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Generalized qualitative Sugeno integrals

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A B S T R A C T

Sugeno integrals are aggregation operations involving a criterion weighting scheme based on the use of set functions called capacities or fuzzy measures. In this paper, we define generalized versions of Sugeno integrals on totally ordered bounded chains, by extending the operation that combines the value of the capacity on each subset of criteria and the value of the utility function over elements of the subset. We show that the generalized concept of Sugeno integral splits into two functionals, one based on a general multiple-valued conjunction (we call integral) and one based on a general multiple-valued implication (we call cointegral). These fuzzy conjunction and implication connectives are related via a so-called semiduality property, involving an involutive negation. Sugeno integrals correspond to the case when the fuzzy conjunction is the minimum and the fuzzy implication is Kleene-Dienes implication, in which case integrals and cointegrals coincide. In this paper, we consider a very general class of fuzzy conjunction operations on a finite setting, that reduce to Boolean conjunctions on extreme values of the bounded chain, and are non-decreasing in each place, and the corresponding general class of implications (their semiduals). The merit of these new aggregation operators is to go beyond pure lattice polynomials, thus enhancing the expressive power of qualitative aggregation functions, especially as to the way an importance weight can affect a local rating of an object to be chosen.

Keywords:
Sugeno integrals
Conjunctions
Implications

1. Introduction

In the setting of Artificial Intelligence (for instance, recommender systems, cognitive robotics, but other fields as well), the use of decision rules based on numerical aggregation functions is not always natural. For instance, probabilities, utilities, importance weights cannot always be easily elicited from the user, by lack of time or lack of precision. Information systems advising persons cannot ask too many questions to users for modeling their preferences, nor collect from them meaningful numbers representing probabilities or criteria importance levels, or yet utility values. Even if they get them, making numerical operations on them looks debatable. Another example is when a referee has to fill a form to assess the merits of a paper for a journal, and numerical ratings are required. What is the precise meaning of these ratings? Does it make sense to compute averages from them?

In such situations it is more natural to resort to a qualitative approach to multicriteria evaluation. The rationale is to refrain from using numbers that look arbitrary or hard to collect, if we can evaluate decisions in a reasoned approach,

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without numerical calculations and address decision problems in the ordinal setting. There are two advantages: (i) a gain in robustness and the need for less data; (ii) qualitative methods lend themselves to a logical representation (which makes proposed choices more easily explainable). There are two possible choices of qualitative settings for representing notions such as utility ratings by several agents, importance levels and likelihood degrees:

- Use distinct non-commensurate scales. This makes the framework very restrictive as impossibility theorems regarding rational aggregation processes are often obtained (e.g., in voting theory).
- Use finite commensurate scales (taking advantage of notions facilitating commensurateness such as certainty equivalents) then one works with a finite ordered set of value classes.

In multi-criteria decision making, Sugeno integrals [39,40] are commonly used as qualitative aggregation functions [25] using finite scales and a commensurateness assumption between them. The definitions of these integrals are based on a monotonic set-function named capacity or fuzzy measure that aims to qualitatively represent the likelihood of sets of possible states of nature, the importance of sets of criteria, etc. These set functions are currently used in many areas such as uncertainty modeling [13,14], multiple criteria aggregation [3,23,24] or in game theory [37]. See also a recent book devoted to capacities in such areas [28]. Moreover, Sugeno integrals naturally lend themselves to a representation in terms of if-then rules involving thresholds [7,15,23], which makes them easy to interpret.

Capacities can be exploited in different ways when aggregating local ratings of objects according to various criteria, and different qualitative integrals (q-integrals, for short) can be obtained. In the case of Sugeno integrals, the capacity is used as a bound that restricts the global evaluation from below or from above. In other cases, the capacity is considered as a tolerance threshold such that overcoming it leads to improving the global evaluation of the object under study. When this threshold is not reached there are two possibilities. Either the local rating remains as it stands or it is modified: improved if the criterion is little important, downgraded if it is important.

These considerations give rise to new aggregation operations in the qualitative setting, such as soft and drastic integrals, studied in a recent paper [16]. Namely, we introduced variants of Sugeno integrals based on Gödel implication and its contrapositive version, using an involutive negation. It models qualitative aggregation methods that extend min and max, based on the idea of tolerance threshold beyond which a criterion is considered satisfied. These variants of integrals are valued in a scale equipped with a residuated implication and an involutive negation. More precisely, in [16], the evaluation scale is both a totally ordered Heyting algebra and a Kleene algebra. These new aggregation operations have been axiomatized in [17] in the setting of a complete bounded chain with an involutive negation.

In the present paper, which extends a conference paper [18], we try to cast this approach in a more general totally ordered algebraic setting, using multivalued conjunction operations that are not necessarily commutative, and implication operations induced from them by means of an involutive negation, with a view to preserve characterization theorems proved in [17] for Gödel implication and its contrapositive version, and non-commutative fuzzy conjunctions obtained via an involutive negation. We show that, once generalized in this way, the concept of Sugeno integral splits into two functionals, one based on a generalized conjunction (we call q-integral) and one based on an implication (we call q-cointegral). These fuzzy conjunction and implication connectives are related via a semiduality property, involving an involutive negation. Sugeno integrals correspond to the case when the fuzzy conjunction is the minimum and the implication is Kleene-Dienes implication, in which case integrals and cointegrals coincide.

The paper is structured as follows. Section 2 presents the main motivations for the study of new aggregation operations, namely the notion of weighted qualitative aggregation function. It studies the way importance weights of criteria affect local ratings of objects with respect to such criteria. Section 3 provides insights on the algebraic setting useful for generalizing Sugeno integrals, namely the possible dependence between fuzzy conjunctions and implications via residuation and a property called semiduality. Section 4 presents the proposed generalizations of Sugeno integrals, emphasizing the existence of distinct integrals and cointegrals respectively defined by means of fuzzy conjunctions and implications. Section 5 presents representation theorems for q-integrals and q-cointegrals.

2. Motivations

After recalling the weighted min and max aggregation functions, this section extends these aggregation functions to other weighting schemes, focusing on the possible effects of weighting criteria on the corresponding ratings. This extension requires more general conjunctions and implications.

We adopt the terminology and notations usual in multi-criteria decision making, where some alternatives are evaluated according to a common set $\mathcal{C} = \{1, \ldots, n\} = [n]$ of criteria. A common evaluation scale L is assumed to provide ratings according to the criteria: each alternative is thus identified with a function $f \in L^{\mathcal{C}}$ which maps every criterion i of $\mathcal{C}$ to the local rating f_i of the alternative with regard to this criterion.

We assume that L is a finite totally ordered set with 1 and 0 as top and bottom, respectively (L may be a subset $\{0 = a_0 < a_1 < \cdots < a_k = 1\}$ of the real unit interval [0, 1] for instance). For any $a \in L$, we denote by $\mathbf{a}_{\mathcal{C}}$ the constant alternative whose ratings equal a for all criteria in $\mathcal{C}$. In addition, we assume that L is equipped with a unary order reversing involutive operation $a \mapsto 1 - a$, that we call *negation*. In the finite setting, there is a unique such negation operation.[1] We respectively

[1] Often called strong negation in the literature, see [31].

denote by $\wedge$ and $\vee$ the minimum and maximum operations induced by $\geq$ on L. Such a finite totally ordered scale defined by $(L, \geq, 1-, 0, 1)$ is called *qualitative* in this paper.

2.1. Weighted minimum and maximum

There are two elementary symmetric qualitative aggregation schemes on a finite totally ordered chain:

- The pessimistic one $\bigwedge_{i=1}^{n} f_i$ which is very demanding since in order to obtain a good evaluation an object needs to satisfy all the criteria.
- The optimistic one, $\bigvee_{i=1}^{n} f_i$ which is very loose since one fulfilled criterion is enough to obtain a good evaluation.

These two aggregation schemes can be slightly generalized by means of importance levels or priorities $\pi_i \in L$, on the criteria $i \in [n]$, thus yielding weighted minimum and maximum [11]. Suppose π_i is increasing with the importance of i. A fully important criterion has importance weight $\pi_i = 1$. A useless criterion has importance weight $\pi_i = 0$. In general these criteria can be eliminated. We also assume $\pi_i = 1$, for at least one criterion i (the most important ones). It is a kind of normalization assumption that ensures that the whole positive part of the scale L is useful. It is typical of possibility theory [12]. Indeed, the set of weights can be viewed as a possibility distribution over criteria, just like the set of numerical weights in a weighted average can be viewed as a probability distribution.

These importance levels can interact with each local evaluation f_i in different manners. Usually, a qualitative weight π_i acts as a saturation threshold that blocks the global score under or above a certain value dependent on the importance level of criterion i. Such weights truncate the evaluation scale from above or from below. Usually, the rating f_i is modified in the form of either $(1 - \pi_i) \vee f_i \in [1 - \pi_i, 1]$, or $\pi_i \wedge f_i \in [0, \pi_i]$. In the first expression, low ratings of unimportant criteria are upgraded; in the second one, high ratings of unimportant criteria are downgraded. The weighted minimum and maximum operations then take the following forms [11]:

$$MIN_\pi(f) = \bigwedge_{i=1}^{n} \big((1 - \pi_i) \vee f_i\big); \quad MAX_\pi(f) = \bigvee_{i=1}^{n} (\pi_i \wedge f_i). \tag{1}$$

In these aggregation operations, a fully important criterion can alone bring the whole global score to 0 ($MIN_\pi(f)$) or to 1 ($MAX_\pi(f)$), according to the chosen attitude (resp. pessimistic or optimistic).

It is well-known that if the range of the local ratings f_i is reduced to $\{0, 1\}$ (Boolean criteria) then letting $A_f = \{i : f_i = 1\}$ be the set of criteria satisfied by alternative f, the function

$$\Pi : A_f \mapsto MAX_\pi(f) = \bigvee \{\pi_i : i \in A_f\}$$

is a possibility measure on $\mathcal{C}$ [42] (i.e, a set function Π that satisfies $\Pi(A \cup B) = \Pi(A) \vee \Pi(B)$ for every $A, B \subseteq \mathcal{C}$), and,

$$N : A_f \mapsto MIN_\pi(f) = \bigwedge \{1 - \pi_i : i \notin A_f\}$$

is a necessity measure [12] (i.e., a set function N that satisfies $N(A \cap B) = N(A) \wedge N(B)$ for every $A, B \subseteq \mathcal{C}$). Note that the well-known duality property

$$\Pi(A) = 1 - N(\overline{A}), \tag{2}$$

where $\overline{A}$ denotes the set complement of A in $\mathcal{C}$, immediately generalizes to the scale L in the following way:

$$MAX_\pi(f) = 1 - MIN_\pi(1 - f). \tag{3}$$

In order to get a maximal global rating with the weighted minimum, we force all criteria with a weight not equal to 0 to be satisfied; and with the weighted maximum we force only one fully important criteria to be satisfied.

Example 1. Let $L = \{0, 0.4, 0.6, 1\} \subset [0, 1]$ and consider three criteria with the importance profile π and the Boolean alternatives f and g:

	1	2	3	MIN_π	MAX_π
π	1	1	0.6		
f	1	1	0	0.4	1
g	1	0	0	0	1

We can see that f violates the least important criterion 3 only, and it makes the global evaluation positive according to the pessimistic criterion, while g is ruled out because it violates one essential criterion. But the optimistic evaluation puts the two decisions on a par because they both satisfy one important criterion.

Note that the form (1) of the criteria is not arbitrary. It we consider weighted aggregations of the form $\wedge_{i=1}^{n} \pi_i \vee f_i$, and $\vee_{i=1}^{n} (1 - \pi_i) \wedge f_i$ for instance, the effect of weights will be radically different. In the case of the former, you get the maximal value 1 for the global rating as soon as all criteria are important, whatever the value of the local ratings, which is not satisfactory. In the second case, it is enough to have one poorly important criterion fully satisfied to get a good

overall rating, again counterintuitive. So, the mathematical form of the combination between a criterion weight and the local rating is affected by the aggregation scheme used and cannot be chosen arbitrarily if intuitively satisfactory results must be obtained.

