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Cauchy problem for the nonlinear Schrödinger equation coupled with the Maxwell equation

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Abstract

In this paper, we study the nonlinear Schrödinger equation coupled with the Maxwell equation. Using energy methods, we obtain a local existence result for the Cauchy problem.

Key words: Schrödinger-Maxwell system, Cauchy problem, symmetric hyperbolic system, energy method

2010 Mathematics Subject Classification. 35L45, 35Q60, 35L70

1 Introduction

In this paper, we consider the following nonlinear Schrödinger equation coupled with Maxwell equation stated in $\mathbb{R}_+ \times \mathbb{R}^3$:

\begin{align*}
    i\psi_t + \Delta \psi &= e\phi\psi + e^2 |A|^2\psi + 2ie\nabla \psi \cdot A + ie\psi \text{div} A - g(|\psi|^2)\psi, \quad (1.1) \\
    A_{tt} - \Delta A &= e \text{Im}(\bar{\psi} \nabla \psi) - e^2 |\psi|^2 A - \nabla \phi_t - \nabla \text{div} A, \quad (1.2) \\
    -\Delta \phi &= \frac{e}{2} |\psi|^2 + \text{div} A, \quad (1.3)
\end{align*}

where $\psi : \mathbb{R}_+ \times \mathbb{R}^3 \to \mathbb{C}$, $A : \mathbb{R}_+ \times \mathbb{R}^3 \to \mathbb{R}^3$, $\phi : \mathbb{R}_+ \times \mathbb{R}^3 \to \mathbb{R}$, $e \in \mathbb{R}$ and $i$ denotes the unit complex number, that is, $i^2 = -1$. In this setting, $\psi$ is an electrically charged field and $(\phi, A)$ represents a gauge potential of an electromagnetic field. System (1.1)-(1.3) describes the interaction of this Schrödinger wave function $\psi$ with the Maxwell gauge potential. The constant $e$ represents the strength of the interaction. For more details and physical backgrounds, we refer to [15].
Since we are interested in the Cauchy Problem, let us consider the following set of initial data:

\[ \psi(0, x) = \psi_0(x), \quad A(0, x) = A_0(x), \quad A_t(0, x) = A_1(x), \quad (1.4) \]

where the regularity of each function is given in Theorem 1.1. It is known that System (1.1)-(1.3) has a so-called gauge ambiguity. Namely if \((\psi, A, \phi)\) is a solution of (1.1)-(1.3), then \(\exp(ie\chi)\psi, A + \nabla \chi, \phi - \chi_t\) is also a solution of (1.1)-(1.3) for any smooth function \(\chi : \mathbb{R}_+ \times \mathbb{R}^3 \to \mathbb{R}\). To push out this ambiguity, we adopt in the sequel the Coulomb gauge:

\[ \text{div} A = 0, \quad (1.5) \]

which is propagated by the set of Equations (1.1)-(1.3). Indeed, if initially

\[ \text{div} A(0, \cdot) = \text{div} A_t(0, \cdot) = 0, \]

then (1.5) holds for all \(t > 0\). (See e.g. [12] for the proof.) In this setting, the last Equation (1.3) can be solved explicitly and the solution is given by

\[ \phi = \frac{e}{2} (-\Delta)^{-1} |\psi|^2, \]

which imposes that

\[ \phi(0, x) = \frac{e}{2} (-\Delta)^{-1} |\psi_0(x)|^2. \]

From (1.5), we also observe that (1.1) can be written as

\[ i \psi_t + L_A \psi - V(x) \psi + g(|\psi|^2) \psi = 0, \quad (1.6) \]

where \(V\) is the non-local potential: \(V(x) = \frac{e^2}{2} (-\Delta)^{-1} |\psi|^2\) and \(L_A\) is the magnetic Schrödinger operator which is defined by \(A = (A_1, A_2, A_3)\) and

\[ L_A \psi := \sum_{j=1}^3 \left( \frac{\partial}{\partial x_j} - ie A_j(x) \right)^2 \psi = \Delta \psi - 2ie \nabla \psi \cdot A - e^2 |A|^2 \psi. \quad (1.7) \]

In this context, the two conserved quantities of the Schrödinger-Maxwell system are the charge \(Q\) and the energy \(E\):

\[ Q(\psi) = \int_{\mathbb{R}^3} |\psi|^2 \, dx, \quad (1.8) \]
\[
E(\psi, A, \phi) = \frac{1}{2} \int_{\mathbb{R}^3} \left( |\nabla \psi - ieA\psi|^2 + |\nabla A|^2 + |\partial_t A|^2 \right) dx \\
+ \frac{e^2}{4} \int_{\mathbb{R}^3} \phi|\psi|^2 dx - \int_{\mathbb{R}^3} G(|\psi|^2) dx,
\]

where \( G(t) = \int_0^t g(s)ds \). To prove that \((1.8)\) is formally conserved, one has to multiply Equation \((1.1)\) by \( \bar{\psi} \), integrate over \( \mathbb{R}^3 \) and take the imaginary part of the resulting equation. In a similar way, the conservation of \((1.9)\) can be proved by multiplying \((1.1)-(1.3)\) by \( \partial_t \bar{\psi} \), \( \partial_t A \), and \( \partial_t \phi \) respectively.

This conserved quantities play a fundamental role if one wants to investigate the stability properties of such system, which is one of our main motivations. Indeed, in a previous paper \([13]\), we have showed that for small \( e > 0 \), System \((1.1)-(1.3)\) admits a unique orbitally stable ground state of the form:

\[
(\psi_{e,\omega}, A_{e,\omega}, \phi_{e,\omega}) := \left( \exp(\im \omega t)u_{e,\omega}, 0, \frac{e}{2} (-\Delta)^{-1}|u_{e,\omega}|^2 \right).
\]

In order to investigate the stability of such standing waves \((\psi_{e,\omega}, A_{e,\omega}, \phi_{e,\omega})\), it is necessary to prove that the Cauchy Problem \((1.1)-(1.3)\) is almost locally well-posed around \((\psi_{e,\omega}, A_{e,\omega}, \phi_{e,\omega})\).

In a previous paper \([12]\), we have proved the local existence of solutions for the nonlinear Klein-Gordon-Maxwell system in Sobolev spaces of high regularity. The method was to convert the Klein-Gordon-Maxwell system into a symmetric hyperbolic system and apply the standard energy estimate. Although our Schrödinger-Maxwell system \((1.1)-(1.3)\) looks similar, especially Equation \((1.2)\) is completely the same, the usual reduction tools does not lead us to a symmetric hyperbolic system, which causes the necessity of a new strategy.

Let us also introduce results concerning the solvability of the Cauchy problem related to \((1.1)-(1.3)\). In \([3]\), \([20]\), the linear Schrödinger equation \((g \equiv 0)\) coupled with the Maxwell equations has been studied. Using the Strichartz estimate, the authors obtained the global well-posedness in the energy space. However, it is not clear that their argument can be applied to the nonlinear case \( g \neq 0 \). We also mention the paper \([19]\), where the Cauchy problem of the Schrödinger-Maxwell system in the Lorentz gauge has been studied by using the energy method. On the other hand, a huge attention has been paid in the magnetic Schrödinger equation \((1.6)\). Especially in \([18]\), the local well-posedness for \((1.6)\) in the energy space has been established in the case \( V \equiv 0 \). However, in this situation, the magnetic potential \( A \) is given and was assumed to be \( C^\infty \), which cannot be expected a priori in our case. We also refer to \([14]\) for the Strichartz estimate for the magnetic Schrödinger operator \((1.7)\) in the case \( A \in L^2_{loc}(\mathbb{R}^3) \).
We mention that if we look for the standing wave (1.10), we are led to the following non-local elliptic problem:

\[-\Delta u + \omega u + \left( \frac{e^{2}}{8\pi |x|} \right) * |u|^2 u = g(|u|^2)u \quad \text{in } \mathbb{R}^3, \]

(1.11)

which is referred as the Schrödinger-Poisson(-Slater) equation. The existence of ground states of (1.11) as well as their orbital stability have been widely studied (see [2], [5], [4], [7], [17] and references therein). Finally, the orbital stability of standing waves for the magnetic Schrödinger equation (1.6) has been considered in [8], [16]. Our study on the solvability of the Cauchy problem for (1.1)-(1.3) and the result established in [13] enable us to generalize these previous results to the full Schrödinger-Maxwell system.

