More Aspects of Arbitrarily Partitionable Graphs

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Abstract

A graph G of order n is arbitrarily partitionable (AP) if, for every sequence $(n_1, \dots, n_p)$ partitioning n, there is a partition $(V_1, \dots, V_p)$ of $V(G)$ such that $G[V_i]$ is a connected graph of order n_i for $i = 1, \dots, p$. The property of being AP is related to other well-known graph notions, such as perfect matchings and Hamiltonian cycles, with which it shares several properties. This work is dedicated to studying two aspects behind AP graphs.

On the one hand, we consider algorithmic aspects of AP graphs, which received some attention in previous works. We first establish the NP-hardness of the problem of partitioning a graph into connected subgraphs following a given sequence, for various new graph classes of interest. We then prove that the problem of deciding whether a graph is AP is in NP for several classes of graphs, confirming a conjecture of Barth and Fournier for these.

On the other hand, we consider the weakening to APness of sufficient conditions for Hamiltonicity. While previous works have suggested that such conditions can sometimes indeed be weakened, we here point out cases where this is not true. This is done by considering conditions for Hamiltonicity involving squares of graphs, and claw- and net-free graphs.

Keywords: arbitrarily partitionable graphs; partition into connected subgraphs; Hamiltonicity.

1. Introduction

1.1. Partitioning Graphs into Connected Subgraphs

Throughout this work, we call a graph of order G an *n-graph*. By a *graph partition* of an n-graph G, it is meant a vertex-partition $(V_1, \dots, V_p)$ of $V(G)$ such that each $G[V_i]$ has particular properties. This work is about partitions into connected subgraphs, which we deal with through the following terminology.

Consider a *sequence* $\pi = (n_1, \dots, n_p)$ being a partition of n, *i.e.*, $n_1 + \dots + n_p = n$. We will sometimes call π an *n-sequence* to make clear which integer it is a partition of. When writing $|\pi|$, we mean the *size* of π (*i.e.*, p), while, by writing $\|\pi\|$, we refer to the sum of its elements (*i.e.*, n). The *spectrum* $\mathrm{sp}(\pi)$ of π is the set of values appearing in π.

By a *realization* of π of G, we mean a partition $(V_1, \dots, V_p)$ of $V(G)$ such that $G[V_i]$ has order n_i and is connected for $i = 1, \dots, p$. In other words, π indicates the number of connected subgraphs we want (p), and their respective order $(n_1, \dots, n_p)$. In case G admits a realization of every n-sequence, and is hence partitionable into arbitrarily many connected subgraphs with arbitrary order, we call G *arbitrarily partitionable* (*AP* for short). AP graphs can also be found in the literature under different names, such as "arbitrarily

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vertex-decomposable graphs" [1, 2, 4] or "fully decomposable graphs" [10] (but these terms might mislead the readers, as the term "decomposition" has multiple other meanings in graph theory).

Although the notion of AP graphs is relatively recent (it was introduced in 2002 by Barth, Baudon and Puech, to deal with a practical resource allocation problem [1]), the problem of partitioning graphs into connected subgraphs is much older. Perhaps the most influential result is the one from the 70's due to Lovász and Győri [21, 17], who independently proved that an n-graph G is k-connected if and only if every n-sequence with size k is realizable in G, even if the k parts are imposed to contain any k vertices chosen beforehand. We refer the interested reader to the literature (such as the Ph.D. thesis of the first author [5]) for many more results of this kind.

As a warm up, let us start by raising notable properties of AP graphs. First, it should be noted that, in every AP n-graph, any realization of the n-sequence $(2, \dots, 2)$ (or $(2, \dots, 2, 1)$ if n is odd) corresponds to a perfect matching (resp. quasi-perfect matching). Note also that every AP graph being spanned by an AP graph is also AP. Since paths are obviously AP, this implies that every traceable graph (*i.e.*, graph having a Hamiltonian path) is AP. So, in a sense, the class of AP graphs lies in between the class of graphs with a (quasi-) perfect matching and the class of traceable graphs (which itself contains the class of Hamiltonian graphs), which are well-studied classes of graphs.

Several aspects of AP graphs have been investigated to date. In this work, we focus on two such aspects, namely the *algorithmic aspects* and the *Hamiltonian aspects*, which are to be developed below. Again, we refer the interested reader to e.g. [5] for an in-depth overview of other aspects of interest (such as the structural aspects of AP graphs).

1.2. Algorithmic Aspects of AP Graphs

From the previous definitions related to graph partitions into connected subgraphs, the following two decision problems naturally arise:

REALIZATION
Instance: An n-graph G, and an n-sequence π.
Question: Is π realizable in G?

AP
Instance: A graph G.
Question: Is G AP?

The NPness of REALIZATION is obvious. Its NP-hardness was investigated by several authors, who established it for numerous restrictions on either G or π. Concerning restrictions on π, REALIZATION remains NP-hard for instances where $\text{sp}(\pi) = \{k\}$ for every $k \geq 3$, as shown by Dyer and Frieze [12]. Note that finding a realization of any sequence π with $\text{sp}(\pi) \subseteq \{1, 2\}$ is equivalent to finding a sufficiently large matching, which can be done in polynomial time using Edmonds' Blossom Algorithm [13]; hence, such instances of REALIZATION can be solved in polynomial time. In [6], the first author showed that REALIZATION remains NP-hard for instances where $|\pi| = k$, for any $k \geq 2$.

Concerning restrictions of REALIZATION on G, Barth and Fournier proved, in [2], that REALIZATION remains NP-hard for trees with maximum degree 3. In [10], Broersma, Kratsch and Woeginger proved that REALIZATION remains NP-hard for split graphs. In [7], the first author proved that REALIZATION remains NP-hard when restricted to graphs with about a third universal vertices (*i.e.*, vertices neighbouring all other vertices).

The status of the AP problem is quite intriguing. A first important point to raise is that, contrarily to what one could naively think, the NP-hardness of Realization does not imply that of AP. That is, in all reductions imagined by the authors above, the reduced n-graph G needs to have a very restricted structure so that a particular n-sequence π is realizable under particular circumstances only; this very restricted structure makes many other n-sequences not realizable in G, implying that it is far from being AP.

It is actually not even clear whether the AP problem is in NP or co-NP. For AP to be in NP, one would need to provide a polynomial certificate attesting that all n-sequences are realizable in G, while the number of such n-sequences is $p(n)$, the partition number of n, which is exponential in n. For AP to be in co-NP, one would need to provide a polynomial-time algorithm for checking that a given sequence is indeed not realizable in G, while the number of possible partitions of G into connected subgraphs is clearly exponential.

On the other hand, though, as pointed out in [2, 6], the AP problem matches the typical structure of Π_2^p problems ("*for every* sequence, *is there* a realization?"), and thus belongs to Π_2^p (recall that Π_2^p problems are, simply put, those decision problems where we are given two sets X, Y of elements, each of these elements ranging in some respective sets of values, and the question is whether, whatever the values of the elements in X are, the elements in Y can always be assigned values so that a particular property holds). However, it is still not known whether AP is Π_2^p-hard. Also, we have no evidence that AP is NP-hard.

Regarding those questions, the main conjecture is due to Barth and Fournier [2]:

Conjecture 1.1 (Barth, Fournier [2])**.** *The* AP *problem is in NP.*

Conjecture 1.1 relies on *polynomial kernels of sequences*, which are presumed to exist for every graph. For an n-graph G, a *kernel* (of n-sequences) is a set $\mathcal{K}$ of n-sequences such that G is AP if and only if all sequences of $\mathcal{K}$ are realizable in G. That is, a kernel for G is a (preferably small) set of sequences attesting the APness of G. We say that $\mathcal{K}$ is a *polynomial kernel* if its size is a polynomial function of n.