In the following we explore various alternative weighting schemes in the ordinal setting. The weighted min and max can be generalized by conjunctive and disjunctive expressions of the form:

$$MIN_{\pi}^{\triangleright}(f) = \bigwedge_{i=1}^{n} (\pi_i \triangleright f_i); \quad MAX_{\pi}^{\otimes}(f) = \bigvee_{i=1}^{n} (\pi_i \otimes f_i). \tag{4}$$

for suitable choices of rating modification operations $\triangleright$ and $\otimes$. In the sequel we study conditions that these operations must satisfy in order for them to make sense. We consider the cases of conjunctive and disjunctive aggregations respectively.

2.2. A generalized rating modification scheme for the conjunctive aggregation

Let $\pi_i \triangleright f_i$ be the result of applying the weight π_i to the rating f_i. Using a conjunctive aggregation, an alternative should not be rejected due to a poor rating on an unimportant criterion, especially the global rating should not be affected by useless criteria, whatever the corresponding rating. The importance weight 0 should turn the local rating on a useless criterion into 1, since $a \wedge 1 = a$, i.e., this extreme value is not taken into account by the conjunctive aggregation. So we must assume that $0 \triangleright f_i = 1$. Moreover, it is clear that if the criterion has maximal importance, a very poor local rating on this criterion should be enough to bring the global evaluation down to 0, which implies $1 \triangleright 0 = 0$. Besides, it is natural that the better the local rating f_i, the greater the modified local rating. However, the less important the criterion, the more lenient should be the modified local rating.

According to the above rationale, the operation $\triangleright$ should formally be a multiple-valued implication (as defined in [1,21] for $L = [0, 1]$):

Definition 1. A fuzzy implication is a two-place operation $\rightarrow$ on L such that:

1. $0 \rightarrow 1 = 1$; $1 \rightarrow 0 = 0$; $1 \rightarrow 1 = 1$; $0 \rightarrow 0 = 1$.
2. $a \rightarrow b$ is increasing in the wide sense with b for all $a \in L$.
3. $a \rightarrow b$ is decreasing in the wide sense with a for all $b \in L$.

As we use a finite scale with an involutive negation, it is natural to consider the special case when the set of possible values for $\pi_i \triangleright f_i$ is restricted to one of the arguments, their negations, and extreme values, i.e., $\{0, 1-\pi_i, 1-f_i, \pi_i, f_i, 1\}$ according to the relative position of f_i with respect to π_i or $1-\pi_i$. In order to respect the properties of the fuzzy implication, we can make the following observations:

- We cannot have $\pi_i \triangleright f_i = \pi_i$ nor $\pi_i \triangleright f_i = 1 - f_i$, for consecutive values of f_i, $\pi_i \neq 0, 1$. Indeed if $a_j \triangleright f_i = a_j$ and $a_{j+1} \triangleright f_i = a_{j+1}$, the modified score would be increasing with π_i, and if $\pi_i \triangleright a_j = 1 - a_j$ and $\pi_i \triangleright a_{j+1} = 1 - a_{j+1}$, then the modified score would be decreasing with f_i, which contradicts the claim that $\triangleright$ should be a fuzzy implication function.
- We cannot have $\pi_i \triangleright 1 = 0$, otherwise we would get $1 \triangleright 1 = 0$, as $\pi_i \triangleright f_i$ decreases with π_i, which means that $\triangleright$ would not generalize implication.

Under these restrictions on the range of $\pi_i \triangleright f_i$, some well-known fuzzy implications [1], used in [16] or [17], are retrieved:

- Rescher-Gaines implication: $\pi_i \triangleright f_i = \pi_i \Rightarrow_{RG} f_i = \begin{cases} 1 & \text{if } \pi_i \leq f_i, \\ 0 & \text{otherwise;} \end{cases}$
- Gödel implication: $\pi_i \triangleright f_i = \pi_i \rightarrow_{G} f_i = \begin{cases} 1 & \text{if } \pi_i \leq f_i, \\ f_i & \text{otherwise;} \end{cases}$
- The contrapositive symmetric of Gödel implication: $\pi_i \triangleright f_i = \pi_i \rightarrow_{GC} f_i = \begin{cases} 1 & \text{if } \pi_i \leq f_i, \\ 1 - \pi_i & \text{otherwise;} \end{cases}$
- Kleene-Dienes implication: $\pi_i \triangleright f_i = \pi_i \rightarrow_{KD} f_i = (1 - \pi_i) \vee f_i$.
- The residuum of the nilpotent minimum: $\pi_i \overline{\Rightarrow} f_i = \begin{cases} 1 & \text{if } \pi_i \leq f_i \\ (1 - \pi_i) \vee f_i & \text{otherwise.} \end{cases}$

Other fuzzy implications, like Łukasiewicz's $\pi_i \rightarrow_{\mathrm{L}} f_i = \min(1 - \pi_i + f_i, 1)$ are possible if $L = \{0, 1/k, 2/k, \ldots, (k-1)/k, 1\}$, but the results may stand outside $\{0, 1-\pi_i, 1, f_i\}$. Note that our framework excludes Zadeh's implication [41] $(1 - a) \vee (a \wedge b)$ as the latter is not decreasing with a.

The corresponding conjunctive aggregation schemes, alternative to MIN_{π} are natural. Using $\rightarrow_{RG}$, the aggregation selects all objects that pass the prescribed thresholds for all criteria and totally rejects the other ones. Using $\rightarrow_{G}$, the aggregation selects all objects that pass the prescribed thresholds for all criteria and ranks the remaining ones according to their worst local ratings (forming a waiting list). Using $\rightarrow_{GC}$, the aggregation selects all objects that pass the prescribed thresholds for all criteria and ranks the remaining ones according to the importance of violated criteria, putting the objects that violate the least important criteria upfront.

Example 2. Consider $L = \{j/10 : j = 0, 1, \ldots, 10\} \subset [0, 1]$, three criteria with the importance profile π and the alternatives f and g:

	1	2	3	$\wedge_i \pi_i \to_{RG} f_i$	$\wedge_i \pi_i \to_G f_i$	$\wedge_i \pi_i \to_{GC} f_i$	$\wedge_i (1 - \pi_i) \vee f_i$
π	1	0.5	0.2				
f	1	0.4	0.3	0	0.4	0.5	0.5
g	1	0.5	0.3	1	1	1	0.5

The excellent global rating of g for all the three new aggregation operations is justified by the fact that it passes the importance thresholds for all criteria while f fails for one of them. Using $\to_{RG}$, f is then completely eliminated. In contrast, the usual weighted minimum cannot tell f from g although the latter Pareto-dominates the former.

2.3. A generalized rating modification scheme for the disjunctive aggregation

Let $\pi_i \otimes f_i$ be the result of applying the weight π_i to the rating f_i. Using a disjunctive aggregation, an alternative should be rejected only due to a poor rating on all important criteria. Again, the global rating should not be affected by useless criteria, whatever the corresponding rating. As the aggregation operation is disjunctive, the local rating on a useless criterion should always be turned into 0, since $a \vee 0 = a$, i.e., 0 does not alter the disjunctive aggregation. So $0 \otimes f_i = 0$. Moreover, it is clear that if the criterion has maximal importance, a very good local rating on this criterion is enough to bring the global evaluation up to 1. So $1 \otimes 1 = 1$. But if $f_i = 0$, criterion i should not participate to the improvement of the global rating, even if the criterion is important. So $\pi_i \otimes 0 = 0$. Besides, it is natural that the better the local rating f_i, the better the modified local rating $\pi_i \otimes f_i$. However, the more important the criterion, the higher should be $\pi_i \otimes f_i$ since a given rating on an important criterion should have a more positive influence on the global evaluation than the same rating on a little important criterion.

According to the above rationale, the operation $\otimes$ should formally be a multiple-valued conjunction:

Definition 2. A fuzzy conjunction is a two-place operation $\otimes$ on L such that:

1. $0 \otimes 1 = 0$; $1 \otimes 0 = 0$; $1 \otimes 1 = 1$; $0 \otimes 0 = 0$.
2. $a \otimes b$ is increasing in the wide sense with a, for all $b \in L$.
3. $a \otimes b$ is increasing in the wide sense with b for all $a \in L$.

Note that we assume neither associativity nor commutativity (e.g., local rating and weight do not play the same role). Moreover the top element can be the identity on the left ($1 \otimes a = a$) or on the right ($a \otimes 1 = a$), in which case we shall speak of *left- or right-conjunctions*, respectively; or an identity on both sides ($1 \otimes a = a \otimes 1 = a$, namely semicopulas [19] when $L = [0, 1]$). Besides, due to monotonicity and limit conditions, $a \otimes 0 = 0 \otimes 1 = 0$.

We shall again study the case when the set of possible values for $\pi_i \otimes f_i$ is limited to $\{0, 1 - \pi_i, \pi_i, f_i, 1 - f_i, 1\}$. It is clear that one cannot assume $\pi_i \otimes f_i \in \{1 - \pi_i, 1 - f_i\}$ for $\pi_i, f_i \neq 0, 1$, as a fuzzy conjunction is increasing in both places. So one is left to have $\pi_i \otimes f_i \in \{0, \pi_i, f_i, 1\}$, which as we shall see leaves the possibility to have $\pi_i \otimes f_i > \pi_i \wedge f_i$, including examples where $\pi_i \otimes f_i = 1$ when $\pi_i < 1, f_i < 1$, even if $\pi_i \otimes f_i$ must extend the Boolean conjunction. In the next subsection, we use a systematic way of producing fuzzy conjunctions from fuzzy implications.

2.4. Semiduality

In order to generate interesting fuzzy conjunctions $\pi_i \otimes f_i$, we can use the joint extension (3) of the De Morgan relationship $\vee_{i=1}^n f_i = 1 - \wedge_{i=1}^n (1 - f_i)$ existing between conjunctive and disjunctive aggregations, and of the well-known duality property between possibility and necessity measures (2). Applying this De Morgan duality to generalized weighted conjunctive aggregation $MIN_\pi^{\triangleright}(f) = \wedge_{i=1}^n \pi_i \triangleright f_i$ yields a family of generalized weighted disjunctive aggregations:

$$MAX_\pi^{\otimes}(f) = 1 - MIN_\pi^{\triangleright}(1 - f) = 1 - \bigwedge_{i=1}^{n} \pi_i \triangleright (1 - f_i) = \bigvee_{i=1}^{n} \pi_i \otimes f_i. \tag{5}$$

In other words, to each connective $\triangleright$ (fuzzy implication) that modifies local ratings in a conjunctive aggregation, corresponds a connective $\otimes = \mathcal{S}(\triangleright)$ (fuzzy conjunction) that modifies local ratings in a disjunctive aggregation, and is of the form $a\mathcal{S}(\triangleright)b := 1 - (a \triangleright (1 - b))$. We shall say that such a pair of operations $(\triangleright, \otimes)$ are *semidual* operations. It is obvious to see that $\triangleright$ is a fuzzy fuzzy implication in the sense of Definition 1 if and only if its semidual is a fuzzy conjunction in the sense of Definition 2. It is also obvious that the connectives can be exchanged, that is, $a \triangleright b = a\mathcal{S}(\otimes)b = 1 - (a \otimes (1 - b))$, i.e., semiduality is an involutive transformation. Note that $a \triangleright b = 1$ if and only if $a\mathcal{S}(\triangleright)(1 - b) = 0$ that is, a and $1 - b$, when positive, are divisors of 0 for the fuzzy conjunction $\otimes = \mathcal{S}(\triangleright)$.

Under semiduality, some fuzzy conjunctions such that $\pi_i \otimes f_i \in \{0, \pi_i, f_i, 1\}$ are recovered from previously mentioned fuzzy implications[2]:

[2] The names we give them are inherited from the corresponding semidual fuzzy implications.

