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Abstract

We are interested in finding and characterizing the symmetry planes of fuzzy objects in 3D space. We introduce first a fuzzy symmetry measure which defines an object symmetry degree with respect to a given plane. It is computed by measuring the similarity between the original object and its reflection. The choice of an appropriate measure of comparison is based on the desired properties. In a second part, a method for finding the best symmetry planes of fuzzy objects is proposed. We then apply these results to the representation of directional relationships.

Keywords: fuzzy object, plane symmetry, symmetry measure, measure of comparison

Notations :
$\mathcal{S}$ is the set containing objects (usually a subset of $\mathbb{R}^3$ or $\mathbb{Z}^3$). $|\mathcal{S}|$ is the cardinality of $\mathcal{S}$ if $\mathcal{S}$ is finite (typically a bounded subset of $\mathbb{Z}^3$). For any finite set X, $|X|$ denotes the cardinality of X.
$\mathcal{F}$ is the set of fuzzy subsets of $\mathcal{S}$.
Capital letters $A, B, \ldots$ denote fuzzy subsets.
μ_A is the membership function of A.
$supp(A)$ is the support of A.
$\top$, $\perp$ denote a t-norm and t-conorm, respectively.

1 Introduction

Symmetry is an important property of objects and also a useful feature for their recognition. However, exact symmetry does not exist in real objects and one has to deal with approximate symmetries. There is a vast literature in mathematics, image processing and computer vision domains dealing with different kinds of symmetry (central, reflection, rotation, skew) of shapes and images. Many works on inexact symmetries quantify the degree of symmetry, using a symmetry measure or distance (see, for example, [9, 10, 15, 18]). Most results on symmetry degree evaluation are obtained for precisely defined objects. Zabrodsky et al. [18] consider shapes with uncertainty, i.e. shapes for which the location of each point is given as a probability distribution.

In this paper we are interested in characterizing approximate symmetries of fuzzy 3D objects. To our knowledge, this problem has not been addressed before. Following a classical approach used for crisp shapes and images, we first introduce a fuzzy symmetry measure which characterizes an object symmetry degree with respect to a given plane (Section 2). For this we use a measure of comparison[1] between the object and its reflection. Various measures of comparison have been proposed in the literature for fuzzy sets. The choice of a measure that is appropriate to our problem is based on its properties (Section 3). In Sec-

[1]We prefer to use the expression "measure of comparison" as in [4] instead of "similarity measure", since different authors assume different properties for the notion of similarity measure.

tion 4, we present an algorithm for finding the best symmetry plane of an object. In Section 5, the results of this paper are applied to the representation of directional spatial relationships.

2 Symmetry measure

2.1 Reflection of a fuzzy object

Let Π be a plane in the 3D space $\mathbb{R}^3$. Given a point x, we denote by $e_\Pi(x)$ its image under the reflection with respect to Π. e_Π is a bijective transformation in $\mathbb{R}^3$. Therefore, one can define the reflection of a fuzzy set as follows.

Definition 2.1 *The reflection of a fuzzy set A is a fuzzy set $e_\Pi(A)$ defined as:*

$$\mu_{e_\Pi(A)}(e_\Pi(x)) = \mu_A(x) \quad \textit{for every } x \in \mathcal{S}.$$

We denote by $e_{\mathbf{u},d}$ the reflection with respect to a plane $\Pi_{\mathbf{u},d}$ which is orthogonal to $\mathbf{u}$ and passing at the signed distance d from the origin. In spherical coordinates a unit vector $\mathbf{u}$ is defined by two angles $\beta \in]-\pi, \pi]$ and $\alpha \in [-\pi/2, \pi/2]$ (see Fig. 1). As vectors $\mathbf{u}$ and $-\mathbf{u}$ define the same plane, we use $\beta \in [0, \pi[$, $\alpha \in]-\pi/2, \pi/2]$ and $d \in \mathbb{R}$.

