Full Waveform Inversion Adjoint Studies MATHIAS 2018
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Full Waveform Inversion Adjoint Studies

MATHIAS 2018

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Outline

The acoustic model
   The continuous model
   The discretized problem
   Assets of Bernstein polynomials

Adjoint Studies
   FWI Introduction
   Adjoint then Discretized
   Discretize then Adjoint

Some Results
   Consistency of the Adjoint Solution
   FWI Preliminary test
   Qualitative Cost Function Gradient Study
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The continuous model

Continuous Problem:

\[
\begin{align*}
\frac{1}{\rho_0 v_p^2} \frac{\partial p}{\partial t} + \frac{\partial v}{\partial x} &= f_p \quad \text{in } \Omega \\
\rho_0 \frac{\partial v}{\partial t} + \frac{\partial p}{\partial x} &= f_v \quad \text{in } \Omega \\
\rho_0 \frac{\partial p}{\partial t} + v_p \frac{\partial p}{\partial x} \cdot n &= 0 \quad \text{on } \Gamma \\
p(t = 0) &= 0 \\
v(t = 0) &= 0
\end{align*}
\]

Figure: 1D Domain Model
The discretized model

Discretized Problem:

\[
\begin{align*}
\frac{\partial \bar{P}}{\partial t} &= A_{pv} \bar{V} + A_{pp} \bar{P} + \bar{F}_p \\
\frac{\partial \bar{V}}{\partial t} &= A_{vp} \bar{P}
\end{align*}
\]

- Discontinuous Galerkin space discretization
- Different time-schemes (RK4, AB3)
- Two polynomial basis (Lagrange and Bernstein)
- Constant velocity ($v_p$) per cells
- Constant density ($\rho_0$) per cells

Figure: 1D Discretized Domain
Bernstein formulation:

\[ B_{ijkl}^N = C_{ijkl}^N \lambda_0^i \lambda_1^j \lambda_2^k \lambda_3^l \quad \text{with:} \quad C_{ijkl}^N = \frac{N!}{i!j!k!l!} \]
Bernstein/DG Properties

Bernstein formulation:

\[ B_{ijkl}^N = C_{ijkl}^N \lambda_0^i \lambda_1^j \lambda_2^k \lambda_3^l \quad \text{with:} \quad C_{ijkl}^N = \frac{N!}{i!j!k!l!} \]

Easy Derivative expression:

\[ \frac{\partial B_{\alpha}^N}{\partial \lambda_p} = NB_{\alpha - e_p}^{N-1} \quad \text{with:} \quad \alpha = (i, j, k, l) \]

\[ P[X^5] \text{ Bernstein basis} \]
Bernstein/DG Properties

Bernstein formulation:

\[ B_{ijkl}^N = C_{ijkl}^N \lambda_0^i \lambda_1^j \lambda_2^k \lambda_3^l \quad \text{with:} \quad C_{ijkl}^N = \frac{N!}{i!j!k!l!} \]

Easy Derivative expression:

\[ \frac{\partial B_{\alpha}^N}{\partial \lambda_p} = NB_{\alpha-e_p}^{N-1} \quad \text{with:} \quad \alpha = (i, j, k, l) \]

Sparse Degree Elevation operator:

\[ B_{\alpha}^{N-1} = \sum_{p=0}^{d} \frac{\alpha_p + 1}{N} B_{\alpha+e_p}^N \]
Bernstein/DG Properties

Unique boundary condition values:

P($X^5$) Lagrange basis

P($X^5$) Bernstein basis

⇒ Same Flux Management
Derivative-Operator Analysis

3D Lagrange D matrix

3D Bernstein D matrix

[1] Chan J. and Warburton T.
GPU-Accelerated Bernstein Bézier Discontinuous Galerkin Methods for Wave Problems
SIAM Journal on Scientific Computing 2017
Figure: Operators NZVs as a function of the order
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FWI Introduction

$v_p \ ?$

$\rho_0 \ ?$

Source

Receiver

Target

Data

Simulated Data

$J(p) = \frac{1}{2} \left| R_p - \text{data} \right|^2$

Find $v_p$ and $\rho_0$ to minimize $J$
FWI Introduction

\[ J(p) = \frac{1}{2} ||R p - data||^2 \]

Find \( v_p \) and \( \rho_0 \) to minimize \( J \)
Continuous Direct Problem
Adjoint Studies

Continuous Direct Problem \[ (*) \] Continuous Adjoint* Problem

Full Waveform Inversion Adjoint Studies
Adjoint Studies

Continuous Direct Problem

Continuous Adjoint* Problem

Discretization

Discretization of the Continuous Adjoint* Problem
Adjoint Studies

Continuous Direct Problem

(*)

Continuous Adjoint* Problem

Discretization

Discrete Direct Problem

(*)

Discretization of the Continuous Adjoint* Problem

(*)

