On the coupling of Spectral Element Method with Discontinuous Galerkin approximation for elasto-acoustic problems
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Hélène Barucq\textsuperscript{1}, Henri Calandra\textsuperscript{2}, Aurélien Citraine\textsuperscript{3,1}, Julien Diaz\textsuperscript{1} and Christian Gout\textsuperscript{3}

\textsuperscript{1} Team project Magique.3D, INRIA.UPPA-CNRS, Pau, France.
\textsuperscript{2} TOTAL SA, CSTJF, Pau, France.
\textsuperscript{3} INSA Rouen-Normandie Université, LMI EA 3226, 76000, Rouen.

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Why using hybrid meshes?

- Useful when the use of unstructured grid is non-sense (e.g. medium with a layer of water)
- Well suited for the coupling of numerical methods in order to reduce the computational cost and improve the accuracy
Elastodynamic system

\begin{equation}
\begin{cases}
\rho(x) \frac{\partial \mathbf{v}}{\partial t}(x, t) = \nabla \cdot \mathbf{\sigma}(x, t)
\\
\frac{\partial \mathbf{\sigma}}{\partial t}(x, t) = \mathbf{C}(x) \varepsilon(\mathbf{v}(x, t))
\end{cases}
\end{equation}

With:
- \( \rho(x) \) the density
- \( \mathbf{C}(x) \) the elasticity tensor
- \( \varepsilon(x, t) \) the deformation tensor
- \( \mathbf{v}(x, t) \), the wavespeed
- \( \mathbf{\sigma}(x, t) \) the strain tensor
Elasticus software

Written in **Fortran** for wave propagation simulation in the **time domain**

**Features**

- Using various types of meshes (**unstructured triangle and tetrahedra**)
- Modelling of various physics (**acoustic, elastic and elasto-acoustic**)

**Discontinuous Galerkin Method (DG)** based on **unstructured triangle and unstructured tetrahedra**
- with various time-schemes: **Runge-Kutta (2 or 4), Leap-Frog**
- with **p-adaptivity, multi-order** computation...
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2. Comparison DG/SEM on structured quadrangle mesh

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4. Comparison between DG/SEM and DG on hybrid meshes

5. 3D extension
1 Numerical Methods

- Discontinuous Galerkin Method (DG)
- Spectral Element Method (SEM)
- Advantages of each method
Use discontinuous functions:

- mesh
- continuous
- discontinuous

Degrees of freedom on each cell:

- P1
- P2
- P3

h adaptivity:

p adaptivity:
Spectral Element Method

General principle

- Finite Element Method (FEM) discretization + Gauss-Lobatto quadrature
- Gauss-Lobatto points as degrees of freedom (exponential convergence on $L^2$-norm)

\[ \int f(x) dx \approx \sum_{j=1}^{N+1} \omega_j f(\xi_j) \]

\[ \varphi_i(\xi_j) = \delta_{ij} \]
Spectral Element Method

Main change with DG

- DG discontinuous, SEM continuous
- Need of defining local to global numbering
- Global matrices required by SEM
- Basis functions computed differently
Advantages of each method

**DG**
- Element per element computation (\( hp \)-adaptivity)
- Time discretization quasi explicit (block diagonal mass matrix)
- Simple to parallelize

**SEM**
- Couples the flexibility of FEM with the accuracy of the pseudo-spectral method
- Reduces the computational cost when using structured meshes in comparison with DG
2 Comparison DG/SEM on structured quadrangle mesh
- DG/SEM comparison on quadrangle mesh
- Description of the test cases
- Comparative tables
Description of the test cases

Physical parameters

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>$P$ wavespeed</td>
<td>1000 m.s$^{-1}$</td>
</tr>
<tr>
<td>Density</td>
<td>1 kg.m$^{-3}$</td>
</tr>
</tbody>
</table>

Second order **Ricker Source** in $P$wave ($f_{peak} = 10$Hz)

General context

- **Acoustic homogeneous** medium
- Four different meshes: 10000 cells, 22500 cells, 90000 cells, 250000 cells
- CFL computed using **power iteration** method
- **Leap-Frog** time scheme
- **Four threads** parallel execution with **OpenMP**
## Comparative tables

Error computed as the difference between an analytical and a numerical solution for each method.

