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Abstract. We investigate the filtration law in rigid non-galilean porous media of incompressible viscous Newtonian fluids. The filtration law is obtained by upscaling the flow from the pore scale. We use the method of multiple scale expansions which gives rigorously the macroscopic behaviour without any prerequisite at the macroscopic scale. For finite Ekman numbers the filtration law is shown to resemble a Darcy’s law, but with a non-symmetric permeability tensor which depends on the angular velocity of the porous matrix. For large Ekman numbers the filtration law is a small correction to the classical Darcy’s law. The corrector is antisymmetric. In this case we recover a similar structure of law to phenomenologic laws introduced in the literature, but with a dissimilar effective coefficient.

rotating porous media / filtration law / Ekman / homogenization

Loi de filtration en milieu poreux tournant

Résumé. *Nous étudions la loi de filtration d’un fluide visqueux newtonien dans un milieux poreux rigide non galiléen. La loi de filtration est obtenue par changement d’échelles à partir de l’échelle des pores. Nous utilisons la méthode des développements asymptotiques à échelles multiples qui donne le comportement macroscopique de façon rigoureuse et sans prérequis à l’échelle macroscopique. Pour un nombre d’Ekman fini, la loi de filtration possède une structure de loi de Darcy, mais avec un tenseur de perméabilité non symétrique et dépendant de la rotation angulaire du milieu poreux. Pour les grands nombres d’Ekman, la loi de filtration est une petite correction de la loi de Darcy. Le terme correcteur est antisymétrique. La structure de loi est alors du type de celle des lois phénoménologiques de la littérature, mais avec un coefficient effectif différent.*

milieux poreux tournants / loi de filtration / Ekman / homogénéisation

Version française abrégée

L’écoulement de fluide dans des milieux poreux tournants concerne de nombreuses applications [1]. Les lois de filtration phénoménologiques utilisées dans la littérature [3] sont de la forme (1). Nous considérons pour simplifier un milieux poreux situé dans une centrifugeuse de rayon r et nous utilisons la méthode des développements asymptotiques à échelles multiples [5,6] qui donne le comportement macroscopique équivalent à partir de la description à l’échelle des pores si la condition de séparation d’échelles (2) est remplie. l mesure la taille des pores et $L = \mathrm{O}(r)$ est une longueur caractéristique macroscopique. Le milieu est supposé périodique.

Le mouvement relatif permanent et lent d’un fluide visqueux newtonien incompressible par rapport à un milieu poreux lui-même en mouvement par rapport à un repère galiléen est décrit par le système (4)–(8),

où $\overrightarrow{\gamma}_e$ and $\overrightarrow{\gamma}_c$ sont les accélérations d'entraînement et de Coriolis, $\overrightarrow{\omega}$ est la vitesse de rotation de la matrice poreuse, O est un point de celle-ci dans la période considérée et M est un point courant du pore. L' équation (4) introduit quatre nombres sans dimension. Le rapport Q des termes de pression et viqueux est $\mathrm{O}(\varepsilon^{-1})$, cf. [4]. Le rapport des parties rotatoire et translationnelle de l'accélération d'entraînement est $A = \mathrm{O}(\omega^2 l/\omega^2\ r) = \mathrm{O}(\varepsilon)$. Le rapport R de l'accélération d'entraînement translationnelle au terme visqueux est pris $\mathrm{O}(1)$ pour assurer la séparation des échelles. Enfin le nombre d'Ekman $E_k = \mu/2\rho\omega l^2$ est supposé $\mathrm{O}(1)$. Les équations sans dimension de l'écoulement sont alors (9)–(11).

Le procédé d'homogénéisation consiste à introduire les développements (12)–(13) dans le système (9)–(11). On obtient alors le premier ordre de la pression (14), puis le premier ordre de la vitesse (20). Enfin le bilan de volume conduit à la description macroscopique (22). La structure obtenue est celle de la loi de Darcy. Toutefois la perméabilité $\boldsymbol{K}$ dépend de la vitesse de rotation ω et n'est pas en général un tenseur symétrique.

