T), \\ \rho_-(x_2)\frac{\partial^2 \mathbf{u}^-}{\partial t^2} - \sum\limits_{i,j=1}^{2}\frac{\partial^2}{\partial t\partial x_i}\left(B_{ij}^-(x_2)\frac{\partial \mathbf{u}^-}{\partial x_j}\right) & \\ \quad - \sum\limits_{i,j=1}^{2}\frac{\partial}{\partial x_i}\left(A_{ij}^-(x_2)\frac{\partial \mathbf{u}^-}{\partial x_j}\right) = \varphi(\mathbf{x},t,\varepsilon) & \text{in } L^- \times (0,T), \\ \sum\limits_{j=1}^{2} A_{2j}^+\frac{\partial \mathbf{u}^+}{\partial x_j} = \mathbf{0} & \text{at } \Gamma_\varepsilon^+ \times (0,T), \\ \mathbf{u}^- = \mathbf{0} & \text{at } \Gamma^- \times (0,T), \\ \mathbf{u}^+ = \mathbf{u}^- & \text{at } \tilde{\Gamma}^0 \times (0,T), \\ \varepsilon^{-3}\left(\sum\limits_{j=1}^{2} A_{2j}^+\frac{\partial \mathbf{u}^+}{\partial x_j}\right) = \sum\limits_{j=1}^{2} A_{2j}^-\frac{\partial \mathbf{u}^-}{\partial x_j} + \sum\limits_{j=1}^{2} B_{2j}^-\frac{\partial^2 \mathbf{u}^-}{\partial t\partial x_j} & \text{at } \tilde{\Gamma}^0 \times (0,T), \\ \mathbf{u}^+|_{t=0} = \left.\frac{\partial \mathbf{u}^+}{\partial t}\right|_{t=0} = \mathbf{0} & \text{in } L_\varepsilon^+, \\ \mathbf{u}^-|_{t=0} = \left.\frac{\partial \mathbf{u}^-}{\partial t}\right|_{t=0} = \mathbf{0} & \text{in } L^-, \\ \mathbf{u}^+,\ \mathbf{u}^- \text{are 1-periodic in } x_1. & \end{cases}$$

Define a weak solution to this problem (A1.12) as a function $\mathbf{u}_\varepsilon(\mathbf{x},t) \in \tilde{U}$ such that for any $\omega \in \tilde{V}$,

$$\text{(A1.13)} \qquad \begin{cases} \int\limits_{D_\varepsilon^+} \rho_+\left(\frac{x_2}{\varepsilon}\right)\ddot{\mathbf{u}}_\varepsilon^+ \cdot \omega^+ + \varepsilon^{-3}\mathcal{I}_{A^+}(\mathbf{u}_\varepsilon^+,\omega^+)_{D_\varepsilon^+} + \int\limits_{D^-} \rho_-(x_2)\ddot{\mathbf{u}}_\varepsilon^- \cdot \omega^- \\ +\mathcal{I}_{A^-}(\mathbf{u}_\varepsilon^-,\omega^-)_{D^-} + \mathcal{I}_{B^-}(\dot{\mathbf{u}}_\varepsilon^-,\omega^-)_{D^-} = \int\limits_{D_\varepsilon^+} \psi\cdot\omega^+ + \int\limits_{D^-} \varphi\cdot\omega^-, \\ \mathbf{u}_\varepsilon|_{t=0} = 0, \\ \dot{\mathbf{u}}_\varepsilon|_{t=0} = 0. \end{cases}$$

THEOREM A.2. *Problem* (A1.12) *admits a unique weak solution* $\mathbf{u}^\varepsilon \in \tilde{U}$, *and there exists a constant* M_4 *independent of* ε *such that*

$$\text{(A1.14)} \qquad \|\mathbf{u}_\varepsilon\|_{\tilde{U}}^2 \leqslant M_4\left(\|\psi\|_{H_T^{0+}}^2 + \|\dot{\psi}\|_{H_{\Gamma_0}^0}^2 + \|\varphi\|_{H_{\Gamma_0}^0}^2 + \|\dot{\varphi}\|_{H_T^{0-}}^2\right).$$

