A relativity theory of traffic along a transit line

Fabien Leurent, Vincent Benezech, François Combes

RelStat12: 12 th Annual International Conference
"Reliability and Statistics in Transportation and Communication"

A RELATIVITY THEORY OF TRAFFIC ALONG A TRANSIT LINE

Fabien Leurent*, Vincent Benezech, François Combes

Université Paris Est, Laboratoire Ville Mobilité Transport, Ecole des Ponts ParisTech, France

Abstract

Along a transit line, vehicle traffic and passenger traffic are jointly subject to variability in travel time and vehicle load hence crowding. The paper provides a stochastic model of passenger physical time and generalized time, including waiting on platform and in-vehicle run time from access to egress station. Five sources of variability are addressed: (i) vehicle headway which can vary between the stations provided that each service run maintains its rank throughout the local distributions of headways; (ii) vehicle order in the schedule of operations; (iii) vehicle capacity; (iv) passenger arrival time; (v) passenger sensitivity to quality of service. The perspective of the operator, which pertains to vehicle runs, is distinguished from the user's one at the disaggregate level of the individual trip. After recalling the basic properties from a previous paper [0], this paper provides additional properties and explores some consequences for models of traffic assignment to a transit network.

Keywords: Vehicle load; headway rank; passenger exposure; wait time; run time; journey time; assignment model

1. Introduction

1.1. Background

The operations of a transit line, and even more of a network of lines, are submitted to variability in a number of ways. On the operator side, vehicle type may not be homogeneous, the passenger load depends on the service schedule and varies along the route, traffic disruptions arise due to causes either internal (such as human error, material incident, passenger incident or accident…) or external (such as adverse weather, malevolent intrusion, conflict with another flow…). On the demand side, the passenger experiences travel conditions along his trip, from service waiting and platform occupancy at the access station up to station egress passing by vehicle occupancy and its journey time, which vary according to the occurrence of the trip in a series of reiterations and also between passengers on a given occurrence. A major issue pertains to service reliability: any disruption causing a large delay induces a significant loss in quality of service, and the frequent reiteration of such events will make the passenger reconsider his travel decision of network route and even of transportation mode. Stated Preferences surveys have shown that frequent significant delays amount to additional travel time in a more than proportional way: for instance, the factor of proportionality was estimated to 1.5 for delays of more than 10 minutes occurring three out of 20 times in Paris suburban railways [1]. Such behavioral patterns must be taken into account in network planning, both within network traffic assignment models and the cost-benefit analysis of transportation projects.

1.2. Objective

The paper's objective is to provide a stochastic model of traffic variability and passenger exposure along a transit service. On the side of transit services that are supplied to passengers, the model assumptions involve the statistical distribution of, first, the local vehicle headways at station nodes, second, the local run times along inter-station links, third, vehicle capacity in terms of seated and standing passengers. On the side of passenger demand, a spatial pattern is assumed for the access-egress matrix of passenger flows, together with a statistical distribution (temporal pattern say on a day-to-day basis) of a volume index.

The model yields the following outcomes: (i) the distribution of vehicle journey times by pair of access-egress stations, together with the distribution of passenger loading; (ii) the distribution of passenger physical time by access-egress pair; (iii) the distribution of passenger generalized time by access-egress pair, assuming that crowding density adds discomfort cost to travel times. Thus the interplay of operations

* Corresponding author. Tel.: +33 1 6415 2111; fax: +33 1 6415 2140.
E-mail address: fabien.leurent@enpc.fr

variability with the spatial pattern and temporal distribution of passenger flows is captured in an explicit and consistent framework.
The framework amounts to a line model that can be used in a larger model of traffic assignment to a transit network, which also deals with route choice.

1.3. Approach

The model pertains to the physics of traffic operations and passenger exposure to travel conditions both of service operations and vehicle load. The main variables of vehicle traffic, passenger traffic and passenger travel are cast into a probabilistic framework in the form of random variables. Variability sources are identified, among which the major one is the heterogeneity of vehicle headways. Analytical properties are established between the main model variables, in the form of functional relationships linking the CDF, PDF, mean and variance of them. This is achieved by postulating the conservation of headway rank by service run and by deriving a series of consequences on the basis of probabilistic calculus. Overall, the model blends up probabilistic analysis taken mostly from the theory of renewal and survival, with traffic analysis at the two levels of transit vehicles and passengers, respectively.
In the basic exposition of the model [0], three steams of related previous work have been identified, namely (i) analysis of vehicle traffic only [2], [3], (ii) focus on passenger waiting on platform [4], (iii) empirical or simulation-based analysis of travel conditions [5], [6]. To the best of our knowledge, our analytical approach to distinguish the operator and user perspectives is original in the context of transit traffic: it may be called a relativity theory of traffic along a transit service.

1.4. Structure

The rest of the paper is organized in six sections. Vehicle traffic is considered first, by focusing on headways and deriving some consequences on journey times by pair of entry-exit stations (Section 2). Then, passenger load by vehicle is characterized with respect to headway rank and the index of demand volume (Section 3). Next, we turn our attention to passenger exposure to in-vehicle crowding, wait time and travel time (Section 4). The consequences of service irregularity and other variations affect not only the physical times but also the "generalized time" which takes into account the discomfort of specific travel states (Section 5). From this stem important consequences for models of traffic assignment to a transit network that are applied to cases that involve both irregularity and congestion (Section 6). Lastly, the conclusion points to the model scope, limitations and potential developments (Section 7).

2. On vehicle headways and journey times

A transit line operated along a single service route in a single direction is considered. The stations are indexed by $m \in M$ and the sections or links between adjacent stations by $a \in A$. Each vehicle run is characterized by a trajectory in space and time. The journey time is made up of the run times on the sections plus the dwell times at the stations. The objective of this section is to model the statistical distribution of vehicle run times between station pairs along the line. The statistical population of interest is the set of runs during a reference period, for instance the morning peak hour of working days. First, we shall model the distribution of vehicle headways (§ 2.1). Second, their propagation between stations is addressed in § 2.2. Then, a postulate is made about the "conservation of headway rank" (§ 2.3), which entails specific properties for the distribution of vehicle headways (§ 2.4) and that of journey times (§ 2.5).

2.1. On vehicle headways

Denote by $\eta_m(i)$ the time between the departure of vehicle i from station m and that of the previous vehicle, $i-1$, and let us call it a *headway* as shorthand for inter-departure time. In the population of vehicle runs, the Cumulated Distribution Function (CDF) of η_m is denoted as H_m with inverse function $\mathrm{H}_m^{(-1)}$. Let us recall classical properties:

i) The service frequency at station m during the reference period, f_m, is the reciprocal of the average headway: $f_m = 1/\mathrm{E}[\eta_m]$.

ii) Service irregularity is related to the deviation of η_m from its average value. It can be assessed by the variance of this distribution, $\mathrm{V}[\eta_m]$, or equivalently by its standard deviation $\sigma[\eta_m]$ or the relative dispersion $\gamma[\eta_m] = \sigma[\eta_m]/\mathrm{E}[\eta_m]$.

Assuming that the incoming passengers at station m arrive independently from one another and from service schedule, their arrivals can be modeled as a Poisson process and, if the process intensity is medium or high, then it can be safely assumed that the number of passengers waiting for a given vehicle is proportional to the headway (neglecting any capacity constraint). Furthermore, the distribution of passenger waiting times at m stems from that of vehicle headway in a specific way (see Section 4).

2.2. Spatial propagation

The instant of departure of vehicle i from station m, $h_m(i)$, is separated from that of the next station, $h_{m+1}(i)$, by the run time along section $a \approx (m, m+1)$ plus the stop time at $m+1$, altogether denoted as $t_a(i)$:

$$h_{m+1}(i) = h_m(i) + t_a(i). \tag{2.1}$$

Note that we also have: $\eta_m(i) = h_m(i) - h_{m-1}(i)$.

So that from vehicle $i-1$ to vehicle i, the headways at service stations satisfy:

$$\eta_m(i) = \eta_{m-1}(i) + \tau_a(i), \tag{2.2}$$

wherein $\tau_a(i) = t_a(i) - t_a(i-1)$ is the difference in travel time along a and m.