Pour les grand nombres d'Ekman, $\overrightarrow{v}^{(0)}$ et $p^{(1)}$ sont recherchés sous la forme (24)–(25), $\eta = E_k^{-1}$. En utilisant les deux premiers ordres de ces développements on obtient l'approximation aux grands nombres d'Ekman (34), valable pour $\varepsilon \ll Ek^{-1} \ll 1$. Le tenseur $\boldsymbol{K}^1$, donné par (35), est antisymétrique, ce qui conduit finalement à une loi de la forme (38). K^0 est la perméabilité à rotation nulle et $\boldsymbol{H}$ est donné par (37). La loi (34) possède la même structure que la loi phénoménologique (1), au coefficient $\boldsymbol{H}$ près.

1. Introduction

Fluid flow in non-galilean porous matrix is concerned with numerous practical applications going from geological applications to industry. A good review of the state of the art is given in [1]. The filtration law in use in rotating porous media, [2,3], is obtained by the direct introduction in the classical Darcy's law of the Coriolis inertia term that appears in Navier–Stokes equations, see equations (4) and (6) below. In Vadasz [3], the author introduces the following isotropic filtration law in dimensionless form

$$\overrightarrow{q} = -k(\overrightarrow{\nabla} p + Ek^{-1}\overrightarrow{e}_\omega \times \overrightarrow{q}) \tag{1}$$

where $\overrightarrow{q}$ is the flowrate vector, k is the permeability, $\overrightarrow{\nabla} p$ is the pressure gradient, $\overrightarrow{e}_\omega$ is a unit vector in the direction of the rotational velocity and Ek is the Ekman number, i.e., the ratio of viscous term to Coriolis term in Navier–Stokes equations. On the other hand, there is to our knowledge no theoretical work on the modelling of fluid flow in a non-galilean porous matrix.

The aim of this paper is to investigate the tensorial filtration law in non-galilean rigid porous matrices for permanent slow flow of an incompressible viscous Newtonian fluid. We use an upscaling technique, i.e., the method of multiple scale expansions to determine the macroscopic flow from its description at the pore scale. Heterogeneous system, for example porous media, may be modelled by an equivalent macroscopic continuous system if the condition of separation of scales is verified,

$$\varepsilon = \frac{l}{L} \ll 1 \tag{2}$$

where l and L are the characteristic lengths of the heterogeneities and of the macroscopic sample or excitation, respectively. The macroscopic equivalent model is obtained from the description at the heterogeneity scale by [4]: (i) assuming the medium to be periodic, without loss of generality; (ii) writing the local description in a dimensionless form; (iii) evaluating the dimensionless numbers with respect to the scale ratio ε; (iv) looking for the unknown fields in the form of asymptotic expansions in powers of ε; (v) solving the successive boundary value problems that are obtained after introducing these expansions in the local dimensionless description. The macroscopic equivalent model is obtained from compatibility conditions which are the necessary conditions for the existence of solutions to the boundary value problems.

In *Section 2*, we give the description of the flow at the pore scale relative to the moving matrix frame. This description is made dimensionless and we evaluate the different dimensionless numbers with respect to the scale ratio ε. *Section 3* is devoted to the upscaling for permanent slow flow. The filtration law resembles a Darcy's law. However, it is shown that the permeability tensor is non-symmetric and depends on the angular velocity of the matrix frame. The case of large Ekman numbers is investigated in *Section 4*. The filtration law that is obtained shows a similar structure to the phenomenological law (1), but with a different effective coefficient in front of the vector product.

2. Local flow description and estimations

Consider the flow of an incompressible liquid through a porous medium. The porous medium is spatially periodic. There are two characteristic length scales in this problem: the characteristic microscopic length scale l of the pores and of the unit cell, and macroscopic length scales that are the macroscopic pressure loss scale and the sample size scale. For simplicity, we assume these macroscopic length scales are of similar order of magnitude, $\mathrm{O}(L)$. Moreover we assume that the two length scales l and L are well separated:

$$l \ll L \tag{3}$$

The unit cell is denoted Ω and is bounded by $\partial\Omega$, the fluid part of the unit cell is denoted Ω_p, and the fluid-solid interface inside the unit cell is Γ. Relatively to the moving porous matrix frame, the momentum balance for the permanent slow incompressible viscous Newtonian liquid flow is

$$\mu\nabla^2\vec{v} - \vec{\nabla}p = \rho(\vec{\gamma}_e + \vec{\gamma}_c) \quad \text{in } \Omega_p \tag{4}$$

where $\vec{v}$ is the velocity vector relative to the matrix frame, p is the pressure, ρ is the density and μ is the viscosity. Gravitational acceleration is included in the pressure term. $\vec{\gamma}_e$ and $\vec{\gamma}_c$ are the convective and the Coriolis accelerations, respectively:

$$\vec{\gamma}_e = \vec{\gamma}(O) + \vec{\omega} \times (\vec{\omega} \times \overrightarrow{OM}) \tag{5}$$

$$\vec{\gamma}_c = 2\vec{\omega} \times \vec{v} \tag{6}$$

where $\vec{\omega}$ is the angular velocity of the porous matrix, O is a fixed point of the porous matrix in the investigated period and M is a current point in Ω_p. Equation (4) is completed by the incompressibility condition and the adherence condition on Γ:

$$\vec{\nabla} \cdot \vec{v} = 0 \quad \text{in } \Omega_p \tag{7}$$

$$\vec{v} = 0 \quad \text{on } \Gamma \tag{8}$$

The independent dimensionless numbers which characterize the liquid flow problem may be related to the magnitude of ε.

We use the local length scale of a pore l as the characteristic length scale for the variations of the differential operators. To fix the ideas we consider a centrifuge of radius $r = \mathrm{O}(L)$ at constant angular velocity $\vec{\omega} = \omega\vec{e}_\omega$, $\omega =$ constant. Then, we are left with four dimensionless numbers: the ratio Q of pressure to viscous forces, the ratio R of the translational convective inertia to the viscous force, the ratio A of the rotational to the translational convective inertia and the ratio *Ek* (the Ekman number) of the viscous force to the Coriolis inertia. We have $A = \mathrm{O}(\omega^2\, l/\omega^2\, r) = \mathrm{O}(\varepsilon)$ and we have $R = \mathrm{O}(1)$. The estimation of R is the consequence of the hypothesis of separation of scales. Higher values would yield non-homogenizable problems, i.e., problems for which equivalent macroscopic description do not exist.

From [4] we have $Q = \mathrm{O}(\varepsilon^{-1})$. We will assume in the following that the Ekman number $\mu/2\rho\omega l^2$ is $\mathrm{O}(1)$. The formal dimensionless set that describes the flow is in the form:

$$\mu\nabla^2\overrightarrow{v} - \varepsilon^{-1}\overrightarrow{\nabla}p = \rho\big(\overrightarrow{\gamma}(O) + Ek^{-1}2\overrightarrow{\omega}\times\overrightarrow{v} + \varepsilon\overrightarrow{\omega}\times(\overrightarrow{\omega}\times\overrightarrow{OM})\big) \quad \text{in } \Omega_p \tag{9}$$

$$\overrightarrow{\nabla}\cdot\overrightarrow{v} = 0 \quad \text{in } \Omega_p \tag{10}$$

$$\overrightarrow{v} = 0 \quad \text{on } \Gamma \tag{11}$$

3. Homogenization

Following the multiple scale expansion technique [5,6], the velocity $\overrightarrow{v}$ and the pressure fluctuation p are looked for in the form of asymptotic expansions of powers of ε:

$$\overrightarrow{v} = \overrightarrow{v}^{(0)}(\overrightarrow{x},\overrightarrow{y}) + \varepsilon\,\overrightarrow{v}^{(1)}(\overrightarrow{x},\overrightarrow{y}) + \varepsilon^2\,\overrightarrow{v}^{(2)}(\overrightarrow{x},\overrightarrow{y}) + \cdots \tag{12}$$

$$p = p^{(0)}(\overrightarrow{x},\overrightarrow{y}) + \varepsilon\,p^{(1)}(\overrightarrow{x},\overrightarrow{y}) + \varepsilon^2\,p^{(2)}(\overrightarrow{x},\overrightarrow{y}) + \cdots \tag{13}$$

Substituting these expansions in the set (9)–(11) gives, by identification of the like powers of ε, successive boundary value problems to be investigated. The lowest order approximation of the pressure verifies:

$$\frac{\partial p^{(0)}}{\partial y_i} = 0, \qquad p^{(0)} = p^{(0)}(\overrightarrow{x}) \tag{14}$$