### Quadrangle mesh 10000 elements:

<table>
<thead>
<tr>
<th></th>
<th>CFL</th>
<th>L2-error</th>
<th>CPU-time</th>
<th>Nb of time steps</th>
</tr>
</thead>
<tbody>
<tr>
<td>DG</td>
<td>1.99e-3</td>
<td>2.5e-2</td>
<td>19.30</td>
<td>500</td>
</tr>
<tr>
<td>SEM</td>
<td>4.9e-3</td>
<td>1.3e-1</td>
<td>0.36</td>
<td>204</td>
</tr>
</tbody>
</table>

### Quadrangle mesh 22500 elements:

<table>
<thead>
<tr>
<th></th>
<th>CFL</th>
<th>L2-error</th>
<th>CPU-time</th>
<th>Nb of time steps</th>
</tr>
</thead>
<tbody>
<tr>
<td>DG</td>
<td>1.33e-3</td>
<td>1.8e-2</td>
<td>100.48</td>
<td>750</td>
</tr>
<tr>
<td>SEM</td>
<td>3.26e-3</td>
<td>7e-2</td>
<td>1.19</td>
<td>306</td>
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</table>
Comparative tables

- Error computed as the difference between an analytical and a numerical solution for each method

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<td>1.3e-1</td>
<td>0.36</td>
<td>204</td>
</tr>
<tr>
<td>SEM(DG CFL)</td>
<td>1.99e-3</td>
<td>4.8e-2</td>
<td>0.73</td>
<td>502</td>
</tr>
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<td>306</td>
</tr>
<tr>
<td>SEM(DG CFL)</td>
<td>1.33e-3</td>
<td>1.2e-2</td>
<td>2.82</td>
<td>751</td>
</tr>
</tbody>
</table>

SEM fifty time faster than DG on a mesh with 22500 cells
3 DG/SEM coupling
- Hybrid meshes structures
- Variational formulation
- Space discretization
Aim at coupling $P_k$ and $Q_k$ structures.

Need to extend or split some of the structures (e.g. neighbour indexes)

Define new face matrices

\[
M_{ij}^{K,L} = \int_{K \cap L} \phi^K_i \phi^L_j, \quad M_{ij}^{K,L} = \int_{K \cap L} \psi^K_i \psi^L_j, \quad M_{ij}^{K,L} = \int_{K \cap L} \phi^K_i \psi^L_j
\]
Global context

- Domain in two parts: $\Omega_{h,1}$ (structured quadrangle + SEM), $\Omega_{h,2}$ (unstructured triangle + DG)
Variational formulation

SEM variational formulation:

\[
\begin{aligned}
\int_{\Omega_{h,1}} \rho \partial_t v_1 \cdot w_1 &= - \int_{\Omega_{h,1}} \sigma_1 \cdot \nabla w_1 + \int_{\Gamma_{out,1}} (\sigma_1 n_1) \cdot w_1 \\
\int_{\Omega_{h,1}} \partial_t \sigma_1 : \xi_1 &= - \int_{\Omega_{h,1}} (\nabla(C\xi_1)) \cdot v_1 + \int_{\Gamma_{out,1}} (C\xi_1 n_1) \cdot v_1
\end{aligned}
\]