Before stating the main result of this paper, we introduce the following notations. As usual, \(L^p(\mathbb{R}^3)\) denotes the usual Lebesgue space:

\[ L^p(\mathbb{R}^3) = \left\{ u \in S'(\mathbb{R}^3) : \|u\|_{L^p} < +\infty \right\}, \]

where

\[ \|u\|_{L^p} = \left( \int_{\mathbb{R}^3} |u(x)|^p \, dx \right)^{\frac{1}{p}} \text{ if } 1 \leq p < +\infty \]

and

\[ \|u\|_{L^\infty} = \text{ess sup} \left\{ |u(x)| : x \in \mathbb{R}^3 \right\}. \]

We define the Sobolev space \(H^s(\mathbb{R}^3)\) as follows:

\[ H^s(\mathbb{R}^3) = \left\{ u \in S'(\mathbb{R}^3) : \|u\|_{H^s(\mathbb{R}^3)} = \int_{\mathbb{R}^3} (1 + |\xi|^2)^s |\mathcal{F}(u)(\xi)|^2 \, d\xi < +\infty \right\}, \]

where \(\mathcal{F}(u)(\xi)\) is the Fourier transform of \(u\). We also introduce the homogeneous Sobolev space \(\dot{H}^1(\mathbb{R}^3)\) as being the completion of \(C_0^\infty(\mathbb{R}^3)\) for the norm \(u \rightarrow \|\xi|\mathcal{F}(u)(\xi)\|_{L^2(\mathbb{R}^3)}\). Recall that the space \(\dot{H}^1(\mathbb{R}^3)\) is continuously embedded into \(L^6(\mathbb{R}^3)\). Finally let \(C(I, E)\) be the space of continuous functions from an interval \(I\) of \(\mathbb{R}\) to a Banach space \(E\). For \(1 \leq j \leq 3\), we set \(\partial_{x_j} = \frac{\partial}{\partial x_j}\) and \(\partial_l = \frac{\partial}{\partial l}\). For \(k \in \mathbb{N}^3\), \(k = (k_1, k_2, k_3)\), we denote \(D^k u = \partial_{x_1}^{k_1} \partial_{x_2}^{k_2} \partial_{x_3}^{k_3} u\) and for a non-negative integer \(s\), \(D^s\) denotes the set of all partial space derivatives of order \(s\). Different positive constants might be denoted by the same letter \(C\). We also denote by \(\text{Re}(u)\) and \(\text{Im}(u)\) the real part and the imaginary part of \(u\) respectively.

We assume that \(g\) satisfies

\[ g \in C^{m+1}(\mathbb{R}, \mathbb{R}) \quad \text{and} \quad g(0) = 0, \]

(1.12)
for some $m \in \mathbb{N}$ with $m \geq 2$, so that the function $W : \mathbb{C} \to \mathbb{C}$ defined by $W(u) := g(|u|^2)u$ satisfies $W \in C^{m+1}(\mathbb{C}, \mathbb{C})$, $W(0) = W'(0) = 0$. Some typical examples of the nonlinear term $g$ are the power nonlinearity $g(s) = \pm s^{p-1/2}$ with $[p] \geq 2m + 3$ ($[p]$ denotes the integer part of $p$), or the cubic-quintic nonlinearity $g(s) = s - \lambda s^2$ for $\lambda > 0$, which frequently appears in the study of solitons in physical literatures. (See [22] for example.) In this setting, we prove the following result.

**Theorem 1.1.** Let $s$ be any integer larger than $\frac{3}{2}$ and assume that $\psi_0 \in H^{s+2}(\mathbb{R}^3, \mathbb{C})$, $A(0) \in H^{s+2}(\mathbb{R}^3, \mathbb{R}^3)$, $A(1) \in H^{s+1}(\mathbb{R}^3, \mathbb{R}^3)$ with $\text{div}A(0) = 0$, $\text{div}A(1) = 0$ and $g$ satisfies (1.12). Then there exist $T^* > 0$ and a unique solution $(\psi, A, \phi)$ to System (1.1)-(1.3) satisfying the initial condition (1.4) such that

$$
\psi \in C([0, T^*]; H^{s+2}(\mathbb{R}^3)) \cap C^1([0, T^*]; H^s(\mathbb{R}^3)),
$$

$$
A \in C([0, T^*]; H^{s+2}(\mathbb{R}^3)) \cap C^1([0, T^*]; H^{s+1}(\mathbb{R}^3)),
$$

$$
\phi \in C([0, T^*]; H^1(\mathbb{R}^3) \cap L^\infty(\mathbb{R}^3)), \quad \nabla \phi \in C([0, T^*]; H^{s+1}(\mathbb{R}^3)),
$$

$$
\phi_t \in C([0, T^*]; H^1(\mathbb{R}^3) \cap L^\infty(\mathbb{R}^3)), \quad \nabla \phi_t \in C([0, T^*]; H^{s+1}(\mathbb{R}^3)).
$$

The proof of Theorem 1.1 is based on energy estimates and particularly, on the strategies developed in [9] and [10]. Note also that to overcome the loss of derivatives embedded in Equation (1.2), we use the original idea of Ozawa and Tsutsumi presented in [21].

The paper is organized as follows. In Section 2, we transform System (1.1)-(1.3) into a system to which we can apply the usual energy method. Section 3 is devoted to the proof of Theorem 1.1.

## 2 Transformation of the equations

In this section, we transform the original System (1.1)-(1.3) into a new symmetric system to which we can apply an energy method. In order to overcome the loss of derivatives contained in Equations (1.1)-(1.2), we introduce the following new unknowns (see [21]):

$$
\Psi = \partial_t \psi \quad \text{and} \quad \Phi = \partial_t \phi.
$$

Let us first derive equations for $\Psi$ and $\Phi$. Differentiating Equation (1.1) with respect to $t$, one obtains

$$
i \Psi_t + \Delta \Psi = e\Phi \psi + e\phi \Psi + e^2|A|^2 \Psi + 2e^2 \psi A \cdot A_t + 2ie \nabla \psi \cdot A + 2ie \nabla \psi \cdot A_t - g'(\psi^2)(\psi^2 \Psi + \psi^2 \Psi) - g(|\psi|^2) \Psi.
$$
Taking advantage of the new unknown $\Psi$, we also transform Equation (1.1) into an elliptic version

$$i\Psi + \Delta \psi = e\phi \psi + e^2|A|^2\psi + 2ie\nabla \psi \cdot A - g(|\psi|^2)\psi.$$ 

Moreover, we derive an equation for $\Phi$ by applying $\partial_t$ on Equation (1.3):

$$-\Delta \Phi = \frac{e}{2}(\overline{\psi} \overline{\psi} + \Psi \overline{\Psi}).$$

Next, in order to ensure the Coulomb condition on $A$ for all $t > 0$, we introduce the projection operator $\mathbb{P}$ on divergence free vector fields:

$$\mathbb{P} : \left(L^2(\mathbb{R}^3)\right)^3 \rightarrow \left(L^2(\mathbb{R}^3)\right)^3,$$

so that if $\text{div} A = 0$, then $\mathbb{P} A = A$. Thus applying $\mathbb{P}$ on Equation (1.2), we derive

$$A_{tt} - \Delta A = \mathbb{P}\left(e \text{ Im}(\overline{\psi} \nabla \psi) - e^2|\psi|^2 A - \nabla \Phi\right). \quad (2.1)$$

