Transitioning from definitions to proof: Exploring non-Euclidean geometry in a college geometry course

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Geometry is the subject where U.S. students are weakest on international assessments, but college geometry is an area of proof that is understudied. Since geometry is secondary students' only exposure to proof, it is vital our secondary teachers can prove effectively in this content area. However, one obstacle to developing deeper understanding of geometric concepts in college geometry courses is that students tend to try recalling prior geometry instruction instead of engaging in any new material within a Euclidean geometric context. A document analysis of student portfolios revealed that although pre-service teachers in this document study began the semester with limited abilities to work with formal definitions, by the end of the semester all were able to propose and justify conjectures on novel surfaces.

Keywords: Geometry, inquiry-based learning, pre-service teachers.

Theoretical background

Geometry arises from a set of undefined terms and axioms through which all other theorems and definitions are constructed. Hence, a thorough understanding of geometry involves a deep understanding of proof; yet, teachers possess a narrow understanding of proof. Studies indicate that pre- and in-service teachers believe proof only helps explain ideas used in mathematical concepts, and they do not recognize the ability of proof to systemize results (Mingus and Grassl, 1999; Knuth, 2002b). Teachers lack the geometry content knowledge required for geometry proofs, and they are convinced by empirical evidence as well (Jones, 1997; Knuth 2002a). Consequently, teachers with inadequate proof and geometry understanding cannot be expected to impart adequate proof and geometry knowledge to students. Furthermore, college geometry is the only undergraduate proofs course that has not been studied in any systematic manner (Speer and Kung, 2016).

Pre-service teachers in undergraduate proof courses do not adequately understand what arguments qualify as proof (Weber, 2001). They lack comprehension of the mathematical language and concepts necessary to proof (Selden, 2012), and they possess an incomplete understanding of definitions and theorems (Weber, 2001; Selden and Selden, 2008). In typical direct-instruction, lecture proof courses, students are expected to develop proficient proof skills with little to no guidance. Without guidance, students will fail and likely cultivate ineffective strategies (Weber, 2001). These ineffective strategies are typically proof schemes dependent upon external and empirical convictions, such as the authoritarian, ritual, and perceptual proof schemes (Harel and Sowder, 2007). In order to successfully write a proof, students need to employ effective strategies or proof schemes with arguments based on axioms and logical deductions, which in turn requires understanding of definitions and the idea of conditionally true statements.

The typical instructor-centered learning environment, is the dominant paradigm, but may not induce the logic and proof techniques needed to construct a proof in all students (Fukawa-Connelly, Johnson, & Keller, 2016). Alternatively, a proof course should consist primarily of student-student and student-teacher interactions (Selden and Selden, 2008). Given the prevalence of lecture-based proof courses in the United States (Fukawa-Connelly, Johnson, & Keller, 2016) and preservice teachers' continuing struggles with geometry, one potential approach to help improve pre-service teachers' is an inquiry-based learning pedagogy where students are active learners, and the instructor is responsible for facilitating students' exploration of the content, particularly definitions (Padraig and McLoughlin, 2009).

However, one of the additional challenges of college geometry is that the material is familiar to students. Rather than investigating the ideas presented in the current assignment, many students rely on recollections from previous geometry courses, especially if the problems seem familiar. Hence, an inquiry-based college geometry course in non-Euclidean geometry seems more likely to help pre-service teachers develop their proving skills and deepen their understanding of the geometric concepts they will eventually teach. This study was guided by the question: how, if at all, an inquiry based non-Euclidean geometry class helped deepen students' understanding of definitions and Euclid's postulates? We argue that the deep exploration of a limited number of non-Euclidean geometry problems helped students to move from primitive geometric knowledge to formalizing definitions and successfully posing and solving conjectures on novel surfaces.

Methodology

An adaptation of Pirie and Kieren's (1994) model for student understanding was used to code students' written assignments for their understanding of definition. Although this eight-level model was originally created to model students' understanding of fractions, it adapts well to geometry, as the purpose was to describe the transition from concrete to abstract reasoning to justification to problem posting. One level, image having, was not used when coding, since students always had access to physical models of whatever non-Euclidean surface they were working with that week, so we could not determine students' facilities for understanding similar exercises without physical models. We also did not include looping back within our standards of evidence because it was not something we observed in the data.

