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Sample covariances of random-coefficient AR(1) panel model

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Abstract

The present paper obtains a complete description of the limit distributions of sample covariances in $N \times n$ panel data when $N$ and $n$ jointly increase, possibly at different rate. The panel is formed by $N$ independent samples of length $n$ from random-coefficient AR(1) process with the tail distribution function of the random coefficient regularly varying at the unit root with exponent $\beta > 0$. We show that for $\beta \in (0, 2)$ the sample covariances may display a variety of stable and non-stable limit behaviors with stability parameter depending on $\beta$ and the mutual increase rate of $N$ and $n$.

Keywords: Autoregressive model; Panel data; Mixture distribution; Long memory; Sample covariance; Scaling transition; Poisson random measure; Asymptotic self-similarity.

2010 MSC: 60F05, 62M10.

1 Introduction

Dynamic panels providing information on a large population of heterogeneous individuals such as households, firms, etc. observed at regular time periods, are often described by simple autoregressive models with random parameters near unity. One of the simplest models for individual evolution is the random-coefficient AR(1) (RCAR(1)) process

\begin{equation}
X(t) = aX(t-1) + \varepsilon(t), \quad t \in \mathbb{Z},
\end{equation}

with standardized i.i.d. innovations $\{\varepsilon(t), t \in \mathbb{Z}\}$ and a random autoregressive coefficient $|a| < 1$ independent of $\{\varepsilon(t), t \in \mathbb{Z}\}$. Granger [8] observed that in the case when the distribution of $a$ is sufficiently dense near unity the stationary solution of RCAR(1) equation in (1.1) may have long memory in the sense that the sum of its lagged covariances diverges. To be more specific, assume that the random coefficient $a \in [0, 1)$ has a density function of the following form

\begin{equation}
\phi(x) = \psi(x)(1-x)^{\beta - 1}, \quad x \in [0, 1),
\end{equation}

where $\beta > 0$ and $\psi(x), x \in [0, 1)$ is a bounded function with $\lim_{x \uparrow 1} \psi(x) =: \psi(1) > 0$. Then for $\beta > 1$ the covariance function of stationary solution of RCAR(1) equation in (1.1) with standardized finite variance innovations decays as $t^{-(\beta - 1)}$, viz.,

\begin{equation}
\gamma(t) := \mathbb{E}(X(0)X(t)) = \frac{a^{|t|}}{1 - a^2} \sim (\psi(1)/2) \Gamma(\beta - 1)t^{-(\beta - 1)}, \quad t \to \infty,
\end{equation}

1
implying $\sum_{t \in \mathbb{Z}} \text{Cov}(X(0), X(t)) = \infty$ for $\beta \in (1, 2]$. The same long memory property applies to the contemporaneous aggregate of $N$ independent individual evolutions $\{X_i(t)\}, i = 1, \ldots, N$ of (1.1) and the limit Gaussian aggregated process arising when $N \to \infty$. Various properties of the RCAR(1) and more general RCAR equations were studied in Gonçalves and Gouriéroux [7], Zaffaroni [29], Celov et al. [3], Oppenheim and Viano [15], Puplinskaitė and Surgailis [22], Philippe et al. [16] and other works.

Statistical inference in the RCAR(1) model was discussed in several works. Leipus et al. [11], Celov et al. [4] discussed nonparametric estimation of the mixing density $\phi(x)$ using empirical covariances of the limit aggregated process. For panel RCAR(1) data, Robinson [26] and Beran et al. [1] discussed parametric estimation of the mixing density. In nonparametric context, Leipus et al. [12] studied estimation of the mixing density. In nonparametric context, Leipus et al. [13] discussed estimation of $\beta$ in (1.2) and testing for long memory in the above panel model. For a $N \times n$ panel comprising $N$ samples $\{X_i(t), t = 1, \ldots, n\}$ of length $n$, $i = 1, \ldots, N$ of independent RCAR(1) processes in (1.1) with mixing distribution in (1.2), Pilipauskaitė and Surgailis [17] studied the asymptotic distribution of the sample mean

$$\bar{X}_{N,n} := \frac{1}{Nn} \sum_{i=1}^{N} \sum_{t=1}^{n} X_i(t),$$

(1.4)
as $N, n \to \infty$, possibly at a different rate. [17] showed that for $0 < \beta < 2$ the limit distribution of this statistic depends on whether $N/n^\beta \to \infty$ or $N/n^\beta \to 0$ in which cases $\bar{X}_{N,n}$ is asymptotically stable with stability parameter depending on $\beta$ and taking values in the interval $(0, 2]$. See Table 2 below. As shown in [17], under the ‘intermediate’ scaling $N/n^\beta \to c \in (0, \infty)$ the limit distribution of $\bar{X}_{N,n}$ is more complicated and is given by a stochastic integral with respect to a certain Poisson random measure.