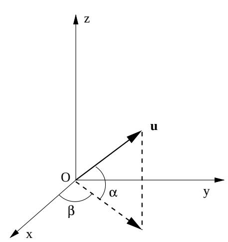


Figure 1: Angles α and β define a unit vector $\mathbf{u}$

We also use notation $e_{\alpha,\beta,d}$ instead of $e_{\mathbf{u},d}$ in 3D, and $e_{\beta,d} = e_{0,\beta,d}$ in 2D.

2.2 Symmetry measure

We want to define a symmetry degree of a fuzzy object with respect to a given plane Π. One option is to compare A and $e_\Pi(A)$. A symmetry measure σ_A can be defined as a measure of comparison between the original object and its reflection:

$$\sigma_A(\Pi) = S(A, e_\Pi(A)),$$

where S is a measure of comparison between fuzzy objects. As before, we use notations $\sigma_A(\mathbf{u}, d) = \sigma_A(\alpha, \beta, d) = \sigma_A(\Pi_{\mathbf{u},d})$.

Various measures of comparison have been proposed in the literature. They possess different properties and the choice of a measure depends on the application and of the concept one wants to describe. Below we discuss the measures that can be used to define a symmetry measure.

3 Deriving symmetry measures from measures of comparison

First we present properties that can be used to distinguish measures of comparison and discuss which of them should be satisfied by a symmetry measure. Then we summarize which of these properties hold for different measures of comparison proposed in the literature and select some of them to define symmetry measures.

3.1 Desired properties of symmetry measures

Measures of similitude

Bouchon-Meunier *et al.* [4] have proposed a classification of measures of comparison between fuzzy sets, in particular M-measures of comparison which are derived from a fuzzy measure M.

Definition 3.1 *[4] An M-measure of comparison is a mapping $S : \mathcal{F} \times \mathcal{F} \to [0, 1]$ such that $S(A, B) = F_S\Big(M(A \cap B), M(B - A), M(A - B)\Big)$ for a given mapping $F_S : \mathbb{R}^+ \times \mathbb{R}^+ \times \mathbb{R}^+ \to [0, 1]$.*

A particular class of measures of comparison is composed of measures of similitude.

Definition 3.2 *[4] An M-measure of similitude is an M-measure of comparison S such that $F_S(u, v, w)$ is non-decreasing in u, non-increasing in v and w.*

M-measures of similitude are well suited for describing symmetries: symmetry is stronger if the measure of intersection between the original object and its reflection increases, and it is weaker if the difference between them increases. Measures of similitude include measures of satisfiability and measures of resemblance.

Definition 3.3 *[4]*

1. *An M-mesure of satisfiability is an M-measure of similitude such that*
   - $F_S(u, v, w)$ *is independent of* w*;*
   - $F_S(0, v, w) = 0$*,* *for all* v*,* w *(exclusivity);*
   - $F_S(u, 0, w) = 1$*,* *for all* $u \neq 0$ *(inclusion).*
2. *An M-measure of resemblance is an M-measure of similitude such that*
   - S *is reflexive, i.e.* $S(A, A) = 1$*.*
   - S *is symmetrical,* *i.e.* $S(A, B) = S(B, A)$*.*

In our case $M(A - e_\Pi(A)) = M(e_\Pi(A) - A)$, therefore measures of satisfiability are also measures of resemblance [4]. Moreover, the exclusivity property entails that the symmetry measure is equal to zero when the plane passes outside the support of the object. The inclusion property, as well as reflexivity, entails that the symmetry degree is equal to 1 when the object coincides with its reflection, i.e. when there is an exact symmetry. The symmetry property implies that the symmetry measure for an object A with respect to a given plane Π is equal to the measure computed for its reflection $e_\Pi(A)$ [2]. Therefore, M-measures of satisfiability seem to be suitable for the definition of measures of symmetry.