- Rescher-Gaines conjunction [17]:
  $\pi_i \star_{RG} f_i = 1 - (\pi_i \Rightarrow_{RG} (1 - f_i)) = \begin{cases} 0 & \text{if } \pi_i \leq 1 - f_i, \\ 1 & \text{otherwise.} \end{cases}$ In this case, we may have that $\pi_i \star_{RG} f_i = 1$ even if $\pi_i < 1, f_i < 1$.
- Gödel conjunction [9]: $\pi_i \otimes_G f_i = 1 - (\pi_i \to_G (1 - f_i)) = \begin{cases} 0 & \text{if } \pi_i \leq 1 - f_i, \\ f_i & \text{otherwise.} \end{cases}$
  It is a non-commutative left-conjunction not upper-bounded by the minimum (since for instance, $\pi_i \otimes_G 1 = 1$ when $\pi_i > 0$) .
- The reciprocal Gödel conjunction [9]:
  $\pi_i \otimes_{GC} f_i = 1 - (\pi_i \to_{GC} (1 - f_i)) = \begin{cases} 0 & \text{if } \pi_i \leq 1 - f_i \\ \pi_i & \text{otherwise.} \end{cases}$ It is a non-commutative right-conjunction not upper-bounded by the minimum (since for instance, $1 \otimes_G f_i = 1$ when $f_i > 0$).
- Kleene-Dienes conjunction: $\pi_i \otimes_{KD} f_i = 1 - (\pi_i \to_{KD} (1 - f_i)) = \pi_i \wedge f_i$, which is the minimum.
- The nilpotent minimum [22]: $\pi_i \overline{\wedge} f_i = 1 - (\pi_i \overline{\to} (1 - f_i)) = \begin{cases} 0 & \text{if } \pi_i \leq 1 - f_i \\ \pi_i \wedge f_i & \text{otherwise.} \end{cases}$

  More general examples of fuzzy conjunctions are

- T-norms, i.e., increasing semi-groups with identity 1 and annihilator 0. In the finite setting it includes $\wedge$, the nilpotent minimum. The Łukasiewicz t-norm $\max(a + b - 1, 0)$ can also be used if $L = \{0, 1/k, 2/k, \ldots, (k-1)/k, 1\}$.
- Weak t-norms [21], i.e., left-conjunctions such that $a \otimes 1 \leq a$.
- The non-commutative left-conjunction $\otimes_{CRG}$ defined by $a \otimes_{CRG} b = 0$ if $a = 0$, and $a \otimes_{CRG} b = b$ if $a \neq 0$ (see [17]).
- The right-conjunction $a \otimes_{RG} b = b \otimes_{CRG} a$ associated with the previous fuzzy conjunction.
- Fuzzy conjunctions having identity 1. On $[0, +\infty)$ they are called pseudo-multiplications after Klement et al. [33] in the definition of universal integrals, and semicopulas [19] on the unit interval.

These fuzzy conjunctions yield disjunctive weighted aggregations with various behaviors:

- Using $\star_{RG}$ the aggregation rejects all objects that have a local rating less than $1 - \pi_i$ for all criteria and fully selects the other ones. The local rating f_i is turned into 0 for a non important criterion, even if this rating is high.
- Using $\otimes_G$ the aggregation rejects all objects that have a local rating less than $1 - \pi_i$ for all criteria and ranks the other ones according to their best local ratings, forming a waiting list.
- Using $\otimes_{GC}$ the aggregation rejects all objects that have a local rating less than $1 - \pi_i$ for all criteria and ranks the other satisfied ones according to the least importance degree of violated criteria, forming a waiting list. Note that $\otimes_G$ and $\otimes_{GC}$ are not commutative, and $a \otimes_{GC} b = b \otimes_G a$.
- Using the nilpotent minimum, the aggregation downgrades to 0 all local ratings less than $1 - \pi_i$, and performs a standard weighted minimum of local ratings on the rest of the criteria.

Example 3. Let us consider $L = \{j/10 : j = 0, 1, \ldots, 10\} \subset [0, 1]$, three criteria with importance vector π and the alternatives f, g and h:

	1	2	3	$\vee_i \pi_i \star_{RG} f_i$	$\vee_i \pi_i \otimes_G f_i$	$\vee_i \pi_i \otimes_{GC} f_i$	$\vee_i \pi_i \wedge f_i$
π	1	0.5	0.2				
f	0	0.4	0.8	0	0	0	0.4
g	0.1	0.6	0.8	1	0.6	1	0.5
h	0	0.6	0.8	1	0.6	0.5	0.5

Option f is eliminated by three operations because it does not pass the requested thresholds. In particular, a grade 0.8 is insufficient to make it for satisfying a poorly important criterion with weight 0.2. Option g is better evaluated because it passes the threshold for the most important criteria 1 and 2. Option h is Pareto-dominated by g and only passes the threshold for criterion 2. h and g receive distinct global ratings with the third aggregation because g passes the threshold on criterion 1 that is more important than criterion 2. The usual weighted maximum, in contrast yields a more lenient global evaluation for f, but cannot distinguish g from h.

Remark 1. The transformation of an aggregation operation A on the unit interval into a fuzzy implication using a negation is the topic of an recent extensive review [34]. However these authors focus on the transformation $A(a, b) \mapsto A(n(a), b)$ for a general negation n, in the spirit of material implication induced by a disjunction. The reader can find many properties induced on fuzzy implications from the properties of the aggregation function by this transformation, which is the counterpart for fuzzy disjunctions of semiduality for fuzzy conjunctions. Namely their results could be equivalently spelled out in terms of semiduality.

3. Algebraic considerations

Let us consider the following transformations that can be applied to some fuzzy conjunction $\star$ on a qualitative scale $(L, \geq, 1 - (\cdot), 0, 1)$:

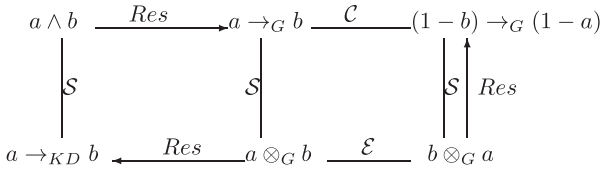


Fig. 1. Connectives induced by the minimum on a finite chain.

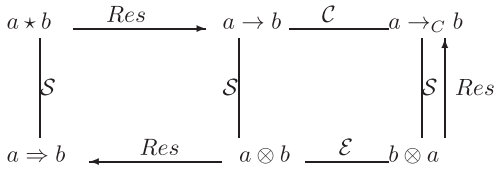


Fig. 2. Residuation and semiduality on a finite chain.

- Residuation: $aRes(\star)b := \bigvee\{x : a \star x \leq b\}$, and we denote $\to = Res(\star)$.
- Semiduality: $a\mathcal{S}(\star)b := 1 - (a \star (1-b))$, and we denote $\Rightarrow = \mathcal{S}(\star)$.
- Contraposition: $a\mathcal{C}(\star)b := (1-b) \star (1-a)$, and we denote $\star_C = \mathcal{C}(\star)$.
- Argument exchange: $a\mathcal{E}(\star)b := b \star a$.
- De Morgan duality: $a\mathcal{D}(\star)b := 1 - (1-a) \star (1-b)$.

Note that semiduality, contraposition, and argument exchange are involutive transformations that are always defined. However, residuation is not defined for any operation. It is well-defined only if the set $\{x\colon a \star x \leq b\}$ is not empty, which means that $\forall a, b \in L$, $\exists x$ s.t. $a \star x \leq b$. This condition holds for fuzzy conjunctions as $a \star 0 = 0 \leq b$.

3.1. Relationships between fuzzy implications and conjunctions

It is known that some fuzzy implications and conjunctions presented in Section 2.2 are interdefinable via such transformations [9]. As an example, the diagram in Fig. 1, where $\star = \wedge$, commutes and involves Gödel and Kleene-Dienes implications.

Note that the diagram also commutes if we start with the nilpotent minimum instead of the minimum, but it collapses to a single transformation since it is well-known that $Res(\overline{\wedge}) = \mathcal{S}(\overline{\wedge}) = \overline{\to}$ and $\mathcal{C}(\overline{\to}) = \overline{\to}$.

This diagram can be generalized to more general fuzzy conjunctions [21]. We consider a qualitative scale $(L, \geq, 0, 1, 1-)$, equipped with a fuzzy conjunction $\star$ (see Definition 2). The diagram on Fig. 2 presents this more general setting where operations $\star, \otimes = \mathcal{S}(Res(\star))$ are fuzzy conjunctions, and operations $\to = Res(\star), \Rightarrow = \mathcal{S}(\star)$ are fuzzy implications (see Definition 1).

Fodor [21] has proved that this diagram commutes if $\star$ is commutative. More precisely in the finite setting (the left-continuity of $\star$ is not needed) Fodor proved that:

- $Res \circ \mathcal{S} \circ Res(\star) = \mathcal{S}(\star)$ if and only if $\{a\colon a \star b \leq c\}$ is never empty. This condition is verified for all fuzzy conjunctions in the sense of Definition 2.
- $Res \circ \mathcal{E} \circ \mathcal{S} \circ Res(\star) = \mathcal{C} \circ Res(\star)$ (that is, on the diagram, $Res \circ \mathcal{E}(\otimes) = \to_C$), if and only if $\star$ is commutative.

There are some useful connections between the properties of the commutative fuzzy conjunction $\star$ at the top left of Fig. 2 and the properties of other connectives in the diagram. The reader can check the following easy results:

Proposition 1. *Let $\star$ be a commutative fuzzy conjunction with identity 1:*

- *$\Rightarrow = \mathcal{S}(\star)$ is a fuzzy implication such that $1 \Rightarrow b = b$, and $a \Rightarrow 0 = 1 - a$.*
- *$\to = Res(\star)$ is a fuzzy implication such that $a \to b = 1$ whenever $a \leq b$, $1 \to b = b$, and if $a > 0$, $a \to 0 = a_\star$ where we denote $a_\star = \bigvee\{x : a \star x = 0\}$.*
- *$\otimes = \mathcal{S}(\to)$ is a left-conjunction such that $a \otimes b = 0$ whenever $b \leq 1 - a$, and $a \otimes 1 = 1 - a_\star$.*
- *$\to_C = \mathcal{C}(\to)$ is a fuzzy implication such that $a \to b = 1$ whenever $a \leq b$, $1 \to_C b = (1-b)_\star$, and $a \to_C 0 = 1 - a$.*

Note that the above proposition applies when $\star$ is any triangular norm on a finite chain (and any left-continuous t-norm on the unit interval, more generally any left-continuous semi-copula).

The properties, neutrality of truth $1 \Rightarrow b = b$ and strong negation principle $a \Rightarrow 0 = 1 - a$ [34], can be related to the properties of its semidual $\otimes$. Namely the following claims are easily verified:

Proposition 2. *Let $\otimes$ be a fuzzy conjunction and $\to$ be its semidual fuzzy implication:*

- *$\otimes$ is a left-conjunction if and only if its semidual verifies $1 \to b = b$.*
- *$\otimes$ is a right-conjunction if and only if its semidual verifies $a \to 0 = 1 - a$.*

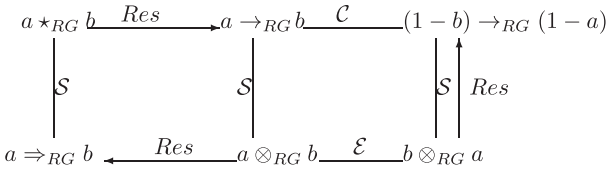


Fig. 3. Connectives associated to the Rescher-Gaines implication on a finite chain.

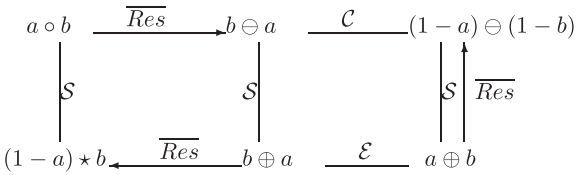


Fig. 4. Connectives induced by De Morgan law.