Note that the existence of a polynomial kernel $\mathcal{K}$ for a given graph class indeed implies the NPness of the AP problem for that class. All results towards Conjecture 1.1 so far are based on proving the existence of such kernels for particular graph classes. The first result of this kind was given by Barth, Baudon and Puech [1], who proved that, for subdivided claws of order n (*i.e.*, trees where the unique vertex of degree more than 2 has degree 3), the set of all n-sequences of the form $(k, \dots, k, r)$ or $(k, \dots, k, k+1, \dots, k+1, r)$ (where $r < k$) is a polynomial kernel. Later on [24], Ravaux showed that, still for subdivided claws of order n, the set of n-sequences π with $|\mathrm{sp}(\pi)| \leq 7$ is an alternative polynomial kernel (although of bigger size, proving that this second set is indeed a kernel for the considered graph class is much easier than proving that the first set is). Concerning other polynomial kernels, Broersma, Kratsch and Woeginger proved in [10] that the set of n-sequences π with $\mathrm{sp}(\pi) \subseteq \{1, 2, 3\}$ is a polynomial kernel for split n-graphs. The first author also provided more examples of polynomial kernels in [7]; in particular, for complete multipartite n-graphs, n-sequences π with $\mathrm{sp}(\pi) \subseteq \{1, 2\}$ form a kernel.

An interesting side aspect is that, from the existence of these polynomial kernels, some of the authors above also derived the polynomiality of the AP problem in certain graph classes. In particular, Barth and Fournier proved that AP is polynomial-time solvable when restricted to subdivided stars [2], while Broersma, Kratsch and Woeginger proved it is polynomial-time solvable when restricted to split graphs [10].

1.3. Hamiltonian Aspects of AP Graphs

As mentioned in Section 1.1, the AP property can be regarded as a weakening of traceability/Hamiltonicity. In particular, all sufficient conditions implying traceability also imply APness. An interesting line of research is thus to investigate whether such conditions can be weakened for the AP property.

To the best of our knowledge, only a few results of this sort can be found in the literature. The first series of such results is related to the following parameter, defined for any given graph G:

$$\sigma_k(G) = \min\{d(v_1) + \cdots + d(v_k) : v_1, \ldots, v_k \text{ are non-adjacent vertices of } G\}.$$

A well-known result of Ore [23] states that every n-graph G ($n \geq 3$) with $\sigma_2(G) \geq n-1$ is traceable. In [22], Marczyk proved that every n-graph G ($n \geq 8$) having a (quasi-) perfect matching and verifying $\sigma_2(G) \geq n-3$ is AP. Later on, this result was improved by Horňák, Marczyk, Schiermeyer and Woźniak [18], who proved that every n-graph G ($n \geq 20$) having a (quasi-) perfect matching and verifying $\sigma_2(G) \geq n-5$ is AP. A similar result for graphs G with large $\sigma_3(G)$ was also claimed by Brandt [9].

The last weakening we are aware of, deals with the number of edges guaranteeing APness. From known results, it can be established that connected n-graphs with more than $\binom{n-2}{2} + 2$ edges are traceable (see [20], Proposition 19). An analogous sufficient condition for APness was given by Kalinowski, Pilśniak, Schiermeyer and Woźniak [20], who proved that, a few exceptions apart, all n-graphs ($n \geq 22$) with more than $\binom{n-4}{2} + 12$ edges are AP.

1.4. Our Results

In this work, we establish more results on AP graphs regarding the two aspects developed above. More precisely:

- Regarding the algorithmic aspects, we provide, in Sections 2 and 3, both positive and negative results. We start, in Section 2, by providing an easy NP-hardness reduction framework, for showing, through slight modifications, that Realization is NP-hard when restricted to many graph classes (see, for instance, Theorems 2.3, 2.4, 2.5). In Section 3, we provide more polynomial kernels for several graph classes excluding particular patterns as induced subgraphs.

- In Section 4, we consider the weakening of more Hamiltonian conditions for AP graphs. Although this line of research seems quite appealing, the results we get show that the distance between traceable graphs and AP graphs is more tenuous than one could hope. This is done by considering the notions of *squares* of graphs, and *claw-free* and *net-free graphs*. More precisely, we show that classical results on Hamiltonicity and these notions do not weaken to the AP property (in the obvious way, to the least).

We conclude this work with Section 5, in which we raise some open questions.

2. An NP-hardness Reduction Framework for Realization

In this section we introduce another yet natural reduction for showing the NP-hardness of Realization. Via several modifications of this reduction, we will, in the next sections, establish the NP-hardness of Realization for several of the graph classes we consider. The reduction is from the 3-Partition problem, which can be stated as follows (see [16], and [11] for more properties of 3-Partition):

3-Partition
Instance: A set $A = \{a_1, \ldots, a_{3m}\}$ of $3m$ elements, a bound $B \in \mathbb{N}^*$, and a size $s : A \to \mathbb{N}^*$ such that:

- $\frac{B}{4} < s(a) < \frac{B}{2}$ for every $a \in A$, and
- $\sum_{a \in A} s(a) = mB$.

Question: Can A be partitioned into m parts $A_1 \cup \cdots \cup A_m$ such that we have $\sum_{a \in A_i} s(a) = B$ for every $i = 1, \ldots, m$?

In some of our proofs, we will use the fact that 3-PARTITION remains NP-complete in the contexts below, which obviously hold:

Observation 2.1. *Let* $<A, B, s>$ *be an instance of* 3-PARTITION *where:*

- $\frac{B}{4} < s(a) < \frac{B}{2}$ *for every* $a \in A$*, and*
- $\sum_{a \in A} s(a) = mB$.

The following instances of 3-PARTITION *are equivalent to* $<A, B, s>$*:*

- $<A, B', s'>$*, where* $s'(a) = s(a) + 1$ *for every* $a \in A$*, and* $B' = B + 3$*;*
- $<A, B'', s''>$*, where (for any* $\alpha \geq 1$*)* $s''(a) = \alpha \cdot s(a)$ *for every* $a \in A$*, and* $B'' = \alpha \cdot B$.

Furthermore, we have:

- $\frac{B'}{4} < s'(a) < \frac{B'}{2}$ *and* $\frac{B''}{4} < s''(a) < \frac{B''}{2}$ *for every* $a \in A$*, and*
- $\sum_{a \in A} s'(a) = mB'$ *and* $\sum_{a \in A} s''(a) = mB''$.

The key argument behind our NP-hardness reduction framework from 3-PARTITION to REALIZATION is the following straightforward equivalence between the two problems:

Theorem 2.2. REALIZATION *is NP-hard when restricted to disconnected graphs.*

Proof. Consider an instance $<A, B, s>$ of 3-PARTITION, where $|A| = 3m$ and $A = \{a_1, \ldots, a_{3m}\}$. We produce an instance $<G, \pi>$ (with G being a disconnected graph) of REALIZATION such that $<A, B, s>$ admits a solution if and only if π is realizable in G.

Consider, as G, the disjoint union of m complete graphs K_B on B vertices. So, we have $|V(G)| = mB = \sum_{a \in A} s(a)$. As π, consider the $|V(G)|$-sequence $(s(a_1), \ldots, s(a_{3m}))$. The equivalence between the two instances is then easy to visualize. Consider any part V_i with size $s(a_i)$ from a realization of π in G. Then V_i includes vertices from one connected component of G only since otherwise $G[V_i]$ would not be connected. Furthermore, since every connected component is complete, actually V_i can be any subset (with the required size) of its vertices. So, basically, in any realization of π in G, each of the connected components of G is covered by three parts with size $s(a_{i_1})$, $s(a_{i_2})$ and $s(a_{i_3})$, and thus $s(a_{i_1}) + s(a_{i_2}) + s(a_{i_3}) = B$. A solution to $<A, B, s>$ can then be directly deduced from a realization of π in G, and conversely by similar arguments. $\square$

Note that, in the reduction given in the proof of Theorem 2.2, we can replace the disjoint union of m complete graphs K_B by the disjoint union of any m AP graphs on B vertices. For instance, one can consider any disjoint union of m traceable graphs of order B.

In the current paper, most of our proofs for showing that REALIZATION is NP-hard for some graph class rely on implicitly getting the situation described in the proof of Theorem 2.2. Namely, we use the fact that if, for some graph G and some $|V(G)|$-sequence π, in any realization of π in G some particular parts $V_1, \ldots, V_k$ have to contain particular subgraphs in such a way that $G - (V_1 \cup \dots \cup V_k)$ is a disjoint union of m AP graphs with the same order B, then we essentially get an instance of Realization that is NP-hard.

