Clearance vs. tolerance for rigid overconstrained assemblies

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Abstract

In order to manage quality, companies need to predict performance variations of products due to the manufacturing components deviations. Usually, to enable the assembly of overconstrained mechanical structure, engineers introduce clearance inside joints. We call mechanical assembly, a set of undeformable components connected together by mechanical joints. This paper presents a solution: firstly, to compute the minimum value of clearance for any given components sizes, and, secondly, to simulate variation of the minimum clearance value when the components dimensions vary between two limits. To achieve this goal, a regularized closure function G is defined. It depends on dimensional parameters, u, representing components dimensions, on positional parameters, p, representing components positions and on clearance parameters, j, representing mechanical joints clearance. A constrained optimization problem is solved to determine the minimum clearance value. An imaginative solution based on numerical integration of an ordinary differential equation is proposed to show the clearance variation. The method is designed to be used during the preliminary phase of overcontrained assemblies design. An advantage is the small number of input data unlike the tolerance analysis dedicated software.

Keywords:
Clearance, Tolerance, Overconstrained, Assembly

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Figure 1: (a) isoconstrained structure and (b) overconstrained structure.

1. Introduction

Shape of every manufactured workpiece is not ideal i.e. similar to the nominal shape defined by the designer. Due to the manufacturing deviations, the expected properties of mechanical assemblies are not exactly realized. To insure the quality of their products, companies need to manage geometric variations during the development and industrialization phases. It should be noted that automotive industry is aware of this necessity because, on one hand, the market is very competitive and, on the other hand, the number of manufactured workpieces is huge. Other industrial domains, like aerospace or building architecture, feel less concerned for many reasons e.g. less economic pressure, small or medium volume of production, etc. However, because of international competition, all industries must control the production quality.

In this paper, we deal with overconstrained rigid assemblies made of undeformable components connected together by mechanical joints. An assembly is called overconstrained if the arrangement of components is impossible for generic dimensions. For instance, as illustrated in figure 1 (a), six rods connected together with four spherical joints form an isoconstrained structure (a tetrahedron). The second structure in figure 1 (b) is made of ten rods connected with four spherical joints. It represents the assembly of two tetrahedrons sharing three rods. So, twelve rods for the two tetrahedrons minus three shared rods give nine rods to obtain a rigid isoconstrained structure. Consequently, the tenths rod's length must have a particular value. This second structure is overconstrained. Usually, in an industrial context, to deal with overconstrained assemblies, there is a need to introduce clearance into the joints to allow mounting as well as interchangeability of workpieces. The actual clearance should be larger than the computed clearance in order to insure components interchangeability.

However, large clearance influences the global properties of the mechanical assembly. The aim of this paper is to present a geometrical model designed to predict, at early phase of development, the size of the minimum clearance needed to insure the assemblability of overconstrained mechanical assemblies in the context of mass production.

Components dimensions vary between two supposedly known limits. These limits are small with respect to the nominal dimensions. The shape of nominal assembly is assumed to be known. Computer-aided tolerancing commercial softwares like VSA edited by Tecnomatix, CETOL 6σ edited by Sigmetrix and 3DCS edited by Dimensional Control Systems offer functionalities to analyze effects of geometric variations on assembly characteristics. This simulation is called tolerance analysis and was strongly studied during the last decades. Mathematical concepts used for tolerance analysis are Small Displacement Torsor by [1] and [2], Direct Linearization Method by [3], Variational Geometric Constraint Network by [4] and some variations of them. A good and complete description is given by [5]. To the best of our knowledge, these concepts are based on the first order linearization of closure equations representing the mechanical assembly. Approximations are made around nominal dimensions of the parts and around the nominal positions of the joints.

However, these tools are dedicated to detailed design of products because they need a significant number of input data, which represents a disadvantage for a fast study. Geometric dimensioning and tolerancing (GD&T) scheme of each workpiece form important part of them. Otherwise, these approaches seem to be effective for assemblies with a small number of parts and with a small number of closed loops. For these reasons, existing solutions are inappropriate in the context of preliminary design, on one hand because they need a big amount of data to run the simulation. On the other hand, because the first order linearization of the closure equations is an incorrect approximation for a large stack of components, and also is a formalization that may hide singularity configuration of the mechanical assembly. The presented solution gives a response to the inadequacy of established solutions for design of structure.

It should be understood that the aim is the existence of an overconsrained assembly, and that simulating its behavior under working condition is a further topic. It is well known that the existence of an overconstrained assembly featuring nominal dimensions and perfect joints is well understood. But dealing with dimensional uncertainty still needs scientific investigation: quantify how dimensional uncertainty will influence the quality of joints. The manufacturing challenge is to get the best possible quality (small clearance) at a reasonable manufacturing cost (not too small tolerance). The "working condition" investigation is the next step, when the behavior of the assembly is simulated by taking into account boundary conditions. This may suggest geometrical or topological changes and then, the whole design cycle is possibly revisited: tolerance/clearence and working load.

Figure 2: Theoretical investigation starts with the closure equation modeling the overconstrained problem. Then, the regularized closure equation transforms the overconstrained problem into an underconstrained problem. Solutions of the underconstrained problem are alternately obtained by minimizing the squared clearance, by finding a zero of the lagrangian gradient or through an ordinary differential.

Detecting dependent/overconstrained subsystems [6], [7] is a valuable concern. The point is that the proposed method overcomes this problem by smoothly spreading dimensional uncertainty over all joints of the assembly. It is significant to understand that if the assembly is isoconstrained (as the one in figure 1 (a)), the method naturally computes a vanishing clearance, meaning that the assembly is always perfectly adjusted despite dimensional uncertainty.

Section 2 compares the present approach with the combinatorial rigidity theory. Section 3 defines the closure equation. Section 4 defines what is an overconstrained solution of the closure equation. Section 5 explains how to introduce clearance parameters in the closure equation. Section 6 states the fundamental clearance minimization problem. Section 7 provides the theoretical solution of the minimization problem. Section 8 extends the theoretical solution by proving second order minimality condition. Section 9 sets up the ordinary differential equation modeling the clearance/tolerance dependency. Section 10 implements the solution on assembly test cases. Finally, algorithmic aspects are detailed in section 12. Useful theorems and lemmas are gathered in the appendix.

The reader who is not interested by mathematical details may advantageously skip the proof of theorem 1 as well as section 8 and the appendix.

2. Combinatorial rigidity

The purpose of this section is to compare the present approach with the combinatorial rigidity theory [8, 9, 10]. This analysis is done in [11]. It features two concerns and is inserted here for completeness.

Combinatorial rigidity investigates structures made of rigid bars connected at their end points by spherical joints. The combinatorial aspect is the logical graph underlying the structure: bars are modeled by edges and joints are modeled by vertices. The theory provides criteria for generic rigidity of such structures.

The first concern is that combinatorial rigidity handles spherical joints only, while many other kinds of joint are needed in mechanical design: cylindrical, prismatic, revolute, coplanar, etc. The principle developed in the present paper is not restricted to spherical joints.

The second concern is that combinatorial rigidity does not explicitly deal with dimensional parameters. Variables of the edge function and its derivative, the rigidity matrix (the key feature of the theory) are vertices coordinates of the graph structure. The theory is not suited to investigate dimensional parameters influence. Indeed, combinatorial rigidity defines genericity from the vertices coordinates point of view, which makes a big difference with the present approach. Consider the 3D assemblies (a) and (b) in figure 1 and the 2D assembly in figure 3. They are all generically rigid from the combinatorial theory point of view: any small perturbation of vertices (joints) coordinates yields an assembly of the same nature. From the mechanical designer point of view, the situation is totally different. Any small perturbation of bars lengths of assembly (a) in figure 1 yields an assembly of the same nature. It is generic in this sense. Conversely, there exists arbitrary small perturbations of bars lengths of 3D assembly (b) in figure 1 that make it impossible. Same thing with the 2D assembly in figure 3. They are not generic in this sense. Bars lengths are not independent, they must fit a special relationship, as investigated in [11]. From the manufacturing point of view, the relevant variables are the mechanical parts dimensions as opposed to vertices coordinates, which is a fundamental difference.