The present paper discusses asymptotic distribution of sample covariances (covariance estimates)

$$\tilde{\gamma}_{N,n}(t, s) := \frac{1}{Nn} \sum_{1 \leq i, i+s \leq N} \sum_{1 \leq k, k+t \leq n} (X_i(k) - \bar{X}_{N,n})(X_{i+s}(k+t) - \bar{X}_{N,n}), \quad (t, s) \in \mathbb{Z}^2,$$

(1.5)
computed from a similar RCAR(1) panel $\{X_i(t), t = 1, \ldots, n, i = 1, \ldots, N\}$ as in [17], as $N, n$ jointly increase, possibly at a different rate, and the lag $(t, s) \in \mathbb{Z}^2$ is fixed, albeit arbitrary. Particularly, for $(t, s) = (0, 0)$, (1.5) agrees with the sample variance:

$$\tilde{\sigma}^2_{N,n} := \frac{1}{Nn} \sum_{i=1}^{N} \sum_{k=1}^{n} (X_i(k) - \bar{X}_{N,n})^2.$$  

(1.6)
The true covariance function $\gamma(t, s) := \mathbb{E}X_i(k)X_{i+s}(k+t)$ of the RCAR(1) panel model with mixing density in (1.2) exists when $\beta > 1$ and is given by

$$\gamma(t, s) = \begin{cases} 
\gamma(t), & s = 0, \\
0, & s \neq 0,
\end{cases}$$

(1.7)
where $\gamma(t)$ defined in (1.3). Note that $\gamma(t)$ cannot be recovered from a single realization of the nonergodic RCAR(1) process $\{X(t)\}$ in (1.1). However, the covariance function in (1.7) can be consistently estimated from the RCAR(1) $N \times n$ panel when $N, n \to \infty$, together with rates. The limit distribution of the sample covariance may exist even for $0 < \beta < 1$ when the covariance itself is undefined. As it turns out, the limit distribution of $\tilde{\gamma}_{N,n}(t, s)$ depends on the mutual increase rate of $N$ and $n$, and is also different for temporal,
or iso-sectional lags \((s = 0)\) and cross-sectional lags \((s \neq 0)\). The distinctions between the cases \(s = 0\) and \(s \neq 0\) are due to the fact that, in the latter case, the statistics in (1.5) involves products \(X_i(k)X_{i+s}(k + t)\) of independent processes \(X_i\) and \(X_{i+s}\), whereas in the former case, \(X_i(k)\) and \(X_i(k + t)\) are dependent r.v.s.

The main results of this paper are summarized in Table 1 below. Rigorous formulations are given in Sec. 3 and 4. For better comparison, Table 2 presents the results of [17] about the sample mean in (1.4) for the same panel model.

<table>
<thead>
<tr>
<th>Mutual increase rate of (N, n)</th>
<th>Parameter region</th>
<th>Limit distribution</th>
</tr>
</thead>
<tbody>
<tr>
<td>(N/n^β \to ∞)</td>
<td>(0 &lt; β &lt; 2, β \neq 1)</td>
<td>asymmetric (β)-stable</td>
</tr>
<tr>
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<td>asymmetric (β)-stable</td>
</tr>
<tr>
<td>(N/n^β \to (0, ∞))</td>
<td>(0 &lt; β &lt; 2, β \neq 1)</td>
<td>‘intermediate Poisson’</td>
</tr>
<tr>
<td>Arbitrary</td>
<td>(β &gt; 2)</td>
<td>Gaussian</td>
</tr>
</tbody>
</table>

\(a)\) temporal lags \((s = 0)\)