Note that Rescher-Gaines conjunction $\star_{RG}$ (defined in Section 2.3) is neither a right nor a left-conjunction, but it is commutative, so we can get another special case of the above commutative diagram. Actually, $\star_{RG}$ is a commutative operation that has no identity. In a numerical setting, it considers that obtaining a rating b for a criterion of importance a is fully acceptable if the average of a and b is strictly greater than 1/2, and fully rejected otherwise.

The (largely ignored, except in the three-valued case, see [6]) three pairs of semidual fuzzy implication functions and conjunctions related to Rescher-Gaines implication, namely $(\Rightarrow_{RG}, \star_{RG})$, $(\to_{RG}, \otimes_{RG})$, and $(\to_{CRG}, \otimes_{CRG})$, are pictured on Fig. 3 where $(\Rightarrow_{RG}$ and $\star_{RG})$ are defined in Sections 2.2 and 2.4 respectively, and the other connectives are such that

$$a \to_{RG} b = \begin{cases} 1 & \text{if } b = 1, \\ 1-a & \text{if } b \neq 1; \end{cases} \quad a \otimes_{RG} b = \begin{cases} 0 & \text{if } b = 0, \\ a & \text{if } b \neq 0; \end{cases} \tag{6}$$

$$a \to_{CRG} b = (1-b) \to_{RG} (1-a); \quad a \otimes_{CRG} b = b \otimes_{RG} a. \tag{7}$$

We can check that

- The semidual of $\star_{RG}$ (Rescher-Gaines implication) is a {0, 1}-valued implication function, hence such that $1 \Rightarrow_{RG} b \neq b$, and $a \Rightarrow_{RG} 0 \neq 1-a$, if $a, b \neq 0, 1$.
- Its residuum $\to_{RG}$ is a fuzzy implication function such that $1 \to_{RG} b \neq b$ for $b \neq 0, 1$, but $a \to_{RG} 0 = 1-a$. As expected, the semidual fuzzy conjunction $\otimes_{RG}$ of $\to_{RG}$ is a right-conjunction: $a \otimes_{RG} 1 = a$.
- Similar results hold for the last pair of residual connectives, but now, $1 \to_{CRG} b = b$, as $\otimes_{CRG}$ is a left-conjunction.

3.2. Relationships between fuzzy disjunctions and differences

Interestingly enough, the diagram in Fig. 2 can be changed into the diagram in Fig. 4 by De Morgan duality with $\mathcal{D}(\star) = \circ$ (a disjunction), $\mathcal{D}(\to) = \ominus$ (a set-difference), $\mathcal{D}(\otimes) = \oplus$ (a non-commutative disjunction).

It can be checked that the disjunctions and differences in Fig. 4 are still exchanged by $\mathcal{S}$, $\mathcal{C}$ and $\mathcal{E}$ as the corresponding fuzzy conjunction and implication functions in Fig. 2. Moreover, the transformation $\overline{Res}$ that links other vertices in Fig. 4 is defined by

$$a\overline{Res}(\bullet)b = \bigwedge \{x \mid a \bullet x \geq b\}$$

To see it :

$$\begin{aligned} b \ominus a &= 1 - (1-a) \to (1-b) = 1 - \bigvee \{x : (1-a) \star x \leq (1-b)\} \\ &= \bigwedge \{1-x : 1 - (1-a) \star x \geq b\} = \bigwedge \{1-x : a \circ (1-x) \geq b\}. \end{aligned}$$

This transformation has been considered in the fuzzy set literature for a long time [10]. $\overline{Res}$ may be called an anti-residuation.

Proposition 3. *Considering a fuzzy conjunction $\star$ and the associated disjunction $\mathcal{D}(\star) = \bullet$, then $\overline{Res}(Res(\star)) = \star$ and $Res(\overline{Res}(\bullet)) = \bullet$, i.e., Res and $\overline{Res}$ are the inverse of each other.*

Proof. Considering Figs. 2 and 4 and assuming the existence of the infimum and of the supremum, let us first prove that $\bigwedge \{y, a \to y \geq b\} = \bigwedge \{y : \bigvee \{x, a \star x \leq y\} \geq b\} = a \star b$.[3]

Let $E(a, b) = \{y : \bigvee \{x, a \star x \leq y\} \geq b\}$. Note that:

[3] It forbids to apply *Res* (resp. $\overline{Res}$) to fuzzy disjunctions (resp. fuzzy conjunctions).

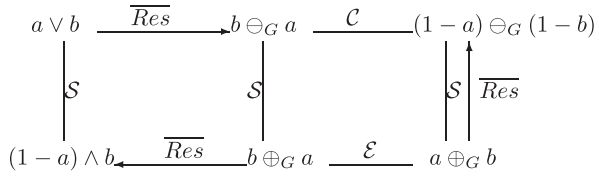


Fig. 5. Connectives induced by the maximum on a finite chain.

1. $a \star b \in E(a, b)$: indeed, take $y = a \star b$ and $x = b$. Then it is true that $\bigvee\{x, a \star x \leq a \star b\} \geq b$.
2. If $y < a \star b$, then $y \notin E(a, b)$. Indeed, it would mean $\exists x \geq b$: $a \star x \leq y < a \star b$, which is impossible since the fuzzy conjunction $\star$ is order-preserving.

So $\bigwedge E(a, b) = a \star b$. □

Example 4. We consider $\circ = \vee$ (see Fig. 5), then,

- $a\overline{Res}(\vee)b = b \ominus_G a = 1 - (1-a) \to_G (1-b) = \begin{cases} 0 & \text{if } b \leq a \\ b & \text{if } b > a \end{cases}$.
- $b \oplus_G a = 1$ if $b \geq 1 - a$ and $b \oplus_G a = b$ if $b < 1 - a$.

Operation $\oplus_G$ is a non-commutative disjunction (it coincides with classical disjunction for Boolean values), and $b \ominus_G a$ expresses a difference (it coincides with $b \wedge \neg a$ for Boolean values). As can be seen, $b \oplus_G a$ is an interesting modifier such that the value b is preserved if b is strictly less than threshold $1 - a$ and is put to 1 otherwise.

The study of differences and disjunctions in relation with conjunctive and disjunctive aggregations, and more generally with qualitative aggregation, is left for further studies.

4. Sugeno-like q-integrals and q-cointegrals

The weighted minimum and maximum can be generalized by assigning relative weights to subsets of criteria via a capacity, which is an inclusion-monotonic map $\gamma : 2^{\mathcal{C}} \to L$ (if $A \subseteq B$, then $\gamma(A) \leq \gamma(B)$), that also satisfies limit conditions $\gamma(\varnothing) = 0$ and $\gamma(\mathcal{C}) = 1$. This general form of weighting allows us to express dependencies between criteria or dimensions.

The Sugeno integral [39,40] of an alternative f can be defined by means of several expressions, among which the two following normal forms [32]:

$$\int_\gamma f = \bigvee_{A \subseteq \mathcal{C}} \left(\gamma(A) \wedge \bigwedge_{i \in A} f_i \right) = \bigwedge_{A \subseteq \mathcal{C}} \left(\gamma(\overline{A}) \vee \bigvee_{i \in A} f_i \right). \tag{8}$$

These expressions, which generalize the conjunctive and disjunctive normal forms in logic, can be simplified as follows [29, 32, Th. 4.1] :

$$\int_\gamma f = \bigvee_{a \in L} \gamma(\{i : f_i \geq a\}) \wedge a = \bigwedge_{a \in L} \gamma(\{i : f_i > a\}) \vee a. \tag{9}$$

The first expression in (8) is the original formulation of Sugeno integral [39], where it is proven to be equivalent to the first equality in (9). Moreover, for the necessity measure N associated with a possibility distribution π, we have $\int_N f = MIN_\pi(f)$; and for the possibility measure Π associated with π, we have $\int_\Pi f = MAX_\pi(f)$ [16,27].

4.1. Motivations and definitions

Let the conjugate capacity γ^c of γ be defined by $\gamma^c(A) = 1 - \gamma(\overline{A})$ for every $A \subseteq \mathcal{C}$. This duality relation extends to Sugeno integral with respect to the conjugate capacity, also extending the De Morgan duality between weighted minimum and maximum given in (3):

$$\int_\gamma f = 1 - \int_{\gamma^c} (1 - f). \tag{10}$$

This equality, proved in [27], stems from Eq. (8). It is interesting to notice that this equality between the two normal forms can be expressed using the conjugate capacity and the Kleene-Dienes implication $\to_{KD} = \mathcal{S}(\wedge)$ (defined in Section 2.2), the semidual of $\wedge$:

$$\int_\gamma f = \bigvee_{A \subseteq \mathcal{C}} \left(\gamma(A) \wedge \bigwedge_{i \in A} f_i \right) = \bigwedge_{A \subseteq \mathcal{C}} \left(\gamma^c(A) \to_{KD} \bigvee_{i \in A} f_i \right). \tag{11}$$

So, Sugeno integral can be generalized with other inner fuzzy conjunction or implication functions linked by semiduality, using expressions we call *q-integrals* and *q-cointegrals*, denoted respectively by $\int_\gamma^{\otimes} f$ and $\int_\gamma^{\rightarrow} f$, for every capacity γ and every $f \in L^{\mathcal{C}}$.

Definition 3. Let $\otimes$ be a fuzzy conjunction in the sense of Definition 2 and $\gamma : 2^{\mathcal{C}} \to L$ be a capacity. A q-integral is the mapping $\int_\gamma^{\otimes} : L^{\mathcal{C}} \to L$ defined by

$$\int_\gamma^{\otimes} f = \bigvee_{A \subseteq C} \left(\gamma(A) \otimes \bigwedge_{i \in A} f_i \right), \text{ for all } f \in L^{\mathcal{C}}.$$

When $\otimes$ is the product and $L = [0, 1]$, this is Shilkret integral [38] proposed in 1971 (and later reintroduced by Kaufmann [30] in 1978, under the name of "admissibility"). Grabisch et al. [27] introduce so-called Sugeno-like integrals, which are similar to q-integrals, where $\otimes$ is a triangular norm. Borzová-Molnárová et al. [2] study this type of integrals in the continuous case as well when $\otimes$ is a semicopula and $L = [0, 1]$. This kind of definition is also proposed by Dvořák and Holčapek [20] assuming $(L, \otimes, 1)$ is a commutative monoid and considering the complete residuated lattice generated by this monoidal operation. In fact, what they study is an extension of Definition 3: Namely, they study fuzzy integrals of this type *extended to algebras of fuzzy sets*, that is, where γ is now a *fuzzy set* function that assigns an importance value $\gamma(\tilde{A})$ to any *fuzzy* subset $\tilde{A}$ of $\mathcal{C}$.

Definition 4. Let $\rightarrow$ be a fuzzy implication in the sense of Definition 1 and let $\gamma : 2^{\mathcal{C}} \to L$ be a capacity. A q-cointegral is a mapping $\int_\gamma^{\rightarrow} : L^{\mathcal{C}} \to L$ defined by

$$\int_\gamma^{\rightarrow} f = \bigwedge_{A \subseteq C} \left(\gamma^c(A) \rightarrow \bigvee_{i \in A} f_i \right), \quad \text{for all } f \in L^{\mathcal{C}}.$$

There are not many works considering q-cointegrals in the above sense. Grabisch et al. [27, p. 302] notice the failure of the duality relation (10) for Sugeno-like integrals that use t-norms other than min, which hints at co-integrals using implications that are semi-duals of t-norms.