Note that any solution to (2.1) satisfying

$$\text{div} A(0, \cdot) = 0 \quad \text{and} \quad \text{div} A_t(0, \cdot) = 0,$$

obviously satisfies

$$\text{div} A(t, \cdot) = 0 \quad \text{for all} \quad t > 0.$$

At this step, we have transformed System (1.1)-(1.3) into

\begin{align*}
 i\Psi + \Delta \psi & = e\phi \psi + e^2|A|^2\psi + 2ie\nabla \psi \cdot A - g(|\psi|^2)\psi, \quad (2.2) \\
 i\Psi_t + e\Phi \psi + e\phi \Psi + e^2|A|^2\Psi + 2e^2\psi A \cdot A_t + 2ie\nabla \Psi \cdot A \\
 & \quad + 2ie\nabla \psi \cdot A_t - g'(||\psi||^2)(||\psi||^2 \Psi + \psi^2 \overline{\Psi}) - g(||\psi||^2)\psi, \quad (2.3) \\
 A_{tt} - \Delta A & = \mathbb{P}\left(e \text{ Im}(\overline{\psi} \nabla \psi) - e^2|\psi|^2 A - \nabla \Phi\right), \\
 -\Delta \phi & = \frac{e}{2}||\psi||^2, \quad (2.5) \\
 -\Delta \Phi & = \frac{e}{2}(\overline{\psi} \overline{\Psi} + \Psi \overline{\Psi}). \quad (2.6)
\end{align*}

In order to take advantage of elliptic regularity properties, we transform Equations (2.2) by adding $-\alpha \psi$ ($\alpha > 0$ will be chosen in Lemma 3.3 below) to both sides of the equation to obtain:

$$(-\Delta + \alpha)\psi = i\Psi - e\phi \psi - e^2|A|^2\psi - 2ie\nabla \psi \cdot A + g(|\psi|^2)\psi + \alpha \psi. \quad (2.7)$$

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For simplicity, introduce $U = (\phi, \Phi)$ and rewrite Equations (2.5) and (2.6) as

$$-\Delta U = F_1(\psi, \Psi),$$

where

$$F_1(\psi, \Psi) = \frac{e}{2} \left( \frac{|\psi|^2}{\psi \overline{\Psi}} + \Psi \overline{\psi} \right).$$

Equation (2.3) is then transformed into

$$i \partial_t \Psi + \Delta \Psi = 2ie \nabla \Psi \cdot A + 2ie \nabla \psi \cdot A_t + F_2(U, \psi, \Psi, A, A_t),$$

where

$$F_2(U, \psi, \Psi, A, A_t) = e\Phi \psi + e\phi \Psi + e^2|A|^2 \Psi + 2e^2 \psi A \cdot A_t - g(|\psi|^2)(|\psi|^2 \Psi + \psi^2 \Psi) - g(|\psi|^2) \Psi.$$

It is then necessary to work with $A_t$ as new unknown. We recall first that $A = (a_1, a_2, a_3)$. To properly write the equations on $A$ and $A_t$, for $j = 1, 2, 3$, $k = 1, 2, 3$ and $\ell = 1, 2, 3$, we introduce

$$p_{j,k} = \partial_{x_k} a_j,$$
$$q_j = \Delta a_j,$$
$$r_j = \partial_t a_j,$$
$$\lambda_{j,k} = \partial_{x_k} \Delta^{-1} \partial_t a_j,$$
$$\mu_{j,k,\ell} = \partial_{x_r} \lambda_{j,k} = \partial_{x_r} \partial_{x_k} \Delta^{-1} \partial_t a_j,$$
$$\nu_{j,k} = \Delta \lambda_{j,k} = \partial_{x_k} \partial_t a_j,$$
$$\tau_{j,k} = \partial_t \lambda_{j,k} = \partial_{x_k} \Delta^{-1} \partial_t^2 a_j,$$

and set $A = (A_1, A_2, A_3)$ with $A_j : \mathbb{R}_+ \times \mathbb{R}^3 \to \mathbb{R}^{24}$ and

$$A_j = (a_j, p_{j,k}, q_j, r_j, \lambda_{j,k}, \mu_{j,k,\ell}, \nu_{j,k}, \tau_{j,k}).$$

We also need to give some details on the projection operator $P$. For that purpose, we introduce the Riesz transform $R_j$ from $L^2(\mathbb{R}^3)$ to $L^2(\mathbb{R}^3)$ which is given by

$$R_j = \partial_{x_j} (-\Delta)^{-\frac{1}{2}} \quad \text{for } j = 1, 2, 3.$$

Then, $P$ can be rewritten as $P = (P_{j,m})_{1 \leq j, m \leq 3}$ where

$$P_{j,m} = \delta_{j,m} + R_j R_m.$$
Now we compute the equations for each component of $A_j$. First by the definitions of $A_j$, one finds that

$$\partial_t a_j = \Delta \Delta^{-1} \partial_t a_j = \sum_{k=1}^{3} \partial_{x_k} (\partial_{x_k} \Delta^{-1} \partial_t a_j) = \sum_{k=1}^{3} \partial_{x_k} \lambda_{j,k},$$

$$\partial_t p_{j,k} = \partial_t \partial_{x_k} a_j = \Delta (\partial_{x_k} \Delta^{-1} \partial_t a_j) = \Delta \lambda_{j,k} = \sum_{\ell=1}^{3} \partial_{x_{\ell}} (\partial_{x_{\ell}} \lambda_{j,k}) = \sum_{\ell=1}^{3} \partial_{x_{\ell}} \mu_{j,\ell},$$

$$\partial_t q_j = \partial_t \Delta a_j = \sum_{k=1}^{3} \partial_{x_k} \Delta (\partial_{x_k} \Delta^{-1} \partial_t a_j) = \sum_{k=1}^{3} \partial_{x_k} \Delta \lambda_{j,k} = \sum_{k=1}^{3} \partial_{x_k} \nu_{j,k},$$

$$\partial_t r_j = \partial_{t}^{2} a_j = \Delta \Delta^{-1} \partial_{t}^{2} a_j = \sum_{k=1}^{3} \partial_{x_k} \partial_t (\partial_{x_k} \Delta^{-1} \partial_t a_j)$$

$$= \sum_{k=1}^{3} \partial_{x_k} \partial_t \lambda_{j,k} = \sum_{k=1}^{3} \partial_{x_k} \tau_{j,k}.$$

Next from Equation (2.4), we have

$$\partial_{t}^{2} a_j = \Delta a_j + \sum_{m=1}^{3} \mathbb{P}_{j,m} (e \text{ Im}(\bar{\psi} \partial_{x_m} \psi) - e^2 |\psi|^2 a_m - \partial_{x_m} \Phi),$$

which provides

$$\partial_t \lambda_{j,k} = \partial_t \partial_{x_k} \Delta^{-1} \partial_t a_j = \partial_{x_k} \Delta^{-1} \partial_t^{2} a_j$$

$$= \partial_{x_k} \Delta^{-1} \left( \Delta a_j + \sum_{m=1}^{3} \mathbb{P}_{j,m} (e \text{ Im}(\bar{\psi} \partial_{x_m} \psi) - e^2 |\psi|^2 a_m - \partial_{x_m} \Phi) \right)$$

$$= \partial_{x_k} a_j + h_{1,j,k}^{1}(\psi, A),$$

$$\partial_t \mu_{j,k,\ell} = \partial_t \partial_{x_k} \partial_{x_\ell} \Delta^{-1} \partial_t a_j = \partial_{x_k} \partial_{x_\ell} \Delta^{-1} (\partial_t^{2} a_j)$$

