AN ABELIAN ANALOGUE OF SCHANUEL'S CONJECTURE AND APPLICATIONS

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ABSTRACT. In this article we study an abelian analogue of Schanuel's conjecture. This conjecture falls in the realm of the generalised period conjecture of Y. André. As shown by C. Bertolin, the generalised period conjecture includes Schanuel's conjecture as a special case. Extending methods of Bertolin, it can be shown that the abelian analogue of Schanuel's conjecture we consider, also follows from André's conjecture. C. Cheng et al. showed that the classical Schanuel's conjecture implies the algebraic independence of the values of the iterated exponential function and the values of the iterated logarithmic function, answering a question of M. Waldschmidt. We then investigate a similar question in the setup of abelian varieties.

1. INTRODUCTION

S. Schanuel proposed the following conjecture while attending a course given by S. Lang at Columbia University in the 1960's. Most of the known results in the transcendental number theory about the values of the exponential function are encompassed by Schanuel's conjecture, and they can be derived as its consequence.

Conjecture 1 (Schanuel)**.** *Let $x_1, \dots, x_n \in \mathbb{C}$ be such that they are linearly independent over $\mathbb{Q}$. Then the transcendence degree of the field*

$$\mathbb{Q}(x_1, \dots, x_n, e^{x_1}, \dots, e^{x_n})$$

over $\mathbb{Q}$ is at least n.

For example, C. Cheng et al. have shown in [5] how to derive from Conjecture 1, the linear disjointness of the two fields constructed over $\mathbb{Q}$ by adjoining repeatedly the algebraic closure of the field generated by the values of the exponential and logarithm functions respectively.

The only known cases of Conjecture 1 are $n = 1$ and $x_1, \dots, x_n \in \overline{\mathbb{Q}}$ for general n. The $n = 1$ case is a consequence of the Hermite-Lindemann theorem, whereas the latter case is known as the Lindemann-Weierstrass theorem. But these two special cases were known much before the inception of this conjecture.

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Schanuel's conjecture has been generalised to various other contexts. The elliptic analogue of Schanuel's conjecture is well-studied. Let Λ be a lattice in $\mathbb{C}$ and $\wp$ denote the associated Weierstrass $\wp$-function,

$$\wp(z) = \wp(\Lambda; z) := \frac{1}{z^2} + \sum_{\omega \in \Lambda \setminus \{0\}} \left(\frac{1}{(z-\omega)^2} - \frac{1}{\omega^2} \right)$$

for $z \in \mathbb{C} \setminus \Lambda$. The Weierstrass $\wp$-function is an elliptic function with double poles at the points of Λ and holomorphic in $\mathbb{C} \setminus \Lambda$. Moreover, for all $z \in \mathbb{C} \setminus \Lambda$, we have the relation

$$\wp'(z)^2 = 4\wp(z)^3 - 60G_4(\Lambda)\wp(z) - 140G_6(\Lambda).$$

Here for $k \geq 2$, $G_{2k}(\Lambda) := \sum_{\omega \in \Lambda \setminus \{0\}} \omega^{-2k}$ is the associated Eisenstein series of weight $2k$. Let $g_2 = 60G_4(\Lambda)$ and $g_3 = 140G_6(\Lambda)$. Then the modular invariant $j(\Lambda)$ is defined by

$$j(\Lambda) := 1728 \frac{g_2^3}{g_2^3 - 27g_3^2}.$$

and the associated Weierstrass ζ-function is defined by

$$\zeta(z) = \zeta(\Lambda; z) := \frac{1}{z} + \sum_{\omega \in \Lambda \setminus \{0\}} \left(\frac{1}{z-\omega} + \frac{1}{\omega} + \frac{z}{\omega^2} \right),$$

where the series above converges absolutely and uniformly in any compact subset of $\mathbb{C} \setminus \Lambda$. Thus it is holomorphic in $\mathbb{C} \setminus \Lambda$. If ω_1, ω_2 denote the fundamental periods of Λ, then the quasi-periods η_1, η_2 are defined by $\eta_i := \zeta(z + \omega_i) - \zeta(z)$ for $i = 1, 2$. With these notations, the elliptic Schanuel conjecture reads as follows (see [3]) :

Conjecture 2 (elliptic Schanuel)**.** *Let Λ be a lattice and $\wp, \zeta$ denote the associated Weierstrass functions. Let K be the field of endomorphisms of Λ and $x_1, \ldots, x_n \in \mathbb{C} \setminus \Lambda$ such that they are linearly independent over K. Then*

$$\operatorname{trdeg}_{\mathbb{Q}} \mathbb{Q}(g_2, g_3, \omega_1, \omega_2, \eta_1, \eta_2, x_1, \ldots, x_n, \wp(x_1), \ldots, \wp(x_n), \zeta(x_1), \ldots, \zeta(x_n)) \geq 2n + \frac{4}{[K : \mathbb{Q}]}.$$

Often the weaker statement

$$\operatorname{trdeg}_{\mathbb{Q}} \mathbb{Q}(g_2, g_3, x_1, \ldots, x_n, \wp(x_1), \ldots, \wp(x_n)) \geq n \tag{1}$$

is also considered for application of the elliptic Schanuel conjecture. Here also the $n = 1$ case is known and it can be deduced as a consequence of a more general theorem of T. Schneider and S. Lang about transcendental values of meromorphic functions. The analogue of Lindemann-Weierstrass theorem when the Weierstrass $\wp$-function with algebraic invariants g_2, g_3 has complex multiplication, was proved independently by the first author [8] and G. Wüstholtz [10].