We illustrate this fact with three easy examples. We start off by considering the class of *subdivided stars* (trees with a unique vertex with degree at least 3). In [2], Barth and Fournier proved that Realization is NP-hard for trees with maximum degree 3 (but having many degree-3 vertices). Using our reduction scheme, we provide an easier proof that Realization is NP-hard for subdivided stars, hence for trees with unbounded maximum degree but only one large-degree vertex.

In the context of AP graphs, subdivided stars, which played a central role towards understanding the structure of AP trees, have been also called *multipodes* (see e.g. [1, 2, 4, 19]). In the next proof, when writing $P_k(a_1, \dots, a_k)$, we refer to the subdivided star with k *branches* where the ith branch has (not counting the center vertex) *order* $a_i \geq 1$.

Theorem 2.3. Realization *is NP-hard when restricted to subdivided stars.*

Proof. The reduction is from 3-Partition. Given an instance $<A, B, s>$ of 3-Partition, we construct a subdivided star G and a $|V(G)|$-sequence π such that $<A, B, s>$ admits a solution if and only if π is realizable in G.

According to Observation 2.1, we may assume that $s(a) > 1$ for every $a \in A$. Let $s_m = \max\{s(a) : a \in A\}$. Consider, as G, the subdivided star $P_{s_m+m}(1, \dots, 1, B, \dots, B)$ with s_m branches of order 1 and m branches of order B. As π, consider $\pi = (s_m + 1, s(a_1), \dots, s(a_{3m}))$.

The keystone of the reduction is that, because no element of π is equal to 1, in every realization of π in G the part containing the center vertex of G necessarily also contains all vertices from the branches of order 1. Since there are s_m branches of order 1 in G, the part containing the center vertex must thus have size at least $s_m + 1$. So basically the part with size $s_m + 1$ of every realization of π in G must include the center vertex of G as well as all the vertices from its s_m branches of order 1.

Once this part is picked, what remains is a forest of m paths P_B (on B vertices) and the sequence $(s(a_1), \dots, s(a_{3m}))$. Hence, finding a realization of π in G is equivalent to the problem of finding a realization of $(s(a_1), \dots, s(a_{3m}))$ in a forest of m paths P_B, while this problem is equivalent to solving $<A, B, s>$ according to the arguments given in the proof of Theorem 2.2. The result then follows. ☐

In the next result, we consider *series-parallel graphs*, for which several NP-hard problems are known to be polynomial-time solvable. These graphs, each of which contains two special vertices (*source* and *sink*), can be defined inductively as follows:

- K_2 is a series-parallel graph, its two vertices being its source and sink, respectively.
- Let G and H be two series-parallel graphs with sources s_G and s_H, respectively, and sinks t_G and t_H, respectively. Then:
  - the *series-composition* of G and H, obtained by identifying t_G and s_H, is a series-parallel graph with source s_G and sink t_H;
  - the *parallel-composition* of G and H, obtained by identifying s_G and s_H, and identifying t_G and t_H (with keeping the graph simple, *i.e.*, omitting all multiple edges, if any is created), is a series-parallel graph with source $s_G = s_H$ and sink $t_G = t_H$.

In the context of AP graphs, a particular class of series-parallel graphs, called *balloons*, has been investigated towards understanding the structure of 2-connected AP graphs (see e.g. [4, 3]). The *balloon* (or k-*balloon*, to make the parameter k clear) $B_k(b_1, \dots, b_k)$ is the series-parallel graph obtained as follows. Start from two vertices r_1 and r_2. Then, for every $i = 1, \dots, k$, we join r_1 and r_2 via a *branch* being a new path with b_i internal vertices having r_1, r_2 as end-vertices. By the *order* of the ith branch, we mean b_i.

Theorem 2.4. Realization *is* NP*-hard when restricted to series-parallel graphs.*

Proof. We use the same reduction scheme as that in the proof of Theorem 2.3. This time, consider, as G, the $(2s_m + m)$-balloon $B_{2s_m+m}(1, \dots, 1, B, \dots, B)$ with $2s_m$ branches of order 1, and m branches of order B. As π, consider

$$\pi = (s_m + 1, s_m + 1, s(a_1), \dots, s(a_{3m})).$$

Because every vertex from a branch of order 1 of G only neighbours r_1 and r_2 (who have degree $2s_m + m$), it has to belong, in every realization of π in G, to a same part as one of r_1 or r_2. Said differently, the at most two parts covering r_1 and r_2 also have to cover all of the vertices from the branches of order 1. Because there are $2s_m$ branches of order 1, these at most two parts must cover at least $2s_m + 2$ vertices. In view of the values in π, in every realization of π in G we necessarily have to use the two parts with size $s_m + 1$ to cover all these vertices. Once these two parts have been picked, what remains is a forest of m paths P_B of order B and the sequence $(s(a_1), \dots, s(a_{3m}))$. We thus have the desired equivalence. □

As mentioned in the introductory section, recall that, by the result of Győri and Lovász [21, 17], all k-connected graphs can always be partitioned into k connected subgraphs with arbitrary orders. In the following result, we prove, generalizing the arguments from the previous proofs, that partitioning a k-connected graph into more than k connected subgraphs is an NP-hard problem.

Theorem 2.5. *For every $k \geq 1$,* Realization *is* NP*-hard when restricted to k-connected graphs.*

Proof. The reduction is similar to that used in the previous two proofs. Let $k \geq 1$ be fixed, and construct G as follows. Add k vertices $r_1, \dots, r_k$ to G, as well as ks_m copies of K_1 and m copies of K_B. Finally, for every $i = 1, \dots, k$, add an edge between r_i and every vertex of $V(G) \setminus \{r_1, \dots, r_k\}$. Note that G is indeed k-connected, and $\{r_1, \dots, r_k\}$ is a k-cutset. As π, consider

$$\pi = (s_m + 1, \dots, s_m + 1, s(a_1), \dots, s(a_{3m})),$$

where the value $s_m + 1$ appears exactly k times at the beginning of π.

Consider any realization of π in G. Because the $s(a_i)$'s are strictly greater than 1, and the ks_m copies of K_1 are joined to the r_i's only, every of these K_1's has to belong to the same part as one of the r_i's. Following these arguments, the k parts with size $s_m + 1$ of a realization of π in G must each include one of the r_i's and s_m of the K_1's. What remains once these k parts have been picked is m vertex-disjoint connected components isomorphic to K_B, as well as the sequence $(s(a_1), \dots, s(a_{3m}))$. This concludes the proof. □

3. Polynomial Kernels for Graphs Without Forbidden Subgraphs

For two graphs G and H, we denote by $G + H$ the *disjoint union* of G and H, which is the disconnect graph with vertex set $V(G) \cup V(H)$ and edge set $E(G) \cup E(H)$. When writing kG for some $k \geq 1$, we refer to the disjoint union $G + \cdots + G$ of k copies of G. If $\mathcal{G}$ is a family of graphs, we write by $k\mathcal{G}$ the class of graphs that are the disjoint union of k members of $\mathcal{G}$. That is, $G \in k\mathcal{G}$ if there exist $G_1, \ldots, G_k \in \mathcal{G}$ such that $G = G_1 + \cdots + G_k$.

We denote by $G \times H$ the *complete join* of G and H, which is the graph with vertex set $V(G) \cup V(H)$ and edge set $E(G) \cup E(H) \cup V(G) \times V(H)$. For a family (set) of graphs $\mathcal{F}$, we say that a graph G is $\mathcal{F}$*-free* if G has no member of $\mathcal{F}$ as an induced subgraph.

For $k \geq 1$, we denote by $\mathcal{G}_k$ the class of all connected k-graphs, while we denote by $\mathcal{G}_{\leq k}$ the class of all connected graphs of order at most k.

In this section, we exhibit polynomial kernels of sequences for some families of $\mathcal{F}$-free graphs. We also discuss some consequences of the existence of these kernels on the complexity of the Realization and AP problems for the considered classes of graphs.

3.1. k-Sequential Graphs

Let $k \geq 1$ be fixed, and $\mathcal{F}$ be a class of graphs. An $\mathcal{F}$*-sequential graph* is a graph that can be defined inductively as follows:

- Graphs in $\mathcal{F}$ are $\mathcal{F}$-sequential graphs;
- For an $\mathcal{F}$-sequential graph G and a graph $H \in \mathcal{F}$, the graphs $G + H$ and $G \times H$ are $\mathcal{F}$-sequential graphs.