The first order approximation of the velocity $\overrightarrow{v}^{(0)}$ and the second order approximation of the pressure $p^{(1)}$ are determined by the following set

$$\mu\frac{\partial^2 v_i^{(0)}}{\partial y_j\partial y_j} - G_i - \frac{\partial p^{(1)}}{\partial y_i} = 2\rho\epsilon_{ijk}\omega_j v_k^{(0)} \quad \text{in } \Omega_p \tag{15}$$

$$\frac{\partial v_i^{(0)}}{\partial y_i} = 0 \quad \text{in } \Omega_p \tag{16}$$

$$v_i^{(0)} = 0 \quad \text{on } \Gamma \tag{17}$$

where $\overrightarrow{v}^{(0)}$ and $p^{(1)}$ are Ω-periodic. $\overrightarrow{G}$ is the macroscopic driving force which is independent of $\overrightarrow{y}$ and which is defined by $\overrightarrow{G} = \overrightarrow{\nabla}_x p^{(0)} + \rho\overrightarrow{\gamma}(O)$. The set (15)–(17) is equivalent to:

$$\forall\,\overrightarrow{u}\in\mathcal{W}, \quad \int_{\Omega_p}\mu\frac{\partial u_i}{\partial y_j}\frac{\partial v_i^{(0)}}{\partial y_j}\,\mathrm{d}y + \int_{\Omega_p}2\rho\epsilon_{ijk}\omega_j v_k^{(0)}u_i\,\mathrm{d}y = -\int_{\Omega_p}u_iG_i\,\mathrm{d}y \tag{18}$$

where $\mathcal{W}$ is the Hilbert space of Ω-periodic, divergence free vectors, where the vectors vanish on Γ, with the scalar product:

$$(\overrightarrow{u},\overrightarrow{v})_{\mathcal{W}} = \int_{\Omega_p}\frac{\partial u_i}{\partial y_j}\frac{\partial v_i}{\partial y_j}\,\mathrm{d}y \tag{19}$$

The above formulation is strongly elliptic, and there exists a unique $\overrightarrow{v}^{(0)}$ which is a linear vector function of $\overrightarrow{G}$:

$$v_i^{(0)} = -k_{ij}G_j \tag{20}$$

where the tensor field $\mathbf{k}$ depends on $\overrightarrow{\omega}$ and $\overrightarrow{y}$.

Finally, the volume balance (10) gives at the second order:

$$\frac{\partial v_i^{(1)}}{\partial y_i} + \frac{\partial v_i^{(0)}}{\partial x_i} = 0 \quad \text{in } \Omega_p \tag{21}$$

By integration on Ω_p, we obtain

$$\frac{\partial \langle v_i^{(0)} \rangle}{\partial x_i} = 0, \quad \langle v_i^{(0)} \rangle = -K_{ij} G_j, \quad K_{ij} = \frac{1}{\Omega} \int_{\Omega_p} k_{ij} \, \mathrm{d}y \tag{22}$$

The filtration tensor $\boldsymbol{K}$ depends on the rotational velocity $\overrightarrow{\omega}$. It is possible to show from the equivalent variational fomulation (18) that $\boldsymbol{K}$ is a positive tensor. However, we obtain from (18):

$$\Omega (K_{pq} - K_{qp}) = 4\rho \epsilon_{ijk} \omega_j \int_{\Omega_p} k_{kp} k_{iq} \, \mathrm{d}y \tag{23}$$

which generally does not cancel out when $\overrightarrow{w} \neq 0$. Therefore, unlike the classical permeability tensor $\boldsymbol{K}(0)$, $\boldsymbol{K}(\overrightarrow{\omega})$ is not a symmetric tensor. Clearly, the divergence operator in the first equation in (22) kills the antisymmetric part of $\boldsymbol{K}$. However, this antisymmetric part is of importance for porous media that are heterogeneous at the macroscopic scale or for macroscopically isotropic or heterogeneous porous media submitted to flux boundary conditions.

4. Large Ekman number

We now consider the case of large Ekman numbers, $\eta = Ek^{-1} \ll 1$, which is satisfied in many practical applications, [3]. We look for $\overrightarrow{v}^{(0)}$ and $p^{(1)}$ in the form of the expansions:

$$\overrightarrow{v}^{(0)} = \overrightarrow{v}^0 + \eta \; \overrightarrow{v}^1 + \eta^2 \; \overrightarrow{v}^2 + \cdots \tag{24}$$

$$p^{(1)} = p^0 + \eta \, p^1 + \eta^2 \, p^2 + \cdots \tag{25}$$

The substitution of expansions (24) and (25) into expansions (12)–(13) shows these expansions are valid up to the term in η^n provided that:

$$\varepsilon^{1/n} \ll \eta \ll 1 \tag{26}$$

Introducing the above expansion in the formulation (18) and identifying like powers of η yields, to the zeroth order in η:

$$\forall \; \overrightarrow{u} \in \mathcal{W}, \;\; \int_{\Omega_p} \mu \frac{\partial u_i}{\partial y_j} \frac{\partial v_i^0}{\partial y_j} \, \mathrm{d}y = -\int_{\Omega_p} u_i G_i \, \mathrm{d}y \tag{27}$$