We also have A. Grothendieck's period conjecture for an abelian variety A, defined over $\overline{\mathbb{Q}}$. It states that the transcendence degree of the period matrix of A is the same as the dimension of the associated Mumford-Tate group. From the work of P. Deligne [6, Cor. 6.4, p.76] one gets that this dimension is at least an upper bound for the transcendence degree of the period matrix. Y. André [2, Chap. 23] suggested a more general conjecture which is now known as the generalised period conjecture. This concerns periods of 1-motives, which are defined over a subfield of $\mathbb{C}$, not necessarily algebraic. In fact, Grothendieck's period conjecture can be seen as a special case of André's conjecture, using Deligne's work. Further, C. Bertolin [3] has shown that this generalised period conjecture includes Schanuel's conjecture as a special case.

We now consider the following weaker version of André's generalised period conjecture. Let $A(\mathbb{C})$ be an abelian variety of dimension g defined over $\overline{\mathbb{Q}}$ and $\exp_A : \mathbb{C}^g \to A(\mathbb{C})$ denote the exponential map, which is periodic with respect to a lattice Λ_A. Let $\omega_1, \dots, \omega_g$ be a basis of the holomorphic differential 1-forms and $\eta_1, \dots, \eta_g$ be a basis of the meromorphic differential 1-forms with residue 0 on A. Next let $\gamma_1, \dots, \gamma_{2g}$ be a basis of the homology of A. So the matrix of period $\tilde{\Lambda}_A$ is the $2g \times 2g$ matrix with entries $\int_{\gamma_j} \omega_i$ and $\int_{\gamma_j} \eta_i$, $i = 1, \dots, g$, $j = 1, \dots, 2g$, while the matrix of the lattice Λ_A is the $g \times 2g$ matrix with entries $\int_{\gamma_j} \omega_i$.

Let $u \in \mathbb{C}^g$ and $y = \exp_A(u)$. The relevant 1-motive here is $M = [\mathbb{Z} \to A]$, $1 \mapsto y$, which is defined over $\overline{\mathbb{Q}}(\exp_A(u))$. Let $MT(M)$ denote its Mumford-Tate group. The periods of the 1-motive M include the periods of A and the components of u i.e. the numbers $\int_0^u \omega_i$, and also the integrals $\int_0^u \eta_i$. Let $\zeta_A(u)$ denote the vector with components $\int_0^u \eta_i$, $i = 1, \dots, g$.

Conjecture 3 (weak abelian Schanuel)**.** *With the notations as above, let $\overline{\mathbb{Q}}(\tilde{\Lambda}_A)$ denote the field generated by the periods and quasi-periods over $\overline{\mathbb{Q}}$. Let $u \in \mathbb{C}^g$ and H be the smallest algebraic subgroup of A containing the point $\exp_A(u)$. Then*

$$\operatorname{trdeg}_{\overline{\mathbb{Q}}(\tilde{\Lambda}_A)} \overline{\mathbb{Q}}(\tilde{\Lambda}_A, \exp_A(u), u, \zeta_A(u)) \geq 2\dim(H). \tag{2}$$

In a discussion, Daniel Bertrand told us that it is possible to deduce Conjecture 3 from André's conjecture based on [1, Proposition 1] and extending the methods of [3]. With due consent, we reproduce an indication of this argument here.

The generalised period conjecture of André for M implies

$$\operatorname{trdeg}_{\overline{\mathbb{Q}}} \overline{\mathbb{Q}}(\tilde{\Lambda}_A, \exp_A(u), u, \zeta_A(u)) \geq \dim MT(M).$$

Since A is defined over $\overline{\mathbb{Q}}$, Grothendieck's conjecture (which is a particular case of André's conjecture) gives $\operatorname{trdeg}_{\overline{\mathbb{Q}}} \overline{\mathbb{Q}}(\tilde{\Lambda}_A) = \dim(MT(A))$. Hence

$$\operatorname{trdeg}_{\overline{\mathbb{Q}}(\tilde{\Lambda}_A)} \overline{\mathbb{Q}}(\tilde{\Lambda}_A, \exp_A(u), u, \zeta_A(u)) \geq \dim MT(M) - \dim MT(A). \tag{3}$$

If $U(M)$ denotes the unipotent radical of $MT(M)$, then $MT(M)/U(M)$ is the (reductive) group $MT(A)$. Hence the right hand side of (3) equals $\dim(U(M))$. Furthermore, by [1, Proposition 1], $U(M)$ is known to be equal to $H^1_{Betti}(H^\circ)$, where H° is the connected component of H containing the trivial element. Thus $\dim(U(M)) = 2\dim(H)$ and we therefore get (2), i.e.

$$\mathrm{trdeg}_{\overline{\mathbb{Q}}(\tilde{\Lambda}_A)} \overline{\mathbb{Q}}(\tilde{\Lambda}_A, \exp_A(u), u, \zeta_A(u)) \geq 2\dim(H).$$

Thus Conjecture 3 follows from the generalised period conjecture of André.