Service operations and exogenous influences may affect the distribution of τ_a and, in turn, that of η_m. The influences on the mean and variance are of crucial interest. By the linearity of expectation:

$$\mathrm{E}[\eta_m] = \mathrm{E}[\eta_{m-1}] + \mathrm{E}[\tau_a], \tag{2.3}$$

whereas, by the bi-linearity of covariance,

$$\mathrm{V}[\eta_m] = \mathrm{V}[\eta_{m-1}] + \mathrm{V}[\tau_a] + 2\,\mathrm{cov}(\eta_{m-1}, \tau_a). \tag{2.4}$$

Formula (2.2) and its consequences (2.3-4) state the propagation of vehicle headways from station to station.

2.3. On the conservation of headway rank

Of course, the conservation of schedule order is assumed along the line, under a First In – First Out discipline. Let us focus on the rank of each run in the "local" distribution of headway, characterized by the fractile $\alpha_m = \mathrm{H}_m(\eta_m)$. In this study, the postulate of conservation of headway rank is made:

$$\forall i, \forall m \neq n, \alpha_m(i) = \alpha_n(i) = \alpha(i). \tag{2.5}$$

This states that if a vehicle run is associated to a relatively low (resp. large) headway at a given station, it is associated to relatively low (resp. large) headways at all the stations of the line. However, local magnitudes may differ, only the rank remains stable.

The postulate is realistic enough in various instances:

- when the operations are regular along the line, the headway at the initial station is maintained from station to station.
- If most of traffic disruptions occur on a given section a, then the main source of variation pertains to τ_a and the rank in its distribution may be assumed to apply on the rest of the line as well.

The most noteworthy consequence is the functional dependency between the headways along the line:

$$\alpha_m = \mathrm{H}_m(\eta_m) = \alpha = \alpha_{m-1} = \mathrm{H}_{m-1}(\eta_{m-1}), \text{ hence} \tag{2.6}$$

$$\eta_m = \mathrm{H}_m^{(-1)} \circ \mathrm{H}_{m-1}(\eta_{m-1}). \tag{2.7}$$

Thus $\tau_a = \eta_m - \eta_{m-1}$ also is a function of η_{m-1}.

Assuming further that the dependency is linear, i.e. $\tau_a = \lambda\eta_{m-1} + \mu$ for some parameters $\lambda \geq 0$ and μ, then it would hold that

$$\mathrm{cov}(\eta_{m-1}, \tau_a) = \sigma[\eta_{m-1}].\sigma[\tau_a]. \tag{2.8}$$

This relationship notably holds for random variables η_{m-1} and τ_a that are distributed along a similar pattern, i.e. when $(\tau_a - \mathrm{E}[\tau_a])/\sigma[\tau_a] \approx (\eta_{m-1} - \mathrm{E}[\eta_{m-1}])/\sigma[\eta_{m-1}]$. This holds notably for perfectly correlated normal variables: in this case a valuable complementary property is that η_m is normal, too, yielding normal variables for headway and section time variation along the line.

2.4. Vehicle journey time with respect to schedule order

Let us turn to the journey time of each vehicle run with respect to its order in the schedule of operations, denoted by i. Let r denote a reference station and $m \geq r$ a subsequent station in the selected direction of traffic, $t_{rm}(i)$ be the journey time of vehicle run i between the instants of departure from r and m, $h_m(i)$ and $h_r(i)$ respectively. It holds that

$$t_{rm}(i) = h_m(i) - h_r(i), \text{ so } t_{rm}(i) = t_{rm}(i-1) + \eta_m(i) - \eta_r(i) \text{ and}$$
$$t_{rm}(i) = t_{rm}(0) + \sum_{j=1}^{i}\left(\eta_m(j) - \eta_r(j)\right), \tag{2.9}$$

wherein vehicle run #0 is an ideal vehicle run of nominal performance which immediately precedes the reference period. By the linearity of expectation, it then follows that

$$\mathrm{E}[T_{rm}(i)] = T_{rm}(0) + i(\mathrm{E}[\eta_m] - \mathrm{E}[\eta_r]). \tag{2.10}$$

Under the assumption that the $\alpha(i)$ are i.i.d., the runs are mutually independent, which implies that:

$$\mathrm{V}[T_{rm}(i)] = i.\mathrm{V}[\eta_m - \eta_r]. \tag{2.11}$$

Under the conservation of headway rank and the assumption of normality, $\eta_m = \eta_r + \sum_{a\in[r,m]}\tau_a$ satisfies that $\sigma[\eta_m] = \sigma[\eta_r] + \sum_{a\in[r,m]}\sigma[\tau_a]$, which entails that

$$\sigma[\eta_m] - \sigma[\eta_r] = \sum_{a\in[r,m]}\sigma[\tau_a] = \sigma[\eta_m - \eta_r]. \tag{2.12}$$

Combining (2.12) and (2.11), we get that

$$\sigma[T_{rm}(i)] = \sqrt{i}\ \sigma[\eta_m - \eta_r]. \tag{2.13}$$

Of course the assumptions of headway rank conservation and of run independence are likely to interfere in practice. However, eqns (2.10) and (2.13) give some insight into the progressive deterioration of the vehicle journey time with respect to the order of the run in the schedule of operations, when submitted to irregularity and random disruptions.

3. Vehicle loading

So far, two sources of variability have been made explicit: headway rank, denoted as α, and the order in the schedule, denoted as i. In this section, two other sources are introduced, namely the level of passenger transport demand, denoted as β, and the vehicle capacity, denoted as κ. Sources α and β jointly influence the vehicle load in passengers. Sources α, β and κ jointly influence the ratio of load to capacity by vehicle run.

This section establishes some analytical properties of the passenger load and load ratio along a transit line, by taking into account the demand (passenger flow) between stations of entry and exit.

3.1. Assumptions about passenger demand

A reference period of given duration is considered for line operations. In fact it refers in some average way to a population of periods, for instance the morning peak hour throughout a series of working days. To depict the variability of periods, let us associate to each period its level β of passenger demand, with CDF B in the population of periods. Within a given period, passenger flow is modeled as a stationary random process, with macroscopic properties as follows: between any pair $r < s$ of stations along the line, the passenger flow arriving at r and destined to s during time interval $[h, h']$ amounts to $\beta q_{rs}(h' - h)$. Thus the set of trip rates $[q_{rs} : r < s]$ describes the spatial structure of passenger demand per unit of time. Across the population of periods, we could define β so as to satisfy that $\mathrm{E}[\beta] = 1$; however we shall keep $\mathrm{E}[\beta]$ in the formulae for the sake of traceability.

3.2. Vehicle loading conditional on β

Assuming that passenger demand is not restrained by vehicle capacity, at each station r of entry a given vehicle run will attract incoming passengers in proportion to its local headway, η_r. On section a, the vehicle load denoted by y_a consists in those passengers having entered at station $r \le a$ and destined to station $s > a$ (with obvious notation for $\le$ and $>$ for position along the line):

$$y_{a,\beta} = \beta \sum_{r \le a, s > a} q_{rs} \eta_r \,. \qquad (3.1)$$

Then, on average:

$$\mathrm{E}[y_{a,\beta}] = \beta \sum_{r \le a, s > a} q_{rs}\, \mathrm{E}[\eta_r]\,. \qquad (3.2)$$

Keeping to the postulate of conservation of headway rank, the vehicle run is characterized by its fractile α so that $\eta_r = \mathrm{H}_r^{(-1)}(\alpha)$. Then

$$y_{a,\beta}(\alpha) = \beta \sum_{r \le a, s > a} q_{rs}\, \mathrm{H}_r^{(-1)}(\alpha)\,. \qquad (3.3)$$

Denote by $\mathrm{Y}_{a,\beta}$ the CDF of y_a conditional on β. Then:

$$\mathrm{Y}_{a,\beta}^{-1} = \beta \sum_{r \le a, s > a} q_{rs}\, \mathrm{H}_r^{(-1)}\,. \qquad (3.4)$$

Furthermore, as in the previous section the sum of totally dependent random variables sharing a Gaussian pattern satisfies that

$$\sigma[y_{a,\beta}] = \beta \sum_{r \le a, s > a} q_{rs} \sigma[\eta_r]\,. \qquad (3.5)$$

3.3. Vehicle loading, overall distribution

Let us now aggregate the analysis with respect to β. Denoting $\xi_a = \sum_{r \le a, s \ge a} q_{rs} \eta_r$ the random variable of reference link flow and by X_a its CDF, it holds generally that:

$$\mathrm{Y}_a(z) = \Pr\{\beta \xi_a \le z\} = \int \mathrm{X}_a(z/\beta)\, \mathrm{dB}(\beta)\,. \qquad (3.6)$$