Additionnal properties
Pappis [11] proposes the following additional properties which are in fact the reverse implications of reflexivity and exclusivity:

$$S(A, B) = 1 \iff A = B,$$

[2]Proofs are omitted here for the sake of conciseness

$$S(A, B) = 0 \iff supp(A) \cap supp(B) = \emptyset.$$

The first property, also called separability for distances, expresses that the symmetry measure is equal to 1 if and only if there is an exact symmetry. The second one expresses that the symmetry measure equals zero if and only if the plane passes outside the support of the object.

Geometrical properties
Intuitively speaking a symmetry measure should be invariant with respect to translation, rotation and scaling. If S is invariant w.r.t. translation (resp. rotation) then so is σ. This is also true for scaling but as the scaling of a fuzzy set in the discrete case is not clearly defined, we will not consider it later on.

Definition 3.4 *The symmetry measure of* A *with respect to* Π *is defined as:*

$$\sigma_A(\Pi) = S(A, e_\Pi(A)),$$

where S *is a measure of comparison with the following properties:*

(P1) *Symmetry:* $S(A, B) = S(B, A)$*;*

(P2) *Reflexivity:* $S(A, B) = 1 \iff A = B$*;*

(P3) $S(A, B) = 0$ *if and only if the supports of* A *and* B *are disjoint;*

(P4) S *is invariant w.r.t. translation;*

(P5) S *is invariant w.r.t. rotation.*

Other properties of measures of comparison considered for instance in [4, 8, 11] are either equivalent to these ones or not interesting for deriving symmetry measures.

3.2 Existing measures

We use here a classification of measures of comparison that is very similar to those used in [19] and [3].

3.2.1 Set-theoretic approach

Most of the measures discussed in this section have been derived from a general measure proposed by Tversky [16] and are based on combinations of μ_A and μ_B using t-norms and t-conorms. They satisfy **(P1)** as t-norms and t-conorms are commutative. The following measure has been used by several authors [4, 5, 11, 19] [3]:

$$S_1(A,B) = \frac{\sum_{x\in\mathcal{S}} \top(\mu_A(x), \mu_B(x))}{\sum_{x\in\mathcal{S}} \perp(\mu_A(x), \mu_B(x))}$$

Property **(P2)** holds if and only if $\top = \min$ and $\perp = \max$. Property **(P3)** is fulfilled for t-norms "minimum" and "product" but is not for "drastic" and "Lukasiewicz" ones [5]. Properties **(P4)** and **(P5)** are fulfilled.

Wang [17] proposed the following measure of comparison:

$$S_2(A,B) = \frac{1}{|\mathcal{S}|} \times \sum_{x\in\mathcal{S}} \frac{\top(\mu_A(x), \mu_B(x))}{\perp(\mu_A(x), \mu_B(x))}$$

with $\frac{0}{0} = 1$. S_2 satisfies **(P2)** if and only if $\top = \min$ and $\perp = \max$ but does not satisfy **(P3)**.

However, it is easy to check that a modified version of S_2 defined as follows:

$$S_3(A,B) = \frac{1}{|supp(A) \cup supp(B)|} \times \sum_{x\in supp(A)\cup supp(B)} \frac{\top(\mu_A(x), \mu_B(x))}{\perp(\mu_A(x), \mu_B(x))}$$

satisfies property **(P3)** for t-norms "minimum" and "product". Properties **(P4)** and **(P5)** are also fulfilled.

Hyung et al. [7] proposed to use a measure of comparison defined as

$$S_4(A,B) = \max_{x\in\mathcal{S}} \top(\mu_A(x), \mu_B(x)).$$

S_4 satisfies property **(P3)** for "minimum" and "product" t-norms but does not satisfy **(P2)**.

3.2.2 L_p distance approach

In this section we use the L_p distance between fuzzy sets A and B:

$$\|A-B\|_p = \left(\sum_{x\in\mathcal{S}} |\mu_A(x) - \mu_B(x)|^p\right)^{\frac{1}{p}}$$

$$\|A-B\|_\infty = \max_{x\in\mathcal{S}}(|\mu_A(x) - \mu_B(x)|).$$

Measures of comparison based on the L_p distance have the following general form:

$$S(A,B) = 1 - \frac{\|A-B\|_p}{K},$$

where K is a normalization coefficient. It is easy to see that properties **(P1)**, **(P2)**, **(P4)** and **(P5)** are fulfilled for measures of this type.