$$= \partial_{x_k} \partial_{x_\ell} \Delta^{-1} \left( \Delta a_j + \sum_{m=1}^{3} \mathbb{P}_{j,m} (e \text{ Im}(\bar{\psi} \partial_{x_m} \psi) - e^2 |\psi|^2 a_m - \partial_{x_m} \Phi) \right)$$

$$= \partial_{x_k} p_{j,k} + h_{2,j,k,\ell}^{2}(\psi, A),$$

$$\partial_t \nu_{j,k} = \partial_{x_k} \partial_{t}^{2} a_j$$

$$= \partial_{x_k} \left( \Delta a_j + \sum_{m=1}^{3} \mathbb{P}_{j,m} (e \text{ Im}(\bar{\psi} \partial_{x_m} \psi) - e^2 |\psi|^2 a_m - \partial_{x_m} \Phi) \right)$$

$$= \partial_{x_k} q_j + h_{3,j,k}^{3}(\psi, A),$$

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where $h_{j,k}^1$, $h_{j,k,\ell}^2$, $h_{j,k}^3$ are non-local functions defined as follows:

$$h_{j,k}^1(\psi, A) = \partial_{x_k} \Delta^{-1} \sum_{m=1}^3 \mathbb{P}_{j,m} (e \text{Im}(\bar{\psi} \partial_{x_m} \psi) - e^2 |\psi|^2 a_m - \partial_{x_m} \Phi),$$

$$h_{j,k,\ell}^2(\psi, A) = \partial_{x_\ell} \partial_{x_k} \Delta^{-1} \sum_{m=1}^3 \mathbb{P}_{j,m} (e \text{Im}(\bar{\psi} \partial_{x_m} \psi) - e^2 |\psi|^2 a_m - \partial_{x_m} \Phi),$$

$$h_{j,k}^3(\psi, A) = \partial_{x_k} \sum_{m=1}^3 \mathbb{P}_{j,m} (e \text{Im}(\bar{\psi} \partial_{x_m} \psi) - e^2 |\psi|^2 a_m - \partial_{x_m} \Phi).$$

Finally one has

$$\partial_t \tau_{j,k} = \partial_{x_k} \Delta^{-1} \partial_t (\partial_t^2 a_j)$$

$$= \partial_{x_k} \Delta^{-1} \partial_t \left( \Delta a_j + \sum_{m=1}^3 \mathbb{P}_{j,m} \left( e \text{Im}(\bar{\psi} \partial_{x_m} \psi) - e^2 |\psi|^2 a_m - \partial_{x_m} \Phi \right) \right).$$

Computing separately each term of the right-hand side of the previous equation, we obtain

$$\partial_t (\bar{\psi} \partial_{x_m} \psi) = \bar{\Psi} \partial_{x_m} \psi + \bar{\psi} \partial_{x_m} \Psi,$$

$$\partial_t (|\psi|^2 a_m) = (\Psi \bar{\psi} + \psi \bar{\Psi}) a_m + |\psi|^2 r_m.$$

Moreover from (2.3) and (2.6), one finds that

$$\partial_t (\partial_{x_m} \Phi) = \partial_t \left( \frac{\epsilon}{2} \partial_{x_m} (-\Delta)^{-1} (\psi \bar{\Psi} + \Psi \bar{\psi}) \right)$$

$$= \frac{\epsilon}{2} \partial_{x_m} (-\Delta)^{-1} (2|\Psi|^2 + \psi \partial_t \bar{\Psi} + \bar{\psi} \partial_t \Psi)$$

$$= e \partial_{x_m} (-\Delta)^{-1} |\Psi|^2 + e \partial_{x_m} (-\Delta)^{-1} \text{Im}(i \partial_t \Psi \bar{\psi})$$

$$= e \partial_{x_m} (-\Delta)^{-1} \left\{ |\Psi|^2 + \text{Im} \left( -\bar{\psi} \Delta \Psi + e \Phi |\psi|^2 + e \phi \bar{\psi} \Psi + e^2 |A|^2 \bar{\psi} \Psi 
+ 2e^2 |\psi|^2 A \cdot A_t + 2ie \bar{\psi} \nabla \Psi \cdot A + 2ie \bar{\psi} \nabla \psi \cdot A_t 
- g'(|\psi|^2) |\psi|^2 (\Psi \bar{\Psi} + \psi \bar{\Psi}) - g(|\psi|^2) \bar{\psi} \Psi \right) \right\},$$

from which we conclude that

$$\partial_t \tau_{j,k} = \partial_{x_k} r_j + h_{j,k}^4(\psi, \Psi, A, R),$$
where \( \mathbf{R} = (r_1, r_2, r_3) \) and

\[
\begin{align*}
\mathcal{H}_{j,k}(\psi, \Psi, \mathbf{A}, \mathbf{R}) &= \frac{\partial}{\partial x} \Delta^{-1} \sum_{m=1}^{3} P_{j,m} \left( e \text{Im}(\overline{\Psi} \partial_{x_m} \psi + \overline{\psi} \partial_{x_m} \Psi) - e^2 (|\psi|^2 r_m + (\Psi \overline{\psi} + \psi \overline{\Psi}) a_m) \right) \\
&- \frac{\partial}{\partial x} \Delta^{-1} \sum_{m=1}^{3} P_{j,m} \left[ e \partial_{x_m} (\Delta)^{-1} \left\{ |\Psi|^2 + \text{Im} \left( -\overline{\Psi} \Delta \Psi + e \Phi |\psi|^2 + e \Phi \overline{\psi} \Psi \\
+ e^2 |\mathbf{A}|^2 \overline{\psi} \Psi + 2 e^2 |\psi|^2 \mathbf{A} \cdot \mathbf{R} + 2 i e \psi \Delta \Psi \cdot \mathbf{A} + 2 i e \overline{\psi} \nabla \cdot \mathbf{R} \\
- g'(|\psi|^2) |\psi|^2 (\overline{\psi} \Phi + \psi \overline{\Phi}) - g(|\psi|^2) \overline{\psi} \Psi \right) \right] \right].
\end{align*}
\]

The equation on \( \mathbf{A}_j \) can be written as a symmetric system of the form

\[
\partial_t \mathbf{A}_j + \mathcal{M}_j(\nabla) \mathbf{A}_j + \mathcal{H}_j(\psi, \Psi, \mathbf{A}, \mathbf{R}) = 0 \quad (j = 1, 2, 3),
\]

where \( \mathcal{H}_j = \{ 0, 0, 0, 0, h_{1,k}^1, h_{2,k}^2, h_{3,k}^3, h_{4,k}^4 \} \), \( \mathcal{M}_j(\nabla) = \sum_{k=1}^{3} \tilde{\mathcal{M}}_j \partial_{x_k} \) are \( 24 \times 24 \) symmetric matrices. Recalling that \( \mathbf{A}_j = (a_j, p_j, q_j, r_j, \lambda_j, \mu_j, \nu_j, \tau_j) \), where \( a_j, q_j, r_j \) are scalar functions, \( p_j, \lambda_j, \nu_j \) and \( \tau_j \) are functions with values in \( \mathbb{R}^3 \) and \( \mu_j \) is a function with values in \( \mathbb{R}^9 \), \( \mathcal{M}_j \) can be simply written by blocks:

\[
\mathcal{M}_j(\nabla) = \begin{pmatrix}
0 & 0 & 0 & 0 & \nabla \cdot & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & \nabla \cdot & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & \nabla & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & \nabla & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & \nabla & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & \nabla & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & \nabla & 0 & 0 & 0 \\
\end{pmatrix},
\]

with

\[
N(\nabla) = \begin{pmatrix}
\nabla & 0 & 0 \\
0 & \nabla & 0 \\
0 & 0 & \nabla \\
\end{pmatrix}.
\]

Note that \( \tilde{\mathcal{M}}_j \) are \( 24 \times 24 \) symmetric matrices whose components are all constants.