Adjoint* of the Discrete Problem
Adjoint Studies

- Continuous Direct Problem

Discretization

- Discrete Direct Problem

Continuous Adjoint* Problem

Discretization

- Discretization of the Continuous Adjoint* Problem

Discretization

- Adjoint* of the Discrete Problem

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AtD : Adjoint then Discretized Strategy

\[ J(p) = \frac{1}{2} \|Rp - data\|^2 \]

\[
\begin{aligned}
\frac{1}{\rho_0 v_p^2} \frac{\partial p}{\partial t} + \nabla.v &= f_p \\
\rho_0 \frac{\partial v}{\partial t} + \nabla p &= 0 \\
p(t = 0) &= 0 \\
v(t = 0) &= 0 \\
\frac{\partial p}{\partial t} + v_p \nabla p.n &= 0 \quad \text{on } \Gamma
\end{aligned}
\]

\[
\begin{aligned}
\frac{1}{\rho_0 v_p^2} \frac{\partial \lambda_1}{\partial t} + \nabla.\lambda_2 &= \frac{\partial J}{\partial p} \\
\rho_0 \frac{\partial \lambda_2}{\partial t} + \nabla \lambda_1 &= 0 \\
\lambda_1(t = T) &= 0 \\
\lambda_2(t = T) &= 0 \\
\frac{\partial \lambda_1}{\partial t} + v_p \nabla \lambda_1.n &= 0 \quad \text{on } \Gamma
\end{aligned}
\]
AtD : Adjoint then Discretized Strategy

\[ J(\tilde{P}) = \frac{1}{2} \| R\tilde{P} - data \|^2 \]

\[
\begin{align*}
\frac{\partial \tilde{P}^n}{\partial t} &= A_{pv} \tilde{V}^n + A_{pp} \tilde{P}^n + \tilde{F}_p \\
\frac{\partial \tilde{V}^n}{\partial t} &= A_{vp} \tilde{P}^n
\end{align*}
\]

\[
\begin{align*}
\frac{\partial \tilde{\Lambda}_1^n}{\partial t} &= +A_{pv} \tilde{\Lambda}_2^n + A_{pp} \tilde{\Lambda}_1^n + \tilde{D}_p \\
\frac{\partial \tilde{\Lambda}_2^n}{\partial t} &= A_{vp} \tilde{\Lambda}_1^n
\end{align*}
\]

Time-steps going Forward

Time-steps going Backward
$\frac{\partial \bar{U}^n}{\partial t} = A\bar{U}^n + \bar{F}^n$ \quad \text{With:} \quad \bar{U} = \begin{pmatrix} \bar{P} \\ \bar{V} \end{pmatrix}, \quad A = \begin{pmatrix} A_{pp} & A_p \\ A_v & 0 \end{pmatrix}, \quad \bar{F} = \begin{pmatrix} \bar{F}_p \\ 0 \end{pmatrix}$

All time scheme can be summed-up such as:

$L\bar{U} = EF$

We are looking for a Discrete Adjoint state satisfying:

$L^*\bar{\Lambda} = -R^*(R\bar{U} - \text{data})$
DTA: Discretize then Adjoint Strategy
Example with RK4

RK4 time-scheme leads to:

\[
\bar{U}^{n+1} = B\bar{U}^n + C_0\bar{F}^n + C_{1/2}\bar{F}^{n+1/2} + C_1\bar{F}^{n+1}
\]

\[
L\bar{U} = E\bar{F} = \bar{G}
\]

\[
\begin{pmatrix}
I & -B & I & & & & \\
-B & I & & & & & \\
& -B & I & & & & \\
& & \ddots & \ddots & & & \\
& & & -B & I & & \\
& & & & & \ddots & \ddots & \\
& & & & & & -B & I
\end{pmatrix}
\begin{pmatrix}
\bar{U}^0 \\
\bar{U}^1 \\
\bar{U}^2 \\
\vdots \\
\bar{U}^n
\end{pmatrix}
= 
\begin{pmatrix}
\bar{G}^0 \\
\bar{G}^1 \\
\bar{G}^2 \\
\vdots \\
\bar{G}^n
\end{pmatrix}
\]

So:

\[
L^* = 
\begin{pmatrix}
I & -B^* & & & & & \\
I & -B^* & & & & & \\
& I & -B^* & & & & \\
& & \ddots & \ddots & & & \\
& & & I & -B^* & & \\
& & & & \ddots & \ddots & \\
& & & & & I & -B^*
\end{pmatrix}
\]
Adjoint test

\[ \langle L\bar{U}, \bar{\Lambda} \rangle = \langle \bar{U}, L^*\bar{\Lambda} \rangle \]
Adjoint test

\[
\langle L \bar{U}, \bar{\Lambda} \rangle = \langle \bar{U}, L^* \bar{\Lambda} \rangle
\]

\[
\begin{align*}
L \bar{U} &= EF = \bar{G} \\
\bar{U}(t = 0) &= 0
\end{align*}
\]

\[
\begin{align*}
L^* \bar{\Lambda} &= -R^* (R \bar{U} - \text{data}) = \bar{D} \\
\bar{\Lambda}(t = T) &= 0
\end{align*}
\]