In reality, demand level β may influence vehicle operations – for instance because the number of boarding and alighting passengers may determine the dwelling time. However, for simplicity, independence is assumed in this model, yielding that:

$$\mathrm{E}[y_a] = \mathrm{E}[\beta] \,.\, \sum_{r \le a, s > a} q_{rs}\, \mathrm{E}[\eta_r]\,. \qquad (3.7)$$

$\mathrm{V}[y_a] = \mathrm{V}[\beta] \,.\, \mathrm{E}[\xi_a]^2 + \mathrm{E}[\beta^2] \,.\, \mathrm{V}[\xi_a]$ due to $\mathrm{V}[XY] = \mathrm{E}[X^2]\mathrm{V}[Y] + \mathrm{E}[Y]^2 \mathrm{V}[X]$, hence

$$\mathrm{V}[y_a] = \mathrm{V}[\beta] \,.\, \left(\sum_{r \le a, s > a} q_{rs}\, \mathrm{E}[\eta_r]\right)^2 + \mathrm{E}[\beta^2] \,.\, \left(\sum_{r \le a, s > a} q_{rs} \sigma[\eta_r]\right)^2 . \qquad (3.8)$$

To gain further insight into the structure of influences, let us add to the assumption of Gaussian headways the approximation of the resulting flow, ξ_a, by a log-normal variable with same mean and standard deviation, $\mathrm{E}[\xi_a]$ and $\sigma[\xi_a]$. Denote by m_a and s_a, respectively, the mean and standard deviation of $\ln \xi_a$. From classical properties of log-normal distributions, these are related to the moments of ξ_a by:

$$\mathrm{E}[\xi_a] = \exp(m_a + \tfrac{1}{2} s_a^2)$$

$$\sigma^2[\xi_a] = (\exp(s_a^2) - 1) . \mathrm{E}[\xi_a]^2$$

Assuming lastly that $\beta \approx \mathrm{LN}(m_\beta, s_\beta)$, then the link load $\beta \xi_a \approx \mathrm{LN}(m_\beta + m_a, \sqrt{s_\beta^2 + s_a^2})$.

3.4. Vehicle loading ratio

Vehicle capacity, denoted as κ, pertains to the number of seats plus a reference number of positions for passenger standing with sufficient comfort (e.g. 4 persons per square meter). Heterogeneous vehicles may be used to operate the transit line, leading to the variability of capacity hence of the ratio of passenger load to capacity. Let us denote that ratio as

$$z_a = y_a / \kappa = \beta \xi_a / \kappa . \tag{3.9}$$

While it is quite natural to assume the independence of β and κ, it would be a wise policy of line operations to assign vehicle types according to the planned headways, by associating larger capacity to larger headways so as to balance the load ratio across the runs. Under such a balancing policy, the load ratio could be analyzed in the same way as vehicle load by replacing $\eta_r(\alpha)$ with $\eta_r(\alpha)/\kappa_\alpha$. On the contrary, a negligent policy may be modeled by assuming independence between κ and α as well as β, yielding straightforward consequences on the mean and variance of load ratio (eqns. (3.10-11) in[0]).

4. Passenger exposition to traffic conditions

Let us come to the perspective of the user at the level of the individual trip, as opposed to the operator's one at the level of the vehicle run.

4.1. User's exposure

Let us recall some basic properties of renewal theory (e.g. [7] pp. 169 *sq*). Denote by $\mathrm{H}_r^{\mathrm{o}}$ the CDF of headway duration η_r and by $\dot{\mathrm{H}}_r^{\mathrm{o}}$ its PDF, with superscript o to mark the operator's perspective. A user willing to board at r arrives on platform at a random instant, which will belong to a headway interval of duration η with a probability proportional to η: in the user's perspective, marked by superscript u,

$$\dot{\mathrm{H}}_r^{\mathrm{u}}(\eta) \propto \eta \dot{\mathrm{H}}_r^{\mathrm{o}}(\eta) . \tag{4.1}$$

By integration, the factor of proportionality amounts to $1/\mathrm{E}[\eta_r^{\mathrm{o}}]$. The moments of η_r^{u} stem from those of η_r^{o} at the next order:

$$\mathrm{E}[(\eta_r^{\mathrm{u}})^k] = \mathrm{E}[(\eta_r^{\mathrm{o}})^{k+1}] / \mathrm{E}[\eta_r^{\mathrm{o}}] . \tag{4.2}$$

Consider now the size of the passenger group that includes the individual user, to board in a vehicle run at station r, n_r^{u}. Its probability density stems from the density $\mathrm{f}^{\mathrm{u}}(\beta,\eta)$ of pair (β,η), which is related to the PDF $\mathrm{f}^{\mathrm{o}}(\beta,\eta)$ in the following way:

$$\mathrm{f}^{\mathrm{u}}(\beta,\eta) \propto \beta\eta\, \mathrm{f}^{\mathrm{o}}(\beta,\eta) , \tag{4.3}$$

wherein f^{o} is the PDF of passenger group sizes from the perspective of the operator. Assuming independence between β and η, then $\mathrm{f}^{\mathrm{o}}(\beta,\eta) = \dot{\mathrm{B}}^{\mathrm{o}}(\beta).\dot{\mathrm{H}}_r^{\mathrm{o}}(\eta)$: thus independence is maintained in the user's perspective, since

$$\mathrm{f}^{\mathrm{u}}(\beta,\eta) \propto \beta\eta\, \dot{\mathrm{B}}^{\mathrm{o}}(\beta).\dot{\mathrm{H}}_r^{\mathrm{o}}(\eta) = \dot{\mathrm{B}}^{\mathrm{u}}(\beta).\dot{\mathrm{H}}_r^{\mathrm{u}}(\eta) . \tag{4.4}$$

in which $\dot{\mathrm{B}}^{\mathrm{u}}(\beta) = \beta \dot{\mathrm{B}}^{\mathrm{o}}(\beta)/\mathrm{E}[\beta^{\mathrm{o}}]$ and $\dot{\mathrm{H}}_r^{\mathrm{u}}(\eta) = \eta \dot{\mathrm{H}}_r^{\mathrm{o}}(\eta)/\mathrm{E}[\eta_r^{\mathrm{o}}]$. As $n_r^{\mathrm{u}} = \beta.\eta$, its CDF is

$$\mathrm{N}_r^{\mathrm{u}}(x) = \Pr\{\beta\eta \leq x\} = \int \mathrm{N}_{r,\beta}^{\mathrm{u}}(x) \mathrm{dB}^{\mathrm{u}}(\beta) = \int \mathrm{H}_r^{\mathrm{u}}(x/\beta) \mathrm{dB}^{\mathrm{u}}(\beta) . \tag{4.5}$$

The independence property enables us to establish the mean and variance of group size as follows:

$$\mathrm{E}[n_r^{\mathrm{u}}] = \mathrm{E}[\beta^{\mathrm{u}}]\mathrm{E}[\eta_r^{\mathrm{u}}] = \frac{\mathrm{E}[(\beta^{\mathrm{o}})^2]}{\mathrm{E}[\beta^{\mathrm{o}}]} \frac{\mathrm{E}[(\eta_r^{\mathrm{o}})^2]}{\mathrm{E}[\eta_r^{\mathrm{o}}]} . \tag{4.6}$$

$$\mathrm{V}[n_r^{\mathrm{u}}] = \mathrm{E}[\beta^{\mathrm{u}2}]\mathrm{V}[\eta_r^{\mathrm{u}}] + \mathrm{V}[\beta^{\mathrm{u}}]\mathrm{E}[\eta_r^{\mathrm{u}}]^2$$
$$= \frac{\mathrm{E}[(\beta^{\mathrm{o}})^3]}{\mathrm{E}[\beta^{\mathrm{o}}]}\left(\frac{\mathrm{E}[(\eta_r^{\mathrm{o}})^3]}{\mathrm{E}[\eta_r^{\mathrm{o}}]} - \frac{\mathrm{E}[(\eta_r^{\mathrm{o}})^2]^2}{\mathrm{E}[\eta_r^{\mathrm{o}}]^2}\right) + \left(\frac{\mathrm{E}[(\eta_r^{\mathrm{o}})^2]}{\mathrm{E}[\eta_r^{\mathrm{o}}]}\right)^2\left(\frac{\mathrm{E}[(\beta^{\mathrm{o}})^3]}{\mathrm{E}[\beta^{\mathrm{o}}]} - \frac{\mathrm{E}[(\beta^{\mathrm{o}})^2]^2}{\mathrm{E}[\beta^{\mathrm{o}}]^2}\right) \tag{4.7}$$