This study took place at a midsized, rural, Hispanic-serving research university in the South, and the students who participated were those enrolled in a ten student college geometry course. The data collected was part of a larger study; this study presents a case study of five pre-service teachers Lindsey, Bradyn, Alexis, Mackenzie, and Chase. We also analyzed the work on one non-preservice teacher, Florencio, because Florencio's papers were different from the other participants. While we wanted to maintain a purposeful sampling of pre-service teachers, Florencio was enough of a disconfirming case that we felt his inclusion was necessary (Patton, 2002). Florencio was an engineering major with one prior proofs class. Lindsey has no formal proof experience, Bradyn had completed discrete mathematics in one attempt with a B, Alexis had completed discrete mathematics and abstract algebra with A's, and Chase, who has a learning disability, had completed discrete mathematics with a C after two attempts and failed another upper level proof class. Mackenzie was a non-traditional student in her final semester; she had completed all other upper level proof classes with a mixture of A, B, and C grades.

Students in the course were provided with course notes (Miller, 2010) that presented open-ended problems related to a specific learning goal. There were fifteen assignments; five of which were focused on formal axiomatic proof, eight on definitions and axioms in various non-Euclidean situations, and two assignments (the midterm and final project) which combined both strands in the same assignment. Four of the assignments were formal (one revision allowed), and the other ten were informal (unlimited revisions). On the midterm (F3), students were given new but similar problems to their assignment and asked to work through them individually, and on the final project (F4), students were asked to discover as many things as they could about the geometry of the surface a cone. For each new assignment, students were assigned a specific problem from the provided course notes and a group. If a group appeared to be making little progress or moving in an unproductive direction, the teacher would use guided questioning to redirect students' thoughts. If multiple groups stopped progressing, the teacher would initiate a whole class discussion.

To determine students' understanding of definitions and postulates, researchers examined the first submission students turned in for each assignment. Researchers also used observations to gain further understanding of students' proof comprehension. As students discussed their ideas within their group, a researcher sat behind them listening and taking notes on their interactions for about ten minutes. The submissions were analyzed by assignment and all the drafts from an individual participant were analyzed at the same time. After this initial reading of blinded assignments, researchers would journal their impressions of the coding and the overall trajectory exhibited in the multiple submissions. These journals were used to operationalize concepts in the literature review, and then they were compared to the standards of evidence table (Table 1).

Level of Understanding	**Identifiers**
Primitive Knowledge (1)	Students are applying prior knowledge of Euclidean geometry, stating given definitions, or providing empirical proofs
Image Making (2)	Students make distinctions and reclassify prior knowledge or use prior knowledge in a new manner
Property Noticing (3)	Students can apply a definition on a previous surface to a novel surface or situation by recognizing commonalities in the learned and novel situation
Formalizing (4)	Students can abstract a method, formula, or common property from previous property noticing
Observing (5)	Students can propose conjectures and provide justification or counterexample
Structuring (6)	The argument is logical and made up of systematic application of axioms and theorems/If any portion of the argument could be clarified, the clarification is not necessary for the argument's validity.
Inventising (7)	Students can pose new questions and solve them, creating new (to the student) knowledge

Table 1: Standards of evidence (Modified from Pirie and Kieren, 1994)

Findings

With the exception of Mackenzie, all pre-service teachers struggled to complete the initial assignments with correct arguments; primitive knowledge from a previous high school geometry course was applied to the problem instead of an argued solution. However, by the fourth inquiry-based task, all participants were able to formalize definitions, and all students were at least able to successfully use definitions and postulates in novel situations to construct proofs. All students followed a similar trajectory throughout the semester and improved, on average, four levels of understanding (Figure 1).