Time-steps going Forward

Time-steps going Backward
Adjoint test

\[ \langle L\bar{U}, \bar{\Lambda} \rangle = \langle \bar{U}, L^* \bar{\Lambda} \rangle \]

\[
\begin{align*}
L\bar{U} &= EF = \tilde{G} \\
\bar{U}(t = 0) &= 0
\end{align*}
\]

\[
\begin{align*}
L^* \bar{\Lambda} &= -R^*(R\bar{U} - \text{data}) = \tilde{D} \\
\bar{\Lambda}(t = T) &= 0
\end{align*}
\]

Time-steps going Forward

Time-steps going Backward

\[ \langle EF, \bar{\Lambda} \rangle = \langle \bar{U}, -R^*(R\bar{U} - \text{data}) \rangle \]
Adjoint test

\[ \langle L\tilde{U}, \tilde{\Lambda} \rangle = \langle \tilde{U}, L^*\tilde{\Lambda} \rangle \]

\[
\begin{align*}
L\tilde{U} &= EF = \tilde{G} \\
\tilde{U}(t = 0) &= 0
\end{align*}
\]

\[
\begin{align*}
L^*\tilde{\Lambda} &= -R^*(R\tilde{U} - \text{data}) = \tilde{D} \\
\tilde{\Lambda}(t = T) &= 0
\end{align*}
\]

Time-steps going Forward

Time-steps going Backward

\[ \langle E\tilde{F}, \tilde{\Lambda} \rangle = \langle \tilde{U}, -R^*(R\tilde{U} - \text{data}) \rangle \]

\[ \langle \tilde{G}, \tilde{\Lambda} \rangle = \langle \tilde{U}, \tilde{D} \rangle \]

Adjoint test succeeds \iff operator \( L^* \) well established
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Adjoint test

Adjoint test passed for:

- Lagrange Operators
- Bernstein Operators
- Runge Kutta 4 time-scheme
- Adams Bashforth 3 time-scheme

- With a canonical space inner-product
  \[ \langle u, v \rangle_X = \sum_i u_i v_i \]

- With a M-space inner product
  \[ \langle u, v \rangle^M_X = \langle Mu, v \rangle_X \]
Adjoint test

Adjoint test passed for:

- Lagrange Operators
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- Runge Kutta 4 time-scheme
- Adams Bashforth 3 time-scheme
- With a canonical space inner-product
  \[ \langle u, v \rangle_X = \sum_i u_i v_i \]
- With a M-space inner product
  \[ \langle u, v \rangle^M_X = \langle Mu, v \rangle_X \]

```
./run
--- Adjoint test ----
inner product UP/DUDP 553123.57586755091
inner product GPGU/QPQU 553123.57586756046
./run
--- Adjoint test ----
inner product UP/DUDP -75077.332007383695
inner product GPGU/QPQU -75077.332007386358
./run
--- Adjoint test ----
inner product UP/DUDP 125669.89223600870
inner product GPGU/QPQU 125669.89223600952
./run
--- Adjoint test ----
inner product UP/DUDP -132852.64215701097
inner product GPGU/QPQU -132852.64215701059
```
Non consistency of the Adjoint solution

With the AtD strategy

With the DtA strategy using the canonical inner-product (Lagrange+RK4)

Adjoint test succeeds!

[1] Sei Alain and Symes William
A Note on Consistency and Adjointness for Numerical Schemes
1997
FWI Preliminary test (for all strategies)

Target model

Initial model (iter=0)
FWI Preliminary test (for all strategies)

Target model

Initial model (iter=0)

Intermediate model (iter=20)

Final model (iter=50)
Figure: Comparison between a Reference Gradient and the FWI Gradient with AtD strategy (Bernstein elements and RK4 time scheme)
Figure: Comparison between a Reference Gradient and the FWI Gradient with DtA strategy (Bernstein elements and RK4 time scheme)
Figure: Comparison between a Reference Gradient and the FWI Gradient with AtD strategy (Lagrange elements and RK4 time scheme)
Figure: Comparison between a Reference Gradient and the FWI Gradient with DtA strategy (Lagrange elements and RK4 time scheme)
Conclusion and Perspectives

Conclusion:
- Adjoint then Discretized strategy works
- Discretized then Adjoint strategy has unexpected results (Gradient formulation? Bug?)
- The adjoint state is not consistent by using the Discretized and Adjoint strategy (but Adjoint test succeeds)

Perspectives:
- Complementary 1D tests
- 2D FWI + tests
- 3D FWI + tests
- Coupling SEM/DG elements (Aurélien Citrain's thesis)
Conclusion:
- Adjoint then Discretized strategy works
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Perspectives:
- Complementary 1D tests
- 2D FWI + tests
- 3D FWI + tests
- Coupling SEM/DG elements (Aurélien Citrain’s thesis)