4.2. Vehicle load by link as experienced by the user

Depending on his entry station e, the user travelling along link $a \geq e$ experiences there a vehicle passenger load as follows, wherein $\eta_{r,e}^{\mathrm{u}}$ depends on the entry station:

$$y_{a,e}^{\mathrm{u}} = \beta \sum_{r\leq a, s>a} q_{rs}\eta_{r,e}^{\mathrm{u}} . \tag{4.8}$$

Given the value η of η_e^{u}, the vehicle run has headway rank $\alpha = \mathrm{H}_e^{\mathrm{o}}(\eta)$. The conservation postulate in the operator's perspective is maintained in the user's perspective, yielding that $\eta_{r,e}^{\mathrm{u}} = \mathrm{H}_r^{\mathrm{o}(-1)}(\alpha)$. Then, conditionally to η:

$$y_{a,e,\beta,\eta}^{\mathrm{u}} = \beta \sum_{r\leq a, s>a} q_{rs}\, \mathrm{H}_r^{\mathrm{o}(-1)} \circ \mathrm{H}_e^{\mathrm{o}}(\eta) . \tag{4.9}$$

Eqn (4.9) must be distinguished clearly from (3.3), in which α is the order of fractile and can be seen as a random variable distributed uniformly on [0, 1].
From the equation above stems the unconditional variable $y_{a,e}^{\mathrm{u}}$. Its CDF is given by:

$$\mathrm{Y}_{a,e}^{\mathrm{u}}(z) = \Pr\{y_{a,e}^{\mathrm{u}} \leq z\} = \int \Pr\{y_{a,e,\beta,\eta}^{\mathrm{u}} \leq z\}\; \mathrm{f}^{\mathrm{u}}(\beta,\eta)\,\mathrm{d}\eta\,\mathrm{d}\beta . \tag{4.10}$$

By successive transformations:

$$\begin{aligned} y_{a,e,\beta,\eta}^{\mathrm{u}} \leq z \;&\Leftrightarrow \sum_{r\leq a,s>a} q_{rs}\, \mathrm{H}_r^{\mathrm{o}(-1)}(\alpha_\eta) \leq z/\beta \\ &\Leftrightarrow \mathrm{X}_a^{\mathrm{o}(-1)}(\alpha_\eta) \leq z/\beta \quad \text{using the inverse CDF of } \mathrm{X}_a^{\mathrm{o}} \text{ on the basis of (3.4) and (3.6)} \\ &\Leftrightarrow \alpha_\eta \leq \mathrm{X}_a^{\mathrm{o}}(z/\beta) \\ &\Leftrightarrow \eta \leq \mathrm{H}_e^{\mathrm{o}(-1)} \circ \mathrm{X}_a^{\mathrm{o}}(z/\beta) \end{aligned}$$

Thus $\Pr\{y_{a,e,\beta}^{\mathrm{u}} \leq z\} = \mathrm{H}_e^{\mathrm{u}} \circ \mathrm{H}_e^{\mathrm{o}(-1)} \circ \mathrm{X}_a^{\mathrm{o}}(z/\beta)$, yielding that

$$\mathrm{Y}_{a,e,\beta}^{\mathrm{u}}(z) = \mathrm{H}_e^{\mathrm{u}} \circ \mathrm{H}_e^{\mathrm{o}(-1)} \circ \mathrm{X}_a^{\mathrm{o}}(z/\beta), \tag{4.11a}$$

$$\mathrm{Y}_{a,e}^{\mathrm{u}}(z) = \int \Pr\{y_{a,e,\beta}^{\mathrm{u}} \leq z\}\; \mathrm{f}^{\mathrm{u}}(\beta)\mathrm{d}\beta = \int \mathrm{H}_e^{\mathrm{u}} \circ \mathrm{H}_e^{\mathrm{o}(-1)} \circ \mathrm{X}_a^{\mathrm{o}}(z/\beta)\; \mathrm{dB}^{\mathrm{u}}(\beta) . \tag{4.11b}$$

4.3. Specific properties

In [0] the distribution of vehicle load has been explored by assuming log-normal headways and log-normal vehicle loads in the operator perspective, yielding log-normal vehicle load in the user perspective. In the general case, by inverting (4.11a), we get that

$$\mathrm{Y}_{a,e,\beta}^{\mathrm{u}(-1)}(\zeta) = \beta\, \mathrm{X}_a^{\mathrm{o}(-1)} \circ \mathrm{H}_e^{\mathrm{o}} \circ \mathrm{H}_e^{\mathrm{u}(-1)}(\zeta) . \tag{4.12}$$

Then, conditional on β, the expectation of any function f of y is

$$\mathrm{E}[\mathrm{f}(y_{a,e,\beta}^{\mathrm{u}})] = \int_0^1 \mathrm{f}(\mathrm{Y}_{a,e,\beta}^{\mathrm{u}(-1)}(\zeta))\mathrm{d}\zeta .$$

Let us change variables by letting $\zeta = \mathrm{F}_e(\alpha)$ where $\mathrm{F}_e = \mathrm{H}_e^{\mathrm{u}} \circ \mathrm{H}_e^{\mathrm{o}(-1)}$. By construct,

$$\begin{aligned} \frac{\mathrm{d}\,\mathrm{F}_e(\alpha)}{\mathrm{d}\alpha} &= \dot{\mathrm{H}}_e^{\mathrm{o}(-1)}(\alpha).\dot{\mathrm{H}}_e^{\mathrm{u}} \circ \mathrm{H}_e^{\mathrm{o}(-1)}(\alpha) \\ &= \dot{\mathrm{H}}_e^{\mathrm{o}(-1)}(\alpha).\frac{\mathrm{H}_e^{\mathrm{o}(-1)}(\alpha)}{\overline{\eta}_e^{\mathrm{o}}}\dot{\mathrm{H}}_e^{\mathrm{o}} \circ \mathrm{H}_e^{\mathrm{o}(-1)}(\alpha) \quad \text{since } \dot{\mathrm{H}}_e^{\mathrm{u}}(x) = \dot{\mathrm{H}}_e^{\mathrm{o}}(x)/\overline{\eta}_e^{\mathrm{o}} \\ &= \dot{\mathrm{H}}_e^{\mathrm{o}(-1)}(\alpha)/\overline{\eta}_e^{\mathrm{o}} \quad \text{since } \mathrm{H}_e^{\mathrm{o}(-1)}(\alpha).\dot{\mathrm{H}}_e^{\mathrm{o}} \circ \mathrm{H}_e^{\mathrm{o}(-1)}(\alpha) = \frac{\mathrm{d}\,\mathrm{H}_e^{\mathrm{o}} \circ \mathrm{H}_e^{\mathrm{o}(-1)}(\alpha)}{\mathrm{d}\alpha} = \frac{\mathrm{d}\alpha}{\mathrm{d}\alpha} = 1 \end{aligned}$$

Then, as $\mathrm{Y}^{\mathrm{u}(-1)}_{a,e,\beta}(\zeta) = \mathrm{Y}^{\mathrm{o}(-1)}_{a,e,\beta}(\alpha)$ if $\zeta = \mathrm{F}_e(\alpha)$,

$$\mathrm{E}[\mathrm{f}(y^{\mathrm{u}}_{a,e,\beta})] = \left[\int_0^1 \mathrm{f}(\mathrm{Y}^{\mathrm{o}(-1)}_{a,e,\beta}(\alpha))\,\mathrm{H}^{\mathrm{o}(-1)}_e(\alpha)\,\mathrm{d}\alpha\right] / \overline{\eta}^{\mathrm{o}}_e . \tag{4.13}$$

To sum up,

$$\mathrm{E}[\mathrm{f}(y^{\mathrm{u}}_{a,e,\beta})] = \mathrm{E}[\mathrm{f}(y^{\mathrm{o}}_{a,e,\beta}).\eta^{\mathrm{o}}_e] / \mathrm{E}[\eta^{\mathrm{o}}_e] . \tag{4.14}$$

Furthermore, as $\mathrm{E}[X.Y] = \mathrm{E}[X].\mathrm{E}[Y] + \mathrm{cov}(X,Y)$, (4.14) is equivalent to

$$\mathrm{E}[\mathrm{f}(y^{\mathrm{u}}_{a,e,\beta})] = \mathrm{E}[\mathrm{f}(y^{\mathrm{o}}_{a,e,\beta})] + \mathrm{cov}(\mathrm{f}(y^{\mathrm{o}}_{a,e,\beta}), \eta^{\mathrm{o}}_e) / \overline{\eta}^{\mathrm{o}}_e . \tag{4.15}$$