The construction of such graphs can equivalently be seen as a sequence of steps, at each of which a new graph of $\mathcal{F}$ is added, its vertices being possibly joined to all vertices added during the previous steps. Following that analogy, for any given vertex v of a sequential graph, we denote by level(v) the *level* of v, where level(v) $= i$ if v belongs to the graph that was added during the ith step. A vertex with level i is called a *join vertex* if a complete join was performed at the end of the ith step.

In what follows, $\mathcal{G}_{\leq k}$-sequential graphs are also called *k-sequential graphs.* It is worthwhile noting that sequential graphs encapsulate known families of graphs; for instance, *threshold graphs* ($\{P_4, C_4, 2K_2\}$-free graphs) are precisely the 1-sequential graphs.

We start by proving that, for every k-sequential n-graph, the set

$$\mathcal{K}_{k,n} = \{\pi : \pi \text{ is an } n\text{-sequence and } \mathrm{sp}(\pi) \subseteq \{1, \ldots, 2k+1\}\}$$

is a kernel. When k is fixed, this provides a polynomial kernel of sequences for k-sequential graphs, which implies that the AP problem is in NP for this class of graphs.

Theorem 3.1. *For every k, n, the set $\mathcal{K}_{k,n}$ is a kernel for k-sequential n-graphs.*

Proof. Let G be a k-sequential n-graph. We need to show that G is AP if and only if all sequences of $\mathcal{K}_{k,n}$ are realizable in G. If we assume that G is AP, then, by definition, all sequences of $\mathcal{K}_{k,n}$ are realizable in G. We thus have to focus on the converse direction only.

Assume that all sequences of $\mathcal{K}_{k,n}$ are realizable in G, and consider any n-sequence $\pi = (n_1, \ldots, n_p) \notin \mathcal{K}_{k,n}$. We have to prove that π admits a realization in G. We build an n-sequence $\pi' \in \mathcal{K}_{k,n}$ in the following way: we consider every element n_i of π in turn, and:

- if $n_i \in \{1, \ldots, 2k+1\}$, then we add n_i to π';
- otherwise, we add all elements of any n_i-sequence $(m_1, \ldots, m_x)$ where $k+1 \leq m_i \leq 2k+1$ for every $i = 1, \ldots, x$ (such exists since $n_i \geq 2k+2$).

Note that indeed π' is an n-sequence, and $\pi' \in \mathcal{K}_{k,n}$. Let thus R' be a realization of π' in G, which exists by assumption.

We obtain a realization R of π in G in the following way. We consider every $n_i \in \pi$ in turn. If $n_i \leq 2k+1$, then there is a corresponding element with value n_i in π', and thus a connected subgraph of order n_i in R', which we add to R. Now, if $n_i \in \pi \geq 2k+2$, then there are corresponding elements $m_1, \dots, m_x$ with value in $\{k+1, .., 2k+1\}$ in π' (such that $m_1 + \dots + m_x = n_i$), and thus connected subgraphs $G_1, \dots, G_x$ of order $m_1, \dots, m_x$ in R'. By the definition of a k-sequential graph (in particular, because all graphs added sequentially to construct G have order at most k), each of the G_i's has to contain a join vertex. The join vertex with maximum level implies that $G_1, \dots, G_x$, in G, form a connected subgraph of order n_i; then we add it to R as the part of size n_i.

Once every n_i has been considered, R is a realization of π in G. □

Regarding Theorem 3.1, it is worth mentioning that, in general, the maximum magnitude $2k+1$ of the elements in $\mathcal{K}_{k,n}$ cannot be lowered. Rephrased differently, there are cases where n-sequences π with $\mathrm{sp}(\pi) \subseteq \{1, \dots, x\}$ for some $x < 2k+1$ do not form a kernel for k-sequential graphs. A straight example is that of $K_2 + K_1$: for this 1-sequential graph, the sequence (3) has no realization, while all sequences with spectrum from $\{1, 2\}$ are realizable. Hence, for 1-sequential graphs, the APness follows from the realizability of sequences with elements in $\{1, 2, 3\}$, and this is, in a sense, best possible.

There are cases, however, where better kernels can be obtained. For instance, when $k \geq 2$, the set

$$\mathcal{K}'_{k,n} = \{\pi : \pi \text{ is an } n\text{-sequence and } \mathrm{sp}(\pi) \subseteq \{1, \dots, \lfloor 3k/2 \rfloor + 1\}\}$$

is a better kernel for k-sequential n-graphs.

Theorem 3.2. *For every $k \geq 2$ and n, the set $\mathcal{K}'_{k,n}$ is a kernel for k-sequential n-graphs.*

Proof. Let G be a k-sequential n-graph. First assume that G is not connected, *i.e.*, the last connected component added during the construction of G was not joined to the rest of the graph. Let H be a connected component of G with smallest order h. By definition, $k \geq h$. Also, $n \geq 2h$ since G is not connected.

Suppose first that $h \geq 5$. If h is odd, then let

$$A = \left\{\frac{h+1}{2}, \frac{h+3}{2}, \dots, h-1\right\}.$$

If h is even, then let

$$A = \left\{\frac{h}{2}+1, \frac{h}{2}+2, \dots, h-1\right\}.$$

Note that every integer at least $2h$ can be expressed as the sum of elements in A. Let thus $\pi \in \mathcal{K}'_{k,n}$ be an n-sequence taking values from A. It can easily be seen that h cannot be expressed as the sum of elements in A; therefore, there is no realization of π in G.

Now assume that $h \leq 4$. As above, we consider some set A of elements, depending on the value of h:

- if $h = 4$, then consider $A = \{3, 5\}$;
- if $h = 3$, then consider $A = \{2, 5\}$;
- if $h = 2$, then consider:

- $A = \{3, 4\}$ if $n \geq 6$;
- $A = \{1, 4\}$ if $n = 5$;
- $A = \{1, 3\}$ if $n = 4$;

- if $h = 1$, then consider $A = \{2, 3\}$.

In every case, it can be noted that every integer at least $2h$ can be expressed as the sum of elements in A, while h cannot. Thus, as in the previous case, there exists an n-sequence of $\mathcal{K}'_{k,n}$ that is not realizable in G. So G has to be connected.

So now assume that G is connected, and that G is not AP. According to Theorem 3.1, graph G is AP if and only if all n-sequences of $\mathcal{K}_{k,n}$, *i.e.*, with spectrum from $\{1, \dots, 2k+1\}$, are realizable in G. To prove the claim, it is sufficient to prove that, under the assumption that all n-sequences of $\mathcal{K}'_{k,n}$ are realizable in G, all n-sequences of $\mathcal{K}_{k,n}$ also are.

Let $\pi \notin \mathcal{K}'_{k,n}$ be an n-sequence of $\mathcal{K}_{k,n}$ not realizable in G. Our goal is to show that there is another n-sequence $\pi' \in \mathcal{K}'_{k,n}$ that is also not realizable in G. Among all possible sequences as π, we choose one that minimizes the maximum element value t appearing in it. Subject to that condition, we also choose such sequence π that minimizes the number of occurrences of t. So $\frac{3k+3}{2} \leq t \leq 2k + 1$, and every n-sequence with maximum element value at most $t - 1$ is realizable in G.