As expected, this is the linear Darcy formulation, [6]:

$$v_i^0 = -k_{ij}^0 G_j \tag{28}$$

which by averaging yields Darcy's law:

$$\langle v_i^0 \rangle = -K_{ij}^0 G_j \tag{29}$$

$$\boldsymbol{K}^0 = \langle \boldsymbol{k}^0 \rangle = \frac{1}{\Omega} \int_{\Omega_p} k^0 \, \mathrm{d}y = \boldsymbol{K}(0) \tag{30}$$

The second rank permeability tensor $\boldsymbol{K}^0$ is symmetric and positive [6].

The first order problem in η is the rotation correction problem:

$$\forall \; \overrightarrow{u} \in \mathcal{W}, \quad \int_{\Omega_p} \mu \frac{\partial u_i}{\partial y_j} \frac{\partial v_i^1}{\partial y_j} \, \mathrm{d}y - \int_{\Omega_p} 2\rho \epsilon_{ijk} \omega_j k_{kl}^0 G_l u_i \, \mathrm{d}y = 0 \tag{31}$$

which is also a linear problem with respect to $\overrightarrow{G}$. Formulation (31) has a unique solution:

$$v_i^1 = -k_{ij}^1 G_j \tag{32}$$

where $\boldsymbol{k}^1$ itself is a linear function of $\overrightarrow{\omega}$. By averaging we obtain:

$$\langle v_i^1 \rangle = -K_{ij}^1 G_j, \qquad K_{ij}^1 = \langle k_{ij}^1 \rangle \tag{33}$$

Tensor $\boldsymbol{K}^1$ depends linearly on $\overrightarrow{\omega}$, like $\boldsymbol{k}^1$. Finally, up to the two first terms the expansion (24) gives:

$$\langle v_i^{(0)} \rangle = \langle v_i^0 + Ek^{-1}\, v_i^1 \rangle = -\big(K_{ij}^0 + Ek^{-1} K_{ij}^1\big) G_i \tag{34}$$

which is valid for $\varepsilon \ll Ek^{-1} \ll 1$. By using formulations (27) and (31) we obtain:

$$K_{pq}^1 = -\frac{1}{\Omega} \int_{\Omega_p} 2\rho \epsilon_{ijk} \omega_j k_{kq}^0 k_{ip}^0 \, \mathrm{d}y \tag{35}$$

It is easy to check that tensor $\boldsymbol{K}^1$ is antisymmetric. Therefore, $\boldsymbol{K}^1$ is associated to a vector. Because $\boldsymbol{K}^1$ is linear with respect to $\overrightarrow{\omega}$, this vector is in the form $\boldsymbol{H} \cdot \overrightarrow{\omega}$ where $\boldsymbol{H}$ is a second order tensor and we have:

$$\boldsymbol{K}^1 \cdot \overrightarrow{G} = \boldsymbol{H} \cdot \overrightarrow{\omega} \times \overrightarrow{G} \tag{36}$$

with

$$H_{lj} = -\rho \epsilon_{plq} \epsilon_{ijk} \frac{1}{\Omega} \int_{\Omega_p} k_{kq}^0 k_{ip}^0 \, \mathrm{d}y \tag{37}$$

Finally the dimentional form of the filtration law for large Ekman numbers (34) is to the second order of approximation in the form:

$$\langle \overrightarrow{v} \rangle = -(\boldsymbol{K}^0 \cdot + \boldsymbol{H} \cdot \overrightarrow{\omega} \times) \overrightarrow{G} \tag{38}$$

Remark that both $\boldsymbol{K}^0$ and $\boldsymbol{H}$ are obtained by averaging and that we need only the local Darcy's velocity field $\boldsymbol{k}^0$ for their determination. Remark also that the filtration law (38) resembles the phenomenogical law (1), with an added coefficient $\boldsymbol{H}$.

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