3. Closure equation

According to [11], two kinds of numerical parameters are involved to define a mechanical assembly. Dimensional parameters are lengths and angles and they specify respective dimensions of rigid bodies. Positional parameters are distances and angles and they specify relative positions of rigid bodies. Noting U the space of dimensional parameters, P the space of positional parameters and E the target space, the closure function is $F : U \times P \to E$. Function F captures the nature of the mechanical system. It involves dimensional and positional parameters according to rigid bodies and joints. Function F is generally analytic or even polynomial, so it is infinitely differentiable. Furthermore, the derivative of F with respect to all its variables is supposed to be full rank. The equation of the mechanical system is the so called closure equation

$$F(u,p) = 0 \tag{1}$$

which is a shortcut to deal with $\dim E$ scalar equations.

Definition 1. *A solution is a couple* $(u_0, p_0) \in U \times P$ *such that* $F(u_0, p_0) = 0$*.*

Definition 2. *A solution* (u_0, p_0) *is rigid if* $p = p_0$ *is an isolated solution of equation* $F(u_0, p) = 0$*. In other words, there exists* $\varepsilon > 0$ *such that if* $|p - p_0| < \varepsilon$ *and* $F(u_0, p) = 0$ *then* $p = p_0$*.*

4. Overconstrained solution

Following again [11], understanding the overconstrained solution is made easier by starting with the generic one. Intuitively, an arbitrary small perturbation of the dimensional parameter does not change the nature of a generic solution. This makes generic systems particularly attractive for manufacturing because the mounting is not sensitive to dimensional uncertainties. The formal definition is as follows.

Definition 3. *The rigid solution* (u_0, p_0) *is generic if there exists* $\varepsilon > 0$ *such that for any* u *such that* $|u - u_0| < \varepsilon$ *there exists* p *such that* (u, p) *is also a rigid solution.*

Conversely...

Definition 4. *...a rigid solution is overconstrained when exists an arbitrary small perturbation* v *of* u_0 *such that the equation* $F(v, p) = 0$ *has no rigid solution.*

Having no rigid solution means having no solution at all or featuring other behavior changes that are investigated in [11]. In short, perturbing the dimensional parameter of an overconstrained solution may unexpectedly lead to unwanted behavior change.

From the manufacturing point of view, overconstrained situations are unrealistic because dimensional parameters never fit the value u_0, meaning that the mechanical system never fits the expected behavior. So, there is a need to change the overconstrained situation into a robust one where it is possible to deal with dimensional uncertainties. Flowchart in figure 2 describes the next steps of theoretical investigation.

Figure 3: Case study number 1: two-dimensional over-constrained structure made of six bars and four joints.

5. Regularized closure equation

Regularizing is to appropriately involve new parameters in the closure function. They are named clearance parameters, noted $j = (j_1, \cdots, j_r) \in J$ where $J = \mathbb{R}^r$, $r = \dim J$, and they behave like positioning parameters. The goal is to deal with generic solutions, while preserving the consistency with the initial (overconstrained) closure function. So, the closure function $F : U \times P \to E$ is transformed into $G : U \times J \times P \to E$ where mapping G is designed in such a way that

$$G(u, 0, p) = F(u, p) \tag{2}$$

for all $(u, p) \in U \times P$. The regularized closure equation is

$$G(u, j, p) = 0. \tag{3}$$

Definition 5. *The triple $(u, j, p) \in U \times J \times P$ is a nominal solution if $u = u_0$, $j = 0$ and $p = p_0$ where (u_0, p_0) is a solution according to definition 1 .*

Thanks to (2), a nominal solution is a solution of (3). From the mechanical point of view, parameters $j \in J$ are degrees of freedom added to the mechanical system by replacing joints with more flexible ones.

Definition 6. *A solution (u_0, j_0, p_0) is mobile if there exists $\varepsilon > 0$ and two smooth functions $j :]-\varepsilon, \varepsilon[\to J$ and $p :]-\varepsilon, \varepsilon[\to P$ of a scalar variable t such that $j(0) = j_0$, $p(0) = p_0$, $|j'(0)|^2 + |p'(0)|^2 \neq 0$ and $G(u_0, j(t), p(t)) = 0$ for all $t \in]-\varepsilon, \varepsilon[$.*

Notice that definition 6 does not say that functions $t \mapsto j(t)$ and $t \mapsto p(t)$ are unique. It includes situations with more than one degree of freedom.

Definition 7. *The mobile solution (u_0, j_0, p_0) is generic if there exists $\varepsilon > 0$ such that for any u such that $|u - u_0| < \varepsilon$ there exists j, p such that (u, j, p) is also a mobile solution.*

Counting equations and parameters, the number $\dim J$ of clearance parameters is adjusted so that $\dim J + \dim P > \dim E$. The issue is that solutions of (3) are highly mobile because of the large number of degrees of freedom. In fact, fixing a dimensional parameter $\bar{u} \in U$, the generic solution of $G(\bar{u}, j, p) = 0$ is a $(\dim J + \dim P - \dim E)$-dimensional submanifold of space $J \times P$. Section 6 deals with defining the appropriate unique solution according to mechanical considerations.

Furthermore, mapping G is designed in such a way that the nominal solution $(u, j, p) = (u_0, 0, p_0)$ is a generic mobile solution of (3) according to definitions 6 and 7. In particular, if $u = u_0$ then $j(0) = 0$ and $p(0) = p_0$. This is the value of the regularization process: it removes the sensitivity of the solution to dimensional parameters uncertainty. In other words, it changes the non generic situation into a generic situation.

The two technical conditions insuring the regularization effectiveness are now explained. They involve the partial derivatives of $G(u, j, p)$ with respect to j and p at $(u, j, p) = (u_0, 0, p_0)$, which, for clarity, are respectively noted

$$A = G_j(u_0, 0, p_0) \tag{4}$$

and

$$B = G_p(u_0, 0, p_0). \tag{5}$$

Firstly, the partial derivative of G with respect to j and p is full rank at the nominal solution $(u, j, p) = (u_0, 0, p_0)$. The linear mapping $G_{(j,p)}(u_0, 0, p_0)$ is bloc defined by two matrices

$$G_{(j,p)}(u_0, 0, p_0) = \begin{pmatrix} A & B \end{pmatrix}.$$

If $G_{(j,p)}(u_0, 0, p_0)$ is full rank, $G_{(j,p)}(u_0, 0, p_0) G_{(j,p)}(u_0, 0, p_0)^T$ is invertible. Since

$$G_{(j,p)}(u_0, 0, p_0) G_{(j,p)}(u_0, 0, p_0)^T = \begin{pmatrix} A & B \end{pmatrix} \begin{pmatrix} A^T \\ B^T \end{pmatrix} = AA^T + BB^T$$

the full rank condition is equivalent to

$$AA^T + BB^T \text{ is invertible.} \tag{6}$$

The second and very important condition involves the partial derivative with respect to p.

$$B^T (AA^T + BB^T)^{-1} B \text{ is invertible} \tag{7}$$

Conditions (6) and (7) are verified in practice and allow the inversion of a key linear mapping, which in turn makes the whole solution feasible, as stated in section 7. Notice that, thanks to (2), $B = G_p(u_0, 0, p_0) = F_p(u_0, p_0)$.

Figure 4: Harmonic perturbations of rod's lengths for case study number 1, the six bars and four joints two-dimensional structure.

6. Optimization problem

The goal is now to set up the mathematical problem that defines a unique solution which is meaningful from the mechanical point of view. Since the point is to deal with a non nominal dimensional parameter, let $u \in U$ be a fixed and arbitrary dimensional parameter, meaning that $u \neq u_0$ is allowed. Among the infinity of j, p solutions of the regularized closure equation (3), the preferred one will minimize the clearance, that is the magnitude of parameter j. Precisely, the preferred solution minimizes the squared norm of parameter j, that is

$$|j|^2 = \sum_{i=1}^{r} j_i^2. \tag{8}$$

This is consistent with a nominal solution because, in this case, $|j|^2 = 0$ which is surely a minimum. Technically, the optimization problem is as follows. Given a (possibly non nominal) dimensional parameter $u \in U$, find the clearance parameters j and the positional parameters p such that the regularized closure equation (3) is satisfied and such that the magnitude of clearance parameter j is as small as possible. This is classically written

$$\min_{G(u,j,p)=0} \frac{1}{2}|j|^2$$

but the "argument of the minimum" formulation is more appropriate to highlight unknowns j and p vs. input parameter u.

$$(j, p) = \text{Argmin}\left\{\frac{1}{2}|j|^2; G(u, j, p) = 0\right\} \tag{9}$$

In fact, (9) is a family of minimization problems parameterized by $u \in U$. This implicitly defines a dependency between u and the optimum (j, p). The hope is to get a positioning p that is close to the nominal positioning p_0 when the dimensional parameter u is close to nominal value u_0. This issue is elucidated in section 7.