For example, Wang [17] and Bouchon-Meunier et al. [4] proposed the following measure

$$S_5(A,B) = 1 - \frac{\|A-B\|_1}{|\mathcal{S}|}.$$

This measure does not satisfy property **(P3)**.

The following measure of comparison proposed by Pappis [11]

$$S(A,B) = 1 - \frac{\sum_{x\in\mathcal{S}} |\mu_A(x) - \mu_B(x)|}{\sum_{x\in\mathcal{S}} [\mu_A(x) + \mu_B(x)]}$$

can be generalized as

$$S_6(A,B) = 1 - \frac{\|A-B\|_p}{\left(\sum_{x\in\mathcal{S}} \mu_A(x)^p + \mu_B(x)^p\right)^{\frac{1}{p}}}$$

Measure S_6 satisfies property **(P3)**.

Pappis [11] also proposed to use the L_∞ distance

$$S_7(A,B) = 1 - \|A-B\|_\infty$$

S_7 does not satisfy property **(P3)**. However, when A and B are normalized fuzzy sets and their supports are disjoint $S_7(A,B) = 0$. But the converse implication is still false.

It is easy to verify that the measure of comparison ($\beta > 0$)

$$S_8(A,B) = e^{-\beta\|A-B\|_p}$$

proposed in [4] does not satisfy property **(P3)** either.

3.2.3 Correlation coefficient

Gerstenkorn [6] introduced a correlation coefficient between fuzzy sets:

$$S_9(A,B) = \frac{C(A,B)}{\sqrt{T(A)T(B)}},$$

[3] Here we deal with the finite discrete case. In the continuous case, the sum is replaced by an integral if it converges.

where

$$C(A,B) = \sum_{x\in\mathcal{S}} [\mu_A(x)\mu_B(x) + (1-\mu_A(x))(1-\mu_B(x))]$$

and

$$T(A) = \sum_{x\in\mathcal{S}} \left[\mu_A(x)^2 + (1-\mu_A(x))^2\right].$$

Measure $S_9(A,B)$ satisfies properties **(P1)** and **(P2)** but does not satisfy **(P3)**.

3.3 Chosen symmetry measures

The measures of comparison S_1, S_3 (for t-norm "minimum") and S_6 satisfy properties **(P1)-(P5)**. Therefore we define three symmetry measures:

$$\sigma_{1,A}(\Pi) = \frac{\sum_{x\in\mathcal{S}} \min(\mu_A(x), \mu_{e_\Pi(A)}(x))}{\sum_{x\in\mathcal{S}} \max(\mu_A(x), \mu_{e_\Pi(A)}(x))}$$

$$\sigma_{2,A}(\Pi) = \frac{1}{|supp(A) \cup supp(e_\Pi(A))|} \times \sum_{x\in supp(A)\cup supp(e_\Pi(A))} \frac{\min(\mu_A(x), \mu_{e_\Pi(A)}(x))}{\max(\mu_A(x), \mu_{e_\Pi(A)}(x))}$$

$$\sigma_{3,A}(\Pi) = 1 - \frac{\|A - e_\Pi(A)\|_p}{\left(\sum_{x\in\mathcal{S}} \mu_A(x)^p + \mu_{e_\Pi(A)}(x)^p\right)^{\frac{1}{p}}}.$$

4 Finding symmetry planes of a fuzzy object

It is of interest to locate automatically symmetry planes of a given object. In this section we show how a symmetry measure can be used for this purpose.