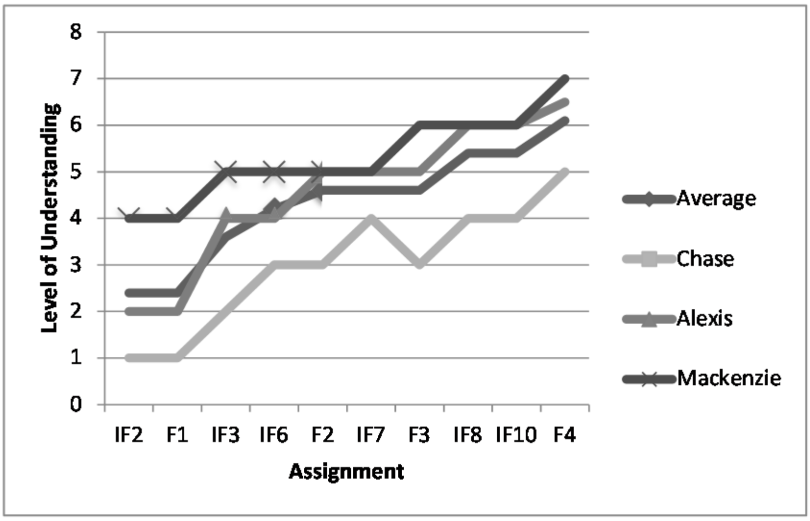


Figure 1: Student levels of understanding throughout the semester

The first definition centered-assignment of the semester was an inverse categorization problem. After finding all possible symmetries on the square, students were asked to use Geogebra to start with a subset of these symmetries, construct all possible quadrilaterals with that set of symmetries, and justify why they had found all cases. Mackenzie was able to categorize on first assignment, but the other preservice teachers either listed the symmetries of each quadrilateral or could not justify if/why they had found all cases (Figure 2). Both of these difficulties indicate students making partial reclassifications of their prior knowledge.

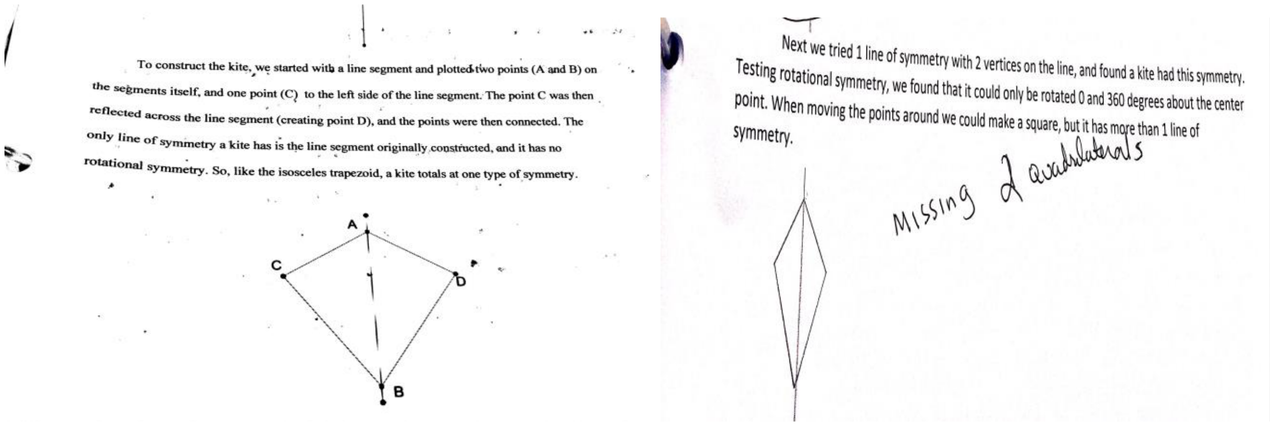


Figure 2: Typical solutions on IF2 Chase (left, primitive knowledge); Florencio (right, image making)

During the middle third of the semester, the two assignments that helped pre-service teachers move towards formalizing their understanding of definitions and counterexamples were IF3 and IF6. In both assignments, students were asked to justify which, if any of Euclid's postulates held on a sphere (IF3) and the hyperbolic plane (IF6). Students were also asked to prove the existence of asymptotic geodesics on the hyperbolic plane. On IF3, the first exposure to the postulates, Alexis was not able to work in the spherical context and reasoned through the justification of the postulate in terms of the more familiar planar geometry (Figure 3). However, by IF6 Alexis was able to provide a counterexample for false postulates (Figure 4).