Clearly, the optimization criterion is based on pure geometry. One could argue that functional or performance considerations are missing. It will be seen in section 7 that a small dimensional perturbation yields a perturbed assembly that remains close to the nominal assembly. So, performance of the perturbed assembly should be close to nominal performance. Nevertheless, a functional or performance criterion $\Phi(u, j, p)$ can be added to the geometrical criterion in order to involve non geometrical aspects, but this is out of the scope of the paper.

7. First order solution of the optimization problem

This section is dedicated to theorem 1. It states that, from the mechanical point of view, despite (u_0, p_0) is an overconstrained solution, a small perturbation u of nominal dimension u_0 yields a unique small perturbation p of nominal positional parameter p_0, the dependency between p and u being smooth. Furthermore, the value of clearance parameter j is unique, meaning that there is no way to mount the assembly with a tighter clearance. One could argue that it is not sure that p is as close as possible to p_0. The answer is very simple: looking for a minimal j yields a unique p, which is necessarily the best possible. In other words, choosing another p yields a non minimum j. Furthermore, conclusion 3 of theorem 1 states that the positional parameter p remains under control despite the perturbation of dimensional parameter u. In addition, it is possible to estimate the first order dependency between j and u in the neighborhood of $j = 0$ and $u = u_0$ as well as the first order dependency between p and u in the neighborhood of p_0 and $u = u_0$

Theorem 1. *Consider an overconstrained solution of closure equation* (1) *and its regularized closure equation* (3) *satisfying* (6) *and* (7)*. Then, there exist a neighborhood Y of u_0, two unique and smooth functions $j = j(u)$ and $p = p(u)$ defined over Y and a constant c such that*

1. $j(u_0) = 0$ *and* $p(u_0) = p_0$
2. *for all* $u \in Y$, $(j(u), p(u))$ *satisfies the stationarity condition of* (9)
3. $|p(u) - p_0| \leq c|u - u_0|$ *for all* $u \in Y$.

The proof is structured as follows. Clearly, (9) is a quadratic minimization problem under non linear equality constraint. Classically, Lagrange multiplier λ is introduced to deal with the constraint. Then, Lagrange function stationarity yields a non linear system featuring the same number of unknowns and equations. Finally, the implicit function theorem 3 provides a smooth dependency between the optimal solution j, p and parameter u.

Proof. The Lagrange function associated with (9) is $L : U \times J \times P \times E \to \mathbb{R}$ defined by

$$L(u, j, p, \lambda) = \frac{1}{2}|j|^2 + \langle \lambda, G(u, j, p) \rangle \tag{10}$$

where the Lagrange multiplier $\lambda \in E$ is an additional unknown. A necessary condition for the existence of a solution to (9) is the stationarity of function (10). This means that its partial derivatives with respect to j, p and λ must vanish, that is

$$\begin{aligned} L_j(u, j, p, \lambda) &= 0 \\ L_p(u, j, p, \lambda) &= 0 \\ L_\lambda(u, j, p, \lambda) &= 0 \end{aligned}$$

where

$$\begin{aligned} L_j(u, j, p, \lambda) &= j + G_j(u, j, p)^T \lambda \\ L_p(u, j, p, \lambda) &= G_p(u, j, p)^T \lambda \\ L_\lambda(u, j, p, \lambda) &= G(u, j, p). \end{aligned} \tag{11}$$

Now, consider the mapping $H : U \times J \times P \times E \to J \times P \times E$ defined by the triple of partial derivatives

$$H(u, j, p, \lambda) = \begin{pmatrix} j + G_j(u, j, p)^T \lambda \\ G_p(u, j, p)^T \lambda \\ G(u, j, p) \end{pmatrix}. \tag{12}$$

By design, the stationary point (j, p, λ) of Lagrange function (10) is a solution of equation

$$H(u, j, p, \lambda) = 0. \tag{13}$$

At this step, it is important to remember that the unknowns of the stationarity problem (13) are j, p, λ and that u is a parameter. Equation (13) defines a dependency between u and j, p, λ. The goal is now to investigate this dependency by applying the implicit function theorem 3 to mapping H at $(u, j, p, \lambda) = (u_0, 0, p_0, 0)$. The first step is to check that $(u_0, 0, p_0, 0)$ is a solution of (13). Indeed, thanks to (2), (12) and to definition 1

$$H(u_0, 0, p_0, 0) = \begin{pmatrix} 0 \\ 0 \\ G(u_0, 0, p_0) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ F(u_0, p_0) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

The second step is to investigate the inversion of the partial derivative $H_{(j,p,\lambda)}$ at $(u_0, 0, p_0, 0)$. By derivating (12)

$$H_{(j,p,\lambda)} = \begin{pmatrix} I + G_{jj}^T\lambda & G_{jp}^T\lambda & G_j^T \\ G_{jp}^T\lambda & G_{pp}^T\lambda & G_p^T \\ G_j & G_p & 0 \end{pmatrix} \tag{14}$$

so, at $(u, j, p, \lambda) = (u_0, 0, p_0, 0)$

$$H_{(j,p,\lambda)}(u_0, 0, p_0, 0) = \begin{pmatrix} I & 0 & A^T \\ 0 & 0 & B^T \\ A & B & 0 \end{pmatrix} \tag{15}$$

with notations (4) and (5). Thanks to lemma 1 and conditions (6) and (7), $H_{(j,p,\lambda)}(u_0, 0, p_0, 0)$ is invertible. Then, according to the implicit function theorem 3, there exist a neighborhood Y of u_0 and three unique and smooth functions $j = j(u)$, $p = p(u)$, $\lambda = \lambda(u)$ defined over Y such that

$$j(u_0) = 0$$
$$p(u_0) = p_0$$
$$\lambda(u_0) = 0$$

which is conclusion 1 of the theorem, and

$$H(u, j(u), p(u), \lambda(u)) = 0$$

for all $u \in Y$, which is conclusion 2 of the theorem. Since $u \mapsto p(u)$ is differentiable, there exists a constant c such that $|p(u) - p_0| \leq c|u - u_0|$ for all $u \in V$, which is conclusion 3 of the theorem. $\square$

The first order dependency between positional parameter p and dimensional parameter u can be estimated by computing $p'(u_0)$. Similarly, the first order dependency between clearance parameter j and dimensional parameter u can be estimated by computing $j'(u_0)$. Indeed, following the proof of theorem 1, the implicit function theorem provides the differential equation

$$\begin{pmatrix} j'(u) \\ p'(u) \\ \lambda'(u) \end{pmatrix} = -H_{(j,p,\lambda)}(u, j, p, \lambda)^{-1} H_u(u, j, p, \lambda).$$

Writing this expression at $u = u_0$ and using (12) and (15) yield

$$\begin{pmatrix} j'(u_0) \\ p'(u_0) \\ \lambda'(u_0) \end{pmatrix} = -\begin{pmatrix} I & 0 & A^T \\ 0 & 0 & B^T \\ A & B & 0 \end{pmatrix}^{-1} \begin{pmatrix} 0 \\ 0 \\ G_u(u_0, 0, p_0) \end{pmatrix}$$

which can be solved by hand in such a way that, noting $D = \left(AA^T + BB^T\right)^{-1}$,

$$p'(u_0) = -\left(\left(B^T DB\right)^{-1} B^T D\right) G_u(u_0, 0, p_0)$$

and

$$j'(u_0) = A^T \left(DB \left(B^T DB\right)^{-1} B^T D - D\right) G_u(u_0, 0, p_0).$$

8. Second order minimality condition

This section investigates the second order minimality condition of the optimization problem (9). It is not useful to numerical solving, but it consolidates the theory by proving that the unique solution stated in theorem 1 is in fact the best possible. Surprisingly, the second order condition requires two different proofs, depending on whether there are many clearance parameters ($\dim J > \dim E$) or few clearance parameters ($\dim J \leq \dim E$).