First assume that the value t appears at least twice in π. Clearly, $3k + 3 \leq 2t \leq 3t - 3$. Let thus π' be an n-sequence obtained from π by removing two elements with value t and adding three new elements r_1, r_2, r_3 such that $k+1 \leq r_1, r_2, r_3 \leq t-1$ and $r_1+r_2+r_3 = 2t$. By our choice of π, there is a realization R of π' in G. Since $r_1, r_2, r_3 \geq k + 1$ note that each of the three parts V_1, V_2, V_3 with size r_1, r_2, r_3 of R contains a join vertex. From this, we can partition $G[V_1 \cup V_2 \cup V_3]$ into two connected subgraphs of order t, which yields a realization of π in G, a contradiction. More precisely, denote by a, b, c the vertices with maximum level of V_1, V_2, V_3. If, say, $\text{level}(a) < \text{level}(b) \leq \text{level}(c)$, then it suffices to move vertices from V_1 to V_2, V_3 so that two connected parts with size t are obtained. Now assume $\text{level}(a) = \text{level}(b) = \text{level}(c)$. Note that V_1 has to contain two vertices u', u'' being not of maximum level. Then we can partition $G[V_1]$ into two convenient subgraphs V_1', V_1'' having the number of vertices we would like to move to V_2 and V_3: start from V_1' and V_1'' containing u' and u'', respectively, add at least one vertex with level $\text{level}(a)$ to V_1' and V_1'' if possible (there might be only one such vertex, but this is not an issue), and add the remaining vertices of V_1 to V_1', V_1'' arbitrarily. Note that $G[V_1']$ and $G[V_1'']$ might be not connected, but, due to the vertices they and V_2, V_3 include, $G[V_2 \cup V_1']$ and $G[V_3 \cup V_1'']$ are connected.

So now assume that the value t appears exactly once in π. Here, consider, as π', the n-sequence obtained by removing the value t from π, and adding two elements with value 1 and $t - 1$, respectively. Again, π' has a realization R in G. Since $t - 1$ is the maximum value appearing in π', we may assume that the part V_1 of R with size $t - 1$ contains a vertex with maximum level. Indeed, if this is not the case, then we can consider a part V_2 containing such a vertex, and move any $|V_1| - |V_2|$ vertices from V_1 to V_2 in such a way that the vertices remaining in V_1 still induce a connected subgraph. This is possible because, by the connectedness of G, the vertices of maximum level are join vertices. We now claim that a part V_1 with size $t - 1$ and a part V_2 with size 1 of R are adjacent, hence yielding a part with size t, and a realization of π in G, a contradiction. This is because $t - 1 > k$ and V_1 includes a join vertex v with maximum level: either the vertex u of V_2 does not have maximum level and is thus adjacent to v, or u has maximum level and is thus adjacent of a vertex in V_1 being not of maximum level. $\square$

We note that the requirement $k \geq 2$ in Theorem 3.2 is best possible, as stars with an even number of vertices are 1-sequential graphs, but they cannot be partitioned following $(2, \dots, 2)$. On the other hand, the value $\lfloor 3k/2 \rfloor + 1$ is best possible for some values of k. Namely, $K_2 + K_3$ is a 2-sequential graph such that all 5-sequences with spectrum from $\{1, 2, 3\}$ are realizable, while $(1, 4)$ is not. Also, $K_3 + K_4$ is a 3-sequential graph such that all 7-sequences with spectrum from $\{1, 2, 3, 4\}$ are realizable, while $(2, 5)$ is not. For $k \geq 4$, we do not know whether the value $\lfloor 3k/2 \rfloor + 1$ can be lowered in Theorem 3.2; we believe this would be an interesting aspect to study further on.

In general, it is worth mentioning that REALIZATION is NP-hard when restricted to k-sequential n-graphs and sequences of $\mathcal{K}_{k,n}$ and $\mathcal{K}'_{k,n}$. Thus, Theorems 3.1 and 3.2 do not imply the polynomiality of the AP problem when restricted to k-sequential graphs. Note, though, that this does not imply the NP-hardness of AP when restricted to those graphs, as there may exist other polynomial kernels for k-sequential n-graphs whose realizability is easy to check.

Theorem 3.3. REALIZATION *is NP-hard when restricted to k-sequential n-graphs and sequences of $\mathcal{K}_{k,n}$ and $\mathcal{K}'_{k,n}$.*

Proof. The proof is similar to that of Theorem 2.5; we prove the result for $k = B$ (where B is part of the input of the given instance of 3-PARTITION). As G, we consider the k-sequential graph constructed, through $s_m + m + 1$ steps, as follows:

- at step i with $i \in \{1, \dots, s_m + 1\}$, we add a new isolated vertex u_i to G;
- at step i with $i \in \{s_m + 2, \dots, s_m + m\}$, we add a new isolated copy of K_B;
- at step $s_m + m + 1$, we add a new vertex v^* joined to all previously-added ones.

Note that indeed G is a k-sequential graph for $k = B$. The sequence π we consider is $(s_m + 1, s(a_1), \dots, s(a_{3m})) \in \mathcal{K}_{k,n}, \mathcal{K}'_{k,n}$. The result follows from the same arguments as earlier: because the $s(a_i)$'s are strictly greater than 1, each u_i has to belong to the same connected part as v^*, which must thus be of size $s_m + 1$. Once this part has been picked, what remains is a disjoint union of m copies of K_B, and the sequence $(s(a_1), \dots, s(a_{3m}))$. $\square$

3.2. $2\mathcal{G}_k$-Free Graphs

Recall that $2\mathcal{G}_k$-free graphs are $\{\bigcup_{G,H \in \mathcal{G}_k} G + H\}$-free graphs. So, in every $2\mathcal{G}_k$-free graph G, for every two disjoint subsets V_1, V_2 of k vertices we take, the graph $G[V_1 \cup V_2]$ is connected as soon as $G[V_1]$ and $G[V_2]$ are. Note, in particular, that $2\mathcal{G}_1$-free graphs are exactly complete graphs (graphs with independence number 1), while $2\mathcal{G}_2$-free graphs are split graphs ($\{2K_2\}$-free graphs).

In what follows, we prove that the set

$$\mathcal{Q}_{k,n} = \{\pi : \pi \text{ is an } n\text{-sequence and } \mathrm{sp}(\pi) \subseteq \{1, \dots, 2k-1\}\}$$

is a kernel for $2\mathcal{G}_k$-free n-graphs, which proves that the AP problem is in NP for these graphs (for fixed k). The proof is essentially a generalization of the proof that a split graph is AP if and only if all sequences with 1's, 2's and 3's are realizable.

Theorem 3.4. *For every k, n, the set $\mathcal{Q}_{k,n}$ is a kernel for $2\mathcal{G}_k$-free n-graphs.*

Proof. The proof goes the same way as that of Theorem 3.1. Consider any n-sequence $\pi = (n_1, \dots, n_p) \notin \mathcal{Q}_{k,n}$. We prove that π is realizable in G. This time, we consider an n-sequence $\pi' \in \mathcal{Q}_{k,n}$ obtained from π as follows. For every element n_i of π:

- if $n_i \in \{1, \ldots, 2k-1\}$, then we add n_i to π';
- otherwise, we add all elements of any n_i-sequence $(m_1, \ldots, m_x)$ where $k \leq m_i \leq 2k-1$ for every $i = 1, \ldots, x$ (such exists since $n_i \geq 2k$).

We obtain a realization R of π in G in the following way. Let R' be a realization of π' in G. Consider every $n_i \in \pi$. If $n_i \leq 2k-1$, then we directly get a connected subgraph of order n_i in R', which we add to R. Otherwise, $n_i \geq 2k$, and there are corresponding elements $m_1, \ldots, m_x$ with value in $\{k, .., 2k-1\}$ in π' (that is, $m_1 + \cdots + m_x = n_i$), and thus connected subgraphs $G_1, \ldots, G_x$ of order $m_1, \ldots, m_x$ in R'. Since all G_i's include a connected subgraph of order k, and G is $2\mathcal{G}_k$-free, necessarily every set $V(G_i) \cup V(G_j)$ for $i \neq j$ induces a connected subgraph in G. So $V(G_1) \cup \cdots \cup V(G_x)$ induces a connected subgraph of order n_i of G, which we add to R as the part of size n_i.

Once every n_i has been considered, R is a realization of π in G. $\square$

The value $2k-1$ in the statement of Theorem 3.4 is best possible for $k = 1, 2$ (consider the graph $K_1 + K_2$). However, it might be not optimal for larger values of k. Let us further mention that the NP-hardness of Realization for $2\mathcal{G}_k$-free n-graphs and sequences of $\mathcal{Q}_{k,n}$ might also be established from the reduction in the proof of Theorem 3.3.

4. Weakening Hamiltonian Properties

In this section, we consider two graph notions behind some known sufficient conditions for Hamiltonicity. Namely, we consider *squares* of graphs and graphs that are *claw-free* and *net-free*. We show that the most obvious way for weakening these sufficient conditions for Hamiltonicity does not yield sufficient conditions for the AP property.