Proof. In order to prove the theorem we apply the Faedo–Galerkin method. Denote by $(\omega_j)_{j\in\mathbb{N}}$ an orthogonal basis of $\tilde{V}$, and for any fixed positive integer n introduce Galerkin's approximate problem which consists in finding a function $\mathbf{u}_n$ defined by

$$\mathbf{u}_n(\mathbf{x},t) = \sum_{l=1}^{n} a_l(t)\omega_l(\mathbf{x}),$$

where $(a_l(t))_{l\in\mathbb{N}}$ satisfy the following system of ordinary differential equations for

$1 \leqslant k \leqslant n$:

$$\begin{cases} \sum_{p=1}^{n} \left(\int\limits_{D_\varepsilon^+} \rho_+ \left(\frac{x_2}{\varepsilon}\right) \omega_p^+ \cdot \omega_k^+ \right) \ddot{a}_k + \varepsilon^{-3} \sum_{p=1}^{n} \left(\mathcal{I}_{A^+}(\omega_p^+, \omega_k^+)_{D_\varepsilon^+} \right) a_k \\ + \sum_{p=1}^{n} \left(\int\limits_{D^-} \rho_-(x_2) \omega_p^- \cdot \omega_k^- \right) \ddot{a}_k + \sum_{p=1}^{n} \left(\mathcal{I}_{A^-}(\omega_p^-, \omega_k^-)_{D^-} \right) a_k + \sum_{p=1}^{n} \left(\mathcal{I}_{B^-}(\omega_p^-, \omega_k^-)_{D^-} \right) \dot{a}_k \\ = \int\limits_{D_\varepsilon^+} \psi \cdot \omega^+ + \int\limits_{D^-} \varphi \cdot \omega^-, \quad k = 0, \dots, n, \\ a_k|_{t=0} = 0, \\ \dot{a}_k|_{t=0} = 0. \end{cases}$$

This system is solvable because the matrix

$$\left(\int_{D_\varepsilon^+} \rho_+ \left(\frac{x_2}{\varepsilon}\right) \omega_p^+ \cdot \omega_k^+ + \int_{D^-} \rho_-(x_2) \omega_p^- \cdot \omega_k^- \right)_{1 \leqslant p,k \leqslant n}$$

is nondegenerate.

Derive next a priori estimates for $\mathbf{u}_n$. Summing up (A1.13) multiplied by $\dot{a}_k(t)$ over $k = 1, \dots, n$ we get

$$\int_{D_\varepsilon^+} \rho_+ \left(\frac{x_2}{\varepsilon}\right) \ddot{\mathbf{u}}_n^+ \cdot \dot{\mathbf{u}}_n^+ + \varepsilon^{-3} \mathcal{I}_{A^+}(\mathbf{u}_n^+, \dot{\mathbf{u}}_n^+)_{D_\varepsilon^+} + \int_{D^-} \rho_-(x_2) \ddot{\mathbf{u}}_n^- \cdot \dot{\mathbf{u}}_n^- + \mathcal{I}_{A^-}(\mathbf{u}_n^-, \dot{\mathbf{u}}_n^-)_{D^-} + \mathcal{I}_{B^-}(\dot{\mathbf{u}}_n^-, \dot{\mathbf{u}}_n^-)_{D^-}$$

$$= \int_{D_\varepsilon^+} \psi \cdot \dot{\mathbf{u}}_n^+ + \int_{D^-} \varphi \cdot \dot{\mathbf{u}}_n^-,$$

so that

$$\frac{1}{2} \frac{d}{dt} \left(\|\sqrt{\rho_+} \dot{\mathbf{u}}_n^+\|_{H^{0+}}^2 + \|\sqrt{\rho_-} \dot{\mathbf{u}}_n^-\|_{H^0}^2 + \varepsilon^{-3} \mathcal{I}_{A^+}(\mathbf{u}_n^+, \mathbf{u}_n^+)_{D_\varepsilon^+} + \mathcal{I}_{A^-}(\mathbf{u}_n^-, \mathbf{u}_n^-)_{D^-} \right) + \mathcal{I}_B(\dot{\mathbf{u}}_n^-, \dot{\mathbf{u}}_n^-)_{D^-}$$