8.1. Additional condition

In addition to the full rank property (6), and if $\dim J < \dim E$, we suppose that it is possible to split the positional parameter p into $p = (q, w) \in Q \times W$ in such a way that the partial derivative of G with respect to (j, q) is invertible at the nominal solution. This means that $\dim J + \dim Q = \dim E$. Noting $p_0 = (q_0, w_0)$, this means that $G_{(j,q)}(u_0, 0, q_0, w_0)$ is invertible. Then, the implicit function theorem yields two unique mappings $(u, w) \mapsto j(u, w)$ and $(u, w) \mapsto q(u, w)$ defined in a neighborhood of (u_0, w_0) such that $j(u_0, w_0) = 0$, $q(u_0, w_0) = q_0$ and

$$G(u, j(u, w), (q(u, w), w)) = 0$$

for all (u, w) in the said neighborhood. The additional condition involves the partial derivative of $j(\cdot)$ with respect to w at the nominal solution:

$$j_w(u_0, w_0)^T j_w(u_0, w_0) \text{ is invertible.} \tag{16}$$

If $\dim J = \dim E$ no splitting is needed and variable p plays the role of variable w. Condition (6) says that linear mapping $G_j(u_0, 0, p_0)$ is invertible and the implicit function theorem yields a unique mappings $(u, p) \mapsto j(u, p)$ defined in a neighborhood of (u_0, p_0) such that $j(u_0, p_0) = 0$ and

$$G(u, j(u, p), p)) = 0$$

for all (u, p) in the said neighborhood. The additional condition (16) is written

$$j_p(u_0, p_0)^T j_p(u_0, p_0) \text{ is invertible.}$$

8.2. Second order theorem

Theorem 2. *Under the conditions of theorem 1 and if, in addition, condition* (16) *is satisfied, then, for all $u \in Y$, $(j(u), p(u))$ satisfies the second order minimality condition of* (9)*.*

8.3. Proof of theorem 2 when $\dim J > \dim E$

Proof. Given $u \in Y$ and since $\dim J > \dim E$, according to lemma 2 the second order minimality condition of (9) is

$$|v|^2 + \langle \lambda(u), G_{jj}(u, j(u), p(u), \lambda(u))vv \rangle > 0 \tag{17}$$

for all $v \in J$ such that $v \neq 0$ and $G_j(u, j(u), p(u), \lambda(u))v = 0$. Note

$$\varphi(u, v) = \langle \lambda(u), G_{jj}(u, j(u), p(u), \lambda(u))vv \rangle$$

so that the left hand side of (17) can be written $|v|^2 + \varphi(u, v)$, that is a perturbation of $|v|^2$ by a quadratic form such that $\varphi(u_0, v) = 0$ for all v. Now,

$$|\varphi(u, v)| \leq \psi(u)|v|^2$$

where $\psi(u) = |\lambda(u)||G_{jj}(u, j(u), p(u), \lambda(u))|$, so that

$$|v|^2 + \varphi(u, v) \geq |v|^2 - |\varphi(u, v)| \geq (1 - \psi(u))|v|^2.$$

Since $\psi(u_0) = 0$, an appropriate restriction of neighborhood Y makes $1-\psi(u)$ larger that $\frac{1}{2}$. Finally, for all u in the restricted Y and all non zero $v \in J$

$$|v|^2 + \varphi(u, v) \geq \frac{1}{2}|v|^2 > 0.$$

The second order minimality condition is satisfied for all u in a neighborhood of u_0. $\square$

Figure 5: Case study number 1. Various shapes of the two-dimensional structure under dimensional perturbation.

8.4. *Proof of theorem 2 when* $\dim J \leq \dim E$

Proof. Using notation of condition (16), the minimization problem (9) is equivalent to

$$w = \operatorname{Argmin}\left\{\frac{1}{2}|j(u,w)|^2\right\}.$$

The proof is to apply lemma 3 using $j(u_0, w_0) = 0$ and the inversion condition (16). $\square$

9. Clearance/tolerance ordinary differential equation

This section makes use of theorem 1 to compute numerical variations of clearance j according to the tolerance, which is the numerical variations of dimension u. The guide line stated by [12] is to deal with an ordinary differential equation, as it is particularly adapted to numerical integration.

9.1. *The ordinary differential equation*

So far, parameter u is taken from neighborhood Y of u_0, Y being an open subset of U. The idea is to consider an arbitrary curve in Y through u_0. Let $u : \mathbb{R} \to U$ be a smooth mapping of a real variable t such that $u(t) \in Y$ for all t and $u(0) = u_0$. Notice that there may exist non zero values of t such that $u(t) = u_0$.

The minimization problem (9) is now written using $u(t)$ instead of u. Given a real number t, find the positional parameters j and p such that the regularized closure equation (3) is satisfied and such that the magnitude of parameter j is as small as possible.

$$(j,p) = \operatorname{Argmin}\left\{\frac{1}{2}|j|^2; G(u(t),j,p) = 0\right\} \tag{18}$$

Now, (18) is a family of minimization problems parameterized by $t \in \mathbb{R}$. The solving technique follows the track of section 7. The Lagrange function associated with (18) is $L : \mathbb{R} \times J \times P \times E \to \mathbb{R}$ defined by

$$L(t,j,p,\lambda) = \frac{1}{2}|j|^2 + \langle \lambda, G(u(t),j,p)\rangle. \tag{19}$$

Mapping $H : \mathbb{R} \times J \times P \times E \to J \times P \times E$ is defined by the triple of partial derivatives

$$H(t, j, p, \lambda) = \begin{pmatrix} j + G_j(u(t), j, p)^T \lambda \\ G_p(u(t), j, p)^T \lambda \\ G(u(t), j, p) \end{pmatrix}. \tag{20}$$

The stationarity equation (13) is transformed into

$$H(t, j, p, \lambda) = 0. \tag{21}$$

The partial derivative of H with respect to (j, p, λ) is invertible at $(t, j, p, \lambda) = (0, 0, p_0, 0)$ just like in section 7 and for the same reason. So, the implicit function theorem 3 applied to mapping H at $(t, j, p, \lambda) = (0, 0, p_0, 0)$ yields a neighborhood $]-\varepsilon, \varepsilon[$ of 0 and three unique and smooth functions $j = j(t)$, $p = p(t)$ and $\lambda = \lambda(t)$ defined over $]-\varepsilon, \varepsilon[$ such that

$$\begin{aligned} j(0) &= 0 \\ p(0) &= p_0 \\ \lambda(0) &= 0 \end{aligned} \tag{22}$$

and

$$H(t, j(t), p(t), \lambda(t)) = 0$$

for all $t \in]-\varepsilon, \varepsilon[$. Furthermore, functions $j(t)$, $p(t)$ and $\lambda(t)$ are solution of ordinary differential equation

$$\begin{pmatrix} j' \\ p' \\ \lambda' \end{pmatrix} = V(t, j, p, \lambda) \tag{23}$$

associated with the initial condition (22) where the vector field $V(\cdot)$ is

$$V(t, j, p, \lambda) = -H_{(j,p,\lambda)}(t, j, p, \lambda)^{-1} H_t(t, j, p, \lambda). \tag{24}$$

Symbol $'$ in (23) is the derivation with respect to real variable t. The partial derivative $H_{(j,p,\lambda)}$ can be detailed in terms of function G by using (14). The partial derivative H_t can be detailed in terms of function G by using (12) and the chain rule.

$$H_t(t, j, p, \lambda) = \begin{pmatrix} (G_{uj}(u(t), j, p)u'(t))^T \lambda \\ (G_{up}(u(t), j, p)u'(t))^T \lambda \\ G_u(u(t), j, p)u'(t) \end{pmatrix} \tag{25}$$

Figure 6: Case study number 1. Zoom of a non-perfect spherical joint of the two-dimensional structure. Red dot is the connection point. Black dots are rods' extremities.