Remark 1. Note that

$$\mathrm{trdeg}_{\overline{\mathbb{Q}}(\tilde{\Lambda}_A)} \overline{\mathbb{Q}}(\tilde{\Lambda}_A, \exp_A(u), u, \zeta_A(u)) \leq \mathrm{trdeg}_{\overline{\mathbb{Q}}(\tilde{\Lambda}_A)} \overline{\mathbb{Q}}(\tilde{\Lambda}_A, \exp_A(u), u) + \dim(H).$$

Hence from (2) we can deduce,

$$\mathrm{trdeg}_{\overline{\mathbb{Q}}(\tilde{\Lambda}_A)} \overline{\mathbb{Q}}(\tilde{\Lambda}_A, u, \exp_A(u)) \geq \dim(H), \tag{4}$$

and therefore

$$\mathrm{trdeg}_{\overline{\mathbb{Q}}(\Lambda_A)} \overline{\mathbb{Q}}(\Lambda_A, u, \exp_A(u)) \geq \dim(H). \tag{5}$$

So we get (4) and (5) as consequence of (2).

Remark 2. Conjecture 3 can further be considered for abelian varieties defined over a subfield of $\mathbb{C}$, not necessarily algebraic. G. Vallée [9] has formulated the relevant statement from André's generalised period conjecture, and his statement includes Conjecture 3 as a special case.

We therefore have supporting evidence for considering Conjecture 3. With this conjecture in place we want to extend the results in C. Cheng et al. [5] to this setting, that is prove the linear disjointness of the two fields defined below. For the sake of completeness we recall the definition of linear disjointness (see [7, Chap. VIII, §3]).

Definition 1. *Let F be a field and F_1, F_2 two of its field extensions contained in a larger field G. Then F_1 is said to be linearly disjoint (resp. free) from F_2 over F if any finite F-linearly (resp. algebraically) independent subset of F_1 is also F_2-linearly (resp. algebraically) independent (as a subset of G).*

Though the above definition is asymmetric, it can be shown that the property of being linearly disjoint (*resp.* free) is actually symmetric for F_1 and F_2. It is easy to see that if F_1 and F_2 are linearly disjoint over F then $F_1 \cap F_2 = F$. Also if F_1 and F_2 are linearly disjoint over F then one can deduce that F_1 and F_2 are free over F (see [7, Chap. VIII, Prop. 3.2]). The converse is true in special cases (see [7, Chap. VIII, Theorem 4.12] and Lemma 1 below). The property of being free is also called as F_1 and F_2 being algebraically independent over F.

We now setup the relevant notations for our theorem. Recall, A is an abelian variety over $\overline{\mathbb{Q}}$ and $\tilde{\Lambda}_A$ denote the matrix of periods. We consider two recursively defined sets $\mathcal{E}, \mathcal{L}$. Let us define [1]

$$\mathcal{E} = \bigcup_{n\geq 0} \mathcal{E}_n \quad \text{and} \quad \mathcal{L} = \bigcup_{n\geq 0} \mathcal{L}_n$$

where $\mathcal{E}_0 = \overline{\mathbb{Q}}$, $\mathcal{L}_0 = \overline{\mathbb{Q}(\tilde{\Lambda}_A)}$ and for $n \geq 1$,

$$\mathcal{E}_n = \overline{\mathcal{E}_{n-1}(\{\text{components of } \exp_A(u) : u \in \mathcal{E}_{n-1}^g\})}$$

and

$$\mathcal{L}_n = \overline{\mathcal{L}_{n-1}(\{\text{components of } u, \zeta_A(u) : \exp_A(u) \in A(\mathcal{L}_{n-1})\})}.$$

Often, as in [9], for a point P of $A(\mathbb{C})$, one uses $\log_A P$ (*resp.* $\tilde{\log}_A P$) to denote the point $u = \exp_A^{-1}(P)$ (*resp.* $(u, \zeta_A(u))$). By convention we take $\exp_A(\tilde{\log}_A P) = \exp_A(\log_A P) = P$. Given a suitable set S, we denote $\exp_A(S)$ (*resp.* $\log_A(S)$, $\tilde{\log}_A(S)$) the set of elements $\exp_A(u)$ for $u \in S$ (*resp.* $\log_A(P)$, $\tilde{\log}_A(P)$ for $P \in S$). Then, by induction, one can see that for $n \geq 1$,

$$\mathcal{E}_n = \overline{\mathbb{Q}(\text{components of } \exp_A(\mathcal{E}_{n-1}^g))} \quad \text{and} \quad \mathcal{L}_n = \overline{\mathbb{Q}(\Lambda_A, \text{components of } \tilde{\log}_A(A(\mathcal{L}_{n-1})))}.$$

Below we state our main theorem about $\mathcal{E}$ and $\mathcal{L}$, where we take the field G in Definition 1 to be $\mathbb{C}$.

Theorem 1. *If Conjecture 3 is true, then $\mathcal{E}$ and $\mathcal{L}$ are linearly disjoint over $\overline{\mathbb{Q}}$. Since $\mathcal{E}, \mathcal{L}$ are algebraically closed, it is equivalent to say that they are algebraically independent over $\overline{\mathbb{Q}}$.*

Proof of this theorem follows the structure of the proof of the main theorem of [5], but a good part of it differs towards the end of our proof. It will be interesting to consider similar problem for semi-abelian varieties to encompass the cases treated here and in [5].