Thus from (2.7), (2.8) and (2.9), we have transformed Equations (1.1)-
(1.3) into the following system:

\[ -\Delta U = F_1(\psi, \Psi), \]
\[ (-\Delta + \alpha)\psi + 2ie\nabla \psi \cdot A = i\Psi - e\phi\psi - e^2|A|^2\psi + g(|\psi|^2)\psi + \alpha\psi, \]
\[ i\partial_t \Psi + \Delta \Psi - 2ie\nabla \psi \cdot A = 2ie\nabla \psi \cdot R + F_2(U, \psi, \Psi, A, R), \]
\[ 0 = \partial_t A_j + M_j(\nabla)A_j + H_j(\psi, \Psi, A, R). \]

### 3 Solvability of the Cauchy Problem

The aim of this section is to prove Theorem 1.1. To this end, we use a fix-point argument on a suitable version of System (2.10)-(2.13). In this procedure, the necessary estimates follow from the application of the usual energy methods.

For \( s \in \mathbb{N} \) with \( s > \frac{3}{2} \), take an initial data

\[ \psi(0) \in H^{s+2}(\mathbb{R}^3, \mathbb{C}), \]
\[ A_0 = (a_{10}, a_{20}, a_{30}) \in H^{s+2}(\mathbb{R}^3, \mathbb{R}^3), \]

and

\[ A_1 = (r_{10}, r_{20}, r_{30}) \in H^{s+1}(\mathbb{R}^3, \mathbb{R}^3), \]

satisfying

\[ \text{div} A_0 = 0, \quad \text{div} A_1 = 0. \]

Let us define \( \Psi(0) \in H^s(\mathbb{R}^3, \mathbb{C}) \) by

\[ \Psi(0) = \psi(0), \quad \Phi(0) = \phi(0) = \frac{e}{2}(-\Delta)^{-1}\psi(0), \]

where \( \phi(0) = \frac{e}{2}(-\Delta)^{-1}|\psi(0)|^2 \). We also put

\[ A_0 = (A_{10}, A_{20}, A_{30}) \in H^s(\mathbb{R}^3), \]

for \( i, j, k, l = 1, 2, 3, \)

\[ A_{j0} = (a_{j0}, p_{j,k0}, q_{j0}, r_{j0}, \lambda_{j,k0}, \mu_{j,k,l0}, \nu_{j,k0}, \tau_{j,k0}), \]

\[ p_{j,k0} = \partial_{x_k} a_{j0}, \quad q_{j0} = \Delta a_{j0}, \quad \lambda_{j,k0} = \partial_{x_k} \Delta^{-1} r_{j0}, \]

\[ \mu_{j,k,l0} = \partial_{x_k} \partial_{x_l} \Delta^{-1} r_{j0}, \quad \nu_{j,k0} = \partial_{x_k} r_{j0}. \]
and
\[
\tau_{j,k}(0) = \partial_{x_k} a_{j(0)} + \partial_{x_k} \Delta^{-1} \sum_{m=1}^{3} P_{j,m} \left( e \text{ Im}(\overline{\psi}_{(0)} \partial_{x_m} \psi(0)) - e^2 |\psi(0)|^2 a_{m(0)} - \partial_{x_m} \Phi(0) \right).
\]

We introduce \( R = 2 \left( \|\psi(0)\|_{H^s} + \|\Psi(0)\|_{H^s} + \|A_{(0)}\|_{H^s} \right) \) and let \( B(R) \) be the ball of radius \( R \) in \( C([0,T];(H^s(\mathbb{R}^3))^2) \) for \( T > 0 \).

We prove the existence of a solution \((U, \psi, \Psi, A_j)\) of \((2.10)-(2.13)\) by the following procedure. Take \((\Psi, A) \in B(R)\) with \( \text{ div } A = 0 \) arbitrarily and construct new functions \( Q \) and \( B = (B_1, B_2, B_3) \) as follows.

First we define \( \psi \in C([0,T]; H^s(\mathbb{R}^3)) \) by
\[
\psi(t, x) := \psi_{(0)}(x) + \int_0^t \Psi(s, x) \, ds.
\]

Then by the construction of \( \psi \), one finds that, for \( T \) small enough,
\[
\|\psi\|_{L^\infty([0,T]; H^s)} \leq R.
\]

Next let \( U \in C([0,T]; \dot{H}^1(\mathbb{R}^3)) \) be a solution to \(-\Delta U = F_1(\psi, \Psi)\).

We note that \( U \in C([0,T]; L^\infty(\mathbb{R}^3)) \) and \( \nabla U \in C([0,T]; H^{s+1}(\mathbb{R}^3)) \). (See Lemma 3.2 below.) Next we introduce the solution \( \chi \in C([0,T]; H^{s+2}(\mathbb{R}^3, \mathbb{C})) \) of the following elliptic equation:
\[
(-\Delta + \alpha) \chi + 2ie \nabla \chi \cdot A = i\Psi - e\phi \psi - e^2 |A|^2 \psi + g(|\psi|^2) \psi + \alpha \psi.
\]

We now consider a linearized version of \((2.12)-(2.13)\). We take \((Q, B) \in H^s(\mathbb{R}^3) \times H^s(\mathbb{R}^3)\) solutions to
\[
\begin{cases}
  i\partial_t Q + \Delta Q - 2ie \nabla Q \cdot A = 2ie \nabla \chi \cdot R + F_2(U, \chi, \Psi, A, R), \\
  Q(0, x) = \Psi_{(0)}.
\end{cases}
\]

\[
\begin{cases}
  \partial_t B_j + M_j(\nabla) B_j + H_j(\chi, \Psi, A, R) = 0, \\
  B_j(0, x) = A_{j(0)}.
\end{cases}
\]

Let
\[
S : (\Psi, A) \mapsto (Q, B).
\]

Our strategy consists in showing that \( S \) is a contraction mapping on \( B(R) \), provided that \( T > 0 \) is sufficiently small and to prove that \( \chi = \psi \), from which we obtain the existence of a solution \((U, \psi, \Psi, A_j)\) of \((2.10)-(2.13)\) and complete the proof of Theorem 1.1.

The proof is divided into 6 steps. We first recall the following classical lemma. (See e.g. [1, Proposition 2.1.1, p. 98] for the proof.)
Lemma 3.1. Let $u, v \in L^\infty(\mathbb{R}^3) \cap H^s(\mathbb{R}^3)$ for $s \in \mathbb{N}$. Then for all $(m_1, m_2) \in \mathbb{N}^3 \times \mathbb{N}^3$ with $|m_1| + |m_2| = s$, one has

$$\|D^{m_1} u D^{m_2} v\|_{L^2} \leq C \left(\|u\|_{L^\infty} \|v\|_{H^s} + \|v\|_{L^\infty} \|u\|_{H^s}\right).$$

• Step 1: Solving the elliptic equation (3.3).

Lemma 3.2. There exists a unique solution $U \in C([0, T]; \hat{H}^1(\mathbb{R}^3))$ of (3.3). Moreover, $U = (\phi, \Phi)$ satisfies the following estimates.

$$\|\nabla \phi\|_{L^\infty([0, T]; H^{s+1})} \leq C_1(R), \quad \|\phi\|_{L^\infty([0, T]; L^\infty)} \leq C_2(R), \quad (3.7)$$

$$\|\nabla \Phi\|_{L^\infty([0, T]; H^{s+1})} \leq C_3(R), \quad \|\Phi\|_{L^\infty([0, T]; L^\infty)} \leq C_4(R), \quad (3.8)$$

where $C_1, C_2, C_3$ and $C_4$ are positive constants depending only on $R$.