9.2. Variations of dimensional parameter u

For numerical investigation purpose, a simple and efficient choice introduced by [12] is to design the curve $t \mapsto u(t)$ as a harmonic function oscillating around the nominal value u_0. In addition, a homotopy effect is used to start from the nominal solution by using the mapping

$$\mu(t) = 1 - e^{-t}.$$

Noting $k = \dim U$ and $u = (u_1, \cdots, u_k)$ the scalar coordinates of u, each scalar function $t \mapsto u_i(t)$ is defined by

$$u_i(t) = u_{0i} + \mu(t)\frac{\Delta_i}{2}\sin(\omega_i t) \tag{26}$$

for $i = 1, \cdots, k$. In formula (26), ω_i is an angular frequency and Δ_i is the interval width, that is the manufacturing uncertainty of dimensional parameter u_i. Angular frequencies ω_i are spread in such a way that when t increases, the oscillating curve $t \mapsto (u_1(t), \cdots, u_k(t))$ "fills" the overall tolerance range $\Delta = \prod_{i=1}^{k} \Delta_i$ as much as possible. Of course, a genuine space-filling curve as described by [13] would be ideal, but (26) is very simple and numerical simulations show that this is efficient. Feeding the ordinary differential equation (23) with the oscillating dimensions (26) and solving over a time interval $[0, T]$ yields oscillating functions $t \mapsto j(t)$, $p \mapsto p(t)$ and $\lambda \mapsto \lambda(t)$ starting with the initial condition (22). The result of interest is the maximum value of clearance parameter $j(t)$ when $t \in [0, T]$.

$$j_{max} = \max\{|j(t)|, t \in [0, T]\} \tag{27}$$

This value does not depend on T provided it is large enough, which is not difficult to set in practice. The quality compromise of mechanical design is as follows. Larger dimensional tolerance Δ_i means smaller manufacturing cost, but also larger j_{max}. Larger j_{max} means bad quality of the manufactured assembly. So, the balance is to adjust the Δ_i to the smallest possible manufacturing cost in such a way that j_{max} fits the quality criteria. Clearly, this can be achieved by using previous simulation through an iterative process.

Figure 7: Clearance variation of the two-dimensional structure due to dimensional harmonic perturbations.

9.3. Family of assemblies

By using $t \mapsto u(t)$ dimensional mapping and $t \mapsto p(t)$ positional mapping, it is easy to set up a graphical animation of the assembly with respect to parameter t. This could mislead the reader because the animation looks like a dancing structure deformed by some physical phenomenon. In fact, it is not one structure under deformation. It is an infinity of structures indexed by t, each of them being defined by dimensions $u(t)$ and positions $p(t)$. It must be understood as a continuous abstraction of the series of thousands of products getting out of the factory.

10. Rod structure test cases

10.1. Framework

The assembly is an overconstrained three-dimensional structure made of n rigid rods linked by m spherical joints. Rod's lengths are noted l_i, $i = 1, \cdots, n$. In a first step, the regularized closure function is set up. In order to anticipate non-perfect connections of several rods at the same junction, m connections points

$$P_i = \begin{pmatrix} x_i \\ y_i \\ z_i \end{pmatrix}$$

$i = 1, \cdots, m$ are introduced. The start point O_i of rod number i is

$$O_i = \begin{pmatrix} s_i \\ t_i \\ r_i \end{pmatrix}. \tag{28}$$

The relative end point E_i of rod number i is

$$E_i = l_i \begin{pmatrix} \cos\beta_i \cos\alpha_i \\ \cos\beta_i \sin\alpha_i \\ \sin\beta_i \end{pmatrix} \tag{29}$$

meaning that the extremities of rod number i are respectively O_i and $O_i + E_i$. The network of rods is defined by a matrix N featuring $2n$ rows and

m columns and filled with 0 and 1. It captures the coincidences of rods extremities O_i and $O_i + E_i$ with connection points P_j, which is gathered in the following equation.

$$\begin{pmatrix} O_1 \\ E_1 + O_1 \\ \vdots \\ O_n \\ E_n + O_n \end{pmatrix} = N \begin{pmatrix} P_1 \\ \vdots \\ P_m \end{pmatrix}. \tag{30}$$

Clearly, the dimensional parameters space is $U = \mathbb{R}^n$ gathering rod's lengths through

$$u = (l_1, \cdots, l_n). \tag{31}$$

Unknowns are connection points and rod's extremities so that the positional parameter space is

$$P = \left(\mathbb{R}^3 \times [0, 2\pi[\times[-\pi, \pi[\right)^n \times \left(\mathbb{R}^3\right)^m$$

and the target space is $E = (\mathbb{R}^3)^{2n}$. Noting $u = (l_1, \cdots, l_n)$ and

$$p = \left(\begin{pmatrix} s_i \\ t_i \\ r_i \end{pmatrix}_{i=1,\cdots,n}, \begin{pmatrix} \alpha_i \\ \beta_i \end{pmatrix}_{i=1,\cdots,n}, \begin{pmatrix} x_i \\ y_i \\ z_i \end{pmatrix}_{i=1,\cdots,m} \right) \tag{32}$$

the closure function is

$$F : \mathbb{R}^n \times \left(\mathbb{R}^3 \times [0, 2\pi[\times[-\pi, \pi[\right)^n \times \left(\mathbb{R}^3\right)^m \to \left(\mathbb{R}^3\right)^{2n}$$

defined by

$$F(u, p) = \begin{pmatrix} O_1 \\ E_1 + O_1 \\ \vdots \\ O_n \\ E_n + O_n \end{pmatrix} - N \begin{pmatrix} P_1 \\ \vdots \\ P_m \end{pmatrix}.$$

For grounding purpose, six positional parameters are set, so that the closure equation $F(u, p) = 0$ features $5n + 3m - 6$ unknown scalar positional parameters and $6n$ scalar equations. A necessary condition for an over-constrained structure is more equations than unknowns, that is $n > 3m - 6$. Introducing the clearance parameters in this context is straightforward because the closure function involves the appropriate geometry. Indeed, choosing $J = E = (\mathbb{R}^3)^{2n}$ and $j \in J$, the regularized closure function is

$$G(u, j, p) = F(u, p) - j.$$

The regularized closure equation $G(u, j, p) = 0$ is clearly under-constrained since it features $11n + 3m - 6$ scalar unknowns and $6n$ scalar equations.

Because of the particular shape of G, the optimization problem (9) leads to a stationarity condition (21)

$$\begin{aligned} j - \lambda &= 0 \\ F_p(u(t), p)^T \lambda &= 0 \\ F(u(t), p) - j &= 0 \end{aligned}$$

which boils down to

$$\begin{aligned} F_p(u(t), p)^T F(u(t), p) = 0 \\ j = F(u(t), p) \\ \lambda = F(u(t), p). \end{aligned}$$

The corresponding differential equation (23) is

$$\begin{aligned} p' &= R(p, t)S(p, t) \\ p(0) &= p_0 \end{aligned}$$

where

$$R(p, t) = -\left(F_{pp}(u(t), p)^T F(u(t), p) + F_p(u(t), p)^T F_p(u(t), p)\right)^{-1}$$

and

$$S(p, t) = \left(F_{up}(u(t), p)u'(t)\right)^T F(u(t), p) + F_p(u(t), p)^T F_u(u(t), p)u'(t)$$

completed with $j(t) = F(u(t), p(t))$ and $\lambda(t) = j(t)$.

The two-dimensional version of the generic framework is easily obtained by setting $z_i = 0$, $r_i = 0$ and $\beta_i = 0$ and by grounding three unknowns. Parameter space and target space are updated accordingly so that

$$P = \left(\mathbb{R}^2 \times [0, 2\pi[\right)^n \times \left(\mathbb{R}^2\right)^m$$

and $E = (\mathbb{R}^2)^{2n}$. Furthermore,

$$p = \left(\begin{pmatrix} s_i \\ t_i \end{pmatrix}_{i=1,\cdots,n}, (\alpha_i)_{i=1,\cdots,n}, \begin{pmatrix} x_i \\ y_i \end{pmatrix}_{i=1,\cdots,m} \right).$$

Figure 8: Case study number 2 is a three-dimensional overconstrained structure.

Figure 9: Case study number 2. Harmonic perturbations of rod's lengths for the three-dimensional structure.

10.2. Case study number 1: two-dimensional structure

As illustrated in figure 3, case study number 1 is a two-dimensional structure made of $n = 6$ rods and $m = 4$ joints the arrangement of which is provided by matrix N.

$$N = \begin{pmatrix} 1&0&0&0\\ 0&1&0&0\\ 0&1&0&0\\ 0&0&1&0\\ 0&0&1&0\\ 0&0&0&1\\ 0&1&0&0\\ 0&0&0&1\\ 1&0&0&0\\ 0&0&1&0\\ 1&0&0&0\\ 0&0&0&1 \end{pmatrix}$$

From the numerical point of view, the nominal configuration is defined by connection points coordinates, which in turn define nominal lengths of all rods. They are respectively

$$P_1^0 = \begin{pmatrix} 0\\ 0 \end{pmatrix} P_2^0 = \begin{pmatrix} 0\\ 3 \end{pmatrix} P_3^0 = \begin{pmatrix} 4\\ 3 \end{pmatrix} P_4^0 = \begin{pmatrix} 4\\ 0 \end{pmatrix}.$$

Dimensional perturbations are defined using (26) by setting $\Delta_i = 1.0$ for $i = 1, \cdots, 6$ and the following angular frequencies.