With this terminology, Eq. (8) actually states that

$$\int_\gamma f \;=\; \int_\gamma^{\wedge} f \;=\; \int_\gamma^{\rightarrow_{KD}} f$$

for every capacity γ and every $f \in L^{\mathcal{C}}$. It means that Sugeno integrals and cointegrals defined by means of the operation $\wedge$ and its semidual Kleene-Dienes implication, respectively, coincide. As already seen using Gödel implication for $\rightarrow_G$ in [16], this is not generally the case. So we study both q-integrals and q-cointegrals separately in the sequel.

In the following when we consider $f \in L^{\mathcal{C}}$, the mapping $i \mapsto (i)$ denotes a permutation on the set of criteria such that $f_{(1)} \leq \cdots \leq f_{(n)}$ and we let $F_{(i)} = \{(i), \ldots, (n)\}$ with the convention $F_{(n+1)} = \varnothing$. For all f in $L^{\mathcal{C}}$ we have $f = \bigvee_{i=1}^n \mathbf{1}_{F_{(i)}} \wedge f_{(i)}$, where $\mathbf{1}_A$ is the characteristic function of A.

4.2. Elementary properties of q-integrals

We first prove that the counterpart of the Sugeno integral expression on the left-hand side of (9), in terms of the nested family of subsets $F_{(i)}$ induced by f, is still valid for fuzzy conjunction-based q-integrals:

Proposition 4. $\int_\gamma^{\otimes} f = \bigvee_{i=1}^n \gamma(F_{(i)}) \otimes f_{(i)}$.

Proof. $\int_\gamma^{\otimes} f = \bigvee_{A \subseteq C} \gamma(A) \otimes \bigwedge_{i \in A} f_i$. Let index $i_A \in A$ be such that $\bigwedge_{i \in A} f_i = f_{i_A}$. It follows that $A \subseteq F_{(i_A)}$ which entails $\gamma(A) \leq \gamma(F_{(i_A)})$ and $\gamma(A) \otimes f_{i_A} \leq \gamma(F_{(i_A)}) \otimes f_{i_A}$. So, if the maximum is attained for set A, it is also attained by set $F_{(i_A)}$. And $\int_\gamma^{\otimes} f = \bigvee_{i=1}^n \gamma(F_{(i)}) \otimes f_{(i)}$, noticing that $f_{(i)} = f_{(i_{F_{(i)}})}$. □

The conjunction-based q-integral is monotonic:

Lemma 1. *For every capacity γ, the map $\int_\gamma^{\otimes} : L^{\mathcal{C}} \to L$ is order-preserving.*

Proof. Directly from the assumption that the map $x \mapsto a \otimes x$ is order-preserving. □

Another simplified form of Sugeno q-integral is valid for conjunction-based q-integrals:

Lemma 2. $\int_\gamma^{\otimes} f = \bigvee_{a \in L} \gamma(\{f \geq a\}) \otimes a$.

Proof. We use Proposition 4. Let $a \in L$

If $a > f_{(n)}$ then $\gamma(\{f \geq a\}) \otimes a = 0 \otimes a = 0$.

If $a \leq f_{(1)}$ then $\gamma(\{f \geq a\}) \otimes a = \gamma(\{f \geq f_{(1)}\}) \otimes a \leq \gamma(\{f \geq f_{(1)}\}) \otimes f_{(1)}$.

If $f_{(i-1)} < a \le f_{(i)}$ then $\gamma(\{f \ge a\}) \otimes a = \gamma(\{f \ge f_{(i)}\}) \otimes a \le \gamma(\{f \ge f_{(i)}\}) \otimes f_{(i)}$. □

Borzová-Molnárová et al. [2] also prove this result when $\otimes$ is a semicopula. We show here it is valid for a larger class of fuzzy conjunctions.

Finally, if γ is a possibility measure we retrieve the $\otimes$-weighted maximum:

Proposition 5. *If γ is a possibility measure Π based on possibility distribution π then $\int_{\Pi}^{\otimes} f = MAX_{\pi}^{\otimes}(f)$.*

Proof. $\bigvee_{A\subseteq C}(\Pi(A) \otimes \bigwedge_{i\in A} f_i) = \bigvee_{A\subseteq C}(\vee_{i\in A}\pi_i) \otimes \bigwedge_{i\in A} f_i = \vee_{i=1}^{n} \pi_i \otimes f_i$ since letting $\pi_{i_A} = \vee_{i\in A}\pi_i$, we do have the inequality $\Pi(A) \otimes \bigwedge_{i\in A} f_i \le \pi_{i_A} \otimes f_{i_A}$, as $\otimes$ is order-preserving. □

The proof of this proposition is the same as for Sugeno integral proper [8, p. 138–139] and [27]. However, if γ is a necessity measure, the expression of the q-integral will not simplify when $\otimes \ne \min$. In other words we do not have that $\int_{N}^{\otimes} f = MIN_{\pi}^{\rightarrow}(f)$ for $\rightarrow = \mathcal{S}(\otimes)$.

4.3. Elementary properties of q-cointegrals

Using semiduality, q-cointegrals can be expressed in terms of q-integrals by extending the duality formula (10):

Proposition 6. *If $\otimes = \mathcal{S}(\rightarrow)$, then $\int_{\gamma}^{\rightarrow} f = 1 - \int_{\gamma^c}^{\otimes}(1-f)$.*

Proof. $1 - \bigvee_{A\subseteq C}(\gamma^c(A) \otimes \bigwedge_{i\in A}(1-f_i)) = \bigwedge_{A\subseteq C} 1 - (\gamma^c(A) \otimes (1 - \bigvee_{i\in A} f_i) = \bigwedge_{A\subseteq C} \gamma^c(A) \rightarrow \bigvee_{i\in A} f_i = \int_{\gamma}^{\rightarrow} f$. □

As in [17] (but now in a more general setting), using semiduality, we derive the following results from the ones on (conjunction-based) q-integrals:

$$\int_{\gamma}^{\rightarrow} f \;=\; \bigwedge_{i=1}^{n} \gamma^c(\overline{F_{(i+1)}}) \rightarrow f_{(i)} \;=\; \bigwedge_{a\in L} \gamma^c(\{f \le a\}) \rightarrow a. \tag{12}$$

We also get that when γ is a necessity measure N based on possibility measure π, the q-cointegral reduces to the $\rightarrow$-based weighted minimum:

$$\int_{N}^{\rightarrow} f = MIN_{\pi}^{\rightarrow}(f) = \wedge_{i=1}^{n} \pi_i \rightarrow f_i$$

To see it, just use the semidual fuzzy conjunction $\otimes = \mathcal{S}(\rightarrow)$:

$$\int_{N}^{\rightarrow} f = 1 - \int_{\Pi}^{\otimes}(1-f) = 1 - MAX_{\pi}^{\otimes}(1-f) = MIN_{\pi}^{\mathcal{S}(\otimes)}(f).$$

However the q-cointegral with respect to a possibility measure $\int_{\Pi}^{\rightarrow} f$ does not reduce to a weighted maximum.

4.4. Some hints on the comparison between q-cointegrals and q-integrals

The equality (8) between q-cointegrals and q-integrals using fuzzy conjunction $\wedge$ and its semidual $\rightarrow_{KD}$ does not extend to conjunction-based q-integrals. For example, using Gödel implication-related connectives, only the inequality: $\int_{\gamma}^{\otimes} f \ge \int_{\gamma}^{\rightarrow} f$ holds for $(\otimes, \rightarrow) \in \{(\otimes_G, \rightarrow_G), (\otimes_{GC}, \rightarrow_{GC})\}$ [16]. This inequality cannot even be generalized to other fuzzy conjunctions.

To get a better insight on the comparison between q-integrals and q-cointegrals, consider the case of a profile $f = aAb$ such that $f_i = a > b$ if $i \in A$ and $f_i = b$ otherwise. Then it is easy to check using Lemma 2, and Eq. (12) that the q-integral and q-cointegral of aAb reduce to

$$\int_{\gamma}^{\otimes}(aAb) = (1 \otimes b) \vee (\gamma(A) \otimes a); \quad \int_{\gamma}^{\rightarrow}(aAb) = (\gamma^c(\overline{A}) \rightarrow b) \wedge (1 \rightarrow a).$$

Suppose for simplicity that $\otimes$ is a left-conjunction and $\rightarrow$ is its semidual. Then the first expression simplifies as follows:

$$\int_{\gamma}^{\otimes}(aAb) = b \vee (\gamma(A) \otimes a).$$

Besides $(\gamma^c(\overline{A}) \rightarrow b) \wedge (1 \rightarrow a) = a \wedge ((1 - \gamma(A)) \rightarrow b) = a \wedge (1 - (1 - \gamma(A)) \otimes (1-b))$. So, using the De Morgan dual $\oplus = \mathcal{D}(\otimes)$ of $\otimes$, we can write the q-cointegral as

$$\int_{\gamma}^{\rightarrow}(aAb) = a \wedge (\gamma(A) \oplus b).$$

These two expressions suggest the following remarks

- Since when $\otimes = \wedge$ the q-integral and q-cointegrals coincide, we notice that if $\otimes$ is a triangular norm, or a semicopula, in fact any fuzzy conjunction dominated by the minimum, then $\int_\gamma^{\rightarrow}(aAb) \geq \int_\gamma^{\otimes}(aAb)$, without equality in general whenever $0 < \gamma(A) < 1$.

  For example, consider the nilpotent minimum $\overline{\wedge}$ [22] and its De Morgan dual $a\underline{\vee}b = \begin{cases} a \vee b & \text{if } a \leq 1-b \\ 1 & \text{otherwise} \end{cases}$ called the nilpotent maximum. Let us show a case where the inequality $\int_\gamma^{\overline{\wedge}} f < \int_\gamma^{\overrightarrow{\rightarrow}} f$ holds, where the fuzzy implication function $\overrightarrow{\rightarrow} = \mathcal{S}(\overline{\wedge}) = Res(\overline{\wedge})$. Suppose $b < a < 1-\gamma(A)$ and $b < a < \gamma(A)$. Then $\int_\gamma^{\overline{\wedge}} aAb = b \vee (\gamma(A)\overline{\wedge}a) = b \vee 0 = b$ while $\int_\gamma^{\overrightarrow{\rightarrow}} aAb = a \wedge (\gamma(A) \oplus b) = a \wedge (\gamma(A) \vee b) = a$.
- However, if $\otimes$ is not upper-bounded by the minimum, then the other inequality $\int_\gamma^{\rightarrow}(aAb) \leq \int_\gamma^{\otimes}(aAb)$ may hold. For instance, this is the case with the semidual of the contrapositive symmetric of Gödel implication, without equality in general [16, p. 773].

5. Characterization results

Sugeno integrals have been characterized by means of a few properties pertaining to co-monotonic minitivity or maxitivity and the corresponding forms of homogeneity. Let us recall the main results.

Two alternatives $f, g \in L^{\mathcal{C}}$ are said to be *comonotonic* if for every $i, j \in [n]$, if $f(i) < f(j)$ then $g(i) \leq g(j)$. Sugeno integral can then be characterized as follows:

Theorem 1. *Let $I : L^{\mathcal{C}} \to L$. There is a capacity γ such that $I(f) = \int_\gamma f$ for every $f \in L^{\mathcal{C}}$ if and only if the following properties are satisfied*

1. $I(f \vee g) = I(f) \vee I(g)$, *for any comonotonic $f, g \in L^{\mathcal{C}}$.*
2. $I(a \wedge f) = a \wedge I(f)$, *for every $a \in L$ and $f \in L^{\mathcal{C}}$ ($\wedge$-homogeneity).*
3. $I(\mathbf{1}_{\mathcal{C}}) = 1$.

Most older formulations of this theorem [4,35] add an assumption of increasing monotonicity of the functional I (if $f \geq g$ then $I(f) \geq I(g)$) to the three conditions (1–3). However, the proof that conditions (1–3) are necessary and sufficient seems to first appear only in the thesis of Rico [36] (later reproduced in the book [26], and also in [28]).

Conditions (1–3) in Theorem 1 can be equivalently replaced by conditions (1'–3') below.