Proof. First we note that the bilinear form

$$a(u, v) := \int_{\mathbb{R}^3} \nabla u \cdot \nabla v \, dx$$

is continuous and elliptic on $\hat{H}^1(\mathbb{R}^3, \mathbb{R}) \times \hat{H}^1(\mathbb{R}^3, \mathbb{R})$. Moreover since $\psi \in H^s(\mathbb{R}^3)$ and $\Psi \in H^s(\mathbb{R}^3)$, a direct computation gives

$$\|\psi\|^2_{L^\infty} = \|\psi\|^2_{L^2} \leq C\|\psi\|^2_{H^1},$$

$$\|\psi\overline{\Psi} + \Psi \overline{\psi}\|_{L^2} \leq C\|\psi\|_{L^3}\|\Psi\|_{L^2} \leq C\|\psi\|_{H^1}\|\Psi\|_{L^2}.$$ 

Then by the Sobolev embedding $L^6(\mathbb{R}^3) \hookrightarrow (\hat{H}^1(\mathbb{R}^3))^*$ and the Lax-Milgram theorem, we deduce that there exists a unique solution $U \in C([0, T]; \hat{H}^1(\mathbb{R}^3))$ of (3.3).

Next for $0 \leq k \leq s$, we apply $D^{k+1}$ to the first line of (3.3), multiply the resulting equation by $D^{k+1}\phi$ and make an integration by parts to obtain

$$\|\nabla(D^{k+1}\phi)\|_{L^2}^2 = \frac{1}{2} \left| \int_{\mathbb{R}^3} D^{k+2}\phi^2 D^{k+1}\phi \, dx \right|$$

$$\leq C \int_{\mathbb{R}^3} |D^k\psi|^2 |D^{k+2}\phi| \, dx.$$ 

Using the Leibniz rule, Lemma 3.1 and the Schwarz inequality, one has

$$\|\nabla \phi(t, \cdot)\|_{H^{k+1}} \leq C\|\psi(t, \cdot)\|^2_{H^k} \quad \text{for all } t \in [0, T]. \quad (3.9)$$

Summing inequalities (3.9) from $k = 0$ to $s$ and recalling the fact that $\|\psi\|_{L^\infty([0, T]; H^s)} \leq R$, we obtain

$$\|\nabla \phi\|_{L^\infty([0, T]; H^{s+1})} \leq C_1(R).$$
where $C_1(R)$ is a constant depending only on $R$.

Finally, the Sobolev embedding $W^{1,6}(\mathbb{R}^3) \hookrightarrow L^\infty(\mathbb{R}^3)$ provides that

$$\|\phi(t, \cdot)\|_{L^\infty} \leq C \left( \sum_{k=1}^{3} \|\partial_{x_k} \phi(t, \cdot)\|_{L^6} + \|\phi(t, \cdot)\|_{L^6} \right) \leq C \left( \sum_{k=1}^{3} \|\nabla(\partial_{x_k} \phi)(t, \cdot)\|_{L^2} + \|\nabla \phi(t, \cdot)\|_{L^2} \right),$$

from which we deduce that there exists a constant $C_2(R)$ depending only on $R$ such that

$$\|\phi(t, \cdot)\|_{L^\infty([0, T]; H^s)} \leq C_2(R),$$

which ends the proof of (3.7). The proof of estimates (3.8) is similar and we omit the details. \qed

• **Step 2:** Solving the elliptic equation (3.4).

**Lemma 3.3.** Suppose that $A \in H^s(\mathbb{R}^3, \mathbb{R}^3)$, $s > \frac{3}{2}$ and $\text{div} A = 0$. Then for sufficiently large $\alpha > 0$, the bilinear form

$$b(u, v) := \int_{\mathbb{R}^3} (\nabla u \cdot \nabla v + \alpha u \overline{v} + 2ie \nabla u \cdot A \overline{v}) \, dx$$

is hermitian, continuous and elliptic on $H^1(\mathbb{R}^3, \mathbb{C}) \times H^1(\mathbb{R}^3, \mathbb{C})$.

As a consequence, there exists a unique solution $\chi(t, \cdot) \in H^1(\mathbb{R}^3, \mathbb{C})$ to (3.4) and there exists a constant $C_5(R)$ such that

$$\|\chi\|_{L^\infty([0, T]; H^{s+2})} \leq C_5(R).$$

**Proof.** First we note that $b$ is hermitian by the condition $\text{div} A = 0$. Indeed, one has

$$2ie \int_{\mathbb{R}^3} (\nabla u \cdot A) \overline{v} \, dx = -2ie \int_{\mathbb{R}^3} \text{div} A u \overline{v} \, dx - 2ie \int_{\mathbb{R}^3} (\nabla \overline{v} \cdot A) u \, dx = 2ie \int_{\mathbb{R}^3} (\nabla v \cdot A) \overline{u} \, dx,$$

from which it follows directly that $b(u, v) = \overline{b(v, u)}$. The continuity is a direct consequence of the Cauchy-Schwarz inequality and the fact that $A \in H^s(\mathbb{R}^3) \hookrightarrow L^\infty(\mathbb{R}^3)$. Finally for all $u \in H^1(\mathbb{R}^3, \mathbb{C})$, we have

$$\left| 2ie \int_{\mathbb{R}^3} (\nabla u \cdot A) \overline{v} \, dx \right| \leq 2e \|A\|_{L^\infty} \|\nabla u\|_{L^2} \|u\|_{L^2} + \frac{1}{2} \|\nabla u\|_{L^2}^2 + 2e^2 \|A\|_{H^s}^2 \|u\|_{L^2}^2 \leq \frac{1}{2} \|\nabla u\|_{L^2}^2 + Ce^2 \|A\|_{H^s}^2 \|u\|_{L^2}^2.$$
Taking $\alpha \geq 2Ce^{2\|A\|_{HS}^2}$, one gets

$$b(u, u) \geq \frac{1}{2} \|\nabla u\|_{L^2}^2 + \frac{\alpha}{2} \|u\|_{L^2}^2.$$  

This shows that $b$ is elliptic on $H^1(\mathbb{R}^3, \mathbb{C}) \times H^1(\mathbb{R}^3, \mathbb{C})$.

Now since $\psi, \Psi \in H^s(\mathbb{R}^3)$ and $\phi \in H^1(\mathbb{R}^3) \cap L^\infty(\mathbb{R}^3)$, it is obvious that

$$i\psi - e\phi \psi - e^2|A|^2\psi - g(|\psi|^2)\psi + \alpha \psi \text{ belongs to } L^2(\mathbb{R}^3) \hookrightarrow (H^1(\mathbb{R}^3))^*.$$  

Then the Lax-Milgram theorem ensures the existence of a unique solution $\chi$ to (3.4) in $H^1(\mathbb{R}^3)$. Using the elliptic regularity theory and recalling that $(\psi, \Psi) \in C([0, T], H^s(\mathbb{R}^3))^2$, $\phi \in C(([0, T]; \dot{H}^1(\mathbb{R}^3) \cap L^\infty(\mathbb{R}^3)))$, one gets

$$\|\chi\|_{L^\infty([0, T]; H^{s+2})} \leq C_5(R),$$  

where $C_5(R)$ is a constant depending only on $R$. This ends the proof of Lemma 3.3. \qed

- **Step 3: Solving the Schrödinger equation (3.5).**

  For convenience, we introduce the real form of Equation (3.5). Denote $\mathcal{R} = (\mathcal{R}_1, \mathcal{R}_2) = (\text{Re} \mathcal{Q}, \text{Im} \mathcal{Q})$ and write

$$\partial_t \mathcal{R} + J\Delta \mathcal{R} - 2e \sum_{j=1}^3 \kappa_j(A) \partial_{x_j} \mathcal{R} = \mathcal{L}_1(\nabla \chi, \mathcal{R}) + \mathcal{L}_2(\mathcal{U}, \chi, \Psi, A, \mathcal{R}),$$  