4.1. Fleischner's Theorem

The *square* G^2 of a given graph G is the graph on vertex set $V(G)$ obtained by adding an edge between every two vertices at distance at most 2 in G. We also say that G^2 was obtained by *squaring* G (*i.e.*, applying the square operation on G).

In this section, we consider a well-known result of Fleischner on squares of graphs [15].

Theorem 4.1 (Fleischner's Theorem)**.** *If G is a 2-connected graph, then G^2 is Hamiltonian.*

Naturally, Fleischner's Theorem yields that the square of every 2-connected graph is AP. Let us point out, however, that this result cannot be weakened to traceability; namely, one can easily come up with connected graphs G such that G^2 is not traceable. Due to the connection between AP graphs and traceable graphs, one could nevertheless wonder whether such a statement holds for the AP property. We here prove that this is not the case. In particular, we show that Realization remains NP-hard when restricted to squared graphs.

To establish that result, we will make use of copies of the gadget H depicted in Figure 1, which will be attached to other graphs in a particular fashion. Namely, let G be a graph with a vertex z. Add a disjoint copy of H to G, and identify z and the (white) vertex u of H. In the resulting graph, we say that there is a *copy* of H *rooted* in z. Equivalently, we say that the graph was obtained by rooting a copy of H at z.

The property of interest of this rooting operation is the following.

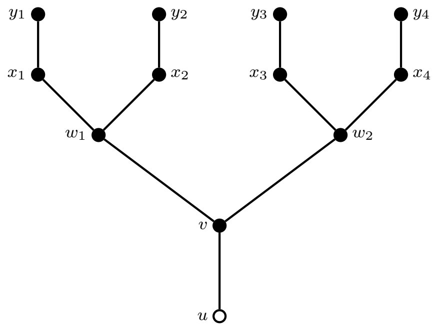

Figure 1: The gadget H needed for the proof of Theorem 4.3.

Lemma 4.2. *Let G be a graph of order $3n$ having a copy of H rooted at u. Then, in any realization R of $\pi = (3, \dots, 3)$ in G^2, the 12 vertices of H are covered by exactly four distinct parts.*

Proof. We deal with the vertices of the copy of H following the terminology indicated in Figure 1. First off, let us note that H^2 itself admits a realization of $(3, \dots, 3)$; for instance,

$$(\{y_1, x_1, x_2\}, \{y_2, w_1, v\}, \{y_3, x_3, x_4\}, \{y_4, w_2, u\})$$

is one such.

Assume now the claim is wrong, and assume there exists a realization R of π in G^2 such that (at least) one of the parts containing u or v contains a vertex of $V(G) \setminus V(H)$ (only these parts can have this property). Note that there cannot be only one such part as it would otherwise cover only one or two vertices of H, while it has order 12 (hence the remaining subgraph of H^2 cannot be partitioned into connected subgraphs of order 3). So there are exactly two parts of R that contain both vertices in $V(H)$ and vertices in $V(G) \setminus V(H)$. Since H^2 is connected, in G^2, to the rest of the graph only through u and v, one of these two parts includes u, w_1 (without loss of generality) and a vertex of $V(G) \setminus V(H)$, while the second part includes v and two vertices of $V(G) \setminus V(H)$ (as otherwise the remaining subgraph of H^2 would have order 8 and thus could not be partitioned into connected subgraphs of order 3). But then we reach a contradiction, as it can easily be checked that $H^2 - \{u, v, w_1\}$ admits no realization of $(3, 3, 3)$. □

We are now ready to prove the following result, which, in a sense, indicates that the natural weakening of Fleischner's Theorem to the AP property does not hold in general.

Theorem 4.3. Realization *is* NP*-hard when restricted to squared bipartite graphs.*

Proof. The proof is by reduction from Realization when restricted to instances where $\pi = (3, \dots, 3)$, which was proved to be NP-hard by Dyer and Frieze [12]. From a given graph G, we construct, in polynomial time, another graph G' such that π is realizable in G if and only if $\pi' = (3, \dots, 3)$ is realizable in G'^2. Furthermore, the graph G' we construct is bipartite.

We start from G' being exactly G. We then consider every edge e of G, and subdivide it in G'; we call v_e the resulting vertex in G'. Finally, for every such vertex v_e in G', we add a copy of the gadget H from Figure 1, and root it at v_e. Note that G' is indeed bipartite (due to the subdivision process and because H is a tree), and, because both G and H have order divisible by 3, so does G'.

The equivalence between partitioning G and G'^2 (following $(3, \dots, 3)$) follows from the fact that, according to Lemma 4.2, in every realization R' of $(3, \dots, 3)$ in G'^2, the 12 vertices

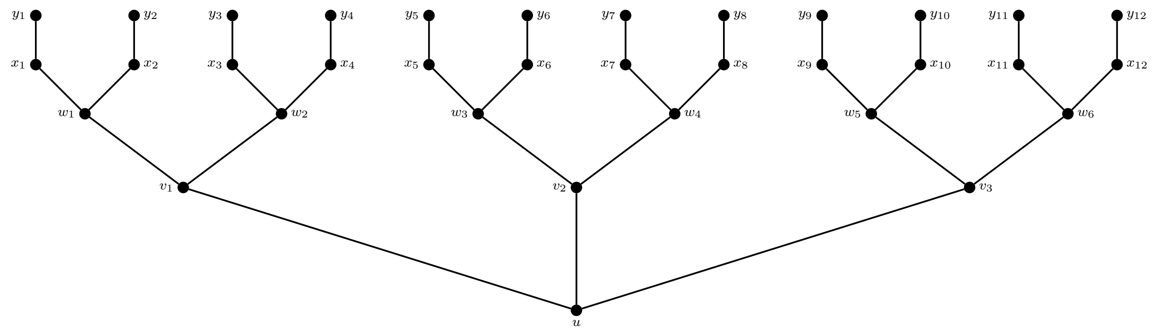

Figure 2: The tree T described in the proof of Theorem 4.5.

from any copy of H are included in exactly four parts. By construction, when removing the copies of H from G'^2, the graph we obtain is exactly G. Hence, when removing from R' the parts covering the copies of H, what remains are parts covering the vertices of G only, and inducing connected subgraphs. These parts thus form a realization of π in G. Hence, a realization of π in G exists if and only if G'^2 admits one of $(3, \dots, 3)$. $\square$

We proved Theorem 4.3 for squared bipartite graphs, but we do think it would be interesting knowing whether Realization remains NP-hard when restricted to squared trees. We leave this question open for now.

Question 4.4. *Is* Realization *NP-hard when restricted to squared trees?*

It is worthwhile pointing out that squared trees without the AP property do exist, which makes Question 4.4 legitimate.

Theorem 4.5. *There exist trees T with $\Delta(T) = 3$ such that T^2 is not AP.*

Proof. We give a single example illustrating the claim, but it naturally generalizes to an infinite family of such trees. Also, considering trees with larger maximum degree might simplify the proof a lot, but we think having the result for subcubic trees is more significant.

Consider, as T, the following tree (see Figure 2):

- T has a degree-3 vertex u with neighbours v_1, v_2, v_3;
- each of v_1, v_2, v_3 has two other degree-3 neighbours, $w_1, \dots, w_6$;
- each of $w_1, \dots, w_6$ has two other degree-2 neighbours, call these $x_1, \dots, x_{12}$;
- each of $x_1, \dots, x_{12}$ has another degree-1 neighbour, call these $y_1, \dots, y_{12}$.

In what follows, we deal with the vertices of T labelled as depicted in Figure 2. Note that $n = |V(T)| = 34$. To prove the claim, we show that T^2 has not realization of the n-sequence $\pi = (1, 3, \dots, 3)$. Towards a contradiction, assume this is not true, and consider a realization $R = (V_1, \dots, V_p)$ of π in T^2, where V_1 is the unique part with size 1.

First, we note that it is not possible that $V_1 = \{u\}$. Indeed, in that case, because $T - \{u\}$ has three connected components of order 11, necessarily, in R, one of the parts with size 3, say V_2, has to contain at least two of v_1, v_2, v_3. No matter which three vertices are contained in V_2, we note that, in all cases, the graph $T^2 - V_1 - V_2$ has at least one connected component of order 10, which thus cannot be partitioned into connected subgraphs of order 3.