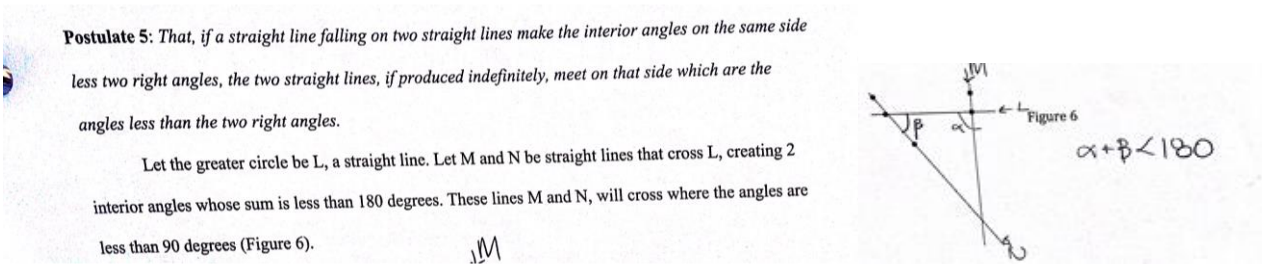


Figure 3: Alexis' IF3 (Image Making) Postulate 5 solution

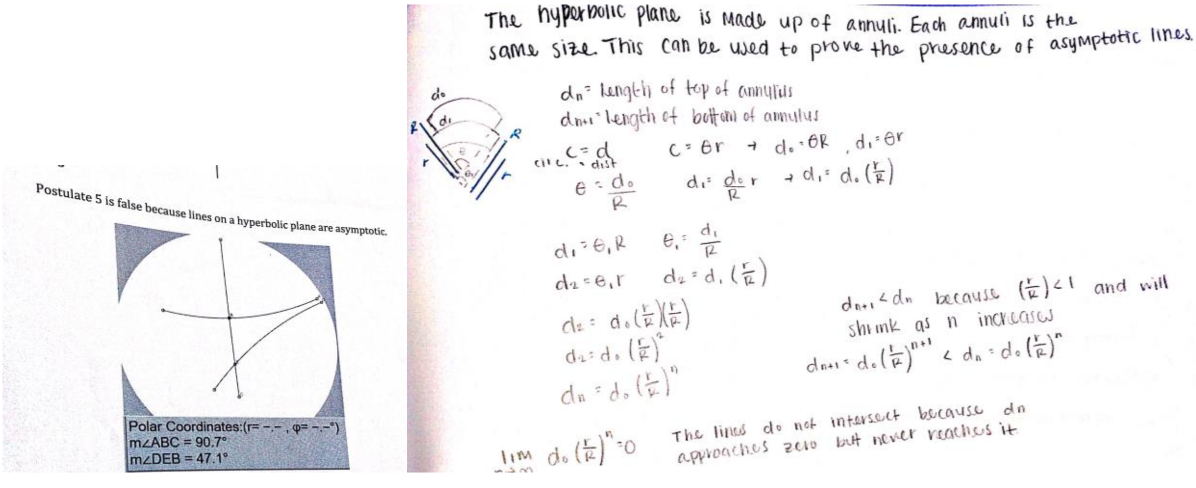
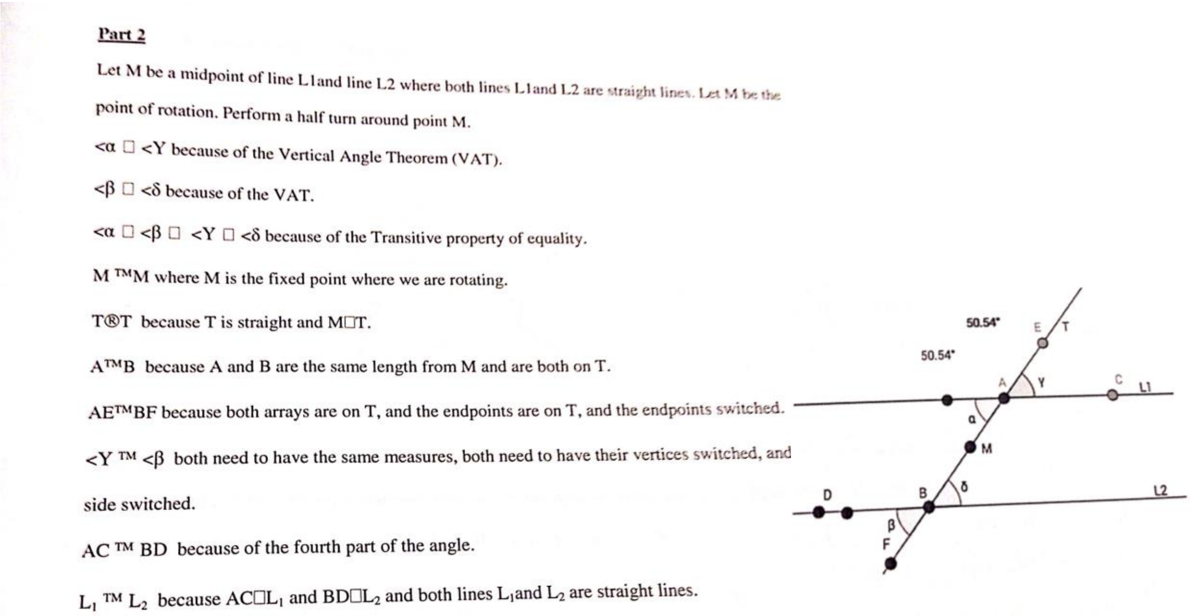