Remark 3. In Theorem 1, one can consider abelian varieties defined over a subfield of $\mathbb{C}$, not necessarily algebraic, as considered by G. Vallée [9]. However, the statement is not true as it stands. In §4.1, we exhibit an elliptic curve such that $\mathcal{E}_1 \cap \mathcal{L}_1 \supsetneq \mathcal{E}_0 \cap \mathcal{L}_0$ for the natural candidate of $\mathcal{E}_0$ and $\mathcal{L}_0$. The difficulty in this case is coming from the fact that we no more have the equality in Grothendieck's period conjecture (also see [2, Chap. 23.4]).

Remark 4. We get an immediate application of Theorem 1 for elliptic curves over $\overline{\mathbb{Q}}$. See §4 for more details.

[1]As previously for the field $\mathbb{Q}$, we denote with a bar $\overline{K}$ the algebraic closure in $\mathbb{C}$ of a subfield $K \subset \mathbb{C}$.

2. Intermediate lemmas

In this section we deduce some intermediate results to prove Theorem 1.

Lemma 1. *Let K_1, K_2 be two sub-fields of $\mathbb{C}$, which are algebraically closed over $K_1 \cap K_2$. Then they are algebraically independent over $K_1 \cap K_2$ if and only if they are linearly disjoint over $K_1 \cap K_2$.*

Proof. It follows immediately from Theorem 4.12 and Proposition 3.2 of [7, Chap. VIII]. □

Lemma 2. *Let A_n be a finite subset of $\mathcal{E}_n$. Then there exists a finite subset $\mathcal{A}$ of $\mathcal{E}_{n-1}$ such that $\mathcal{A} \cup A_n$ is algebraic over $\overline{\mathbb{Q}}$(components of $\exp_A(\mathcal{A}^g)$).*

Proof. Since $A_n \subset \mathcal{E}_n$, for each $x \in A_n$, there exists a finite subset C_x of $\mathcal{E}_{n-1}$ such that x is algebraic over $\overline{\mathbb{Q}}$(components of $\exp_A(C_x^g)$). Let $A_{n-1} := \cup_{x \in A_n} C_x \subset \mathcal{E}_{n-1}$. Then A_{n-1} is a finite subset of $\mathcal{E}_{n-1}$.

We repeat the process to get sets $\{A_i\}_{0 \le i \le n-2}$ such that for each i, A_i is a finite subset of $\mathcal{E}_i$ and A_{i+1} is algebraic over $\overline{\mathbb{Q}}$(components of $\exp_A(A_i^g)$). We take $\mathcal{A} := \cup_{0 \le i \le n-1} A_i \subset \mathcal{E}_{n-1}$. Then $\mathcal{A} \cup A_n$ is algebraic over $\overline{\mathbb{Q}}$(components of $\exp_A(\mathcal{A}^g)$). □

Lemma 3. *Let C be a finite subset of $\mathcal{L}_n$. Then there exists a finite set $\mathcal{C} \subset \mathcal{L}_n^{2g}$ with $\exp_A(\mathcal{C}) \subset A(\mathcal{L}_{n-1})$ such that the set {components of $\exp_A(\mathcal{C})$} $\cup$ C is algebraic over the field $\overline{\mathbb{Q}}(\tilde{\Lambda}_A$, components of $\mathcal{C})$.*

Proof. Since $C \subset \mathcal{L}_n$, for each $y \in C$, there exists a finite subset D_y of $A(\mathcal{L}_{n-1})$ such that y is algebraic over $\overline{\mathbb{Q}}(\tilde{\Lambda}_A$, components of $\tilde{\log}_A(D_y))$. Define $B_{n-1} := \tilde{\log}_A(\cup_{y \in C} D_y)$, so that C is algebraic over $\overline{\mathbb{Q}}(\tilde{\Lambda}_A$, components of $B_{n-1})$. Then $B_{n-1} \subset \tilde{\log}_A(A(\mathcal{L}_{n-1})) \subset \mathcal{L}_n^{2g}$. Hence $\exp_A(B_{n-1}) \subset A(\mathcal{L}_{n-1})$, i.e. the components of $\exp_A(B_{n-1})$ is a finite subset of $\mathcal{L}_{n-1}$.

We repeat this process for the components of $\exp_A(B_{n-1})$ in place of C and so on, to get sets $\{B_i\}_{0 \le i \le n-2}$ such that for each i, $\exp_A(B_i) \subset A(\mathcal{L}_i)$ and components of $\exp_A(B_{i+1})$ is algebraic over $\overline{\mathbb{Q}}(\tilde{\Lambda}_A$, components of $B_i)$. We set $\mathcal{C} := \cup_{0 \le i \le n-1} B_i$ to complete the proof. □

3. Proof of Theorem 1

In view of Lemma 1, we show that $\mathcal{E}$ and $\mathcal{L}$ are algebraically independent over $\overline{\mathbb{Q}}$. It is enough to prove that $\mathcal{E}_m$ and $\mathcal{L}_n$ are algebraically independent over $\overline{\mathbb{Q}}$ for all m, n.