$$= \int_{D_\varepsilon^+} \psi \cdot \dot{\mathbf{u}}_n^+ + \int_{D^-} \varphi \cdot \dot{\mathbf{u}}_n^- \leqslant \frac{\gamma_1}{2} \|\psi\|_{H^{0+}}^2 + \frac{1}{2\gamma_1} \|\dot{\mathbf{u}}_n^+\|_{H^{0+}}^2 + \frac{\gamma_2}{2} \|\varphi\|_{H^{0-}}^2 + \frac{1}{2\gamma_2} \|\dot{\mathbf{u}}_n^-\|_{H^{0-}}^2.$$

Integrate the last inequality from 0 to $t \leqslant T$; taking into account (A1.2) we get

$$\|\sqrt{\rho_+} \dot{\mathbf{u}}_n^+\|_{H^{0+}}^2 + \|\sqrt{\rho_-} \dot{\mathbf{u}}_n^-\|_{H^{0-}}^2 + \varepsilon^{-3} \nu \|\nabla(\mathbf{u}_n^+)\|_{H^{0+}}^2 + \nu \|\nabla(\mathbf{u}_n^-)\|_{H^{0-}}^2 + 2\nu \|\nabla(\dot{\mathbf{u}}_n^-)\|_{H_T^{0-}}^2$$

$$\leqslant \gamma_1 \|\psi\|_{H_T^{0+}}^2 + \frac{1}{\gamma_1} \|\dot{\mathbf{u}}_n^+\|_{H_T^{0+}}^2 + \gamma_2 \|\varphi\|_{H_T^{0-}}^2 + \frac{1}{\gamma_2} \|\dot{\mathbf{u}}_n^-\|_{H_T^{0-}}^2.$$

Integrate the last inequality from 0 to T. Since (A1.2) for $\gamma_1 = \gamma_2 > \frac{2T}{\kappa_-}$, we have

$$\|\dot{\mathbf{u}}_n^+\|_{H_T^{0+}}^2 + \|\dot{\mathbf{u}}_n^-\|_{H_T^{0-}}^2 \leqslant M_1 (\|\psi\|_{H_{\Gamma_0}^0}^2 + \|\varphi\|_{H_T^{0-}}^2) \tag{A1.15}$$

and

$$\nu \|\nabla(\mathbf{u}_n^+)\|_{H_T^{0+}}^2 + \nu \|\nabla(\mathbf{u}_n^-)\|_{H_T^{0-}}^2 + 2\nu T \|\nabla(\dot{\mathbf{u}}_n^-)\|_{H_T^{0-}}^2 \leqslant M_2 (\|\psi\|_{H_{\Gamma_0}^0}^2 + \|\varphi\|_{H_T^{0-}}^2). \tag{A1.16}$$

Applying the Poincaré–Friedrichs inequality to $\mathbf{u}_n^-$ and $\mathbf{u}_n^+$ (vanishing at $t = 0$) we derive from (A1.15)

$$\|\mathbf{u}_n^+\|_{H_T^{0+}}^2 + \|\mathbf{u}_n^-\|_{H_T^0}^2 \leqslant M_3 (\|\psi\|_{H_{\Gamma_0}^0}^2 + \|\varphi\|_{H_T^0}^2). \tag{A1.17}$$

Consider (A1.13) with $\dot{\psi}$ and $\dot{\varphi}$ instead of ψ and φ. Then we get the same inequality as (A1.15):

$$\|\ddot{\mathbf{u}}_n^+\|^2_{H_T^{0+}} + \|\ddot{\mathbf{u}}_n^-\|^2_{H_T^{0-}} \leqslant M_1(\|\dot{\psi}\|^2_{H^0_{\Gamma_0}} + \|\dot{\varphi}\|^2_{H_T^{0-}}).$$

So

$$\|\mathbf{u}_n\|^2_{\tilde{U}} \leqslant M_4 \left(\|\psi\|^2_{H_T^{0+}} + \|\dot{\psi}\|^2_{H^0_{\Gamma_0}} + \|\varphi\|^2_{H_T^{0-}} + \|\dot{\varphi}\|^2_{H_T^{0-}} \right).$$

The next reasonings, including the existence proof, are similar to the proof of Theorem A.1. The proof is complete. ∎

Theorem 1.1 is a direct consequence of this theorem.