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<tbody>
<tr>
<td>(N/n^{2β} \to ∞)</td>
<td>(1 &lt; β &lt; 3/2)</td>
<td>Gaussian</td>
</tr>
<tr>
<td></td>
<td>(1/2 &lt; β &lt; 1)</td>
<td>symmetric ((2β))-stable</td>
</tr>
<tr>
<td>(N/n^{2β} \to 0)</td>
<td>(3/4 &lt; β &lt; 3/2)</td>
<td>symmetric ((4β/3))-stable</td>
</tr>
<tr>
<td>(N/n^{2β} \to (0, ∞))</td>
<td>(3/4 &lt; β &lt; 3/2)</td>
<td>‘intermediate Poisson’</td>
</tr>
<tr>
<td>Arbitrary</td>
<td>(β &gt; 3/2)</td>
<td>Gaussian</td>
</tr>
</tbody>
</table>

\(b)\) cross-sectional lags \((s \neq 0)\)

Table 1: Limit distribution of sample covariances \(\tilde{γ}_{N,n}(t, s)\) in (1.5)

<table>
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</tr>
</tbody>
</table>

Table 2: Limit distribution of the sample mean \(\bar{X}_{N,n}\) in (1.4)

**Remark 1.1.** (i) \(β\)-stable limits in Table 1 a) arising when \(N/n^β \to 0\) and \(N/n^β \to ∞\) have different scale parameters and hence the limit distribution of temporal sample covariances is different in the two cases.

(ii) ‘Intermediate Poisson’ limits in Tables 1-2 refer to infinitely divisible distributions defined through certain stochastic integrals w.r.t. Poisson random measure. A similar terminology was used in [19].

(iii) It follows from our results (see Theorem 4.1 below) that a scaling transition similar as in the case of the sample mean [17] arises in the interval \(0 < β < 2\) for temporal sample covariances and product random
fields $X_u(u)X_v(u + t), (u, v) \in \mathbb{Z}^2$ involving temporal lags, with the critical rate $N \sim n^\beta$ separating regimes with different limit distributions. For ‘cross-sectional’ product fields $X_u(u)X_{v+s}(u + t), (u, v) \in \mathbb{Z}^2, s \neq 0$ involving cross-sectional lags, a similar scaling transition occurs in the interval $0 < \beta < 3/2$ with the critical rate $N \sim n^{2\beta}$ between different scaling regimes, see Theorem 3.1. The notion of scaling transition for long-range dependent random fields in $\mathbb{Z}^2$ was discussed in Puplinskaitė and Surgailis [23], [24], Pilipauskaite and Surgailis [19], [20].

(iv) The limit distributions of cross-sectional sample covariances in the missing intervals $0 < \beta < 1/2$ and $0 < \beta < 3/4$ of Table 1 b) are given in Corollary 3.1 below. They are more complicated and not included in Table 1 b) since the term $Nn(\bar{X}_{N,n})^2$ due to the centering by the sample mean in (1.5) may play the dominating role.

(v) We expect that the asymptotic distribution of sample covariances in the RCAR(1) panel model with common innovations (see [18]) can be analyzed in a similar fashion. Due to the differences between the two models (the common and the idiosyncratic innovation cases), the asymptotic behavior of sample covariances might be quite different in these two cases.

(vi) The results in Table 1 a) are obtained under the finite 4th moment conditions on the innovations, see Theorems 4.1 and 4.2 below. Although the last condition does not guarantee the existence of the 4th moment of the RCAR(1) process, it is crucial for the limit results, including the CLT in the case $\beta > 2$. Scaling transition for sample variances of long-range dependent Gaussian and linear random fields on $\mathbb{Z}^2$ with finite 4th moment was established in Pilipauskaitė and Surgailis [20]. On the other side, Surgailis [28], Horvath and Kokoszka [10] obtained stable limits of sample variances and autocovariances for long memory moving averages with finite 2nd moment and infinite 4th moment. Finally, we mention the important work Davis and Resnick [5] on limit theory for sample covariance and correlation functions of moving averages with infinite variance and short memory.

The rest of the paper is organized as follows. Sec. 2 presents some preliminary facts, including the definition and properties of the intermediate processes appearing in Table 1. Sec. 3 contains rigorous formulations and the proofs of the asymptotic results for cross-sectional sample covariances (1.5), $s \neq 0$ and the corresponding partial sums processes. Analogous results for temporal sample covariances and partial sums processes are presented in Sec. 4. The main application of these results is the consistency of the autocovariance estimator for RCAR(1) process and its convergence rates. Some auxiliary proofs are given in Appendix.

2 Preliminaries

This section contains some preliminary facts which will be used in the following sections.

