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ADDENDUM: NON-COOPERATIVE FISHER–KPP SYSTEMS: TRAVELING WAVES AND LONG-TIME BEHAVIOR

LÉO GIRARDIN

ABSTRACT. The solution to a problem left open in the original paper [1] and solved afterwards is presented. Additionally, a few conflicting notations and a typo are corrected.

1. EXTINCTION IN THE CRITICAL CASE $\lambda_{PF}(\mathbf{L}) = 0$

The critical case $\lambda_{PF}(\mathbf{L}) = 0$, for which extinction still occurs, was unsolved at the time of writing of [1]. It was solved later on thanks to an hint of Adrian Lam, who pointed out that the argument used to establish the upper estimates of [1, Theorem 1.2] can actually be used again to solve the extinction case.

Consequently, [1, Theorem 1.3] should be corrected as follows. The conjecture following [1, Theorem 1.3] and the discussion in [1, Section 4.1.1] can now safely be ignored.

Theorem. *[Extinction or persistence dichotomy]*

- (1) Assume $\lambda_{PF}(\mathbf{L}) < 0$. Then all bounded nonnegative classical solutions of (E_{KPP}) set in $(0, +\infty) \times \mathbb{R}$ converge asymptotically in time, exponentially fast, and uniformly in space to $\mathbf{0}$.
- (2) Assume $\lambda_{PF}(\mathbf{L}) > 0$. Then there exists $\nu > 0$ such that all bounded positive classical solutions $\mathbf{u}$ of (E_{KPP}) set in $(0, +\infty) \times \mathbb{R}$ satisfy, for all bounded intervals $I \subset \mathbb{R}$,

$$\left(\liminf_{t \rightarrow +\infty} \inf_{x \in I} u_i(t, x) \right)_{i \in [N]} \geq \nu \mathbf{1}_{N,1}.$$

Consequently, all bounded nonnegative nonzero classical solutions of (S_{KPP}) are valued in

$$\prod_{i=1}^N [\nu, g_i(0)].$$

- (3) Assume $\lambda_{PF}(\mathbf{L}) = 0$ and

$$\text{span}(\mathbf{n}_{PF}(\mathbf{L})) \cap \mathbf{K} \cap \mathbf{c}^{-1}(\{\mathbf{0}\}) = \{\mathbf{0}\}.$$

Then all bounded nonnegative classical solutions of (E_{KPP}) set in $(0, +\infty) \times \mathbb{R}$ converge asymptotically in time and uniformly in space to $\mathbf{0}$.

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Proof. The third case is proved as follows.

Let $\mathbf{u}$ be a bounded nonnegative classical solution. Using its boundedness, we can define for all $T > 0$

$$C_T = \inf \{C > 0 \mid C \mathbf{n}_{PF}(\mathbf{L}) \geq \mathbf{u}(T, x) \text{ for all } x \in \mathbb{R}\}.$$

Subsequently, fix temporarily $T > 0$ and let $\mathbf{v} = C_T \mathbf{n}_{PF}(\mathbf{L}) - \mathbf{u}$. Since $\lambda_{PF}(\mathbf{L}) = 0$, $\mathbf{v}$ satisfies

$$\mathbf{v}(T, x) \geq \mathbf{0} \text{ for all } x \in \mathbb{R},$$

$$\partial_t \mathbf{v} - \mathbf{D} \partial_{xx} \mathbf{v} - \mathbf{L} \mathbf{v} = \mathbf{c}[\mathbf{u}] \circ \mathbf{u} \text{ in } (0, +\infty) \times \mathbb{R}.$$

Since the nonnegativity of $\mathbf{c}(H_2)$ implies $\mathbf{c}[\mathbf{u}] \circ \mathbf{u} \geq \mathbf{0}$, it follows from the maximum principle that

$$\mathbf{v} \geq \mathbf{0} \text{ in } [T, +\infty) \times \mathbb{R},$$

that is

$$C_T \mathbf{n}_{PF}(\mathbf{L}) \geq \mathbf{u} \text{ in } [T, +\infty) \times \mathbb{R},$$

whence $C_{T'} \leq C_T$ for all $T' \geq T$. In other words, $(C_T)_{T>0}$ is a nonincreasing family.

Next, let us verify that it is in fact a decreasing family if and only if $\mathbf{u}$ is nonzero.

Of course, if $\mathbf{u} = \mathbf{0}$, then $C_T = 0$ for all $T > 0$.