1'. $I(f \wedge g) = I(f) \wedge I(g)$, for any comonotonic $f, g \in L^{\mathcal{C}}$.
2'. $I(b \vee f) = b \vee I(f)$, for every $a \in L$ and $f \in L^{\mathcal{C}}$ ($\vee$-homogeneity).
3'. $I(\mathbf{0}_{\mathcal{C}}) = 0$.

The existence of these two equivalent characterizations is due to the possibility of writing Sugeno integral in conjunctive and disjunctive forms (8) equivalently. As this identity is no longer valid for more general forms of this integral, we shall characterize q-integrals w.r.t. a fuzzy conjunction and q-cointegrals w.r.t. a fuzzy implication separately in the following. The aim is to look for minimal properties of fuzzy conjunctions and implications that ensure the validity of such characterizations in the style of Theorem 1.

5.1. Characterization of q-integrals associated to a fuzzy conjunction

We first verify that q-integrals satisfy comonotonic maxitivity and a generalized form of $\wedge$-homogeneity.

Lemma 3. *For any fuzzy conjunction and any two comonotonic functions $f, g \in L^{\mathcal{C}}$, we have $\int_\gamma^{\otimes}(f \vee g) = \int_\gamma^{\otimes} f \vee \int_\gamma^{\otimes} g$.*

Proof. The inequality $\int_\gamma^{\otimes}(f \vee g) \geq \int_\gamma^{\otimes} f \vee \int_\gamma^{\otimes} g$ follows from Lemma 1. Let us prove the other inequality. Let $a \in L$. For any two comonotonic functions $f, g \in L^{\mathcal{C}}$ we have either $\{f \geq a\} \subseteq \{g \geq a\}$ or $\{g \geq a\} \subseteq \{f \geq a\}$.

If $\{f \geq a\} \subseteq \{g \geq a\}$ then $\{f \vee g \geq a\} = \{f \geq a\} \cup \{g \geq a\} = \{g \geq a\}$ and $\gamma(\{f \vee g \geq a\}) \otimes a = \gamma(\{g \geq a\}) \otimes a \leq (\gamma(\{g \geq a\}) \otimes a) \vee (\gamma(\{f \geq a\}) \otimes a)$. By symmetry, the inequality is also true when $\{g \geq a\} \subseteq \{f \geq a\}$; hence $\int_\gamma^{\otimes}(f \vee g) \leq \vee_{a \in L}((\gamma(\{g \geq a\}) \otimes a) \vee (\gamma(\{f \geq a\}) \otimes a)) = \int_\gamma^{\otimes} f \vee \int_\gamma^{\otimes} g$. □

Lemma 4. *For every $f \in L^{\mathcal{C}}$ and every $\ell \in \{1, \ldots, n-1\}$, the maps $\mathbf{1}_{F_{(\ell)}} \wedge f_{(\ell)}$ and $\bigvee_{i=\ell+1}^{n} \mathbf{1}_{F_{(i)}} \wedge f_{(i)}$ are comonotonic.*

Proof. We represent both maps as vectors of components ordered according to $(1), \ldots, (n)$, so that $F_{(\ell)} = \{(\ell), \ldots, (n)\}$. In consequence, $\mathbf{1}_{F_{(\ell)}} \wedge f_{(\ell)}(i) = 0$ if $i \leq \ell$ while $\mathbf{1}_{F_{(\ell)}} \wedge f_{(\ell)} = (0, \ldots, 0, f_{(\ell)}, \ldots, f_{(\ell)})$; and $\bigvee_{i=\ell+1}^{n} \mathbf{1}_{F_{(i)}} \wedge f_{(i)} = (0, \cdots, 0, 0, f_{(\ell+1)}, \ldots, f_{(n)})$. Hence it is easy to check that the two maps are comonotonic. □

Lemma 5. *For any capacity γ, any $B \subseteq \mathcal{C}$ and any $a \in L$ we have $\int_\gamma^{\otimes}(a \wedge \mathbf{1}_B) = \gamma(B) \otimes a$. In particular $\int_\gamma^{\otimes} \mathbf{1}_{\mathcal{C}} = 1$.*

Proof. Using Lemma 2, we already established that the q-integral of aAb reduces to $\int_\gamma^{\otimes}(aBb) = (1 \otimes b) \vee (\gamma(B) \otimes a)$. If $b = 0$, we immediately conclude since $\int_\gamma^{\otimes}(aB0) = \gamma(B) \otimes a$ and $aB0 = a \wedge \mathbf{1}_B$. □

Note that contrary to when $\otimes = \wedge$, we do not have $\int_\gamma^\otimes (a \wedge f) = \int_\gamma^\otimes (f) \otimes a$; namely if $\{i : f_{(i)} \geq a\} = B$, $\int_\gamma^\otimes (a \wedge f) = (\gamma(B) \otimes a) \bigvee \vee_{i \notin B} (\gamma(F_{(i)}) \otimes f_{(i)})$ clearly differs from $\int_\gamma^\otimes (f) \otimes a = \vee_{i=1}^n (\gamma(F_{(i)}) \otimes f_{(i)} \otimes a)$, in general.

We can now prove our first characterization result.

Theorem 2. *Let $I : L^{\mathcal{C}} \to L$ be a mapping. There are a capacity γ and a fuzzy conjunction $\otimes$ such that $I(f) = \int_\gamma^\otimes f$ for every $f \in L^{\mathcal{C}}$ if and only if*

1. *$I(f \vee g) = I(f) \vee I(g)$, for any comonotonic $f, g \in L^{\mathcal{C}}$.*
2. *There are a capacity $\lambda : 2^{\mathcal{C}} \to L$ and a binary operation $\star$ on L such that $I(a \wedge \mathbf{1}_A) = \lambda(A) \star a$ for every $a \in L$ and every $A \subseteq \mathcal{C}$.*
3. *$I(\mathbf{1}_{\mathcal{C}}) = 1$ and $I(\mathbf{0}_{\mathcal{C}}) = 0$.*

In that case, we have $\gamma = \lambda$ and $\otimes = \star$.

Proof. Necessity is obtained by previous Lemmas 3 and 5. For sufficiency, assume that I is a mapping that satisfies conditions 1 and 2 and let $f \in L^{\mathcal{C}}$. We have $I(f) = I(\bigvee_{i=1}^n f_{(i)} \wedge \mathbf{1}_{F_{(i)}})$. Using Lemma 4, $I(f) = \bigvee_{i=1}^n I(f_{(i)} \wedge \mathbf{1}_{F_{(i)}}) = \bigvee_{i=1}^n \lambda(F_{(i)}) \star f_{(i)}$. Now we must prove that $\star$ is a fuzzy conjunction. Namely

- $\star$ is increasing in the wide sense in both places. If $a \geq b$, then $a \wedge \mathbf{1}_A \geq b \wedge \mathbf{1}_A$ pointwise, and the two profiles are comonotonic. Thus by condition 1, we find that $I((a \wedge \mathbf{1}_A) \vee (b \wedge \mathbf{1}_A)) = I(a \wedge \mathbf{1}_A) \vee I(b \wedge \mathbf{1}_A) = I(a \wedge \mathbf{1}_A)$, since $(a \wedge \mathbf{1}_A) \vee (b \wedge \mathbf{1}_A) = a \wedge \mathbf{1}_A$. Using condition 2, it reads $\lambda(A) \star a \geq \lambda(A) \star b$. Likewise using $A \subset B$, we can prove with the same argument that $\lambda(B) \star a \geq \lambda(A) \star a$. Indeed, $\mathbf{1}_A \wedge a$ and $\mathbf{1}_B \wedge a$ are comonotonic and $(\mathbf{1}_A \wedge a) \vee (\mathbf{1}_B \wedge a) = \mathbf{1}_B \wedge a$. Condition 1 implies $I((\mathbf{1}_A \wedge a) \vee (\mathbf{1}_B \wedge a)) = I(\mathbf{1}_B \wedge a) = I(\mathbf{1}_A \wedge a) \vee I(\mathbf{1}_B \wedge a)$. So $I(\mathbf{1}_B \wedge a) \leq I(\mathbf{1}_A \wedge a)$, which reads $\lambda(B) \star a \geq \lambda(A) \star a$. So, since λ is a capacity, $\star$ is order-preserving on the left place as well.
- $I(1 \wedge \mathbf{1}_{\mathcal{C}}) = 1 \star 1 = I(\mathbf{1}_{\mathcal{C}}) = 1$
- $I(0 \wedge \mathbf{1}_A) = \lambda(A) \star 0 = I(\mathbf{0}_{\mathcal{C}}) = 0$,
- $I(a \wedge \mathbf{1}_\emptyset) = 0 \star a = I(\mathbf{0}_{\mathcal{C}}) = 0$.

So we can let $\gamma = \lambda$ and $\otimes = \star$ and get $I(f) = \int_\lambda^\otimes f$. □

This result is not a generalization of Theorem 1, since Theorem 2 assumes λ is a capacity and characterizes operation $\star$. A counterpart of Theorem 1 would assume a fuzzy conjunction $\otimes$, i.e., respecting the three conditions of Definition 2. It is easy to prove, following the same steps as in [26,28,36] for the proof of Theorem 1, that due to the monotonicity and extremal properties of $\otimes$, the set-function $A \mapsto I(\mathbf{1}_A) = \gamma(A) \otimes 1$ is a capacity $\hat{\gamma}$ that generally differs from γ, and contrary to universal integrals [33], $\int_\gamma^\otimes \mathbf{1}_A = \hat{\gamma}(A) \neq \gamma(A)$, generally, since 1 is not necessarily an identity on the right. The following counter-example shows that we may have $\hat{\gamma}(A) \neq \gamma(A)$ and $\int_\gamma^\otimes \neq \int_{\hat{\gamma}}^\otimes$.

Example 5. $\mathcal{C} = \{1, 2\}$, $L = \{0, 0.2, 0.8, 1\}$ and $\otimes_G$ is the Gödel conjunction. We have $\gamma(A) \otimes_G 1 = 1$ if $\gamma(A) > 0$ and 0 otherwise so $\gamma(A) \otimes_G 1 \neq \gamma(A)$ if $0 < \gamma(A) < 1$. Let us consider γ such that $\gamma(\{1\}) = 0$ and $\gamma(\{2\}) = 0.3$. If f is defined by $f_1 = 0$ and $f_2 = 0.8$ we obtain $\int_\gamma^{\otimes_G} f = (1 \otimes_G 0) \vee (0.3 \otimes_G 0.8) = 0.8$ and $\int_{\hat{\gamma}}^{\otimes_G} f = 1$ since $\hat{\gamma}(\{2\}) = 1$.

Corollary 1. *If the fuzzy conjunction $\otimes$ is a right-conjunction, then under the assumptions of Theorem 2, the functional I is of the form $I(f) = \int_\gamma^\otimes f$ where $\gamma(A) = I(\mathbf{1}_A)$.*

Proof. Due to Theorem 2, we know that $I(\mathbf{1}_A) = \gamma(A) \otimes 1$ for a capacity γ. As $\otimes$ is a right-conjunction, $\gamma(A) \otimes 1 = \gamma(A) = I(\mathbf{1}_A)$. □

Due to Proposition 1, operation $\otimes$ can be built from exchanging arguments to a fuzzy conjunction that is the semidual of the residuum of another fuzzy conjunction that has a two-sided identity 1, such as a semicopula.

In the case when the functional I is fully maxitive, we can extend the representation Theorem 2 if we slighly restrict the notion of a fuzzy conjunction:

Definition 5. A binary operation $\odot$ is said to be *x-right-cancellative* if and only if for every $a, b \in L$, if $a \odot x = b \odot x$ then $a = b$.