Now suppose that there exists a pair $(m, n) \in \mathbb{N}^2$ such that $\mathcal{E}_m$ and $\mathcal{L}_n$ are not algebraically independent over $\overline{\mathbb{Q}}$. We choose such a pair (m, n) with the property that if $(a, b) < (m, n)$, then $\mathcal{E}_a$ and $\mathcal{L}_b$ are algebraically independent over $\overline{\mathbb{Q}}$. Here the ordering '$<$' is the ordering on $\mathbb{N}^2$ where $(a, b) < (m, n)$ if and only if either $a \le m$ and $b < n$, or $a < m$ and $b \le n$. Clearly $m, n \ge 1$.

As $\mathcal{E}_m$ and $\mathcal{L}_n$ are not algebraically independent over $\overline{\mathbb{Q}}$, there exists an element ℓ of $\mathcal{L}_n \setminus \overline{\mathbb{Q}}$ which is algebraic over $\mathcal{E}_m$ i.e. there exists a finite subset $\{e_1, \ldots, e_k\}$ of non-zero elements of $\mathcal{E}_m$ such that ℓ is algebraic over $\overline{\mathbb{Q}}(e_1, \ldots, e_k)$.

Now from Lemma 2, we know that there exists a finite subset $\mathcal{A}$ of $\mathcal{E}_{m-1}$ such that $\mathcal{A} \cup \{e_1, \ldots, e_k\}$ is algebraic over $\overline{\mathbb{Q}}$(components of $\exp_A(\mathcal{A}^g)$). Similarly by Lemma 3, we have a finite set $\mathcal{C} \subset \mathcal{L}_n^{2g}$ with $\exp_A(\mathcal{C}) \subset A(\mathcal{L}_{n-1})$ such that {components of $\exp_A(\mathcal{C})\} \cup \{\ell\}$ is algebraic over $\overline{\mathbb{Q}}(\tilde{\Lambda}_A$, components of $\mathcal{C}$). Let $|\mathcal{A}^g| = n_1$ and $|\mathcal{C}| = n_2$.

Define $\mathbf{u_1} := (u : u \in \mathcal{A}^g) \in TA^{n_1}$ by concatenating elements of $\mathcal{A}^g$ one after another. Here TA^{n_1} denotes the tangent space of the abelian variety A^{n_1}. Let A_1 be the smallest algebraic subgroup of A^{n_1} containing the point $\exp_{A^{n_1}}(\mathbf{u_1})$. Similarly define $\mathbf{u_2} := (u : u \in \mathcal{C}) \in TA^{n_2}$ Let A_2 be the smallest algebraic subgroup of A^{n_2} containing the point $\exp_{A^{n_2}}(\mathbf{u_2})$.

Let

$$K_1 := \overline{\mathbb{Q}(\mathbf{u_1}, \exp_{A^{n_1}}(\mathbf{u_1}), \zeta_{A^{n_1}}(\mathbf{u_1}))} \text{ and } K_2 := \overline{\mathbb{Q}(\tilde{\Lambda}_A, \mathbf{u_2}, \exp_{A^{n_2}}(\mathbf{u_2}), \zeta_{A^{n_2}}(\mathbf{u_2}))}$$

Then $\overline{\overline{\mathbb{Q}}(e_1, \ldots, e_k)} \subset K_1 = \overline{\mathbb{Q}(\exp_{A^{n_1}}(\mathbf{u_1}), \zeta_{A^{n_1}}(\mathbf{u_1}))}$ by Lemma 2 and $\overline{\mathbb{Q}}(\ell) \subset K_2 = \overline{\mathbb{Q}(\tilde{\Lambda}_A, \mathbf{u_2}, \zeta_{A^{n_2}}(\mathbf{u_2}))}$ by Lemma 3. By Conjecture 3, we get

$$\operatorname{trdeg}_{\overline{\mathbb{Q}}} K_1 \geq 2\dim(A_1) \text{ and } \operatorname{trdeg}_{\overline{\mathbb{Q}}(\tilde{\Lambda}_A)} K_2 \geq 2\dim(A_2)$$

for $i = 1, 2$. Since $\exp_{A^{n_1}}(\mathbf{u_1}) \in A_1$ and $\mathbf{u_2} \in TA_2$, we get that

$$\operatorname{trdeg}_{\overline{\mathbb{Q}}} K_1 \leq 2\dim(A_1)$$

and

$$\operatorname{trdeg}_{\overline{\mathbb{Q}}(\tilde{\Lambda}_A)} K_2 \leq 2\dim(TA_2) = 2\dim(A_2).$$

Hence

$$\operatorname{trdeg}_{\overline{\mathbb{Q}}} K_1 = 2\dim(A_1) \text{ and } \operatorname{trdeg}_{\overline{\mathbb{Q}}(\tilde{\Lambda}_A)} K_2 = 2\dim(A_2).$$

We want to show that

$$\operatorname{trdeg}_{\overline{\mathbb{Q}}(\tilde{\Lambda}_A)} K_1 K_2 = 2\dim(A_1 \times A_2) = \operatorname{trdeg}_{\overline{\mathbb{Q}}} K_1 + \operatorname{trdeg}_{\overline{\mathbb{Q}}(\tilde{\Lambda}_A)} K_2. \tag{6}$$

Adding $\operatorname{trdeg}_{\overline{\mathbb{Q}}} \overline{\mathbb{Q}}(\tilde{\Lambda}_A)$ to both sides of (6), we would get

$$\operatorname{trdeg}_{\overline{\mathbb{Q}}} K_1 K_2 = \operatorname{trdeg}_{\overline{\mathbb{Q}}} K_1 + \operatorname{trdeg}_{\overline{\mathbb{Q}}} K_2.$$

This would prove that the fields K_1 and K_2 are algebraically independent over $\overline{\mathbb{Q}}$. We will thus get a contradiction to our assumption.