Appendix B. Residual estimates.

THEOREM B.1. *Let* $\mathbf{f}^-(\mathbf{x},t) \in H_T^{4J+\mu\ -}$, $\mathbf{f}^+(x_1,t) \in H_{\Gamma^0}^{4J+\mu}$, $\mu \geqslant 2$. *Then for all* j *such that* $0 \le j \le \min\{J/2, J-2\}$ *we have*

$$\mathbf{v}_j^- \in H_T^{4J-3j+\mu\ -}, \quad \mathbf{w}_j^{(1)} \in H_{\Gamma^0}^{4J-3j+\mu}, \quad \mathbf{w}_{4+j}^{(2)} \in H_{\Gamma^0}^{4J-3j+\mu-2}$$

and there exists a positive constant M *such that*

$$\text{(B2.1)}\qquad \begin{aligned} \|\mathbf{v}_j^-\|_{H_T^{4J-3j+\mu\ -}} &\leqslant M \left(\|\mathbf{f}^-\|_{H_T^{4J+\mu\ -}} + \|\mathbf{f}^+\|_{H^{4J+\mu}_{\Gamma_0}} \right), \\ \|\mathbf{w}_j^{(1)}\|_{H^{4J-3j+\mu}_{\Gamma_0}} &\leqslant M \left(\|\mathbf{f}^-\|_{H_T^{4J+\mu\ -}} + \|\mathbf{f}^+\|_{H^{4J+\mu}_{\Gamma_0}} \right), \\ \|\mathbf{w}_{4+j}^{(2)}\|_{H^{4J-3j+\mu-2}_{\Gamma_0}} &\leqslant M \left(\|\mathbf{f}^-\|_{H_T^{4J+\mu-2\ -}} + \|\mathbf{f}^+\|_{H^{4J+\mu}_{\Gamma_0}} \right). \end{aligned}$$

Proof. Indeed, applying the same reasoning as in [7] we get

$$\mathbf{v}_0^- \in H_T^{4J+\mu\ -}, \quad \|\mathbf{v}_0^-\|_{H_T^{4J+\mu\ -}} \leqslant M \left(\|\mathbf{f}^-\|_{H_T^{4J+\mu\ -}} + \|\mathbf{f}^+\|_{H^{4J+\mu}_{\Gamma_0}} \right).$$

Then we get $\mathbf{w}_0^{(1)}$ as the trace of $\mathbf{v}_0^-$ on Γ_0 and $\mathbf{w}_4^{(2)}$ as the trace of $l^-\mathbf{v}_0^-$ on Γ_0. So

$$\mathbf{w}_0^{(1)} \in H_{\Gamma^0}^{4J+\mu}, \quad \mathbf{w}_4^{(2)} \in H_{\Gamma^0}^{4J+\mu-2},$$

$$\|\mathbf{w}_0^{(1)}\|_{H^{4J+\mu}_{\Gamma_0}} \leqslant M \left(\|\mathbf{f}^-\|_{H_T^{4J+\mu\ -}} + \|\mathbf{f}^+\|_{H^{4J+\mu}_{\Gamma_0}} \right),$$

$$\|\mathbf{w}_4^{(2)}\|_{H^{4J+\mu-2}_{\Gamma_0}} \leqslant M \left(\|\mathbf{f}^-\|_{H_T^{4J+\mu-2\ -}} + \|\mathbf{f}^+\|_{H^{4J+\mu}_{\Gamma_0}} \right).$$