So we may assume that u belongs to a part of R with size 3. Then $V_1 \neq \{u\}$ includes a vertex from one of the three connected components of $G - \{u\}$. The other two connected

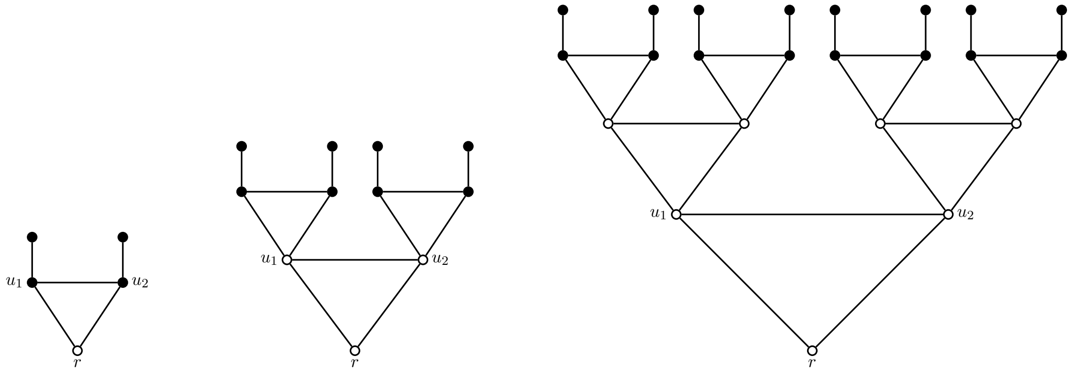

Figure 3: The graphs S_1 (left), S_2 (middle) and S_3 (right).

components, together with u, induce two copies, both rooted at u, of the gadget depicted in Figure 1. According to Lemma 4.2, each of these two gadgets must entirely be covered by parts of size 3 of R. This is not possible, since they share the same root u; a contradiction. □

4.2. Forbidding Claws and Nets

Another condition guaranteeing Hamiltonicity of graphs is the absence of two induced subgraphs, the *claw* and the *net*. The claw is the complete bipartite graph $K_{1,3}$, while the net Z_1 is the graph obtained by attaching a pendant vertex to every vertex of a triangle.

Theorem 4.6 (e.g. [14]). *Every 2-connected (resp. connected) $\{K_{1,3}, Z_1\}$-free graph is Hamiltonian (resp. traceable).*

One could again wonder how Theorem 4.6 could be weakened to the AP property. In this section, we point out that such a sufficient condition for APness cannot be obtained by just dropping any of $K_{1,3}$ or Z_1 from the equation.

Let us first point out that the reduction in the proof of Theorem 2.2 yields disconnected graphs that are $\{K_{1,3}, Z_1\}$-free. From this, we directly get that Realization is NP-hard for such disconnected graphs. This is not satisfactory, however, as, in the context of AP graphs, it makes more sense considering connected graphs.

The counterpart of that result for connected net-free graphs, though, follows directly from the proof of Theorem 2.3, as subdivided stars are clearly net-free graphs.

Theorem 4.7. Realization *is NP-hard when restricted to connected net-free graphs.*

Unfortunately, the similar result for claw-free graphs does not follow immediately from another of the reductions we have introduced in the previous sections. Below, we thus provide another reduction for establishing such a claim (upcoming Theorem 4.9). We actually even establish the NPness of Realization for *line graphs* (graphs of edge adjacencies), a well-known subclass of claw-free graphs.

The proof is another implementation of the reduction framework introduced in Section 2, which relies on the use of the following infinite family $\mathcal{S}$ of claw-free gadgets. $\mathcal{S}$ contains graphs $S_1, S_2, \dots$ defined inductively as follows (see Figure 3 for an illustration). Each S_i contains a unique degree-2 vertex which we call the *root* of S_i. S_1 is the graph obtained by considering a triangle ru_1u_2r, then joining u_1 to a pendant vertex, and then joining u_2 to a pendant vertex. The root of S_1 is r. Now consider any $i \geq 2$ such that S_{i-1} can be constructed. Then S_i is obtained from a triangle ru_1u_2r by adding two disjoint

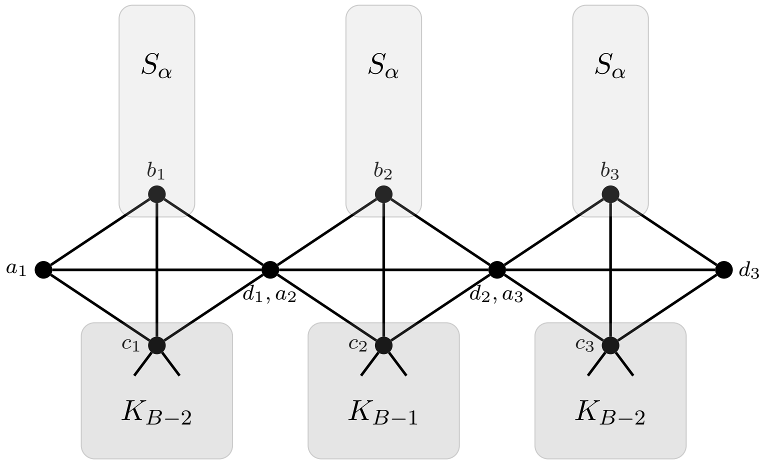

Figure 4: Illustration of the reduced graph constructed in the proof of Theorem 4.9, for $m = 3$.

copies S' and S'' of S_{i-1}, identifying the root of S' and u_1, and similarly identifying the root of S'' and u_2. The root of S_i is r.

For every $i \geq 1$, let n_i denote the number of vertices of S_i. So $n_1 = 5$, and, for every $i \geq 2$, we have $n_i = 2n_{i-1} + 1$. More precisely, we have $n_i = 5 \times 2^{i-1} + 2^{i-1} - 1$. To every member $S_i \in \mathcal{S}$, we associate a set I_i of integers defined as follows:

- $I_1 = \{3\}$, and
- $I_i = \{n_{i-1} + 1, \dots, 2n_{i-1} - 1\}$ of integers, for every $i \geq 2$.

Note that $|I_1| < |I_2| < \dots$. Furthermore, every S_i has the following property regarding I_i:

Observation 4.8. *Let $i \geq 1$ be fixed. For every $\alpha \in I_i$, the graph S_i has no subset $V_\alpha \subset V(S_i)$ such that $S_i[V_\alpha]$ is a connected α-graph and $S_i - V_\alpha$ is a connected graph containing the root of S_i.*

Proof. Assume such a part V_α exists. Let r denote the root of S_i. Note that every non-leaf vertex of S_i different from r is a cut-vertex. Under all assumptions, this yields that, by the value of α, necessarily the two neighbours r' and r'' of r belong to V_α. Since $r'r''$ is a cut-edge of S_i, this means that V_α has to cover all vertices different from r; but this is not possible due to the value of α. This is a contradiction. □

We are now ready to prove the NP-hardness of Realization for claw-free graphs.

Theorem 4.9. Realization *is NP-hard when restricted to connected claw-free graphs.*

Proof. We follow the lines of the proofs of Theorems 2.3, 2.4 and 2.5. Let $<A, B, s>$ be an instance of 3-Partition, where we use the same terminology as in these proofs. We may assume that $s(a_1) \leq \dots \leq s(a_{3m})$. Free to modify this instance following Observation 2.1, we can assume that there is an α such that $s(a_1), \dots, s(a_{3m}) \in I_\alpha$.

We construct G as follows (see Figure 4 for an illustration). We add m disjoint copies of K_4 to the graph, where the vertices of the ith copy are denoted by a_i, b_i, c_i, d_i. For every $i = 1, \dots, m-1$, we then identify the vertices d_i and a_{i+1}, so that the K_4's form a kind of path connected via cut-vertices. For every $i = 1, \dots, m-1$, we then add a copy of S_α to the graph, and we identify its root with b_i. Finally, we consider every $i = 1, \dots, m$, and:

- for $i = 1$ or $i = m$, we add a complete graph K_{B-2} to the graph, and we identify one of its vertices and c_i;
- for $i \in \{2, \dots, m-1\}$, we add a complete graph K_{B-1} to the graph, and we identify one of its vertices and c_i.