(3.10)

$$\mathcal{R}(0, x) = (\text{Re} \Psi_0(x), \text{Im} \Psi_0(x)),$$

where

$$J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad \kappa_j(A) = \begin{pmatrix} a_j & 0 \\ 0 & a_j \end{pmatrix},$$

$$\mathcal{L}_1(\nabla \chi, \mathcal{R}) = \begin{pmatrix} \text{Im} (2ie\nabla \chi \cdot \mathcal{R}) \\ -\text{Re} (2ie\nabla \chi \cdot \mathcal{R}) \end{pmatrix},$$

$$\mathcal{L}_2(\mathcal{U}, \chi, \Psi, A, \mathcal{R}) = \begin{pmatrix} \text{Im} F_2(\mathcal{U}, \chi, \Psi, A, \mathcal{R}) \\ -\text{Re} F_2(\mathcal{U}, \chi, \Psi, A, \mathcal{R}) \end{pmatrix}.$$  

Now for $\varepsilon > 0$, we consider a long-wave type regularization of (3.10) (see [11]):

$$\partial_t (1 - \varepsilon \Delta) \mathcal{R}_\varepsilon + J\Delta \mathcal{R}_\varepsilon - 2e \sum_{j=1}^3 \kappa_j(A) \partial_{x_j} \mathcal{R}_\varepsilon = \mathcal{L}_1 + \mathcal{L}_2,$$  

(3.11)
with $\mathcal{R}_\varepsilon(0) = (1 - \varepsilon\Delta)^{-1}(\text{Re}\,\Psi_0, \text{Im}\,\Psi_0)$. Since Equation (3.11) is linear and contains differential operator in space of at most zero order, one can show that there exists a unique solution $\mathcal{R}_\varepsilon \in C([0, T]; H^s(\mathbb{R}^3))$ to Equation (3.11). Furthermore, we have the following estimate.

**Lemma 3.4.** Let $\mathcal{R}_\varepsilon$ be the unique solution of Equation (3.11). Then there exist constants $C_6(R), C_7(R)$ independent of $\varepsilon$ such that

$$
\|\mathcal{R}_\varepsilon\|_{L^\infty([0, T]; H^s)} \leq e^{C_6(R)T}\|\Psi_0\|_{H^s} + \left( e^{C_7(R)T} - 1 \right)^{\frac{1}{2}}.
$$

*Proof.* We first begin with the $L^2$-estimate. We multiply (3.11) by $\mathcal{R}_\varepsilon$ and integrate over $\mathbb{R}^3$. Since $\mathcal{J}$ is skew-symmetric, one obtains

$$
\frac{\partial}{\partial t} \left( \frac{1}{2} \int_{\mathbb{R}^3} (|\mathcal{R}_\varepsilon|^2 + \varepsilon |\nabla \mathcal{R}_\varepsilon|^2) \, dx \right) = 2\varepsilon \int_{\mathbb{R}^3} \sum_{j=1}^3 K_j(A) \partial_{x_j} \mathcal{R}_\varepsilon \cdot \mathcal{R}_\varepsilon \, dx + \int_{\mathbb{R}^3} L_1 \cdot \mathcal{R}_\varepsilon \, dx + \int_{\mathbb{R}^3} L_2 \cdot \mathcal{R}_\varepsilon \, dx. \quad (3.12)
$$

For $j = 1, 2, 3$, we have from $\|\partial_{x_j} a_j\|_{H^s} \leq R$ that

$$
\left| \int_{\mathbb{R}^3} K_j(A) \partial_{x_j} \mathcal{R}_\varepsilon \cdot \mathcal{R}_\varepsilon \, dx \right| = \left| \frac{1}{2} \int_{\mathbb{R}^3} a_j \partial_{x_j} |\mathcal{R}_\varepsilon|^2 \, dx \right| = \left| \frac{1}{2} \int_{\mathbb{R}^3} \partial_{x_j} a_j |\mathcal{R}_\varepsilon|^2 \, dx \right| 
\leq \frac{1}{2} \|\partial_{x_j} a_j\|_{L^\infty} \|\mathcal{R}_\varepsilon\|_{L^2}^2 \leq C(R)\|\mathcal{R}_\varepsilon\|_{L^2}^2. \quad (3.13)
$$

Since $\Psi, \mathcal{R} \in H^s$ and $A \in H^{s+1}$, using Lemmas 3.2-3.3, one can also compute as follows:

$$
\left| \int_{\mathbb{R}^3} L_1(\nabla \chi, \mathcal{R}) \cdot \mathcal{R}_\varepsilon \, dx \right| \leq C(R)\|\mathcal{R}_\varepsilon\|_{L^2},
$$

$$
\left| \int_{\mathbb{R}^3} L_2(\mathcal{U}, \chi, \Psi, A, \mathcal{R}) \cdot \mathcal{R}_\varepsilon \, dx \right| \leq C(R)\|\mathcal{R}_\varepsilon\|_{L^2}. \quad (3.14)
$$

Collecting (3.12)-(3.14), we derive

$$
\frac{\partial}{\partial t} \|\mathcal{R}_\varepsilon\|_{L^2}^2 \leq C(R)\|\mathcal{R}_\varepsilon\|_{L^2}^2 + C(R).
$$

By the Gronwall inequality and from

$$
\|\mathcal{R}_\varepsilon(0, \cdot)\|_{L^2} = \|(1 - \varepsilon\Delta)^{-1}(\text{Re}\,\Psi_0, \text{Im}\,\Psi_0)\|_{L^2} \leq \|\Psi_0\|_{L^2},
$$

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it follows that
\[ \| R_\varepsilon(t, \cdot) \|_{L^2}^2 \leq e^{C(R)t} (\| R_\varepsilon(0, \cdot) \|_{L^2}^2 + 1 - e^{-C(R)t}) \]
\[ \leq e^{C(R)t} \| \Psi_0 \|_{L^2}^2 + e^{C(R)t} - 1 \]
\[ \leq \left( e^{C(R)t} \| \Psi_0 \|_{L^2} + (e^{C(R)t} - 1)^{\frac{1}{2}} \right)^2 \]
for all \( t \in [0, T] \).

Next we perform the \( H^s \)-estimate. We apply \( D^s \) on (3.11), multiply the resulting equation by \( D^s R_\varepsilon \), integrate over \( \mathbb{R}^3 \) and use the Gronwall inequality. We limit our attention to non-trivial terms. Recalling that \( \chi \in C([0, T]; H^{s+2}) \) and using Lemma 3.3, we obtain
\[ \left| \int_{\mathbb{R}^3} D^s L_1(\nabla \chi, R) \cdot D^s R_\varepsilon \, dx \right| \leq C(R) \| R_\varepsilon \|_{H^s}. \]

Moreover, one gets
\[ \left| \int_{\mathbb{R}^3} K_j(A) \partial_{x_j} D^s R_\varepsilon \cdot D^s R_\varepsilon \, dx \right| = \frac{1}{2} \int_{\mathbb{R}^3} a_j \partial_{x_j} \| D^s R_\varepsilon \|_{L^2}^2 \, dx \]
\[ = \frac{1}{2} \int_{\mathbb{R}^3} \partial_{x_j} a_j \| D^s R_\varepsilon \|_{L^2}^2 \, dx \]
\[ \leq \frac{1}{2} \| \partial_{x_j} a_j \|_{L^\infty} \| R_\varepsilon \|_{H^s}^2. \]

Arguing similarly as above, one finds that
\[ \| R_\varepsilon \|_{L^\infty([0, T]; H^s)} \leq e^{C_6(R)T} \| \Psi_0 \|_{H^s} + (e^{C_7(R)T} - 1)^{\frac{1}{2}}, \]
which ends the proof of Lemma 3.4.