While solving the problem for $\mathbf{v}_1^-$ we have the derivative of $\mathbf{w}_4^{(2)}$ with respect to x_1 in the right-hand side of (2.7), so

$$\mathbf{v}_1^- \in H_T^{4J+\mu-3\ -}, \quad \|\mathbf{v}_1^-\|_{H_T^{4J+\mu-3\ -}} \leqslant M \left(\|\mathbf{f}^-\|_{H_T^{4J+\mu\ -}} + \|\mathbf{f}^+\|_{H^{4J+\mu}_{\Gamma_0}} \right)$$

and

$$\mathbf{w}_1^{(1)} \in H_{\Gamma^0}^{4J+\mu-3}, \quad \mathbf{w}_5^{(2)} \in H_{\Gamma^0}^{4J+\mu-5},$$

$$\|\mathbf{w}_1^{(1)}\|_{H^{4J+\mu-3}_{\Gamma_0}} \leqslant M \left(\|\mathbf{f}^-\|_{H_T^{4J+\mu\ -}} + \|\mathbf{f}^+\|_{H^{4J+\mu}_{\Gamma_0}} \right),$$

$$\|\mathbf{w}_5^{(2)}\|_{H^{4J+\mu-5}_{\Gamma_0}} \leqslant M \left(\|\mathbf{f}^-\|_{H_T^{4J+\mu-2\ -}} + \|\mathbf{f}^+\|_{H^{4J+\mu}_{\Gamma_0}} \right).$$

In the same way we get (B2.1) for $j \leqslant J-2$.

According to (2.3) the maximum number of derivatives of $\mathcal{R}^{(\mathcal{J})}$ is contained in the summand $D_{0,J+6}\mathbf{w}^{(2)(J-2)}$, $\mathbf{w}_{4+j}^{(2)} \in H_{\Gamma^0}^{4J-3j+\mu-2}$; for $j \leqslant J-2$ we have $4J-3j+\mu-2 \geqslant J+4+\mu \geqslant J+6$ for $\mu \geqslant 2$. ∎

Theorem 1.2 is a direct consequence of Theorems A.1 and B.1.

Remark B.1. So for $\mathbf{u}_+^{(J)}, \mathbf{u}_-^{(J)}$ we have the residual $\mathcal{S}^{(J)}$ (2.4) for $(1.1)_1$, which is estimated as

$$\|\mathcal{S}^{(J)}\|_{H_T^{\mu-2}{}^-} \leqslant M\varepsilon^{J-2}\left(\|\mathbf{f}^-\|_{H_T^{4J+\mu}{}^-} + \|\mathbf{f}^+\|_{H_{\Gamma_0}^{4J+\mu}}\right). \tag{B2.2}$$

The remaining conditions in (1.1) for $\mathbf{u}_+^{(J)}, \mathbf{u}_-^{(J)}$, including $(1.1)_{10}$ and $(1.1)_{11}$, are satisfied exactly.

THEOREM B.2. *The following estimate holds:*

$$\|\mathbf{u}^{(J)} - \mathbf{u}_\varepsilon\|_{H_T^1} = O(\varepsilon^{J-2}).$$

Proof. We substitute the difference $\mathbf{u}^{(J)} - \mathbf{u}_\varepsilon$ to the left-hand side operator of $(1.1)_1$ and get $\mathcal{S}^{(J)}$, which is estimated as (B2.2) due to Remark B.1. The operator $(1.1)_2$ of $\mathbf{u}^{(J)} - \mathbf{u}_\varepsilon$ is equal to zero since (2.7). Now we apply Theorem A.2 with right-hand sides $\psi = \mathcal{S}^{(J)}$ for the first equation and $\varphi = 0$ for the second. Note that

$$\|\dot{\mathcal{S}}^{(J)}\|_{H_T^{\mu-3}{}^-} \leqslant M\varepsilon^{J-2}\left(\|\mathbf{f}^-\|_{H_T^{4J+\mu}{}^-} + \|\mathbf{f}^+\|_{H_{\Gamma_0}^{4J+\mu}}\right).$$