This general property enables us to derive the conditional moments of $y^{\mathrm{u}}_{a,e,\beta}$. At the first order:

$$\begin{aligned}
\mathrm{cov}(y^{\mathrm{o}}_{a,e,\beta}, \eta^{\mathrm{o}}_e)/\overline{\eta}^{\mathrm{o}}_e &= \beta \sum_{r\le a<s} q_{rs}\,\mathrm{cov}(\eta^{\mathrm{o}}_r, \eta^{\mathrm{o}}_e)\frac{1}{\overline{\eta}^{\mathrm{o}}_e} \text{ by the bilinearity of covariance}\\
&= \beta \sum_{r\le a<s} q_{rs}\,\mathrm{cov}\Big(\sigma^{\mathrm{o}}_r \frac{\eta^{\mathrm{o}}_r - \overline{\eta}^{\mathrm{o}}_r}{\sigma^{\mathrm{o}}_r}, \sigma^{\mathrm{o}}_e \frac{\eta^{\mathrm{o}}_e - \overline{\eta}^{\mathrm{o}}_e}{\sigma^{\mathrm{o}}_e}\Big)\frac{1}{\overline{\eta}^{\mathrm{o}}_e}\\
&= \beta \sum_{r\le a<s} q_{rs}\sigma^{\mathrm{o}}_r\,\gamma^{\mathrm{o}}_e\,\mathrm{V}[\eta'] \text{ since } \frac{\eta^{\mathrm{o}}_r - \overline{\eta}^{\mathrm{o}}_r}{\sigma^{\mathrm{o}}_r} = \frac{\eta^{\mathrm{o}}_e - \overline{\eta}^{\mathrm{o}}_e}{\sigma^{\mathrm{o}}_e} \equiv \eta' \text{ due to total correlation}\\
&= \gamma^{\mathrm{o}}_e\,\sigma[y^{\mathrm{o}}_{a,e,\beta}] \text{ because } \mathrm{V}[\eta'] = 1 \text{ by construct}
\end{aligned}$$

So:

$$\mathrm{E}[y^{\mathrm{u}}_{a,e,\beta}] = \overline{y}^{\mathrm{o}}_{a,e,\beta} + \gamma^{\mathrm{o}}_e\,\sigma[y^{\mathrm{o}}_{a,e,\beta}] = \overline{y}^{\mathrm{o}}_{a,e,\beta}(1 + \gamma^{\mathrm{o}}_a\,\gamma^{\mathrm{o}}_e) \text{ wherein } \gamma^{\mathrm{o}}_a \equiv \sigma[y^{\mathrm{o}}_{a,e,\beta}] / \overline{y}^{\mathrm{o}}_{a,e,\beta} . \tag{4.16}$$

At the second order,

$$\begin{aligned}
\mathrm{cov}((y^{\mathrm{o}}_{a,e,\beta})^2, \eta^{\mathrm{o}}_e)/\overline{\eta}^{\mathrm{o}}_e &= \beta^2 \textstyle\sum_{i,j} q_i q_j\,\mathrm{cov}(\eta^{\mathrm{o}}_i \eta^{\mathrm{o}}_j, \eta^{\mathrm{o}}_e)/\overline{\eta}^{\mathrm{o}}_e\\
&= \beta^2\,\mathrm{E}\Big[\textstyle\sum_{i,j} q_i q_j [(\eta^{\mathrm{o}}_i - \overline{\eta}^{\mathrm{o}}_i)(\eta^{\mathrm{o}}_j - \overline{\eta}^{\mathrm{o}}_j) + \overline{\eta}^{\mathrm{o}}_i(\eta^{\mathrm{o}}_j - \overline{\eta}^{\mathrm{o}}_j) + \overline{\eta}^{\mathrm{o}}_j(\eta^{\mathrm{o}}_i - \overline{\eta}^{\mathrm{o}}_i)].\eta'\gamma^{\mathrm{o}}_e\Big]\\
&= \gamma^{\mathrm{o}}_e\,\beta^2\,\mathrm{E}\Big[\textstyle\sum_{i,j} q_i q_j (\sigma^{\mathrm{o}}_i \sigma^{\mathrm{o}}_j \eta'^2 + \overline{\eta}^{\mathrm{o}}_i \sigma^{\mathrm{o}}_j \eta' + \sigma^{\mathrm{o}}_i \overline{\eta}^{\mathrm{o}}_j \eta').\eta'\Big]\\
&= \gamma^{\mathrm{o}}_e \Big[\mathrm{V}[y^{\mathrm{o}}_{a,e,\beta}]\,\mathrm{E}[\eta'^3] + 2\overline{y}^{\mathrm{o}}_{a,e,\beta}\sigma[y^{\mathrm{o}}_{a,e,\beta}]\,\mathrm{V}[\eta']\Big]\\
&= \gamma^{\mathrm{o}}_e\,(\overline{y}^{\mathrm{o}}_{a,e,\beta})^2 (\gamma^{\mathrm{o}\,2}_a\,\mathrm{E}[\eta'^3] + 2\gamma^{\mathrm{o}}_a)
\end{aligned}$$

So:

$$\mathrm{E}[(y^{\mathrm{u}}_{a,e,\beta})^2] = \mathrm{E}[(y^{\mathrm{o}}_{a,e,\beta})^2] + (\overline{y}^{\mathrm{o}}_{a,e,\beta})^2 \gamma^{\mathrm{o}}_e\,(\gamma^{\mathrm{o}\,2}_a\,\mathrm{E}[\eta'^3] + 2\gamma^{\mathrm{o}}_a) , \tag{4.17a}$$

$$\mathrm{V}[y^{\mathrm{u}}_{a,e,\beta}] = \mathrm{V}[y^{\mathrm{o}}_{a,e,\beta}].[1 + \gamma^{\mathrm{o}}_e\,\mathrm{E}[\eta'^3] - \gamma^{\mathrm{o}\,2}_e] . \tag{4.17b}$$

Assuming that ξ^{o}_a and β are statistically independent, then the unconditional moments of the vehicle load in the user perspective are derived from their conditional counterparts in the following way:

$$\mathrm{E}[y^{\mathrm{u}}_{a,e}] = \mathrm{E}[\beta^{\mathrm{u}}]\,\overline{y}^{\mathrm{o}}_{a,e,\beta}(1 + \gamma^{\mathrm{o}}_a\,\gamma^{\mathrm{o}}_e) , \tag{4.18a}$$

$$\mathrm{V}[y^{\mathrm{u}}_{a,e,\beta}] = \mathrm{V}[\beta^{\mathrm{u}}].[\overline{\xi}^{\mathrm{o}}_{a,e}.(1 + \gamma^{\mathrm{o}}_a\,\gamma^{\mathrm{o}}_e)]^2 + \mathrm{E}[\beta^{\mathrm{u}\,2}].\mathrm{V}[\xi^{\mathrm{o}}_{a,e}].[1 + \gamma^{\mathrm{o}}_e\,\mathrm{E}[\eta'^3] - \gamma^{\mathrm{o}\,2}_e] . \tag{4.18b}$$

Formula (4.18a) was obtained as an approximation in [0] under the heuristic assumptions of first normal then log-normal distribution of headways: it turns out that it is an exact property in the general case.