ω_1	ω_2	ω_3	ω_4	ω_5	ω_6
13	7	19	23	17	11

Harmonic perturbations of rod's lengths are illustrated in figure 4. The resulting clearance variation is illustrated in figure 7. Typical perturbed structures are illustrated in figures 5 and 6. Each structure is distorted in order to minimize the widths of non-perfect connections. Red dots are connexion points, black dots are rod's extremities.

10.3. Case study number 2: three-dimensional structure

As illustrated in figure 8, case study number 2 is a three-dimensional structure made of $n = 10$ rods and $m = 5$ joints the arrangement of which is

Figure 10: Case study number 2. Various shapes corresponding to dimensional perturbations.

Figure 11: Case study number 2. Clearance variation of three-dimensional structures due to dimensional harmonic perturbations.

provided by matrix N.

$$N = \begin{pmatrix} 1&0&0&0&0 \\ 0&1&0&0&0 \\ 1&0&0&0&0 \\ 0&0&0&1&0 \\ 0&1&0&0&0 \\ 0&0&1&0&0 \\ 0&1&0&0&0 \\ 0&0&0&1&0 \\ 0&0&1&0&0 \\ 0&0&0&1&0 \\ 1&0&0&0&0 \\ 0&0&1&0&0 \\ 0&0&0&0&1 \\ 1&0&0&0&0 \\ 0&0&0&0&1 \\ 0&1&0&0&0 \\ 0&0&0&0&1 \\ 0&0&1&0&0 \\ 0&0&0&0&1 \\ 0&0&0&1&0 \end{pmatrix}$$

From the numerical point of view, the nominal configuration is defined by connection points coordinates, which in turn define nominal lengths of all rods. They are respectively

$$P_1^0 = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} P_2^0 = \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix} P_3^0 = \begin{pmatrix} 0 \\ 2 \\ 0 \end{pmatrix} P_4^0 = \begin{pmatrix} -2 \\ 1 \\ 1 \end{pmatrix} P_5^0 = \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix}.$$

Dimensional perturbations are defined using (26) by setting $\Delta_i = 0.4$ for $i = 1, \cdots, 10$ and the following angular frequencies.

ω_1	ω_2	ω_3	ω_4	ω_5	ω_6	ω_7	ω_8	ω_9	ω_{10}
2	3	5	7	11	13	17	19	23	29

Harmonic perturbations of rod's lengths are illustrated in figure 9. The resulting clearance variation is illustrated in figure 11. Typical structure deformation is illustrated in figure 10. Here again, each structure is distorted in order to minimize the widths of non-perfect connections. Red dots are connexion points, black dots are rod's extremities.

11. Case study number 3

As opposed to section 10, this case study involves cylindrical joints rather than spherical joints. The assembly is made of two parts (part 1 and part

Figure 12: Typical part and its dimensional parameters for case study number 3. Three dimensional and skeleton geometry is illustrated.

Figure 13: Assembly and dimensional parameters of case study number 3.

2) which are instances of the typical part illustrated in figure 12. Part 1 and part 2 are connected through two cylindrical joints, as illustrated in figure 13. Clearly, the assembly is overconstrained and can be mounted provided some assembly condition is satisfied, for example $\beta_1 = \beta_2 = 0$ and $\alpha_1 + \alpha_2 = \pi$. It turns out to be a mobile mechanism if, for example, $\beta_1 = \beta_2 = 0$ and $\alpha_1 = \alpha_2 = \frac{\pi}{2}$, but this configuration is out of the scope of the study.

11.1. Designing the perfect/non-perfect cylindrical joint

Consider a rigid line segment $[A, B]$ and a line defined by a point C and a normalized vector V. Capturing that the line segment slides on the line is to specify that points A and B are located on the said line. Formally, there exist two real numbers v and w such that $A = C + vV$ and $B = C + wV$. This suggests a closure equation like

$$\begin{aligned} C + vV - A = 0 \\ C + wV - B = 0 \end{aligned}$$

and a corresponding closure function

$$F = \begin{pmatrix} C + vV - A \\ C + wV - B \end{pmatrix}.$$

Here again, the regularized closure function is $G = F - j$. Indeed, if $j = 0$ the line segment is rigorously aligned with the line, thus modeling a perfect joint, while if $j \neq 0$ the line segment is allowed to leave the line depending on neighboring conditions, thus modeling a non-perfect joint.

11.2. Closure equation

Let $D \in SE(3)$ be the relative position of part 2 with respect to part 1. It is defined by a rotation R including three angles a translation vector T including three scalar parameters. As illustrated in figure 14, the closure equation is obtained by writing that the line segment $[A_2, B_2]$ of part 2 at

Figure 14: Closure equation features of case study number 3.

position D is aligned with line C_1, V_1 of part 1 and to write that the line segment $[A_1, B_1]$ of part 1 is aligned with line C_2, V_2 of part 2 at position D.

$$\begin{gathered} C_1 + v_1 V_1 - D(A_2) = 0 \\ C_1 + w_1 V_1 - D(B_2) = 0 \\ D(C_2 + v_2 V_2) - A_1 = 0 \\ D(C_2 + w_2 V_2) - B_1 = 0. \end{gathered}$$

Choosing a self explanatory coordinate system where

$$C_1 = C_2 = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \qquad V_1 = V_2 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

$$A_i = \begin{pmatrix} l_i \cos\alpha_i + d \sin\alpha_i \cos\beta_i \\ l_i \sin\alpha_i - d\cos\alpha_i\cos\beta_i \\ -d\sin\beta_i \end{pmatrix} \qquad B_i = \begin{pmatrix} l_i \cos\alpha_i - d \sin\alpha_i \cos\beta_i \\ l_i \sin\alpha_i + d\cos\alpha_i\cos\beta_i \\ d\sin\beta_i \end{pmatrix}$$

this leads to the following closure function $F : \mathbb{R}^6 \times \mathbb{R}^4 \times SE(3) \to \mathbb{R}^{12}$

$$F(l_1, \alpha_1, \beta_1, l_2, \alpha_2, \beta_2, v_1, w_1, v_2, w_2, D) = \begin{pmatrix} C_1 + v_1 V_1 - D(A_2) \\ C_1 + w_1 V_1 - D(B_2) \\ D(C_2 + v_2 V_2) - A_1 \\ D(C_2 + w_2 V_2) - B_1 \end{pmatrix}$$

where the dimensional parameters are l_1, α_1, β_1, l_2, α_2, β_2 and the positional parameters are v_1, w_1, v_2, w_2, D. The closure equation $F = 0$ features ten positional parameters and twelve equations, meaning some overconstraint, as expected.

The regularized closure function is defined by $G : \mathbb{R}^6 \times \mathbb{R}^4 \times SE(3) \times \mathbb{R}^{12} \to \mathbb{R}^{12}$ with $G(u, j, p) = F(u, p) - j$ where $u = (l_1, \alpha_1, \beta_1, l_2, \alpha_2, \beta_2)$, $p = (v_1, w_1, v_2, w_2, D)$ and $j = (j_1, \cdots, j_{12})$.

11.3. Nominal configuration and uncertainties

Nominal dimensions are $l_1^0 = 1.0$, $l_2^0 = 1.5$, $\alpha_1^0 = \frac{3\pi}{5}$, $\alpha_2^0 = \frac{2\pi}{5}$, $\beta_1^0 = 0$ and $\beta_2^0 = 0$. Nominal positional parameters are $r_0 = \frac{l_2^0 \cos\alpha_1^0 - l_1^0}{\sin\alpha_1^0}$, $v_1^0 = r_0 - d$,

Figure 15: Dimensional perturbations for case study number 3.