DG variational formulation:

\[
\begin{aligned}
\int_{\Omega_{h,2}} \rho \partial_t v_2 \cdot w_2 &= - \int_{\Omega_{h,2}} \sigma_2 \cdot \nabla w_2 + \int_{\Gamma_{out,2}} (\sigma_2 n_2) \cdot w_2 + \int_{\Gamma_{int}} \{\{\sigma_2\}\}[w_2] \cdot n_2 \\
\int_{\Omega_{h,2}} \partial_t \sigma_2 : \xi_2 &= - \int_{\Omega_{h,2}} (\nabla(C\xi_2)) \cdot v_2 + \int_{\Gamma_{out,2}} (C\xi_2 n_2) \cdot v_2 + \int_{\Gamma_{int}} \{\{v_2\}\}[C\xi_2] \cdot n_2
\end{aligned}
\]
Add the average of the solution of each part at the interface + put \( \sigma_\ast n_\ast = 0 \)

\[
\begin{align*}
\int_{\Omega_{h,1}} \rho \partial_t v_1 \cdot w_1 &= - \int_{\Omega_{h,1}} \sigma_1 \cdot \nabla w_1 + \frac{1}{2} \int_{\Gamma_{1/2}} (\sigma_1 + \sigma_2) n_1 \cdot w_1 \\
\int_{\Omega_{h,1}} \partial_t \sigma_1 : \xi_1 &= - \int_{\Omega_{h,1}} (\nabla (C \xi_1)) \cdot v_1 + \frac{1}{2} \int_{\Gamma_{1/2}} (C \xi_1 n_1) \cdot (v_1 + v_2) \\
\int_{\Omega_{h,2}} \rho \partial_t v_2 \cdot w_2 &= - \int_{\Omega_{h,2}} \sigma_2 \cdot \nabla w_2 + \int_{\Gamma_{int}} \{\sigma_2\}[[w_2]] \cdot n_2 - \frac{1}{2} \int_{\Gamma_{1/2}} (\sigma_1 + \sigma_2) n_1 \cdot w_2 \\
\int_{\Omega_{h,2}} \partial_t \sigma_2 : \xi_2 &= - \int_{\Omega_{h,2}} (\nabla (C \xi_2)) \cdot v_2 + \int_{\Gamma_{int}} \{v_2\}[[C \xi_2]] \cdot n_2 \\
&- \frac{1}{2} \int_{\Gamma_{1/2}} (C \xi_2 n_1) \cdot (v_1 + v_2)
\end{align*}
\]
Energy study

Goal: Show that our coupling preserves the energy

- We set $\xi_1 = \sigma_1$, $\xi_2 = \sigma_2$, $w_1 = v_1$, $w_2 = v_2$
- We add the equations of the two parts variational formulation

$$\frac{d}{dt} E = 0$$
Space discretization: SEM part

- $\varphi_i$: SEM basis functions
- $\psi_i$: DG basis functions

\[ \begin{cases} 
M_{v_1} \frac{\partial}{\partial t} v_{h,1} + R_{\sigma_1} \sigma_{h,1} + R_{\sigma_2}^{2,1} \sigma_{h,2} = 0 \\
M_{\sigma_1} \frac{\partial}{\partial t} \sigma_{h,1} + R_{v_1} v_{h,1} + R_{v_2}^{2,1} v_{h,2} = 0 
\end{cases} \]

- $M_{ij} = \int_\Omega \varphi_i \varphi_j \approx \sum_{e \in \text{supp}(\varphi_i) \cap \text{supp}(\varphi_j)} \sum_{k=1}^{(r+1)^d} \omega_k \varphi_i(\xi_k) \varphi_j(\xi_k) = \sum_{e \in \text{supp}(\varphi_i) \cap \text{supp}(\varphi_j)} \omega_i \delta_{i,j}$ the mass matrix

- $R_{p_{ij}} = \int_\Omega \varphi_i \frac{\partial \varphi_j}{\partial p}$ stiffness matrix

Matrix of DG/SEM coupling:

\[ R_{\sigma_2,ij}^{2,1} = \frac{1}{2} \int_{\partial \Omega_1 \cap \partial \Omega_2} \psi_i \varphi_j \]
Space discretization : DG part

\[
\begin{align*}
\rho M_{v_2} \partial_t v_{h,2} + R_{\sigma_2} \sigma_{h,2} - R^{1,2}_{\sigma_1} \sigma_{h,1} &= 0 \\
M_{\sigma_2} \partial_t \sigma_{h,2} + R_{v_2} v_{h,2} - R^{1,2}_{v_1} v_{h,1} &= 0
\end{align*}
\]