Define $\mathbf{u_3} := (\mathbf{u_1}, \mathbf{u_2})$. Let B be the smallest algebraic subgroup of $A^{n_1+n_2}$ containing the point $\exp_{A^{n_1+n_2}}(\mathbf{u_3})$. Then by Conjecture 3, we get

$$\operatorname{trdeg}_{\overline{\mathbb{Q}}(\tilde{\Lambda}_A)} K_1 K_2 \geq 2\dim(B).$$

Thus we are reduced to prove that $\dim(B) = \dim(A_1) + \dim(A_2)$. If they are torsion subgroups then we have nothing to prove. So we assume that at least one of A_1 and A_2 is not a torsion subgroup.

We first assume A to be simple. For $\mathbf{u_i} \in TA_i \hookrightarrow TA^{n_i}$, we choose a basis and write $\mathbf{u_i} = (u_{i1}, \ldots, u_{in_i})$ with $u_{ij} \in TA$ for $i = 1, 2$ and $j = 1, \ldots, n_i$. We consider any of the defining relation for TB,

$$\sum_{1 \le j \le n_1} \delta_{1j} x_{1j} - \sum_{1 \le j \le n_2} \delta_{2j} x_{2j} = 0, \tag{7}$$

where $\delta_{ij} \in End(A)$ for $i = 1, 2$ and $j = 1, \ldots, n_i$. Let $u := \sum_{1 \le j \le n_1} \delta_{1j} u_{1j} = \sum_{1 \le j \le n_2} \delta_{2j} u_{2j}$. Then $u \in \mathcal{E}^g_{m-1} \cap \mathcal{L}^g_n$ as each $u_{1j} \in \mathcal{A}^g$ and $u_{2j} \in \mathcal{C}$. Thus $u \in \overline{\mathbb{Q}}^g$, by the choice of m, n.

On the other hand $\exp_A(u) = \sum_{1 \le j \le n_1} \delta_{1j} \exp_A(u_{1j}) = \sum_{1 \le j \le n_2} \delta_{2j} \exp_A(u_{2j})$. For similar reason $\exp_A(u) \in A(\mathcal{E}_m) \cap A(\mathcal{L}_{n-1})$. Again $\mathcal{E}_m \cap \mathcal{L}_{n-1} = \overline{\mathbb{Q}}$, and hence $\exp_A(u) \in A(\overline{\mathbb{Q}})$. Thus,

$$\mathrm{trdeg}_{\overline{\mathbb{Q}}} \overline{\mathbb{Q}}(u, \exp_A(u)) = 0$$

Now if H is the smallest algebraic subgroup of A containing $\exp_A(u)$, then $\dim(H) = 0$, by Conjecture 3, i.e. H is torsion subgroup. In particular, $nu \in \overline{\mathbb{Q}}^g$ and $\exp_A(nu) = 0$, for a suitable integer n.

Thus, $\exp_A(nu) = \sum_{1 \le j \le n_1} \delta_{1j} \exp_A(nu_{1j}) = \sum_{1 \le j \le n_2} \delta_{2j} \exp_A(nu_{2j}) = 0$. Now A_i is the smallest algebraic subgroup of A^{n_i} containing the point $\exp_{A^{n_i}}(\mathbf{u_i})$ for $i = 1, 2$. Thus, $\sum_{1 \le j \le n_i} \delta_{ij} \exp_A(nx_{ij}) = 0$ for any point $\exp_{A^{n_i}}(\mathbf{x_i}) \in A_i$ for $i = 1, 2$. Since at least one of A_1 or A_2 is not a torsion subgroup, there exists $i \in \{1, 2\}$, such that $\sum_{1 \le j \le n_i} \delta_{ij} n x_{ij} = 0$ on TA_i. Hence $\sum_{1 \le j \le n_i} \delta_{ij} x_{ij} = 0$ on TA_i, and therefore defining relations for TB separate into disjoint relations defining TA_1 and TA_2. Thus we have $\dim(B) = \dim(A_1) + \dim(A_2)$.

Now we treat the case when A is not a simple abelian variety. In this case, we would like to write down the generic form of a defining relation for TB and we show that it is a collection of relations of the form (7). Then the proof will follow as above.

We suppose that $A^{n_1+n_2}$ is isogenous to $V_1^{r_1} \times \cdots \times V_l^{r_l}$, where for $1 \le i \ne j \le l$, V_i is an abelian variety not isogenous to V_j. Thus, the tangent space $TA^{n_1+n_2}$ has the form

$$\underbrace{TV_1 \oplus \cdots \oplus TV_1}_{r_1 \text{ times}} \oplus \cdots \oplus \underbrace{TV_l \oplus \cdots \oplus TV_l}_{r_l \text{ times}}.$$

Now $B \subset A_1 \times A_2 \subset A^{n_1+n_2}$. Hence, $A^{n_1+n_2}/B$ can be written in the form $V_1^{s_1} \times \cdots \times V_l^{s_l}$, where for each $1 \le i \le l$, $s_i \le r_i$. Now B is the kernel of the natural map from $A^{n_1+n_2} \to A^{n_1+n_2}/B$. So for this we find the corresponding map $V_1^{r_1} \times \cdots \times V_l^{r_l} \to V_1^{s_1} \times \cdots \times V_l^{s_l}$, for which B is isogenous to the kernel.