$w_1^0 = r_0 + d$, $v_2^0 = \frac{l_2^0 - l_1^0 \cos\alpha_1^0}{\sin\alpha_1^0} - d$, $w_2^0 = v_2^0 + 2d$. Nominal position D_0 is made of rotation R_0 and translation T_0 as follows

$$R_0 = \begin{pmatrix} \cos(\pi - \alpha_2^0) & -\sin(\pi - \alpha_2^0) & 0 \\ \sin(\pi - \alpha_2^0) & \cos(\pi - \alpha_2^0) & 0 \\ 0 & 0 & 1 \end{pmatrix} \qquad T_0 = \begin{pmatrix} r_0 \\ l_2^0 \\ 0 \end{pmatrix}.$$

Dimensional perturbations are defined using (26) by setting $\Delta = 0.1$ and the following frequencies

ω_1	ω_2	ω_3	ω_4	ω_5	ω_6
5	7	11	3	13	2

respectively associated with l_1, l_2, α_1, α_2, β_1 and β_2, as illustrated in figure 15. It is noticeable that nominal parts are planar while perturbed parts are non planar due to non zero angles β_1 and β_2. Figure 16 illustrates typical configurations of perturbed assemblies resulting from dimensional perturbations and optimal relative positioning. Notice the non-perfect joints. Figure 17 illustrates the clearance variation caused by dimensional perturbations.

12. Algorithm

Flowchart of figure 19 describes the implementation procedure. The algorithmic aspect is now detailed. It includes three main steps: pre-processing, solving, post-processing. Pre-processing is to compute the nominal dimensions u_0 and positions p_0, and to generate the equations of the problem, mainly mappings $G(\cdot)$, $H(\cdot)$ or $V(\cdot)$. Solving is to compute the clearance and position mappings $j(\cdot)$ and $p(\cdot)$ over a sampling $0 < t_1 < \cdots < t_K = T$ of the investigation interval $[0, T]$. The output of the solving step is a table $Table(\cdot)$ so that, for $k = 1, \cdots, K$, $Table(k)$ is the couple of clearance and position solution $(j(t), p(t))$ for $t = t_k$. Three alternate solving techniques are presented. They are respectively based on minimization, zero finding and ODE integration. The solving step advantageously makes use of a software library. Post-processing is to compute the maximum clearance values by considering the output data of the solving step.

12.1. Preprocessing

As explained in section 10, the test case is defined by the nominal positions P_i^0 of connections and by the rods end connections through matrix N. For clarity, this matrix is noted as a stack of rows

$$N = \begin{pmatrix} N_1 \\ \vdots \\ N_{2n} \end{pmatrix}.$$

The nominal position parameter p_0 is computed by setting points O_i^0 and vectors E_i^0 so that equation (30) is satisfied, and by using definitions (28), (29), (32). The nominal dimensions are rods lengths l_i, as defined by (31).

Input : Connections nominal position P_i^0, matrix N.
Output: Nominal dimension u_0, nominal position p_0.

for $i := 1$ **to** n **do**

$$O_i^0 := N_{2i-1} \begin{pmatrix} P_1^0 \\ \vdots \\ P_n^0 \end{pmatrix}$$

$$E_i^0 := N_{2i} \begin{pmatrix} P_1^0 \\ \vdots \\ P_n^0 \end{pmatrix} - O_i^0$$

$l_i := \|E_i^0\|$

end

Algorithm 1: Computing nominal dimensions and positions.

Mappings defining the equations of the problem are automatically generated by using symbolic computation. The nominal geometry of the assembly, its topology, dimensional uncertainties and frequencies are captured in a text file, as illustrated in figure 18, which feeds a Maple program generating the appropriate $G(\cdot)$ mapping. If needed, mappings $H(\cdot)$ and $V(\cdot)$ are deduced from mapping $G(\cdot)$ by using symbolic computation again.

12.2. Solving: minimize

The shortest way is a single loop so that the minimization problem (18) is solved at each step k by using a program from a mathematical software library.

Figure 16: Case study number 3. Typical configurations of the non-nominal assembly. Notice the non-perfect cylindrical joints and the non-planar configurations.

Input : Nominal dimension u_0, nominal position p_0, mapping $G(\cdot)$.
Output: Clearance and position mappings $Table(k) = (j(t_k), p(t_k))$.

for $k := 1$ **to** K **do**
  $(j, p) := \text{Argmin}\left\{\frac{1}{2}|j|^2; G(t_k, j, p) = 0, \{j, p\}\right\}$
  $Table(k) := (j, p)$
end

Algorithm 2: Minimization algorithm for solving 2D and 3D test cases.

The drawback is that the library program is responsible for initializing the searching, which can lead to unpredictable results. A variant is to control the initialization by providing the minimizer with the previous solution. The initial guess is written as the second argument of Argmin function. 3D and 2D test cases are successfully solved by using this algorithm.

Input : Nominal dimension u_0, nominal position p_0, mapping $G(\cdot)$.
Output: Clearance and position mappings $Table(k) = (j(t_k), p(t_k))$.

$(j_1, p_1) := (0, p_0)$
for $k := 1$ **to** K **do**
  $(j, p) := \text{Argmin}\left\{\frac{1}{2}|j|^2; G(t_k, j, p) = 0, \{j, j_1\}, \{p, p_1\}\right\}$
  $Table(k) := (j, p)$
  $(j_1, p_1) := (j, p)$
end

Algorithm 3: Minimization algorithm with controlled initialization for solving 2D and 3D test cases.

12.3. Solving: zero finder

Next algorithm is to solve the non linear system (13) by using a zero finding program of the software library, Newton-Raphson or the like. Unknowns are clearance variable j, positioning variable p and Lagrange parameter λ. The zero finder is initialized by using the previous solution. 3D and 2D test cases are successfully solved by using this algorithm.

Figure 17: Clearance variation for case study number 3.

Figure 18: Input text file for symbolic computation.

Input : Nominal dimension u_0, nominal position p_0, mapping $H(\cdot)$.
Output: Clearance and position mappings $Table(k) = (j(t_k), p(t_k))$.

$(j_1, p_1, \lambda_1) := (0, p_0, 0)$
for $k := 1$ **to** K **do**
  Solve $\{H(t_k, j, p, \lambda) = 0, \{j, j_1\}, \{p, p_1\}, \{\lambda, \lambda_1\}\}$
  $Table(k) := (j, p)$
  $(j_1, p_1, \lambda_1) := (j, p, \lambda)$
end

Algorithm 4: Zero finder algorithm for solving 2D and 3D structures and the two sliders assembly.

12.4. Solving: ODE integration

Finally, if, through symbolic or numerical computation, the inversion (24) can be performed, the vector field (24) of differential equation (23) is used to feed a standard library ODE solver, say Runge and Kutta or the like. Then, a single line of computer program is enough to get mappings $t \mapsto j(t)$ and $t \mapsto p(t)$. For symbolic inversion purpose, only the 2D test case is solved by this method.

Input : Nominal dimension u_0, nominal position p_0, mapping $V(\cdot)$.
Output: Clearance and position mappings $t \mapsto j(t)$ and $t \mapsto p(t)$.

$$\text{ODESolve}\left\{\begin{pmatrix} j'(t) \\ p'(t) \\ \lambda'(t) \end{pmatrix} = V(t, j, p, \lambda), \begin{pmatrix} j(0) \\ p(0) \\ \lambda(0) \end{pmatrix} := \begin{pmatrix} 0 \\ p_0 \\ 0 \end{pmatrix}, \{t \in [0, T]\}\right\}$$

Algorithm 5: ODE numerical integration for solving 2D structure test cases.

12.5. Post-processing

Whatever the method used for solving, it provides (a numerical approximation of the) clearance mapping $t \mapsto j(t)$. The extremal clearance is computed according to (27) and illustrated in figures 7, 11, and 17. Local analysis at each joint can be performed if needed by using the appropriate coordinates of mapping $j(\cdot)$.

Figure 19: The implementation procedure involves a preprocessing step for data initialization, three alternate solving techniques and a postprocessing step for result analysis.

13. Conclusion

Overconstrained structures are represented by a set of closure equations F involving two kinds of numerical parameters: dimensional and positional. In an industrial context, engineers introduce clearance into joints to enable mechanical structure assembly. Clearance parameters are called j. Closure function F is transformed into regularized closure function G depending on dimensional, positional and clearance parameters. From an algebraic point of view, G function is highly underconstrained.

For specific rod's lengths, the unique solution is computed by solving the following optimization problem: minimize the squared norm of clearance parameters with respect to the G function. Due to manufacturing deviations, lengths of rods vary between two known limits. In the dimensional parameters space, hyper-volume is scanned by a set of harmonic functions parameterized by a single parameter t. Then, minimization, zero finding or numerical integration of an ordinary differential equation handles the simulation.

The number of input data of the proposed tool is very small: connectivity matrix of the assembly structure, coordinates of the spherical joints center and manufacturing precision class of the parts. First simulation results are quickly obtained but with a questionable degree of confidence. But the longer the simulation is, the more reliable it is. Implementation is prototyped with mathematical software using standard toolboxes as well as formal computation capabilities.