4.1 Modes of the function σ_A

One has $\sigma_A(\Pi) = 1$ for an exact symmetry plane Π of a fuzzy set A. Let us say that a set A has a local symmetry plane $\Pi_{\alpha,\beta,d}$ if a symmetry measure σ_A has a local maximum in (α, β, d). Thus, to find the local symmetry planes of A one has to find the local maxima of σ_A. Figure 2 shows the shape of $\sigma_{1,A}$ for a synthetic 2D fuzzy object. This function has four modes which correspond to four axes of local symmetry: one axis of exact symmetry ($\beta = 0$ degrees, $d = 0$), two axes of strong but not exact symmetry ($\beta = 45$ or 135 degrees, $d = 0$) and one axis of a weak symmetry ($\beta = 90$ degrees, $d = 0$).

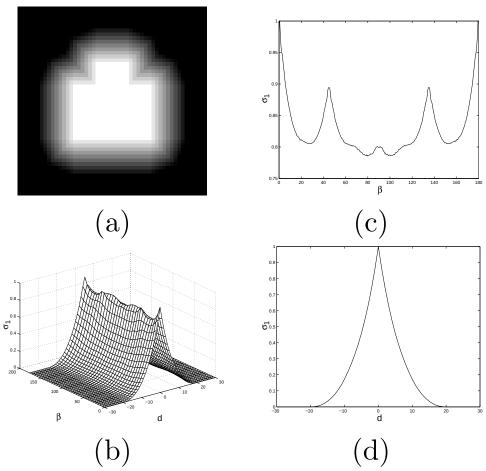


Figure 2: (a) A 2D fuzzy set A. (b) $\sigma_{1,A}$. (c) $\sigma_{1,A}$ for $d = 0$. (d) $\sigma_{1,A}$ for $\beta = 0$.

Figure 3 shows another 2D fuzzy object. In the direction defined by $\beta = 0$, the maximum of $\sigma_{1,A}$ is obtained for $d = 10$ which is the position of the symmetry plane of the alpha-cut of level 0.5. This result fits well with intuition.

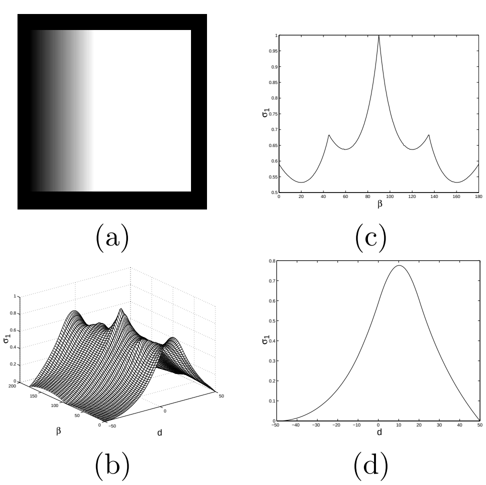


Figure 3: (a) A 2D fuzzy set A. (b) $\sigma_{1,A}$. (c) $\sigma_{1,A}$ for $d = 0$. (d) $\sigma_{1,A}$ for $\beta = 0$.

It can be useful to perform some preliminary smoothing on σ_A to exclude some non-significant local maxima. They can appear due to the object itself or due to artefacts generated by discretization. These artefacts can appear, for example, when steps on α or β become small. We do not study in this paper the influence of discretization on the symmetry measure.

4.2 Efficient search of symmetry planes

Whereas the computation of σ_A for a sufficiently small step is feasible in the 2D case, it is a far too expensive operation for 3D objects. In many cases, one only wants to locate the best symmetry plane of an object corresponding to the largest symmetry measure value. We propose a method that expresses the problem of finding the best symmetry plane as an optimization problem in the parametric space $]-\pi/2,\pi/2]\times[0,\pi[\times\mathbb{R}$.