Note that G is claw-free (it is actually a line graph). The $|V(G)|$-sequence π we consider for the reduction is $\pi = (mn_\alpha + m - 1, s(a_1), \dots, s(a_{3m}))$.

Recall that $n_\alpha > s(a_{3m})$, and that α was chosen so that $s(a_1), \dots, s(a_{3m}) \in I_\alpha$. For this reason, by Observation 4.8, in any realization of π in G, the part V_1 with size $mn_\alpha + m - 1$ has to contain the vertices of all S_α's added to G, and, because $G[V_1]$ must be connected, also all vertices $d_1, \dots, d_{m-1}$. Then $G - V_1$ is a disjoint union of traceable B-graphs, and we have to find a realization of $(s(a_1), \dots, s(a_{3m}))$ in it. This is equivalent to finding a solution to $<A, B, s>$. $\square$

5. Conclusion

In this work, we have first considered the algorithmic complexity of the Realization and AP problems. On the one hand, we have mainly established, along all sections, the NP-hardness of Realization for more classes of graphs with various structure. On the other hand, we have provided, in Section 3, new kernels of sequences showing that the AP problem is in NP for a few more classes of graphs. However, we are still far from a proof that 1) every graph has a polynomial kernel of sequences (which would establish the full NPness of AP), and that 2) the AP problem is complete for some complexity class (NP or Π_2^p being candidate classes). More efforts should thus be dedicated to these points.

One particular appealing case is the one of *cographs* ($\{P_4\}$-free graphs), which was mentioned in [10] by Broersma, Kratsch and Woeginger. It can easily be noted that the reduction in our proof of Theorem 3.3 yields cographs, so Realization is NP-hard for these graphs. It is still open, though, whether there is a polynomial kernel of sequences for cographs. Note that Theorem 3.1 makes a step in that direction, as 1-sequential graphs (threshold graphs) form a subclass of cographs.

The second line of research we have considered in this work is the weakening, to APness, of well-known sufficient conditions for Hamiltonicity (or traceability). It would be interesting if there were such a weakening for every condition for Hamiltonicity, as it would emphasize the relationship between Hamiltonicity and APness. However, previous investigations and some of our results seem to indicate that this connection is not as tight as one could expect.

We believe, however, that it would be nice dedicating more attention to this direction; let us thus raise an open question which might be interesting. As mentioned in the introductory section, Ore's well-known condition for Hamiltonicity can be weakened to APness. In particular, all n-graphs G with $\sigma_2(G) \geq n - 2$ having a (quasi-) perfect matching are AP. This result implies one direction of upcoming Question 5.1, which, if true, would stand as a result *à la* Bondy-Chvátal.

Namely, for a graph G, the *k-closure* of G is the (unique) graph obtained by repeatedly adding an edge between two non-adjacent vertices with degree sum at least k. A celebrated result of Bondy and Chvátal states that an n-graph is Hamiltonian if and only if its n-closure is Hamiltonian [8]. Analogously, an n-graph is traceable if and only if its $(n-1)$-closure is traceable. However, it is not true that every n-graph is AP if and only if its $(n-2)$-closure is AP: In the complete bipartite graph $K_{n/2-1,n/2+1}$, every two non-adjacent vertices have degree sum at least $n-2$, so its $(n-2)$-closure is complete and thus AP; however, note that $K_{n/2-1,n/2+1}$ has no perfect matching (realization of $(2,\dots,2)$) and is thus no AP. This is actually not the only exception. Indeed, let G be a graph of order $n = 4k+2$ consisting of two complete graphs $K_{n/2}$ with a common vertex and a pendant edge attached to this vertex. Then the $(n-2)$-closure of this graph is complete, but G has not realization of $(n/2, n/2)$. We wonder whether there are many such exceptions, in the following sense:

Question 5.1. *Is there an "easy" class of graphs $\mathcal{G}$, such that if $G \notin \mathcal{G}$ is an n-graph, then G is AP if and only if the $(n-2)$-closure of G is AP?*

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References

[1] D. Barth, O. Baudon, J. Puech. Decomposable trees: a polynomial algorithm for tripodes. *Discrete Applied Mathematics*, 119:205-216, 2002.

[2] D. Barth, H. Fournier. A degree bound on decomposable trees. *Discrete Mathematics*, 306(5):469-477, 2006.

[3] O. Baudon, J. Bensmail, F. Foucaud, M. Pilśniak. Structural properties of recursively partitionable graphs with connectivity 2. *Discussiones Mathematicae Graph Theory*, 37:89-115, 2017.

[4] O. Baudon, F. Gilbert, M. Woźniak. Recursively arbitrarily vertex-decomposable graphs. *Opuscula Mathematica*, 32(4):689-706, 2012.

[5] J. Bensmail. *Partitions and decompositions of graphs*. Ph.D. thesis, Université de Bordeaux, France, 2014.

[6] J. Bensmail. On the complexity of partitioning a graph into a few connected subgraphs. *Journal of Combinatorial Optimization*, 30:174–187, 2015.

[7] J. Bensmail. On three polynomial kernels of sequences for arbitrarily partitionable graphs. *Discrete Applied Mathematics*, 202:19-29, 2016.

[8] J.A. Bondy, V. Chvátal. A method in graph theory. *Discrete Mathematics*, 15(2):111-135, 1976.

[9] S. Brandt. Finding vertex decompositions in dense graphs. In *Proceedings 15th Workshop on Graph Theory - Colourings, Independence and Dominations*, 2013.

[10] H. Broersma, D. Kratsch, G.J. Woeginger. Fully decomposable split graphs. *European Journal of Combinatorics*, 34(3):567-575, 2013.

[11] M. Dell'Amico, S. Martello. Reduction of the Three-Partition Problem. *Journal of Combinatorial Optimization*, 3(1):17-30, 1999.

[12] M.E. Dyer, A.M. Frieze. On the complexity of partitioning graphs into connected subgraphs, *Discrete Applied Mathematics*, 10:139–153, 1985.

[13] J. Edmonds. Paths, trees, and flowers. *Canadian Journal of Mathematics*, 17:449–467, 1965.

[14] R. Faudree, E. Flandrin, Z. Ryjáček. Claw-free graphs – A survey. *Discrete Mathematics*, 164(1-3):87-147, 1997.

[15] H. Fleischner. In the square of graphs, Hamiltonicity and pancyclicity, Hamiltonian connectedness and panconnectedness are equivalent concepts. *Monatshefte für Mathematik*, 82(2):125-149, 1976.

[16] M.R. Garey, D.S. Johnson. *Computers and Intractability; A Guide to the Theory of NP-Completeness*. W.H. Freeman & Co., 1990.

[17] E. Győri. On division of graphs to connected subgraphs. In *Proceedings 5th Hungarian Combinational Colloquium*, 485–494, 1978.

[18] M. Horňák, A. Marczyk, I. Schiermeyer, M. Woźniak. Dense Arbitrarily Vertex Decomposable Graphs. *Graphs and Combinatorics*, 28:807-821, 2012.

[19] M. Horňák, Zs. Tuza, M. Woźniak. On-line arbitrarily vertex decomposable trees. *Discrete Applied Mathematics*, 155:1420-1429, 2007.

[20] R. Kalinowski, M. Pilśniak, I. Schiermeyer, M. Woźniak. Dense arbitrarily partitionable graphs. *Discussiones Mathematicae Graph Theory*, 36(1):5-22, 2016.

[21] L. Lovász. A homology theory for spanning trees of a graph. *Acta Mathematica Academiae Scientiarum Hungaricae*, 30(3-4):241–251, 1977.

[22] A. Marczyk. An Ore-type condition for arbitrarily vertex decomposable graphs. *Discrete Mathematics*, 309:3588–3594, 2009.

[23] O. Ore. Note on hamilton circuits. *American Mathematical Monthly*, 67:55, 1960.

[24] R. Ravaux. *Graphes arbitrairement partitionnables: propriétés structurelles et algorithmiques.* Ph.D. thesis, Université de Versailles Saint-Quentin, France, 2009. 2009.