Theorem 3. *Let $I : L^{\mathcal{C}} \to L$ be a mapping. There are a possibility measure Π and a fuzzy conjunction $\otimes$ such that $I(f) = \int_\Pi^\otimes f$ for every $f \in L^{\mathcal{C}}$ if and only if I satisfies the following properties:*

1. *$I(f \vee g) = I(f) \vee I(g)$, for any $f, g \in L^{\mathcal{C}}$.*
2. *There are a capacity $\lambda : 2^{\mathcal{C}} \to L$ and a binary operation $\star$ on L such that $I(a \wedge \mathbf{1}_A) = \lambda(A) \star a$ for every $a \in L$ and every $A \subseteq \mathcal{C}$.*
3. *$I(\mathbf{1}_{\mathcal{C}}) = 1$ and $I(\mathbf{0}_{\mathcal{C}}) = 0$.*

In that case, we have $\star = \otimes$ and, if $\star$ is 1-right-cancellative, we have $\Pi = \lambda$.

Proof. A q-integral with respect to a possibility measure satisfies the requested properties. Let us prove the converse. According to the previous theorem, there exists a capacity λ and a fuzzy conjunction $\star$ such that $I(f) = \int_\lambda^\star f$. For all $a \neq 0$, we have $(a \wedge \mathbf{1}_A) \vee (a \wedge \mathbf{1}_B) = (a \wedge \mathbf{1}_{A \cup B})$ and by condition 1, $I((a \wedge \mathbf{1}_A) \vee (a \wedge \mathbf{1}_B)) = I(a \wedge \mathbf{1}_A) \vee I(a \wedge \mathbf{1}_B)$. Now, since

$I(a \wedge \mathbf{1}_A) = \lambda(A) \star a$ for all a, we get $(\lambda(A) \star a) \vee (\lambda(B) \star a) = \lambda(A \cup B) \star a$. By order-preservingness of $\star$, $(\lambda(A) \star a) \vee (\lambda(B) \star a) = (\lambda(A) \vee \lambda(B)) \star a$ for all a. In particular it holds for $a = 1$, hence the cancellativeness property for value 1 allows us to conclude $\lambda(A \cup B) = \lambda(A) \vee \lambda(B)$. □

Since the fuzzy conjunction $\otimes$ is not supposed to be commutative, there is a companion q-integral defined as follows.

Definition 6. Let $\otimes$ be a non-commutative fuzzy conjunction, and let $\mathcal{E}(\otimes)$ be the fuzzy conjunction obtained from the former by exchanging arguments, i.e., $a\mathcal{E}(\otimes)b = b \otimes a$. Let $\gamma\colon 2^C \to L$ be a capacity. The q-integral $\int_\gamma^{\mathcal{E}(\otimes)}$ is the mapping $L^C \to L$ defined by

$$\int_\gamma^{\mathcal{E}(\otimes)} f = \bigvee_{A \subseteq C} \left(\bigwedge_{i \in A} f_i \otimes \gamma(A) \right).$$

This q-integral generally differs from $\int_\gamma^{\otimes} f$ [16], but it satisfies the same properties since we only use the increasingness property of the fuzzy conjunction in both places, and the limit conditions $0 \otimes 1 = 1 \otimes 0 = 0$ and $1 \otimes 1 = 1$. More precisely we have

- $\int_\gamma^{\mathcal{E}(\otimes)} f = \bigvee_{i=1}^n f_{(i)} \otimes \gamma(F_{(i)})$,
- for every capacity γ, the map $\int_\gamma^{\mathcal{E}(\otimes)} : L^C \to L$ is order-preserving,
- $\int_\gamma^{\mathcal{E}(\otimes)} f = \bigvee_{a \in L} a \otimes \gamma(\{f \geq a\})$,
- for any comonotonic $f, g \in L^C$, we have $\int_\gamma^{\mathcal{E}(\otimes)}(f \vee g) = \int_\gamma^{\mathcal{E}(\otimes)} f \vee \int_\gamma^{\mathcal{E}(\otimes)} g$.

Lemma 6. $\int_\gamma^{\mathcal{E}(\otimes)} a \wedge \mathbf{1}_A = a \otimes \gamma(A)$.

Proof. $\int_\gamma^{\mathcal{E}(\otimes)} a \wedge \mathbf{1}_A = \bigvee_{B \subseteq A} a \otimes \gamma(B) = a \otimes \gamma(A)$. □

Theorem 4. *Let $I : L^C \to L$ be a mapping. There are a capacity γ and a fuzzy conjunction $\otimes$, such that $I(f) = \int_\gamma^{\mathcal{E}(\otimes)} f$ for every $f \in L^C$ if and only if*

1. *$I(f \vee g) = I(f) \vee I(g)$, for any comonotonic $f, g \in L^C$.*
2. *There are a capacity $\lambda\colon 2^C \to L$ and a binary operation $\star$ on L such that $I(a \wedge \mathbf{1}_A) = a \star \lambda(A)$.*
3. *$I(\mathbf{1}_C) = 1$ and $I(\mathbf{0}_C) = 0$.*

In that case γ is defined by $\gamma = \lambda$, and $\otimes = \star$.

Proof. Necessity is obtained by previous Lemmas. For sufficiency, assume that I is a mapping that satisfies conditions 1 and 2 and let $f \in L^C$. We have $I(f) = I(\bigvee_{i=1}^n f_{(i)} \wedge \mathbf{1}_{F_{(i)}}) = \bigvee_{i=1}^n I(f_{(i)} \wedge \mathbf{1}_{F_{(i)}}) = \bigvee_{i=1}^n f_{(i)} \star \lambda(F_{(i)})$. We follow the same lines as in the proof of Theorem 2, to prove that $\star$ is a fuzzy conjunction. Then, we get $I(f) = \int_\lambda^{\mathcal{E}(\star)} f$. □

Clearly using a left-conjunction in the above theorem we conclude that $\lambda(A) = I(\mathbf{1}_A)$.

The results of this section generalise and simplify our previous results presented in [17,18]. They indicate that the algebraic construction underlying Sugeno integrals needs very few properties for the inner aggregation operation.

5.2. q-cointegrals defined from fuzzy implications

We consider a fuzzy implication function $\rightarrow$, which can always be assumed to be of the form $a \rightarrow b = 1 - (a \otimes (1 - b))$ for a fuzzy conjunction $\otimes$. This semiduality property and the generalized form of identity (10) - see Proposition 6 - linking Sugeno integrals and cointegrals lead to state the following characterization result.

Theorem 5. *Let $I : L^C \to L$ be a mapping. There are a capacity γ and a fuzzy implication function $\rightarrow$ such that $I(f) = \int_\gamma^{\rightarrow} f$ for every $f \in L^C$ if and only if the following properties are satisfied.*

1. *$I(f \wedge g) = I(f) \wedge I(g)$, for any comonotonic $f, g \in L^C$.*
2. *There are a capacity $\rho : 2^C \to L$ and a binary operation $\rhd$ such that $I(a \vee \mathbf{1}_A) = \rho^c(\overline{A}) \rhd a, \forall a \in L$.*
3. *$I(\mathbf{1}_C) = 1$ and $I(\mathbf{0}_C) = 0$.*

In that case $\rho = \gamma$, and $\rhd = \rightarrow$.

Proof. Let us define I' by $I'(f) = 1 - I(1 - f)$.

For any comonotonic $f, g \in L^C$, $I'(f \vee g) = 1 - I(1 - (f \vee g)) = 1 - I((1 - f) \wedge (1 - g)) = 1 - I(1 - f) \wedge I(1 - g) = I'(f) \vee I'(g)$.

Likewise, $I'(a \wedge \mathbf{1}_A) = 1 - I(1 - (a \wedge \mathbf{1}_A)) = 1 - I((1 - a) \vee \mathbf{1}_{\overline{A}}) = 1 - (\rho^c(A) \rhd (1 - a)) = \rho^c(A)\mathcal{S}(\rhd)a$.

Hence $I'(f) = \int_{\rho^c}^{\otimes} f$, where $\otimes = \mathcal{S}(\rhd)$ is a fuzzy conjunction, applying Theorem 2. So $I(f) = 1 - \int_{\rho^c}^{\otimes}(1 - f) = \int_\rho^{\rightarrow} f$. □

Note that the homogeneity condition $I(a \vee \mathbf{1}_A) = \rho^c(\overline{A}) \to a$ for q-cointegrals is better understood if we express the latter expression $(1 - \rho(\overline{A})) \to a$ as $\rho(\overline{A}) \oplus a$, where $\oplus$ is the De Morgan dual of $\otimes = \mathcal{D}(\otimes) = \mathcal{D}(\mathcal{S}(\to))$, which means it is indeed the natural counterpart, using a disjunction, of the homogeneity condition $I(a \wedge \mathbf{1}_A) = \lambda(A) \otimes a$ for q-integrals.

Clearly, if the fuzzy implication function is such that $a \to 0 = 1 - a$, then $I(\mathbf{1}_A) = 1 - \rho^c(\overline{A}) = \rho(A)$ (for instance $\to$ is the symmetric contrapositive of a residual fuzzy implication induced by a conjunction having two-sided identity 1, such as a semicopula, after Proposition 1).

In the case when the functional I is fully minitive, we prove the following:

Theorem 6. *Let $I : L^{\mathcal{C}} \to L$ be a mapping. There are a necessity measure N and a fuzzy implication function $\to$ such that $I(f) = \int_N^{\to} f$ for every $f \in L^{\mathcal{C}}$ if and only if I satisfies the following properties:*

1. *$I(f \wedge g) = I(f) \wedge I(g)$, for any $f, g \in L^{\mathcal{C}}$.*
2. *There are a capacity $\rho : 2^{\mathcal{C}} \to L$ and a binary operation $\rhd$ such that $I(a \vee \mathbf{1}_A) = \rho^c(\overline{A}) \rhd a$, for every $a \in L$ and every $A \subseteq \mathcal{C}$.*
3. *$I(\mathbf{1}_{\mathcal{C}}) = 1$ and $I(\mathbf{0}_{\mathcal{C}}) = 0$.*

In that case, we have that $\rhd = \to$ is a fuzzy implication function and, if the fuzzy implication $\to$ is 0-right-cancellative, $\rho = N$ is a necessity measure.

Proof. A q-cointegral with respect to a necessity measure satisfies the requested properties. Let us prove the converse. According to the previous theorem, there exists a capacity ρ and a fuzzy implication $\to$ such that $I(a \vee \mathbf{1}_A) = \rho^c(\overline{A}) \to a$ for all a. For all $a \neq 0$, $(a \vee \mathbf{1}_A) \wedge (a \vee \mathbf{1}_B) = (a \vee \mathbf{1}_{A \cap B})$; so $I((a \vee \mathbf{1}_A) \wedge (a \vee \mathbf{1}_B)) = I(a \vee \mathbf{1}_A) \wedge I(a \vee \mathbf{1}_B) = \rho^c(\overline{A \cap B}) \to a$. Again, as $\to$ is decreasing in the first place, $(\rho^c(\overline{A}) \to a) \wedge (\rho^c(\overline{B}) \to a) = (\rho^c(\overline{A}) \vee \rho^c(\overline{B})) \to a$ for all a. In particular it holds for $a = 0$, hence the 0-right cancellativeness property allows us to conclude $\rho^c(A \cup B) = \rho^c(A) \vee \rho^c(B)$. So, ρ is a necessity measure. □

The result holds for those fuzzy implications such that $a \to 0 = 1 - a$, like the contrapositive form of Gödel implication, but also semiduals of triangular norms, residuated fuzzy implications (w.r.t. a triangular norm). In that case, $I(\mathbf{1}_A) = N(A)$. We turn to the companion q-cointegral defined from a q-cointegral by contrapositive symmetry.