It should be understood that the geometrical model is not restricted to rods and spherical joints and that equations can be generated through symbolic computation. Any mechanical joint can be taken into account provided it can be handled by a (regularized) closure function.

Three perspectives of this work are being considered. The first one is to develop a data post processing i.e. computing clearance size into each joint, maximum joint displacement in a specific direction, etc. Even further, keep in mind that the method provides a family of assemblies indexed by t and sharing the same topology. Then, a natural post processing is to simulate some appropriate physical behavior over each assembly in order to understand how dimensional uncertainty and clearance optimization influence the said physical behavior. This context is precisely the one of [14] addressing how to reuse computation from one model of the family to another one, which is a promising track. The second perstective is to deal with more practical cases, including overconstrained mobile mechanisms such as deployable structures. The third perstective is a fundamental investigation about how to systematically introduce clearance parameters into perfect joints.

Appendix A. Theorems and lemmas

The appendix gathers the implicit function theorem 3 and the proofs of lemmas 1 2 and 3. In the following, X, Y, Z, U, P, J and E are finite dimensional euclidean spaces.

Theorem 3. *Let $f : X \times Y \to Z$ be a smooth function and let (x_0, y_0) such that $f(x_0, y_0) = 0$. Suppose that the linear mapping $f_y(x_0, y_0)$ from Y to Z is invertible. Then, there exists a unique function $x \mapsto \varphi(x)$ defined over a neighborhood V of x_0 onto a neighborhood W of y_0 such that $\varphi(x_0) = y_0$ and, for all (x, y) in $V \times W$, $f(x, y) = 0$ if and only if $y = \varphi(x)$. In addition, $\varphi'(x) = -f_y\left(x, \varphi(x)\right)^{-1} f_x(x, \varphi(x))$ for all x in V.*

Lemma 1. *Consider linear mappings $A : J \to E$ and $B : P \to E$ and suppose that $AA^T + BB^T \in \mathcal{L}(E)$ is invertible. Consider the bloc defined linear mapping $M : J \times P \times E \to J \times P \times E$*

$$M = \begin{pmatrix} I & 0 & A^T \\ 0 & 0 & B^T \\ A & B & 0 \end{pmatrix}$$

Then, M is invertible if and only if $B^T(AA^T+BB^T)^{-1}B \in \mathcal{L}(P)$ is invertible.

Proof. Because of the finite dimension of spaces J, P and E it is enough to prove that M is injective, meaning that if $Mw = 0$ then $w = 0$. So, let $\begin{pmatrix} j \\ p \\ \lambda \end{pmatrix} \in J \times P \times E$ and suppose that $M \begin{pmatrix} j \\ p \\ \lambda \end{pmatrix} = 0$. According to the bloc structure of M, this means

$$\begin{aligned} j + A^T\lambda &= 0 \\ B^T\lambda &= 0 \\ Aj + Bp &= 0. \end{aligned}$$

The first equation implies $Aj + AA^T\lambda = 0$ and, with the third one, $AA^T\lambda = Bp$. The second equation implies $BB^T\lambda = 0$. So $(AA^T + BB^T)\lambda = Bp$ and, by hypothesis, $\lambda = (AA^T + BB^T)^{-1}Bp$. Using again the second equation, $B^T(AA^T + BB^T)^{-1}Bp = 0$. Now, if $B^T(AA^T + BB^T)^{-1}B$ is invertible, then $p = 0$ so $\lambda = 0$ and $j = 0$ meaning that M is invertible. Conversely, if $B^T(AA^T + BB^T)^{-1}B$ is not invertible, there exists $p^* \neq 0$ such that $B^T(AA^T + BB^T)^{-1}Bp^* = 0$. Then, considering $\lambda^* = (AA^T + BB^T)^{-1}Bp^*$ and $j^* = -A^T\lambda^*$, linear mapping M is not invertible because $\begin{pmatrix} j^* \\ p^* \\ \lambda^* \end{pmatrix} \neq 0$ and $M \begin{pmatrix} j^* \\ p^* \\ \lambda^* \end{pmatrix} = 0$, which ends the proof. □

Lemma 2. *Consider a smooth mapping $G : J \times P \to E$ with $\dim J > \dim E$ and the minimization problem*

$$(j, p) = \text{Argmin}\left\{\frac{1}{2}|j|^2; G(j, p) = 0\right\}.$$

Consider the associated Lagrange function $L : J \times P \times E \to \mathbb{R}$

$$L(j, p, \lambda) = \frac{1}{2}|j|^2 + \langle \lambda, G(j, p)\rangle.$$

If (j^*, p^*, λ^*) is a stationary point of Lagrange function, and if

$$|v|^2 + \langle \lambda^*, G_{jj}(j^*, p^*)vv\rangle > 0$$

for all $v \in J$ such that $v \neq 0$ and $G_j(j^*, p^*)v = 0$, then j^* is minimum.

Proof. Since (j^*, p^*, λ^*) is a stationary point of Lagrange function, $j^* + G_j(j^*, p^*)^T\lambda^* = 0$ and $G(j^*, p^*) = 0$. Let $v \neq 0$ be an arbitrary vector of J such that $G_j(j^*, p^*)v = 0$. Consider a smooth curve $s \mapsto j(s)$ such that $j(0) = j^*$, $j'(0) = v$ and $G(j(s), p^*) = 0$ for all s. By elementary computation,

$$\left[\frac{d^2}{ds^2}\left(\frac{1}{2}|j(s)|^2\right)\right]_{s=0} = |v|^2 + \langle \lambda^*, G_{jj}(j^*, p^*)vv\rangle$$

that is non negative, so j^* is a minimum. □

Lemma 3. *Consider a smooth mapping $f : U \times X \to E$ with $\dim X < \dim E$ and the minimization problem*

$$x = \operatorname{Argmin} \left\{ \frac{1}{2} |f(u,x)|^2 ; x \in X \right\}.$$

Suppose that exists $(u_0, x_0) \in U \times X$ such that $f(u_0, x_0) = 0$ and that the linear mapping

$$f_x(u_0, x_0)^T f_x(u_0, x_0) \in \mathcal{L}(X)$$

is invertible. Then, there exists a neighborhood V of u_0 and a unique mapping $u \mapsto x(u)$ defined over V such that $x(u_0) = x_0$ and $x(u)$ satisfies the stationary condition and the second order minimality condition.

Proof. The existence and uniqueness of mapping $u \mapsto x(u)$ is proven by applying the implicit function theorem to mapping $h : U \times X \to X$ defined by $h(u,x) = f_x(u,x)^T f(u,x)$. Indeed, $h(u_0, x_0) = 0$ and $h_x(u_0, x_0) = f_x(u_0, x_0)^T f_x(u_0, x_0)$ is invertible. Note $g(u,x) = \frac{1}{2}|f(u,x)|^2$. Then,

$$g_x(u, x(u)) = f_x(u, x(u))^T f(u, x(u)) = h(u, x(u)) = 0$$

for all u close to u_0, which is the stationary condition. Furthermore, for all $\xi \in X$,

$$g_{xx}(u,x)\xi\xi = |f_x(u,x)\xi|^2 + \langle f(u,x), f_{xx}(u,x)\xi\xi \rangle.$$

The linear mapping $f_x(u,x(u))^T f_x(u,x(u))$ is invertible for all u close to u_0, so there exists $a_0 > 0$ such that $|f_x(u,x(u))\xi|^2 \geq a_0 |\xi|^2$ for all $\xi \in X$ and all u close to u_0. Noting

$$\varphi(u,\xi) = \langle f(u,x(u)), f_{xx}(u,x(u))\xi\xi \rangle$$

then

$$|\varphi(u,\xi)| \leq \psi(u)|\xi|^2$$

where $\psi(u) = |f(u,x(u))||f_{xx}(u,x(u))|$ is such that $\psi(u_0) = 0$. Consequently, for all u close to u_0 and all $\xi \in X$

$$\begin{aligned} g_{xx}(u,x(u))\xi\xi &= |f_x(u,x(u))\xi|^2 + \varphi(u,\xi) \\ &\geq a_0|\xi|^2 - |\varphi(u,\xi)| \\ &\geq (a_0 - \psi(u))|\xi|^2 \\ &\geq \frac{a_0}{2}|\xi|^2 \end{aligned}$$

which is the second order minimality condition. □

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