Figure 4: IF 6 (Formalizing) Postulate 5 solution

In the final third of the semester, the goal of all assignments was to integrate pre-service teachers' improved proof schemes with more formal uses of definitions. IF5 and F3 were major proofs assignments that took students most of the middle third of the semester. With their improved proof schemas, and understanding of the surfaces, were more easily able to construct proofs for parallel lines that were independent of the surface upon which the lines were drawn (IF8 and IF10). Most students were able to construct a generally correct proof, with some minor disordering of steps and missing justifications. This shows participants possessed a more structural understanding of symmetry than the understanding demonstrated in IF2. Bradyn, like Florencio, was image making on the first assignment related to symmetry, but by the time symmetry was used to construct proofs related to parallel transported lines, Bradyn was much more successful (Figure 5). Although Bradyn's language is not quite standard and he had trouble typesetting his proof, his overall structure is systematic and he has a transformational understanding of symmetry not present in his initial write-ups.



Part 2

Let M be a midpoint of line L1and line L2 where both lines L1and L2 are straight lines. Let M be the point of rotation. Perform a half turn around point M.

<α □ <Y because of the Vertical Angle Theorem (VAT).

<β □ <δ because of the VAT.

<α □ <β □ <Y □ <δ because of the Transitive property of equality.

M ™M where M is the fixed point where we are rotating.

T®T because T is straight and M□T.

A™B because A and B are the same length from M and are both on T.

AE™BF because both arrays are on T, and the endpoints are on T, and the endpoints switched.

<Y ™ <β both need to have the same measures, both need to have their vertices switched, and side switched.

AC ™ BD because of the fourth part of the angle.

L_1 ™ L_2 because AC□L_1 and BD□L_2 and both lines $L_1$and L_2 are straight lines.

Figure 5: Bradyn's second proof in IF8 (observing)

The final formal assignment asked students to discover (and prove) as many things as they could about the geometry of an infinite cone. Groups were expected to prove 2-4 conjectures. Given the open nature of F4, one group chose to only investigate properties of a cone where group members had successfully revised an assignment on another surface. This limited their levels of understanding to observation. The other two groups each had at least one investigation of a conjecture about a novel concept with at least one new or newly-modified definition, which is summarized in Table 2.

Participant	Project Summary	Code
Alexis (+3 others)	Using a novel group-invented definition of straightness to investigate self-intersecting lines on cones (conjecture: no formula possible), angle sums of triangles with self-intersecting sides	Structuralizing, Inventising
Mackenzie, Lindsey (+1 other)	Holonomy, internal angle sums of a triangle with no self-intersecting sides, triangle congruence theorems, non-intersecting lines that are not parallel transports	Inventising
Bradyn, Chase (+1 other)	Postulates (some cone angles), collected data for self-intersecting lines	Observing

Table 2: Summary of final project

Discussion

Regardless of prior proof course grades or experience all pre-service teachers struggled to complete the initial assignments with correct arguments; primitive knowledge from a previous high school geometry course was applied to the problem instead of an argued solution. However, by the fourth inquiry-based task, all participants except for Chase were able to formalize definitions, and all students were at least able to successfully use definitions and postulates in novel situations to construct proofs by the end of the semester.

By centering the college geometry course around understanding core geometric concepts on several different surfaces, participants were forced to engage in understanding each new situation rather than simply applying their prior Euclidean geometry knowledge to a more familiar problem. As a result, students developed more advanced understanding definitions and counterexamples. All participants got to at least formalizing definitions and seven of the ten students in the class ended the semester at either the structuring or Inventising level.

The structure of the course maximized students' opportunities to reify their understanding of definitions and postulates. The use of multiple non-Euclidean contexts was key to helping students develop better understanding of formal definitions. By switching surfaces, operationalizing definitions and determining if they were applicable stayed a problem and not an exercise. Further, the repetition of the postulates and determining geodesics in particular were important because these concepts were presented in enough slightly different concepts to allow students to develop deeper levels of understanding of the definition than a single context would allow. Students also reported that the chance to revise their written work was a valuable way to help them reflect on the current surface and how it compared to their prior work.

There are two potential limitations for this study, which could be remedied by further inquiry. First, we did not collect students' interview data; our analysis is a document study coupled with observations of students' in-class discussions. Teaching experiments or task-based interviews of students' understanding of definition in similar inquiry based classes, and comparative data to students in axiomatic Euclidean courses would also be of use. We also had non-native English speaking students in this class, and further research on their experience with proof and geometry is still needed. Finally, although most students followed the same trajectory, Chase was about two weeks behind everyone else. However, Chase has dyslexia; and there is a dearth of literature about undergraduates with learning disabilities; more work is needed to understand how to support such students' learning.

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