4.4. Run time

In section 2.3 some statistical properties of run time have been established for vehicles: schedule order i determines the mean and variance of run time $T_{rs}(i)$. Any user that arrives at station r at a given instant

h will board a vehicle of order i which is random due to irregularity, so he will get a random run time. The precise definition of $i(h)$ as a random variable is difficult except for Markovian vehicle runs which would yield a Poisson distribution but at the price of assuming a large amount of variability. For simplicity, let us assume here that $i(h)$ has a uniform discrete distribution derived from $i \approx 1+\mathrm{int}[(h-h_0)/\mathrm{E}[\eta_r]]$ on the reference period $[h_0, h_1]$. Let $I = i(h_1)$ and $1/I$ be the elemental probability of $i \in \{1,..I\}$. Let also $\Delta\mathrm{E} = \mathrm{E}[\eta_s] - \mathrm{E}[\eta_r]$. The average run time is

$$\mathrm{E}[t_{rs}^{\mathrm{u}}] = \tfrac{1}{I}\textstyle\sum_{i=1}^{I} \mathrm{E}[t_{rs}(i)] = t_{rm}(0) + (\mathrm{E}[\eta_s] - \mathrm{E}[\eta_r])\tfrac{1}{I}\textstyle\sum_{i=1}^{I} i = t_{rm}(0) + \tfrac{I+1}{2}\Delta\mathrm{E} \quad . \tag{4.19}$$

By the law of total variance, the variance of the run time is made of an interclass part plus an intra-class part in the following way, in which $\Delta\sigma = \sigma[\eta_s - \eta_r]$:

$$\begin{aligned} \mathrm{V}[t_{rs}^{\mathrm{u}}] &= \tfrac{1}{I}\textstyle\sum_{i=1}^{I} (\mathrm{E}[t_{rs}(i)] - \mathrm{E}[t_{rs}^{\mathrm{u}}])^2 + \tfrac{1}{I}\textstyle\sum_{i=1}^{I} \sigma^2[t_{rs}(i)] \\ &= \tfrac{1}{I}\Delta\mathrm{E}^2 \textstyle\sum_{i=1}^{I} (i - \tfrac{I+1}{2})^2 \qquad + \tfrac{1}{I}\textstyle\sum_{i=1}^{I} i(\Delta\sigma)^2 \\ &= \Delta\mathrm{E}^2 \tfrac{I^2-1}{24} + (\Delta\sigma)^2 \tfrac{I+1}{2} = \tfrac{I+1}{2}\left(\Delta\mathrm{E}^2 \tfrac{I-1}{12} + (\Delta\sigma)^2\right) \end{aligned} \tag{4.20}$$

4.5. Wait time

The user wait time on the station platform, w_e, amounts to the residual span (or lifetime) of the on-going headway interval. From survival theory, its PDF is

$$\dot{\mathrm{W}}_e(x) = \left(1 - \mathrm{H}_e^{\mathrm{o}}(x)\right)/\mathrm{E}[\eta_e^{\mathrm{o}}] \tag{4.21}$$

This leads to the following relationships between the moments of the two variables:

$$\mathrm{E}[{w_e}^k] = \frac{\mathrm{E}[(\eta_e^{\mathrm{o}})^{k+1}]}{(k+1)\,\mathrm{E}[\eta_e^{\mathrm{o}}]} = \mathrm{E}[(\eta_e^{\mathrm{u}})^k]/(k+1) \tag{4.22}$$

So it holds that

$$\mathrm{E}[w_e] = \mathrm{E}[(\eta_e^{\mathrm{o}})^2]/2\mathrm{E}[\eta_e^{\mathrm{o}}] = \tfrac{1}{2}\mathrm{E}[\eta_e^{\mathrm{u}}], \tag{4.23}$$

$$\mathrm{V}[w_e] = \tfrac{1}{3}\mathrm{E}[(\eta_e^{\mathrm{u}})^2] - \tfrac{1}{4}\mathrm{E}[\eta_e^{\mathrm{u}}]^2 = \tfrac{1}{4}\mathrm{V}[\eta_e^{\mathrm{u}}] + \tfrac{1}{12}\mathrm{E}[(\eta_e^{\mathrm{u}})^2]. \tag{4.24}$$

Furthermore, η_e^{u} is correlated to w_e and so are the headway rank and all derived variables such as $y_{a,e}^{\mathrm{o/u}}$. For instance, $\mathrm{E}[w_e.\eta_e^{\mathrm{u}}] = \mathrm{E}[(\eta_e^{\mathrm{u}})^2]/2$ so $\mathrm{cov}[w_e, \eta_e^{\mathrm{u}}] = \mathrm{V}[\eta_e^{\mathrm{u}}]/2$.

4.6. Travel time

The travel time of a user between stations r and s is composed by the wait time at r, w_r, plus the run time between the two stations, t_{rs}^{u}:

$$\tilde{t}_{rs}^{\mathrm{u}} = w_r + t_{rs}^{\mathrm{u}}. \tag{4.25}$$

By the linearity of expectation,

$$\mathrm{E}[\tilde{t}_{rs}^{\mathrm{u}}] = \mathrm{E}[w_r] + \mathrm{E}[t_{rs}^{\mathrm{u}}]. \tag{4.26}$$

There may be some correlation between the two components. However independence may be assumed as a crude approximation, yielding:

$$\mathrm{V}[\tilde{t}_{rs}^{\mathrm{u}}] = \mathrm{V}[w_r] + \mathrm{V}[t_{rs}^{\mathrm{u}}]. \tag{4.27}$$

4.7. Platform crowding

A related issue pertains to the number of passengers waiting on platform at a given station r. At any instant, this number is proportional to the level of the incoming flow, $\beta.\sum_{s>r} q_{rs}$, times the time elapsed since the departure of the last vehicle. From survival theory [7], the latter is the random variable $\eta_r^{\mathrm{u}} - w_r$. Thus the passenger stock amounts to

$$S_r = \beta .(\sum_{s>r} q_{rs})(\eta_r^{\mathrm{u}} - w_r) . \tag{4.28}$$

Independence of β and η_r implies that $\eta_r^{\mathrm{u}} - w_r$ is independent of β, yielding

$$\mathrm{E}[S_r] = \tfrac{1}{2}\mathrm{E}[\eta_r^{\mathrm{u}}]\mathrm{E}[\beta](\sum_{s>r} q_{rs}) \tag{4.29a}$$

$$\mathrm{V}[S_r] = (\sum_{s>r} q_{rs})^2 \left[\mathrm{E}[\beta^2]\mathrm{V}[\eta_r^{\mathrm{u}}] + \mathrm{V}[\beta].\mathrm{E}[\eta_r^{\mathrm{u}}]^2\right]. \tag{4.29b}$$

The perspective of either the operator or the user is specified by setting the adequate distribution of β.

5. On passenger generalized time

To a trip-maker, the "generalized time" of travel is a comprehensive disutility to capture both the physical travel time and the quality of service during the trip. Each physical state (e.g. sitting in-vehicle) or transition (e.g. vehicle egress) within the trip sequence, is associated with a specific penalty factor: from 1 for sitting in-vehicle to 2 for standing in-vehicle under dense crowding or more for waiting in crowd with no traffic information. The physical time spent in a given state is multiplied by its penalty factor to yield the generalized time of that state. This is aggregated along the trip sequence to yield the generalized time of the trip. It is used in discrete choice models of network route or transportation mode. It is also the basis to evaluate the benefits and costs of a transport plan to the community.

5.1. The formation of generalized time

The notion of generalized time involves penalty factors that vary across the individual trip-makers. Small persons resent standing in a crowd more than tall ones do. In general, old persons move and walk more slowly than younger ones. People are more or less sensitive to fatigue. Let ε denote the particular sensitivity of a given individual. Wait time w_r and link time t_a are transformed into generalized times, denoted as $\omega_{r\varepsilon}$ and $\theta_{a\varepsilon}$, respectively. The generalized travel time amounts to

$$\lambda_{rs,\varepsilon} = \omega_{r\varepsilon} + \sum_{a\in]r,s[} \theta_{a\varepsilon} . \tag{5.1}$$

To model the dependency of ω and θ on the crowding density, assume that

$$\omega_{r\varepsilon} = w_r\, \psi_{r\varepsilon}(S_r) , \tag{5.2}$$

$$\theta_{a\varepsilon} = t_a^{\mathrm{u}}\, \varphi_{a\varepsilon}(y_a^{\mathrm{u}}, \kappa) . \tag{5.3}$$

Formulae (5.1-3) provide a basis to analyze the influence of passenger flow on travel disutility. Taking wait time and link time as random variables, then so are $\omega_{r\varepsilon}$, $\theta_{a\varepsilon}$ and $\lambda_{rs,\varepsilon}$ conditionally to ε. From the previous section, w_r and S_r are correlated. Link loads y_a^{u} along successive links are correlated, too. Furthermore, platform variables and link loads are correlated due to headway rank. As all the correlations are positive, the generalized travel time conditionally to ε is subject to large relative dispersion.