The optimization procedure needs a starting point. We suggest to use the ellipsoid of inertia to define candidates for this starting point. The ellipsoid of inertia has already been used in [10] to define the symmetry plane of an object. Here it is only taken as an initialization. The directions of axes are defined as the eigenvectors of the covariation matrix:

$$\begin{pmatrix} m_{200} & m_{110} & m_{101} \\ m_{110} & m_{020} & m_{011} \\ m_{101} & m_{011} & m_{002} \end{pmatrix}$$

Here m_{pqr} defines a central moment of order $p+q+r$

$$m_{pqr}(A) = \sum_{\mathcal{S}} \mu_A(x,y,z)(x-x_c)^p(y-y_c)^q(z-z_c)^r,$$

where $\mathbf{c} = (x_c, y_c, z_c)$ is the object center of mass. If a 3D object possesses an exact plane of symmetry it passes through its center of mass and is orthogonal to one of the ellipsoid axes. Let us denote by $\mathbf{u}_1$, $\mathbf{u}_2$ and $\mathbf{u}_3$ the eigenvectors of the covariation matrix. We consider then three planes orthogonal to these vectors and passing through the center of mass: $\Pi_1 = \Pi_{\mathbf{u}_1,\mathbf{u}_1\cdot\mathbf{c}}$, $\Pi_2 = \Pi_{\mathbf{u}_2,\mathbf{u}_2\cdot\mathbf{c}}$ and $\Pi_3 = \Pi_{\mathbf{u}_3,\mathbf{u}_3\cdot\mathbf{c}}$. Our initial symmetry plane Π_i maximizes the symmetry measure, i.e. $\sigma_A(\Pi_i) = \max\{\sigma_A(\Pi_1), \sigma_A(\Pi_2), \sigma_A(\Pi_3)\}$. This is only possible when the eigenvectors are different. Otherwise, one gets an ellipsoid of revolution.

Then, the best symmetry plane is found using an optimization technique in the parametric space $]-\pi/2,\pi/2]\times[0,\pi[\times\mathbb{R}$. We use the *Nelder-Mead downhill simplex* method [13] which was also used in [1] but with a different initialization. This method is often used when one does not know the function derivatives. It is accurate and robust under a good starting point. However, it is a local optimization method and, in general, one has no guaranty to find the global maximum.

Figure 4 shows the symmetry plane found by this method. The image is a fuzzy segmentation of the lateral ventricles in an MR image. The symmetry measure $\sigma_{1,A}$ of the object with respect to this plane is 0.73. We also applied this method on gray-level images (but with a different symmetry measure) to compute the brain symmetry plane in MR images [14]. It has shown very good results.

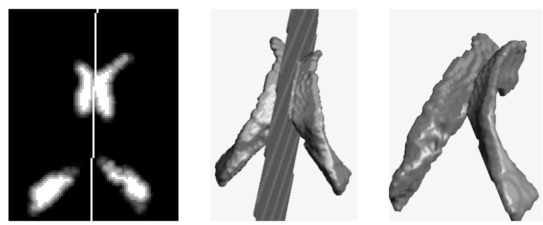


Figure 4: Left, one slice of a 3D object with its symmetry plane. Right, 3D renderings of the alpha-cut of level 0.5 of this object (with and without the symmetry plane).

5 Application to the representation of directional spatial relationships

Spatial relationships can be very useful for scene recognition and interpretation. Here we deal with directional relationships such as *to the left of*, *above* etc. There are basically two ways to define directional relationships: with respect either to an extrinsic frame or to an instrinsic frame which is defined by the reference object. The latter case often occurs when the reference object presents an approximate reflectional symmetry. The frame can then be partially defined by the symmetry plane of the reference object. For example, in the common cases of a human body or a car, relationships *left* and *right* refer to directions which are orthogonal to the symmetry plane of this object. A typical example can be found in neuroanatomy: the brain is an approximately symmetrical scene in which directions are defined with respect to an intrinsic frame. This frame is partially defined by the mid-saggital plane which approximately corresponds to the brain symmetry plane.

The results of this paper can be easily integrated in the representation proposed in [2]. Let us recall the principle of this method (the relationship is defined between a reference object R and a target object A):

1. Definition of a fuzzy landscape around the reference object. This landscape is a fuzzy subset of $\mathcal{S}$ such that the membership value of each point corresponds to the degree of satisfaction of the spatial relation.