Such a map is expressed as a block diagonal matrix of order $(s_1 + \cdots + s_l, r_1 + \cdots + r_l)$. This matrix has diagonal blocks of order (s_i, r_i) with entries from $End(V_i)$, for each $1 \le i \le l$.

Now such a matrix acts on an element $(x_{11},\ldots,x_{1r_1},\ldots,x_{l1},\ldots,x_{lr_l})$ of $TA^{n_1+n_2}$, written as a column. Under this action an element of TB is mapped to the zero vector i.e.

$$\begin{pmatrix} (\delta_{1jk})_{s_1\times r_1} & (0)_{s_1\times r_2} & \cdots & (0)_{s_1\times r_l} \\ (0)_{s_2\times r_1} & (\delta_{2jk})_{s_2\times r_2} & \cdots & (0)_{s_2\times r_l} \\ \vdots & \vdots & \ddots & \vdots \\ (0)_{s_l\times r_1} & (0)_{s_l\times r_2} & \cdots & (\delta_{ljk})_{s_l\times r_l} \end{pmatrix} \begin{pmatrix} \mathbf{x}_1 \\ \mathbf{x}_2 \\ \vdots \\ \mathbf{x}_l \end{pmatrix} = \mathbf{0}, \tag{8}$$

where for $1 \le i \le l$, $\mathbf{x}_i = \begin{pmatrix} x_{i1} \\ \vdots \\ x_{ir_i} \end{pmatrix}$. Since the matrix is block diagonal, we get that a defining relation for TB is given as the relations of the form

$$\sum_{k=1}^{r_i} \delta_{ijk} x_{ik} = 0$$

for some $\delta_{ijk} \in End(V_i)$ with $1 \le i \le l, 1 \le j \le s_i$ and $1 \le k \le r_i$. This completes the proof.

3.1. **A special case.** Let us consider the following two sub-fields of $\mathcal{E}$ and $\mathcal{L}$:

$$\mathcal{E}' = \bigcup_{n\ge 0} \mathcal{E}'_n \quad \text{and} \quad \mathcal{L}' = \bigcup_{n\ge 0} \mathcal{L}'_n$$

where $\mathcal{E}'_0 = \overline{\mathbb{Q}}$, $\mathcal{L}'_0 = \overline{\mathbb{Q}}(\Lambda_A)$ and for $n \ge 1$,

$$\mathcal{E}'_n = \overline{\mathcal{E}'_{n-1}(\{\text{components of } \exp_A(u) : u \in \mathcal{E}'^{g}_{n-1}\})}$$

and

$$\mathcal{L}'_n = \overline{\mathcal{L}'_{n-1}(\{\text{components of } u : \exp_A(u) \in A(\mathcal{L}'_{n-1})\})}.$$

In fact, by induction, one can see that for $n \ge 1$,

$$\mathcal{E}'_n = \overline{\mathbb{Q}(\text{components of } \exp_A(\mathcal{E}'^{g}_{n-1}))} \quad \text{and} \quad \mathcal{L}'_n = \overline{\mathbb{Q}(\Lambda_A, \text{components of } \exp_A^{-1}(A(\mathcal{L}'_{n-1})))}.$$

Arguments as in our proof of Theorem 1 immediately yields the following :

Theorem 2. *If (5) is true, then $\mathcal{E}'$ and $\mathcal{L}'$ are linearly disjoint over $\overline{\mathbb{Q}}$.*

4. An application

Now we consider the two recursively defined sets $\mathcal{E},\mathcal{L}$ related to the Weierstrass $\wp$-function associated to a lattice Λ with algebraic invariants g_2, g_3, defined as follows

$$\mathcal{E} = \bigcup_{n\ge 0} \mathcal{E}_n \quad \text{and} \quad \mathcal{L} = \bigcup_{n\ge 0} \mathcal{L}_n$$

where $\mathcal{E}_0 = \overline{\mathbb{Q}}, \mathcal{L}_0 = \overline{\mathbb{Q}(\Lambda)}$ and for $n \ge 1$,

$$\mathcal{E}_n = \overline{\mathcal{E}_{n-1}(\{\wp(x) : x \in \mathcal{P}_{n-1} \setminus \Lambda\})} \quad \text{and} \quad \mathcal{L}_n = \overline{\mathcal{L}_{n-1}(\{x : \wp(x) \in \mathcal{L}_{n-1} \cup \{\infty\}\})}.$$

In fact, by induction, one can see that for $n \geq 1$,

$$\mathcal{E}_n = \overline{\mathbb{Q}(\wp(\mathcal{E}_{n-1} \setminus \Lambda))} \text{ and } \mathcal{L}_n = \overline{\mathbb{Q}(\wp^{-1}(\mathcal{L}_{n-1} \cup \{\infty\}))}.$$

As a corollary to Theorem 1, we obtain the following result.