- \( M^K_{ij} = \int_K \psi^K_i \psi^K_j \) \quad mass matrix,
- \( R^K_{p_{ij}} = \int_K \psi^K_i \frac{\partial \psi^K_j}{\partial p} \) \quad stiffness matrix,
- \( R^{K,L}_{p_{ij}} = \int_{\partial K \cap \partial L} \psi^K_i \psi^L_j n_K \cdot e_p \) \quad the mass-face matrix

Two new matrices which come from the DG/SEM coupling \( R^{1,2}_x \). Block composed :

\[
R^{1,2}_{v_1} = R^{1,2}_{\sigma_1} = -\frac{1}{2} \int_{\partial \Omega_2 \cap \partial K_1} \psi^K_j \varphi_i
\] (1)
Comparison between DG/SEM and DG on hybrid meshes

- Experimentation context
- Comparative tables
Context

- Acoustic homogeneous medium
- 54000 triangles
- 21000 quadrangles
- Using Leap-Frog time scheme
- Parallel computation using OpenMP
- Done with different orders of discretization
### Comparative tables

#### $P_1 - Q_3$ computation:

<table>
<thead>
<tr>
<th></th>
<th>CFL</th>
<th>$L_2$-error</th>
<th>CPU-time</th>
</tr>
</thead>
<tbody>
<tr>
<td>DG</td>
<td>1e-5</td>
<td>0.03</td>
<td>7343.92</td>
</tr>
<tr>
<td>DG/SEM</td>
<td>1e-5</td>
<td>0.03</td>
<td>823.22</td>
</tr>
</tbody>
</table>

#### $P_3 - Q_1$ computation:

<table>
<thead>
<tr>
<th></th>
<th>CFL</th>
<th>$L_2$-error</th>
<th>CPU-time</th>
</tr>
</thead>
<tbody>
<tr>
<td>DG</td>
<td>3e-5</td>
<td>0.009</td>
<td>3078.15</td>
</tr>
<tr>
<td>DG/SEM</td>
<td>3e-5</td>
<td>0.01</td>
<td>2951</td>
</tr>
</tbody>
</table>

#### $P_2 - Q_3$ computation:

<table>
<thead>
<tr>
<th></th>
<th>CFL</th>
<th>$L_2$-error</th>
<th>CPU-time</th>
</tr>
</thead>
<tbody>
<tr>
<td>DG</td>
<td>1e-5</td>
<td>0.002</td>
<td>9452.73</td>
</tr>
<tr>
<td>DG/SEM</td>
<td>1e-5</td>
<td>0.003</td>
<td>1393.80</td>
</tr>
</tbody>
</table>

#### $P_3 - Q_2$ computation:

<table>
<thead>
<tr>
<th></th>
<th>CFL</th>
<th>$L_2$-error</th>
<th>CPU-time</th>
</tr>
</thead>
<tbody>
<tr>
<td>DG</td>
<td>1e-5</td>
<td>5.4e-4</td>
<td>9951.60</td>
</tr>
<tr>
<td>DG/SEM</td>
<td>1e-5</td>
<td>0.007</td>
<td>3122</td>
</tr>
</tbody>
</table>
Only deal with a simple case of 3D hybrid meshes: one hexahedra has only two tetrahedra as neighbour

- Extend SEM in 3D (basis functions...)
- Require introducing a new matrix which handles the rotation cases between two elements
Conclusion and perspectives

Conclusion

1. Build a variational formulation for DG/SEM coupling and find a CFL condition that ensures stability
2. As expected, SEM is more efficient on structured quadrangle mesh than DG
3. Show the utility of using hybrid meshes and method coupling (reduce computational cost,...)

Perspectives

- Implement DG/SEM coupling on the code (2D) ✓
- Develop DG/SEM coupling in 3D ✓
- Add a local time-stepping scheme
- Develop PML in the hexahedral part
Thank you for your attention!

Questions?