Now we argue as in [6], [11] and we perform the limit \( \varepsilon \to 0 \). By Lemma 3.4, we know that \( R_\varepsilon \) is uniformly bounded in \( L^\infty([0, T], H^s) \). From (3.11), one also has
\[ \partial_t R_\varepsilon = -(1 - \varepsilon \Delta)^{-1} J \Delta R_\varepsilon + 2 \varepsilon (1 - \varepsilon \Delta)^{-1} \sum_{j=1}^3 K_j(A) \partial_{x_j} R_\varepsilon \]
\[ + (1 - \varepsilon \Delta)^{-1} L_1 + (1 - \varepsilon \Delta)^{-1} L_2. \]
This implies that
\[ \| \partial_t R_\varepsilon \|_{H^{s-2}} \leq C(R) \| R_\varepsilon \|_{H^s} + \| L_1 \|_{H^{s-2}} + \| L_2 \|_{H^{s-2}} \leq C \]
for all \( t \geq 0 \) and \( \varepsilon \in (0, 1] \).

Thus passing to a subsequence, we may assume that
\( R_\varepsilon \to R \in L^\infty([0, T], H^s), \partial_t R_\varepsilon \to \partial_t R \in L^\infty([0, T], H^{s-2}) \) in the weak star topology.
From (3.11), one can see that $R$ is a solution of Equation (3.10) and satisfies $$\|R\|_{L^\infty([0,T];H^s)} \leq e^{C_6 T} \|\Psi_0\|_{H^s} + (e^{C_7 T} - 1)^{\frac{1}{2}}.$$ Moreover since $R_\epsilon(0) \to (\text{Re } \Psi_0, \text{Im } \Psi_0)$, we get $R(0) = (\text{Re } \Psi_0, \text{Im } \Psi_0)$. We then deduce the existence of a solution $Q$ to Equation (3.5) satisfying $$\|Q\|_{L^\infty([0,T];H^s)} \leq e^{C_6 T} \|\Psi_0\|_{H^s} + (e^{C_7 T} - 1)^{\frac{1}{2}}.$$ (3.15)

**Step 4: Solving the symmetric system (3.6).**

First we note that it is straightforward to prove the existence of a unique solution $B_j$ to Equation (3.6). (We refer to [1, Proposition 1.2, P. 115] for the proof.) Furthermore, by using the Fourier transform $F$, one has directly $$R_j u = F^{-1} \left( \frac{\xi_j}{|\xi|} F(u) \right) \text{ for } j = 1, 2, 3,$$ from which we deduce that $R_j$ and hence $P_{j,m}$ are bounded from $L^2(\mathbb{R}^3)$ to $L^2(\mathbb{R}^3)$. As a consequence, using the fact $\Psi \in H^s$, $A \in H^{s+2}$, $R \in H^{s+1}$ and $\chi \in H^{s+2}$, one can prove that $$\|H_j(\chi, \Psi, A, R)\|_{L^\infty([0,T];H^s)} \leq C(R).$$ Thus applying the energy estimate to (3.6), recalling that $M_j(\nabla) = \sum_{k=1}^{3} \tilde{M}_j \partial_{x_k}$ is symmetric and using the fact $\tilde{M}_j$ consists of constant elements, we get $$\frac{\partial}{\partial t}\|B_j\|_{H^s}^2 \leq \|B_j\|_{H^s}^2 + C(R).$$ Then by the Gronwall inequality, we obtain the following estimate.

**Lemma 3.5.** Let $B_j$ be the unique solution of (3.6). Then there exists a constant $C_8(R) > 0$ such that $$\|B_j\|_{L^\infty([0,T];H^s)} \leq e^{T} \|A_{j(0)}\|_{H^s} + C_8(R)(e^T - 1)^{\frac{1}{2}}.$$ (3.16)

Collecting (3.15) and (3.16), we can state the following result.

**Proposition 3.6.** There exists $\hat{T} > 0$ such that for $0 < T \leq \hat{T}$, $\mathcal{S}$ maps $B(R)$ into itself.

**Step 5: Contraction mapping.**

Now we establish the following result.
Proposition 3.7. There exists $T^* \in (0, \hat{T}]$ such that $S$ is a contraction mapping in the $L^\infty([0, T^*]; L^2(\mathbb{R}^3))$-norm.

Proof. The proof is based on the fact that $s > \frac{3}{2}$ and on the fact that all the functions of Equations (3.3)-(3.6) are Lipschitz with respect to their arguments. The proof is classical and we omit the details. □

From Propositions 3.6, 3.7 and by the contraction mapping principle, it follows that there exists a unique $(\Psi, A) \in B(R)$ such that

$$S(\Psi, A) = (\Psi, A),$$

that is, $\Psi$ is the unique solution of the Schrödinger equation:

$$
\begin{cases}
\begin{align*}
&i\partial_t \Psi + \Delta \Psi - 2ie\nabla \Psi \cdot A = 2ie\nabla \chi \cdot R + F_2(U, \chi, \Psi, A, R), \\
&\Psi(0, x) = \Psi(0),
\end{align*}
\end{cases}
$$

and $A_j \ (j = 1, 2, 3)$ is the unique solution to the symmetric system:

$$
\begin{cases}
\begin{align*}
&\partial_t A_j + M_j(\nabla) A_j + H_j(\chi, \Psi, A, R) = 0, \\
&A_j(0, x) = A_{j(0)}.
\end{align*}
\end{cases}
$$

Since $A(0, x) = A_{(0)}$, we also have $A(0, x) = A_{(0)}$ and $A_t(0, x) = A_{(1)}$.

• Step 6: Proof of Theorem 1.1 completed.

Let us now go back to the original problem (1.1)-(1.3). To this end, we first remark that $\partial_t \psi = \Psi$ by (3.2). Applying $\partial_t$ on Equation (3.4), comparing the resulting equation with Equation (2.12) and recalling that $R = \partial_t A$, one obtains

$$-\Delta(\partial_t \chi - \partial_t \psi) + 2ie\nabla(\partial_t \chi - \partial_t \psi) \cdot A = 0. \quad (3.17)$$

By Lemma 3.3, we know that the bilinear form

$$b(u, v) = \int_{\mathbb{R}^3} (\nabla u \cdot \nabla v + \alpha u v + 2ie\nabla u \cdot A v) \, dx$$

is hermitian, continuous and elliptic on $H^1(\mathbb{R}^3, \mathbb{C}) \times H^1(\mathbb{R}^3, \mathbb{C})$, from which we deduce that Equation (3.17) has a unique solution. Since obviously 0 is a solution to Equation (3.17), one has $\partial_t \chi = \partial_t \psi$. Moreover from (3.2), it follows that $\psi(0, x) = \psi(0)$ and hence $\phi(0, x) = \phi(0)$ by the uniqueness of the solution of (3.3). Thus substituting $t = 0$ into (3.4), we get

$$-\Delta \chi(0) + 2ie\nabla \chi(0) \cdot A_{(0)} = i\Psi(0) - e\phi(0)\psi(0) - e^2 |A_{(0)}|^2 \psi(0) + g(|\psi(0)|^2) \psi(0).$$

By the definition of $\Psi(0)$ given in (3.1), one finds that

$$-\Delta (\chi(0) - \psi(0)) + 2ie\nabla (\chi(0) - \psi(0)) \cdot A_{(0)} = 0,$$
yielding that $\chi(0) = \psi(0)$ by Lemma 3.3.

Since $\partial_t \chi = \partial_t \psi$ and $\chi(0) = \psi(0)$, it follows that $\chi = \psi$. As a consequence, $\psi$ is the unique solution to Equation (1.1). Then from (3.4) and (3.5), we conclude that $A$ and $\phi$ are the unique solutions to (1.2) and (1.3) respectively. Moreover by the uniqueness of (3.3), one finds that $\partial_t \phi = \Psi$. Finally by Lemmas 3.2, 3.3 and from (3.15), (3.16), $(\psi, A, \phi)$ has the desired regularity as stated in Theorem 1.1.

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