2. Comparison of the object A to the fuzzy landscape. This is done using a fuzzy pattern matching approach.

It is shown that the fuzzy landscape associated to the direction defined by the unit vector $\mathbf{u}_{\alpha,\beta}$ ($\alpha \in]-\pi/2, \pi/2]$ and $\beta \in [0, 2\pi[$) corresponds to the fuzzy dilation of R by the following structuring element $B_{\alpha,\beta}$:

$$\mu_{B_{\alpha,\beta}}(x) = f(\arccos \frac{\mathbf{v}_x \cdot \mathbf{u}_{\alpha,\beta}}{\|\mathbf{v}_x\|}) \quad \text{for all } x \in \mathcal{S}, \text{ and } \mu_{B_{\alpha,\beta}}(o) = 1,$$

where f is a decreasing function on $[0, \pi]$ e.g. $f(\theta) = \max[0, 1-(2/\pi)\theta]$, o is the center of the structuring element and $\mathbf{v}_x$ is a vector from o to x.

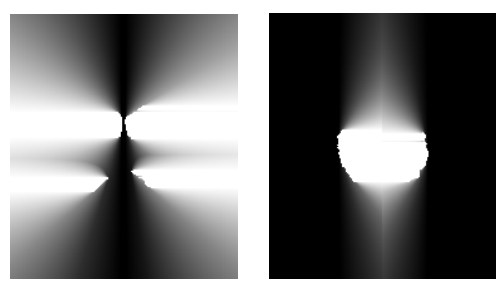
The algorithm presented in Section 4 can be used directly to define directional relationships. The plane orientation is defined by a normal vector $\mathbf{u}$ which can be used directly for the computation of the fuzzy landscape.

When not only the reference object but the whole scene is approximately reflection symmetrical, the scene is often better described by relationships *outside* and *inside* than *left* and *right*. Indeed, *left* and *right* would have to be duplicated to describe properly both halfs of the scene. In this context, *A is outside R* means that the right (resp. left) part of A is to the right (resp. left) of the right (resp. left) part of R (it has a directional meaning and not a topological one as it could have in other situations). Similarly, *A is inside R* means that the right (resp. left) part of A is to the left (resp. right) of the right (resp. left) part of R. Again, typical examples of such descriptions can be found in neuroanatomy e.g. *the caudate nuclei are outside the lateral ventricles*. Fuzzy landscapes corresponding to these relationships can be derived as follows. We denote by R_r (resp. R_l) the right (resp. left) part of R, by Π_r (resp. Π_l) the half-space to the right (resp. left) of the symmetry plane Π of R and by $L_r(R)$ (resp. $L_l(R)$) the fuzzy landscape representing the relationship *to the right (resp. left) of R*. Then the fuzzy landscapes $L_o(R)$ and $L_i(R)$ representing respectively the relationships *outside* and *inside* R can be defined as follows[4]:

$$L_o(R) = L_r(R_r) \cup L_l(R_l)$$

$$L_i(R) = (L_l(R_r) \cap \Pi_r) \cup (L_r(R_l) \cap \Pi_l)$$

Figure 5 shows an example of such fuzzy landscapes.



(a) (b)

Figure 5: Fuzzy landscapes for the relationships (a) *outside the lateral ventricles* and (b) *inside the putamen*.

6 Conclusion

In this paper, we studied approximate plane symmetries of fuzzy objects. Three symmetry measures were derived from measures of

[4] Unions and intersections are fuzzy

comparison. They show good properties. Using these measures, we also proposed an algorithm for finding the best symmetry plane of a 3D fuzzy object. This method was applied to the definition of spatial relationships in symmetrical scenes.

We are now working on the use of symmetry measures to represent imprecisely located symmetry planes as fuzzy subsets of the parametric space representing orientations and directions. It would also be interesting to use symmetry measures as features for scene recognition. They could be used, for example, to define symmetry attributes in a fuzzy attributed graph [12].

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