5.2. In-vehicle discomfort

Let us focus on in-vehicle time and the influence of crowding density on its specific penalty factor. A well-known model is the so-called BPR function [8]:

$$\varphi_{a1}(y_a^{\mathrm{u}}, \kappa) = 1 + c_a .(y_a / \kappa)^{b_a} , \tag{5.4}$$

in which exponent b_a takes positive values such as 1 or 4, whereas factor c_a takes positive values between 0 and 3 typically. Formulae (5.4) and (5.3) state that crowding discomfort inflicts a specific additional cost of $t_a^{\mathrm{u}}\, c_a\, (y_a / \kappa)^{b_a}$ to the physical link time. In the operator's perspective (resp. the user's one), the average additional cost is evaluated as

$$\mathrm{SC}^{\mathrm{o/u}} = \mathrm{E}[t_a^{\mathrm{o/u}}\, c_a\, (y_a^{\mathrm{o/u}} / \kappa)^{b_a}] = c_a\, \mathrm{E}[t_a^{\mathrm{o/u}}]\mathrm{E}[(y_a^{\mathrm{o/u}} / \kappa)^{b_a}] . \tag{5.5}$$

Assuming that capacity is homogeneous, the two notions differ by a ratio of

$$\mathrm{SC^u/SC^o} = \mathrm{E}[y_a^{\mathrm{u}\;b_a}]/\mathrm{E}[y_a^{\mathrm{o}\;b_a}]. \tag{5.6}$$

Using the log-normal approximation, $y^b \approx \mathrm{LN}(bm_y, bs_y)$ so

$$\mathrm{SC^u}/\mathrm{SC^o} = \exp[bs_\beta^2 + bs_a s_e^\mathrm{o}] = \rho^b \quad \text{wherein} \quad \rho = \exp(s_a s_e^\mathrm{o} + s_\beta^2). \tag{5.7}$$

5.3. Numerical instance

To fix ideas, let us assume that $\gamma_a = \gamma_e^\mathrm{o} = 0.3$ and $\gamma_\beta = 0.2$, yielding $s_a = s_e^\mathrm{o} \approx 0.3$ and $s_\beta \approx 0.2$. Then $\rho = 1.13$ and the ratio of sur-costs is varied from 1.13 to 1.65 as b is changed from 1 to 4. Fig. 1 depicts the variation of the disutility factor φ_{a1} with respect to the apparent occupancy ratio, $\mathrm{E}[y_a^\mathrm{o}/\kappa]$. For a given apparent ratio, the experienced crowding density is equal to the disutility factor at $b=1$ and $c=1$, minus one: it differs from the apparent ratio in a significant yet not major amount. Irregularity also affects the base travel time, $\mathrm{E}[t_a]$. Between stations r and s, from (4.14) the related additional cost amounts to $\mathrm{ST} = (I+1)(\mathrm{E}[\eta_s] - \mathrm{E}[\eta_r])/2$. Denoting by f_r the service frequency delivered at station r during a reference period of length H, $I = f_r$ and $\mathrm{E}[\eta_r] = H/f_r$ while $\mathrm{E}[\eta_s] = H/f_s$. Then, $\mathrm{ST} \approx H.(f_r/f_s - 1)/2$. For instance, along the line A of the regional railways in the Paris area, at the morning peak hour westwards, the service frequency is reduced from $f_r = 30$/hour upstream of the centre, to $f_s = 27$/hour downstream. The resulting additional time is about 3' per trip. The train capacity is about 2,000 passengers and the apparent occupancy ratio of 83% upstream. The additional cost per trip, from nominal quality of service of $T_0 = 15'$ to personal experience, amounts to $(T_0 + \mathrm{ST}).(1 + c(\mathrm{E}[y_a^\mathrm{u}]/\kappa)^b) - T_0 \approx 19.6$' if $b = 2$ and $c = 1$, whereas a naive evaluation by the operator would yield $T_0(1 + c(\mathrm{E}[y_a^\mathrm{o}]/\kappa)^b) - T_0 \approx 10.3$' only.

The discrepancy between the two evaluations would be much larger for larger values of exponent b. This demonstrates the need for accurate estimations of penalty functions and a consistent, user-oriented evaluation of vehicle crowding in the cost-benefit assessment of transport plans.

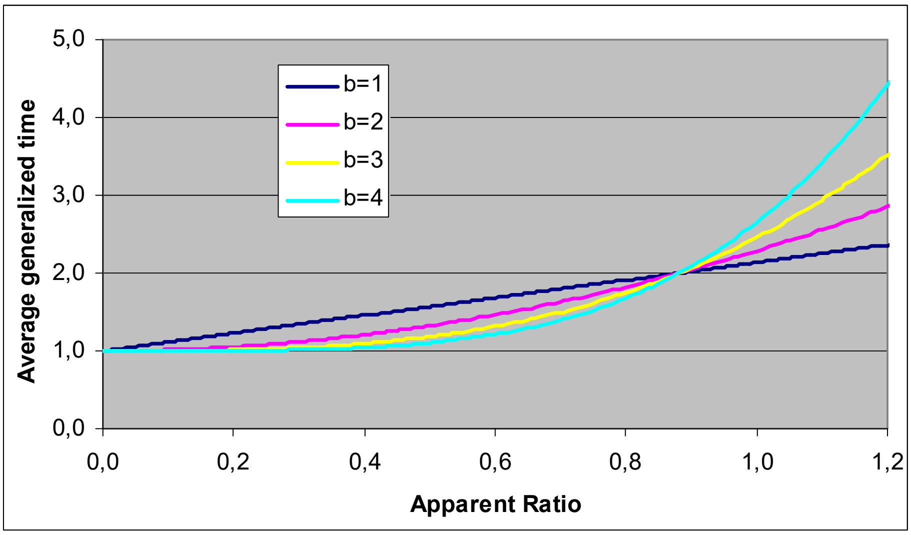

Fig. 1. Generalized time versus Occupancy ratio, according to variability parameter

6. Consequences for traffic assignment models

A model of traffic assignment to a transit network is purported to yield flows of passenger trips by network link – either a service link or a pedestrian link of service access, egress or transfer – on the basis of the performance of transit services and of a matrix of passenger trip flows by origin-destination pair of travel demand zones. Local travel conditions are perceived by a network user at the level of the network

path by serial composition of the links along the path. The travel conditions aggregated along the path determine its quality of service, hence its attractiveness and value to the user. By assumption, every user chooses a path of minimum generalized time to him. Thus, the relationship between trip flow and quality of service is a key component in the assignment model.
The most widely used model is a "static" model with Markovian transit services, such that path choice amounts to the split of trip flow at any network node between competitive services that are available and attractive there in proportion to their respective frequency [8]. In most applications, trip flows are considered only at the service level, rather than by vehicle. Out of the variety of capacity constraints and associated congestion effects [9], crowding discomfort requires to characterize not only the passenger load by vehicle but also the associated in-vehicle capacity. It has been addressed in [8] by assuming a constant vehicle load – which is not consistent with the assumption of Markovian services, which entails an exponential distribution of vehicle headways.
This section is purported to provide the properties of exponential headways (§ 6.1) and to derive their consequences in the evaluation of crowding discomfort (§ 6.2). Then, it states the requirements for stochastic modeling in the framework of dynamic transit assignment models (§ 6.3).

6.1. Exponential headways and their properties

An exponential distribution of headways at station r, denoted $\eta_r \approx \mathrm{EXP}(f_r / H)$, has a parameter, f_r / H, which is the local frequency of service by time period of length H. For simplicity, let us assume here not only the conservation of headway rank between stations, but also the homogeneity of local frequency, yielding $\eta_r = \eta_e$ all along the service route. Then $y^{\mathrm{o}}_{a,e,\beta} = \beta Q_a \eta_e$, wherein $Q_a = \sum_{r \le a < s} q_{rs}$.
As $\mathrm{H}_e(x) = 1 - \exp(-f.x / H)$, it holds that $\mathrm{H}_r^{(-1)}(\alpha) = -\ln(1-\alpha) H / f$.
As a lemma, let us evaluate $\mathrm{E}[(y^{\mathrm{o}}_{a,e,\beta})^n]$ for $n \ge 1$:

$$\mathrm{E}[(y^{\mathrm{o}}_{a,e,\beta})^n] = \int_0^1 (\mathrm{Y}^{\mathrm{o}(-1)}_{a,e,\beta}(\alpha))^n \,\mathrm{d}\alpha = \int_0^1 (\beta Q_a \,\mathrm{H}_r^{\mathrm{o}(-1)}(\alpha))^n \,\mathrm{d}\alpha$$

$$= \left(\frac{\beta Q_a H}{f}\right)^n \int_0^\infty x^n \exp(-x)\,\mathrm{d}x \quad \text{by the change of variables } x = -\ln(1-\alpha)$$