So (A1.14) gives

$$\|\mathbf{u}^{(J)} - \mathbf{u}_\varepsilon\|_{H_T^1} \leqslant M_4 M\varepsilon^{J-2}\left(\|\mathbf{f}^-\|_{H_T^{4J+\mu}{}^-} + \|\mathbf{f}^+\|_{H_{\Gamma_0}^{4J+\mu}}\right),$$

which proves Theorem B.2 for $\mu \geqslant 4$. ∎

COROLLARY B.3. *The following estimates hold:*

$$\|\mathbf{u}_+ - \mathbf{v_0}|_{x_2=0}\|_{L^2(D_\varepsilon^+\times(0,T))} = O(\varepsilon\sqrt{\varepsilon}), \quad \|\mathbf{u}_- - \mathbf{v_0}\|_{H^1(D^-\times(0,T))} = O(\varepsilon).$$

Proof. The assertion of this statement follows from Theorem B.2, evident estimates

$$\|\mathbf{u}_+^{(J)} - \mathbf{v_0}|_{x_2=0}\|_{L^2(D_\varepsilon^+\times(0,T))} = O(\varepsilon\sqrt{\varepsilon}); \quad \|\mathbf{u}_-^{(J)} - \mathbf{v_0}\|_{H^1(D^-\times(0,T))} = O(\varepsilon)$$

for $J > 2$, and the triangle inequality. ∎

REFERENCES

[1] Z. ABDESSAMAD, I. KOSTIN, G. PANASENKO, AND V. P. SMYSHLYAEV, *Memory effect in homogenization of viscoelastic Kelvin-Voigt model with time dependent coefficients*, Math. Models Methods Appl. Sci., 19 (2009), pp. 1603–1630, https://doi.org/10.1142/S0218202509003905.

[2] D. L. ANDERSON, *Theory of the Earth*, Blackwell Scientific Publications, Boston, 1989.

[3] N. BAKHVALOV AND G. PANASENKO, *Homogenisation: Averaging Processes in Periodic Media*, Math. Appl. (Soviet Ser.) 36 , Kluwer, Dordrecht, London, Boston, 1989, https://doi.org/10.1007/978-94-009-2247-1.

[4] L. C. EVANS, *Partial Differential Equations*, Grad. Stud. Math. 19, AMS, Providence, RI, 1998, https://doi.org/10.1090/gsm/019.

[5] C. FUMIHIRO AND K. TAKASHI, *Newmark's method and discrete energy applied to resistive MHD equation*, Vietnam J. Math., 30 (2002), suppl., pp. 501–520.

[6] F. HECHT, *New development in FreeFem++*, J. Numer. Math., 20 (2012), pp. 251–265, https://doi.org/10.1515/jnum-2012-0013.

[7] I. MALAKHOVA-ZIABLOVA, G. PANASENKO, AND R. STAVRE, *Asymptotic analysis of a thin rigid stratified elastic plate viscous fluid interaction problem*, Appl. Anal., 95 (2016), pp. 1467–1506, https://doi.org/10.1080/00036811.2015.1132311.

[8] M. Meyers and K. Chawla, *Mechanical Behavior of Materials*, Cambridge University Press, Cambridge, New York, Melbourne, Madrid, Cape Town, Singapore, Sao Paolo, 2009, https://doi.org/10.1108/aeat.2009.12781bae.001.

[9] G. Panasenko and R. Stavre, *Asymptotic analysis of a viscous fluid-thin plate interaction: Periodic flow*, Math. Models Methods Appl. Sci., 24 (2014), pp. 1781–1822, http://doi.org/10.1142/S0218202514500079.

[10] G. Panasenko and R. Stavre, *Asymptotic analysis for the Kelvin-Voigt model for a thin laminate*, C.R. Mécanique, 343 (2015), pp. 365–370, https://doi.org/10.1016/j.crme.2015.04.001.

[11] P.-A. Raviart and J.-M. Thomas, *Introduction à l'analyse numérique des équations aux dérivees partielles*, Masson, Paris, 1983.