Corollary 1. *If Conjecture 2 is true, then $\mathcal{E}$ and $\mathcal{L}$ are linearly disjoint over $\overline{\mathbb{Q}}$.*

In fact, in view of Theorem 2, the conclusion of the above corollary also holds if (1) is true. We know that when g_2, g_3 are algebraic, $\omega \in \Lambda \setminus \{0\}$ is transcendental. Hence as a corollary to Corollary 1 we obtain that $\omega \notin \mathcal{E}$. In particular, ω can not be a value of the $\wp$ function iterated at an algebraic point.

4.1. **An example.** We now exhibit an example in the elliptic case which shows that if the curve is not defined over $\overline{\mathbb{Q}}$, then the conclusion of our Theorem 1 (or, Corollary 1) does not follow from the same hypothesis.

For a fixed real algebraic irrational number α, our aim is to find out a lattice $\mathbb{Z} + \mathbb{Z}\tau$ such that $\wp(\tau; \alpha) = \wp(\mathbb{Z} + \mathbb{Z}\tau; \alpha)$ can be written as a polynomial in τ with algebraic coefficients.

We first claim that we can find τ_0 such that $\frac{d}{d\tau}\wp(\tau;\alpha)|_{\tau=\tau_0} \neq \frac{\wp(\tau_0;\alpha)}{\tau_0}$. If not, then we have $\frac{d\wp(\tau;\alpha)}{\wp(\tau;\alpha)}|_{\tau=\tau_0} = \frac{d\tau}{\tau}|_{\tau=\tau_0}$ for all τ_0 in the complex upper half plane. Thus $d(\log(\wp(\tau;\alpha)))|_{\tau=\tau_0} = d(\log\tau)|_{\tau=\tau_0}$ i.e. $\wp(\tau_0;\alpha) = c_\alpha \tau_0$ for all τ_0, where c_α is a constant depending on α. But this is not possible, as can be checked from the q-expansion of $\wp$.

So we choose τ_0 such that $\frac{d}{d\tau}\wp(\tau;\alpha)|_{\tau=\tau_0} \neq \frac{\wp(\tau_0;\alpha)}{\tau_0}$ and denote the ratio $\frac{\wp(\tau_0;\alpha)}{\tau_0}$ by λ_0. We now choose λ close to λ_0 such that $\lambda \in \overline{\mathbb{Q}}$ and $\frac{d}{d\tau}\wp(\tau;\alpha)|_{\tau=\tau_0} \neq \lambda$. Consider the function $f(\tau) = \wp(\tau;\alpha) - \lambda\tau$. Then we get that $f'(\tau)|_{\tau=\tau_0} \neq 0$. Hence f has a local inverse at τ_0, say g, which is defined in a neibourghood of $f(\tau_0)$. Choose $\beta \in \overline{\mathbb{Q}}$ sufficiently close to $f(\tau_0)$ and set $\tau_1 = g(\beta)$. Then $\beta = f(\tau_1) = \wp(\tau_1;\alpha) - \lambda\tau_1$. Thus $\wp(\tau_1;\alpha)$ can be written as a polynomial in τ_1 with algebraic coefficients.

The elliptic Schanuel conjecture implies that τ_1 is transcendental. Indeed, if τ_1 is a quadratic irrational, then the associated j invariant is algebraic. Hence g_2, g_3 are algebraically related and η_1, η_2, satisfying Masser's relation, are algebraically dependent over $\overline{\mathbb{Q}(g_2, g_3)}$. Now from Conjecture 2 we get a contradiction by taking $n = 1$ and $x_1 = \alpha$. If τ_1 is algebraic of degree larger than 2, then Conjecture 2 gives a contradiction again for $n = 1$ and $x_1 = \alpha$.

Now for this choice of τ_1, we see that τ_1 belongs to both $\mathcal{E}_1$ and $\mathcal{L}_1$, where the tower of fields $\mathcal{E}_i$'s and $\mathcal{L}_i$'s are constructed as in the beginning of this section, but with $\overline{\mathbb{Q}}$ replaced by the corresponding field of definition $\overline{\mathbb{Q}(g_2, g_3)}$. However, we show below that τ_1 is transcendental over $\mathbb{Q}(g_2, g_3)$. This gives $\mathrm{trdeg}_{\mathcal{E}_0}\mathcal{E}_1 \cap \mathcal{L}_1 \geq 1$, which implies $\mathcal{E}_1 \cap \mathcal{L}_1 \neq \mathcal{E}_0$ and therefore $\mathcal{E}_1$ and $\mathcal{L}_1$ are not linearly disjoint over $\mathcal{E}_0$.

To prove that τ_1 is transcendental over $\mathbb{Q}(g_2, g_3)$, note that $1, \tau_1$ and α are $\mathbb{Q}$ linearly independent. Then the elliptic Schanuel conjecture yields $\mathrm{trdeg}_{\mathbb{Q}}\mathbb{Q}(g_2, g_3, 1, \tau_1, \alpha, \wp(\tau_1;\alpha)) \geq 3$.

Now from our construction we see that $\mathrm{trdeg}_{\mathbb{Q}}\mathbb{Q}(g_2, g_3, 1, \tau_1, \alpha, \wp(\tau_1; \alpha)) = \mathrm{trdeg}_{\mathbb{Q}}\mathbb{Q}(g_2, g_3, \tau_1)$. Thus τ_1 is transcendental over $\mathbb{Q}(g_2, g_3)$.

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