Now, assume that there exist $T > 0$ and $T' > T$ such that, for all $t \in [T, T']$, $C_t = C_T$. Let $\mathbf{v} = C_T \mathbf{n}_{PF}(\mathbf{L}) - \mathbf{u}$. By optimality of $C_{T'}$,

$$\min_{i \in [N]} \inf_{x \in \mathbb{R}} v_i(T', x) = 0.$$

If there exists $x \in \mathbb{R}$ such that

$$\mathbf{v}(T', x) = C_{T'} \mathbf{n}_{PF}(\mathbf{L}) - \mathbf{u}(T', x) \in \partial \mathbf{K},$$

then by the strong maximum principle

$$\mathbf{v} = \mathbf{0} \text{ in } [T, T'] \times \mathbb{R}.$$

This leads to $\mathbf{c}[\mathbf{u}] \circ \mathbf{u} = \mathbf{0}$ on one hand and to $\mathbf{u} = C_T \mathbf{n}_{PF}(\mathbf{L})$ on the other hand, whence by assumption on $\mathbf{c}$ we deduce $C_T = 0$. Therefore $\mathbf{u} = \mathbf{0}$ in $[T, T'] \times \mathbb{R}$ and then in $(0, +\infty) \times \mathbb{R}$.

On the contrary, if such an $x \in \mathbb{R}$ does not exist, then by optimality of $C_{T'}$, there exists $\mathbf{w} \in \partial \mathbf{K}$ and $(x_n)_{n \in \mathbb{N}}$ such that, as $n \rightarrow +\infty$,

$$x_n \rightarrow \pm\infty \text{ and } \mathbf{v}(T', x_n) \rightarrow \mathbf{w}.$$

Defining the sequence

$$\mathbf{u}_n : (t, x) \mapsto \mathbf{u}(t, x + x_n)$$

and using classical parabolic estimates to extract a locally uniform limit $\mathbf{u}_\infty$, we find that $\mathbf{v}_\infty = C_T \mathbf{n}_{PF}(\mathbf{L}) - \mathbf{u}_\infty$ satisfies

$$\partial_t \mathbf{v}_\infty - \mathbf{D} \partial_{xx} \mathbf{v}_\infty - \mathbf{L} \mathbf{v}_\infty \geq \mathbf{0} \text{ in } [T, T'] \times \mathbb{R},$$

$$\mathbf{v}_\infty \geq \mathbf{0} \text{ in } [T, T'] \times \mathbb{R},$$

$$\mathbf{v}_\infty(T', 0) = \mathbf{w} \in \partial \mathbf{K},$$

and then again by the strong maximum principle we find $\mathbf{v}_\infty = \mathbf{0}$ and subsequently $\mathbf{u} = \mathbf{0}$.

Hence either $\mathbf{u} = \mathbf{0}$ or the family $(C_T)_{T>0}$ is decreasing. Let

$$C_\infty = \lim_{T \rightarrow +\infty} C_T \geq 0.$$

Assuming by contradiction that $C_\infty > 0$, defining the sequence

$$\mathbf{u}_n : (t, x) \mapsto \mathbf{u}(t + n, x)$$

and its locally uniform limit $\mathbf{u}_\infty$, we can repeat the argument and obtain that the family $(D_T)_{T>0}$, where

$$D_T = \inf \{ D > 0 \mid D \mathbf{n}_{PF}(\mathbf{L}) \geq \mathbf{u}_\infty(T, x) \text{ for all } x \in \mathbb{R} \},$$

is decreasing, which directly contradicts the fact that $D_T = C_\infty$ for all $T > 0$. This ends the proof. $\square$

2. CONFLICTING NOTATIONS, TYPOS

2.1. Conflicting notations. In [1, Section 1.5.1], “The cane toads equation with non-local competition”, the notation θ_N denotes the mesh size of the partition $\frac{\bar{\theta}-\underline{\theta}}{N-1}$. However in [1, Section 1.5.2], “The cane toads equation with non-local mutations and competition”, it denotes both the mesh size and the last point of the partition $(\theta_i)_{i \in [N]}$. In order to solve this conflict, the mesh size should be denoted $\delta\theta$ instead.

Similarly, in [1, Section 1.5.3], “The Gurtin–MacCamy equation with diffusion and overcrowding effect”, the mesh size should be denoted δa instead of a_{N+1} .

2.2. Typo. On page 153, first line, A_ε should be replaced by A .

REFERENCES

- [1] Girardin Léo, 2018, Non-cooperative Fisher–KPP systems: traveling waves and long-time behavior, *Nonlinearity* **31**, 108–164