Definition 7. Let $\gamma\colon 2^{\mathcal{C}} \to L$ be a capacity. The contrapositive q-cointegral of $\int_\gamma^{\to}$ is the co-integral $\int_\gamma^{\to_C}$, where $\to_C = \mathcal{C}(\to)$. It is if the form:

$$\int_\gamma^{\to_C} f = \bigwedge_{A \subseteq C} \left(\left(1 - \bigvee_{i \in A} f_i \right) \to (1 - \gamma^c(A)) \right).$$

This q-integral generally differs from $\int_\gamma^{\to} f$ [16] but it satisfies the same properties since $\mathcal{C}(\to)$ is also a fuzzy implication. We can check that

$$\int_\gamma^{\to_C} a \wedge \mathbf{1}_A = (1 - a) \to (1 - \gamma^c(A)),$$

and the following variant of the previous Theorem 5 holds:

Theorem 7. *Let $I : L^{\mathcal{C}} \to L$ be a mapping. There are a capacity γ and a fuzzy implication $\to$ such that $I(f) = \int_\gamma^{\to_C} f$ for every $f \in L^{\mathcal{C}}$ if and only if*

1. *$I(f \vee g) = I(f) \vee I(g)$, for any comonotonic $f, g \in L^{\mathcal{C}}$.*
2. *There are a capacity $\lambda\colon 2^{\mathcal{C}} \to L$ and a binary operation $\rhd$ such that $I(a \wedge \mathbf{1}_A) = (1 - a) \rhd \lambda(A)$.*
3. *$I(\mathbf{1}_{\mathcal{C}}) = 1$ and $I(\mathbf{0}_{\mathcal{C}}) = 0$.*

In that case, we have that $\rhd = \to$ is a fuzzy implication and, if the fuzzy implication $\to$ is 0-right-cancellative, γ is defined by $\gamma = \lambda^c$.

Of course, if the fuzzy implication is such that $1 \to b = b$, we get $I(\mathbf{1}_A) = \lambda(A)$.

To summarize, what the above representation results express is that

- Comonotonic maxitivity is specific to q-integrals and comonotonic minitivity is specific to q-cointegrals.
- q-integrals and q-cointegrals do not satisfy the same homogeneity conditions, the former being homogeneous with respect to the meet $\wedge$ and the latter with respect to the join $\vee$.
- q-integrals simplify if computed w.r.t. a possibility measure and q-cointegral simplify w.r.t. a necessity measure but not conversely.

6. Conclusion

In this paper, we have proposed an algebraic setting with minimal properties, for defining generalized forms of Sugeno integrals, where the inside operation is either a not-necessarily commutative multivalued conjunction or a multivalued implication. We have highlighted the existence of q-integrals and q-cointegrals that are distinct, in contrast with the case of

standard Sugeno integrals. The properties in the algebraic setting were chosen to be minimal while preserving representation theorems by means of comonotonic minitive or maxitive functionals: q-integrals are comonotonic maxitive, cointegrals are comonotonic minitive while standard Sugeno integrals are both. This paper leads to several perspectives and open questions:

- A natural question is to find necessary and sufficient conditions for a fuzzy conjunction $\otimes$ to ensure the equality between q-integrals and their semidual q-cointegrals. It can be conjectured that only the pair formed by the minimum and its semidual satisfy this property, up to a rescaling of the arguments of the operation. Indeed as in the case of Sugeno integrals, the q-integral and q-cointegral formats are generalized disjunctive and conjunctive normal forms, moving from one to the other requires mutual distributivity of $\wedge$ and $\vee$, and we know, from studies on fuzzy conjunctions and disjunctions [31], that such distributivity conditions hold for $\wedge$ and $\vee$ only (this argument was actually pointed out by Grabisch et al. [27], a long time ago).
- We have highlighted the point that operations defining q-integrals and q-cointegrals are pairs of semidual operations, and that using residuations there are always six interdefinable connectives (three fuzzy conjunctions and three fuzzy implications) if one of the fuzzy conjunctions is commutative. Studying these interdependencies when starting with a non-commutative fuzzy conjunction remains an open problem: when does the generation process stop?
- The comparison between q-integrals and q-cointegrals should be carefully studied, especially in the scope of using them for multiattribute decision evaluation processes in the qualitative setting. Namely, what attitudes of the decision-maker can they capture? it would be worth pursuing the work of Chateauneuf and Rico [5] in this generalized setting.
- Some modification schemes, different from fuzzy conjunctions and implications, are not taken into account in this paper. But they seem natural, for instance the one such that if $f_i > \pi_i$ the local rating is f_i, and it is 0 otherwise. They are related to the commutative diagram of Fig. 4 relating multivalued disjunctions and set-difference operations obtained from the diagram relating fuzzy conjunctions and implications via De Morgan-like transformations, that also involve a transformation dual to residuation. Studying the corresponding counterparts of Sugeno integrals is also a perspective to this work.

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References

[1] M. Baczyński, B. Jayaram, Fuzzy Implications, Springer, 2008.
[2] J. Borzová-Molnárová, L. Halčinová, O. Hutník, The smallest semicopula-based universal integrals I properties and characterizations, Fuzzy Sets Syst. 271 (2015) 1–15.
[3] D. Bouyssou, T. Marchant, M. Pirlot, A conjoint measurement approach to the discrete Sugeno integral, in: S. Brams (Ed.), The Mathematics of Preference, Choice and Order Essays in Honor of Peter C. Fishburn, Studies in Choice and Welfare, Springer-Verlag Berlin Heidelberg, 2009, pp. 85–109.
[4] L.M. de Campos, M.J. Bolaños, Characterization and comparison of Sugeno and Choquet integrals, Fuzzy Sets Syst. 52 (1992) 61–67.
[5] A. Chateauneuf, M. Grabisch, A. Rico, Modeling attitudes toward uncertainty through the use of the Sugeno integral, J. Math. Econ. 44 (2008) 1084–1099.
[6] D. Ciucci, D. Dubois, A map of dependencies among three-valued logics, Inf. Sci. 250 (2013) 162–177. Corrigendum, Information Sciences 256: 234–235 2014
[7] D. Dubois, C. Durrieu, H. Prade, A. Rico, Y. Ferro, Extracting decision rules from qualitative data using Sugeno integral: a case-study, in: S. Destercke, T. Denoeux (Eds.), Symbolic and Quantitative Approaches to Reasoning with Uncertainty Proc. ECSQARU, Springer, 2015, pp. 14–24.
[8] D. Dubois, H. Prade, Fuzzy Sets and Systems, Theory and Applications, Academic Press, 1980.
[9] D. Dubois, H. Prade, A theorem on implication functions defined from triangular norms, Stochastica 8 (1984) 267–279.
[10] D. Dubois, H. Prade, Fuzzy set-theoretic differences and inclusions and their use in the analysis of fuzzy equations, Control Cybern. (Warsaw, Poland) 13 (1984) 129–145.
[11] D. Dubois, H. Prade, Weighted minimum and maximum operations, Inf. Sci. 39 (1986) 205–210.
[12] D. Dubois, H. Prade, Possibility Theory, Plenum Press, New York, 1988.
[13] D. Dubois, H. Prade, R. Sabbadin, Qualitative decision theory with Sugeno integrals, in: M. Grabisch (Ed.), Fuzzy Measures and Integrals Theory and Applications, Heidelberg, Physica Vrlg, 2000, pp. 314–322.
[14] D. Dubois, H. Prade, R. Sabbadin, Decision–theoretic foundations of qualitative possibility theory, Eur. J. Oper. Res. 128 (2001) 459–478.
[15] D. Dubois, H. Prade, A. Rico, The logical encoding of Sugeno integrals, Fuzzy Sets Syst. 241 (2014) 61–75.
[16] D. Dubois, H. Prade, A. Rico, Residuated variants of Sugeno integrals, Inf. Sci. 329 (2016) 765–781.
[17] D. Dubois, A. Rico, H. Prade, B. Teheux, Characterizing variants of qualitative Sugeno integrals in a totally ordered Heyting algebra, in: Proc. of 9th Conference of the European Society for Fuzzy Logic and Technology Eusflat,Gijon, 2015, pp. 865–872.
[18] D. Dubois, A. Rico, H. Prade, B. Teheux, Generalized Sugeno integrals, in: J.P. Carvalho (Ed.), Information Processing and Management of Uncertainty in Knowledge-Based Systems, Proc. 16th Int. Conf. IPMU 2016, Eindhoven, Part II. CCIS 610,, Springer, 2016, pp. 363–374.
[19] F. Durante, C. Sempi, Semicopulae, Kybernetika 41 (3) (2005) 315–328.
[20] A. Dvořák, M. Holčapek, Fuzzy measures and integrals defined on algebras of fuzzy subsets over complete residuated lattices, Inf. Sci. 185 (2012) 205–229.
[21] J. Fodor, On fuzzy implication operators, Fuzzy Sets Syst. 42 (1991) 293–300.
[22] J. Fodor, Contrapositive symmetry of implication operators, Fuzzy Sets Syst. 69 (1991) 141–156.
[23] S. Greco, B. Matarazzo, R. Slowinski, Axiomatic characterization of a general utility function and its particular cases in terms of conjoint measurement and rough-set decision rules, Eur. J. Oper. Res. 158 (2004) 271–292.
[24] M. Grabisch, The application of fuzzy integrals in multicriteria decision making, Eur. J. Oper. Res. 89 (3) (1996) 445–456.
[25] M. Grabisch, C. Labreuche, A decade of application of the Choquet and Sugeno integrals in multi-criteria decision aid, Ann. Oper. Res. 175 (2010) 247–286.
[26] M. Grabisch, J.-L. Marichal, R. Mesiar, E. Pap, Aggregation Functions, Cambridge University Press, 2009.

[27] M. Grabisch, T. Murofushi, M. Sugeno, Fuzzy measure of fuzzy events defined by fuzzy integrals, Fuzzy Sets Syst. 50 (3) (1992) 293–313.
[28] M. Grabisch, Set Functions, Games and Capacities in Decision-Making, Springer, 2016.
[29] A. Kandel, W.J. Byatt Fuzzy sets, Fuzzy algebra and fuzzy statistics, Proc. IEEE 68 (1978) 1619–1639.
[30] A. Kaufmann, Le calcul des admissibilités. Une idée nouvelle à partir de la théorie des sous-ensembles flous, in: Proc. Colloque International sur la Théorie et les Applications des Sous-Ensembles Flous, Marseilles, I, 1978.
[31] E.P. Klement, R. Mesiar, E. Pap, Triangular Norms, Springer, 2000.
[32] J.-L. Marichal, On Sugeno integrals as an aggregation function, Fuzzy Sets Syst. 114 (2000) 347–365.
[33] E. Klement, R. Mesiar, E. Pap, A universal integral as common frame for Choquet and Sugeno integral, IEEE Trans. Fuzzy Syst. 18 (2010) 178–187.
[34] A. Pradera, G. Beliakov, H. Bustince, B. De Baets, A review of the relationships between implication, negation and aggregation functions from the point of view of material implication, Inf. Sci. 329 (2016) 357–380.
[35] D. Ralescu, M. Sugeno, Fuzzy integral representation, Fuzzy Sets Syst. 84 (1996) 127–133.
[36] A. Rico, Modélisation des Préférences pour l'Aide à la Décision par l'Intégrale de Sugeno, Université Paris I Sorbonne, 2002 Ph.d. thesis.
[37] D. Schmeidler, Cores of exact games, J. Math. Anal. Appl. 40 (1972) 214–225.
[38] N. Shilkret, Maxitive measure and integration, Indagationes Mathematicae 33 (1971) 109–116.
[39] M. Sugeno, Theory of Fuzzy Integrals and its Applications, Tokyo Institute of Technology, 1974 Phd thesis.
[40] M. Sugeno, Fuzzy measures and fuzzy integrals: a survey., in: M.M. Gupta (Ed.), Fuzzy Automata and Decision Processes, North-Holland, 1977, pp. 89–102.
[41] L.A. Zadeh, Outline of a new approach to the analysis of complex systems and decision processes, IEEE Trans. Syst. Man Cybern. 3 (1973) 28–44. 1
[42] L.A. Zadeh, Fuzzy sets as a basis for a theory of possibility, Fuzzy Sets Syst. 1 (1978) 3–28.