Integrating by parts, it comes out that

$$\int_0^\infty x^n \exp(-x)\,\mathrm{d}x = \left[x^n(-\exp(-x))\right]_0^\infty + n\int_0^\infty x^{n-1}\exp(-x)\,\mathrm{d}x = n\int_0^\infty x^{n-1}\exp(-x)\,\mathrm{d}x$$

So, by recursion, $\int_0^\infty x^n \exp(-x)\,\mathrm{d}x = n!$ if $n \ge 1$, yielding that

$$\mathrm{E}[(y^{\mathrm{o}}_{a,e,\beta})^n] = n!\,\theta_a^n \text{, wherein } \theta_a = \beta Q_a H / f. \tag{6.1}$$

Then, from eqn (4.14),

$$\begin{aligned}\mathrm{E}[(y^{\mathrm{u}}_{a,e,\beta})^n] &= \mathrm{E}[(y^{\mathrm{o}}_{a,e,\beta})^n\, \eta_e^{\mathrm{o}}] / \overline{\eta}_e^{\mathrm{o}} = (\overline{\eta}_e^{\mathrm{o}})^{-1} \int_0^1 (\beta Q_a \,\mathrm{H}_r^{\mathrm{o}(-1)}(\alpha))^n \,\mathrm{H}_r^{\mathrm{o}(-1)}(\alpha)\,\mathrm{d}\alpha \\ &= f(H\beta Q_a)^{-1} \int_0^1 (\beta Q_a \,\mathrm{H}_r^{\mathrm{o}(-1)}(\alpha))^{n+1}\,\mathrm{d}\alpha = (n+1)!\,\theta_a^{n+1} / \theta_a \\ &= (n+1)!\,\theta_a^n = (n+1)\,\mathrm{E}[(y^{\mathrm{o}}_{a,e,\beta})^n]\end{aligned} \tag{6.2}$$

This entails the following consequences:

$\mathrm{E}[y^{\mathrm{u}}_{a,e,\beta}] = 2\theta_a = 2\,\mathrm{E}[y^{\mathrm{o}}_{a,e,\beta}]$

$\mathrm{E}[(y^{\mathrm{u}}_{a,e,\beta})^2] = 6\theta_a^2 = 3\,\mathrm{E}[(y^{\mathrm{o}}_{a,e,\beta})^2]$

$\mathrm{V}[y^{\mathrm{u}}_{a,e,\beta}] = 2\theta_a^2 = 2\,\mathrm{V}[y^{\mathrm{o}}_{a,e,\beta}]$

This is consistent with formula (4.16), since $\gamma_e^o = \gamma_a^o = 1$ so the conversion ratio is $1 + \gamma_e^o \gamma_a^o = 2$.

6.2. Interaction with congestion law

In [8], crowding discomfort is modeled by multiplying the in-vehicle time by a penalty factor as follows:

$$\chi_a = 1 + c_a(\bar{y}_a^{\mathrm{o}} / \kappa_a)^{b_a} . \tag{6.3}$$

This entails a "naïve" evaluation of the average passenger sur-time on link a as follows:

$$\chi_a - 1 = c_a(\theta_a / \kappa_a)^{b_a} . \tag{6.3}$$

The naïve evaluation is consistent neither with the operator nor with the user perspective under irregular headways. Under exponential headways and a fixed β, the exact consequences of the congestion law in the population of passenger trips are

$$\mathrm{E}[c_a(y_a^{\mathrm{u}} / \kappa_a)^{b_a}] = (b_a + 1)! c_a (\theta_a / \kappa_a)^{b_a} = (\chi_a - 1)(b_a + 1)!$$

Thus the ratio of exact to naïve sur-time amounts to $(b+1)!$, which varies from 2 to 120 when b is varied from 1 to 4 – the instance value taken by analogy to motorway traffic.
So it is most important to deal with crowding discomfort in consistency with headway variability. This principle should apply not only to the simulation of network flows but also in any survey of the costs of discomfort across a population of transit passengers.

6.3. On stochastic features and dynamic assignment

A dynamic model of traffic assignment to a transit network is able to capture the temporal variation of both demand flows and service times, hence of inter-run headways. A precise determination of headway η will lead to a precise determination of passenger loads.

However, even in a dynamic framework it is still necessary to deal with the following stochastic features:

- Randomness in any headway around its scheduled value: what is the degree of reliability of the nominal timetable?
- Randomness in vehicle capacity – factor κ : are there several vehicle types operated on a given service and, if so, is the vehicle type selected ex-ante for each planned service run?
- Randomness in passenger flow rate – factor β .

In the authors' opinion, the probabilistic model presented here can be used to capture the stochastic features in an enhanced static setting; it could also be adapted to the dynamic setting by considering any headway from the nominal schedule as a random variable influenced by the operating policy.

7. Conclusion

A model of traffic along a transit line has been provided at both levels of traffic unit, the vehicle versus the passenger. The perspectives of the operator and the user have been identified. Based on a powerful postulate, the conservation of headway rank, it has been shown that service irregularity and demand variations, as well as other factors such as vehicle order in schedule, vehicle size and passenger sensitivity to quality of service, affect the passenger conditions of travel significantly. Crowding density above a ratio of say 80% exerts major influence on generalized travel time. The operator perspective is plagued with bias that must be corrected to represent passenger conditions objectively.
The model captures a set of variability sources. Analytical formulae have been established to assess their respective effects. The main postulate is the conservation of headway rank. Gaussian or log-normal approximations have been made to yield convenient approximations; in the authors' opinion their effect is innocuous.
The established properties will be useful in models of traffic assignment to a transit network, as they pertain to travel conditions hence to the leg quality of service, which determines the passenger travel choice of a network route.
Further work is required to analyze transit lines serviced by a set of routes: vehicle type and load will depend on the route and the joint operations. On the passenger side, between some station pairs a subset of routes will be used, yielding reduced waiting time but more diverse in-vehicle conditions. Another research topic pertains to the feedback of vehicle load on the operating conditions, as in the assignment model of [10].

Acknowledgements

This research was supported by the Research and Education Chair on “The socio-economics and modeling of urban public transport”, operated by Ecole des Ponts ParisTech in partnership with the Transport Organizing Authority in the Paris Region (STIF), to which we are grateful.

References

[0] Leurent F., Benezech V., Combes F. (2012) A stochastic model of passenger generalized time along a transit line. Paper presented at the XVth EWGT Congress, Paris, September 2012. Forthcoming in Elsevier Procedia - Social and Behavioral Sciences.

[1] Kroes E., Kouwenhoven M., Duchateau H., Debrincat L., Goldberg J. (2005) On the value of punctuality on suburban trains to and from Paris. Proceedings of the European Transport Conference.

[2] Carey M., Kwiencinski A. (1994) Stochastic approximation to the effects of headways on knock-on delays of trains. Transportation Research Part B, vol 28/4: 251-267.

[3] Meester L.E., Muns S. (2007) Stochastic delay propagation in railway networks and phase-type distributions. Transportation Research Part B, vol 41: 218-230.

[4] Bowman L.A., Turnquist M.A. (1981) Service frequency, schedule reliability and passenger wait times at transit stops. Transportation Research Part A, General, vol 15: 465-471.

[5] Islam M.K., Vandebona U. (2011) Impact of Occupancy Profile for Waiting Time of Bus Transit Users. Proceedings of the TRB Annual Congress, January, Washington DC.

[6] Xue Y., Jing J., Lai J., Ran B., Yang D. (2011) Empirical Characteristics of Transit Travel Time Distribution for Commuting Routes. Proceedings of the TRB Annual Congress, January, Washington DC.

[7] Kleinrock L. (1975) Queuing systems, Vol. I. Wiley Interscience.

[8] Spiess H., Florian M. (1989) Optimal strategies: a new assignment model for transit networks. Transportation Research B 23: 83-102.

[9] Leurent F (2011) Transport capacity constraints on the mass transit system: a systemic analysis, European Transport Research Review, 3/1: 11-21.

[10] Leurent F., Chandakas E., Poulhès A. (2011) User and service equilibrium in a structural model of traffic assignment to a transit network. In Zak J. (ed) The State of the Art in the European Quantitative Oriented Transportation and Logistics Research – 14th Euro Working Group on Transportation & 26th Mini Euro Conference & 1st European Scientific Conference on Air Transport. Elsevier Procedia - Social and Behavioral Sciences, 20: 495-505.