2.1. Double stochastic integrals and quadratic forms. Let $B_i, i = 1, 2$ be independent standard Brownian motions on the real line. Let

$$I_i(f) := \int_{\mathbb{R}} f(s)dB_i(ds), \quad I_{ij}(g) := \int_{\mathbb{R}^2} g(s_1, s_2)dB_i(s_1)dB_j(s_2), \quad i, j = 1, 2$$

(2.1)
denote Itô-Wiener stochastic integrals (single and double) w.r.t. $B_i, B_j$. The integrals in (2.1) are jointly defined for any (non-random) integrands $f \in L^2(\mathbb{R}), g \in L^2(\mathbb{R}^2)$; moreover, $EI_i(f) = EI_{ij}(g) = 0$ and

In (2.2) we could add that $EI_{ij}(g)I_{i',j'}(g') = 0$ if $(i, j) \neq (i', j')$ and $(i, j) \neq (j', i')$. Furthermore, $EI_{ij}(g)I_{i',j'}(g') = \langle g, g' \rangle$ if $i \neq j$ and either $(i, j) = (i', j')$ or $(i, j) = (j', i')$. 

4
where $\langle f, f' \rangle = \int_{\mathbb{R}} f(s)f'(s)ds$, $\langle g, g' \rangle = \int_{\mathbb{R}^2} g(s_1, s_2)g'(s_1, s_2)ds_1ds_2$ denote scalar products (norms) in $L^2(\mathbb{R})$ and $L^2(\mathbb{R}^2)$, respectively, and $\text{sym}$ denotes the symmetrization, see, e.g., ([6] sec.11.5, 14.3). Note that for $g(s_1, s_2) = f_1(s_1)f_2(s_2)$, $f_i \in L^2(\mathbb{R})$, $i = 1, 2$ we have $I_{ii}(g) = I_i(f_i)I_i(f_2) - \langle f_1, f_2 \rangle$, $I_{12}(g) = I_i(f_1)I_j(f_2)$, in particular, $I_{12}(g) = \|f_1\|\|f_2\|Z_1Z_2$, where $Z_i \sim \mathcal{N}(0,1), i = 1, 2$ are independent standard normal r.v.s.

Let $\{\epsilon_i(s), s \in \mathbb{Z}\}, i = 1, 2$ be independent sequences of standardized i.i.d.r.v.s, $E\epsilon_i(s) = 0, E\epsilon_i(s)\epsilon_i(s') = 1$ if $(i, s) = (i', s')$, $E\epsilon_i(s)\epsilon_i(s') = 0$ if $(i, s) \neq (i', s')$, $i, i' = 1, 2$, $s, s' \in \mathbb{Z}$. Consider the centered quadratic form

$$Q_{ij}(h) = \sum_{s_1, s_2 \in \mathbb{Z}} h(s_1, s_2)[\epsilon_i(s_1)\epsilon_j(s_2) - E\epsilon_i(s_1)\epsilon_j(s_2)], \quad i, j = 1, 2,$$

where $h \in L^2(\mathbb{Z}^2)$. For $i = j$ we additionally assume $E\epsilon_i^4(0) < \infty$. Then the sum in (2.3) converges in $L^2$ and

$$\text{var}(Q_{ij}(h)) \leq (1 + E\epsilon_i^4(0))\sum_{s_1, s_2 \in \mathbb{Z}} h^2(s_1, s_2),$$

see ([6], 4.5.4)). With any $h \in L^2(\mathbb{Z}^2)$ and any $\alpha_1, \alpha_2 > 0$ we associate its extension to $L^2(\mathbb{R}^2)$, namely,

$$\tilde{h}^{(\alpha_1, \alpha_2)}(s_1, s_2) := (\alpha_1\alpha_2)^{1/2}h([\alpha_1s_1], [\alpha_2s_2]), \quad (s_1, s_2) \in \mathbb{R}^2,$$

with $\|\tilde{h}^{(\alpha_1, \alpha_2)}\|^2 = \sum_{s_1, s_2 \in \mathbb{Z}} h^2(s_1, s_2)$. We shall use the following criterion for the convergence in distribution of quadratic forms in (2.3) towards double stochastic integrals (2.1).

**Proposition 2.1.** ([6], Prop.11.5.5.) Let $i, j = 1, 2$ and $Q_{ij}(h_{\alpha_1, \alpha_2}), \alpha_1, \alpha_2 > 0$ be a family of quadratic forms as in (2.3) with coefficients $h_{\alpha_1, \alpha_2} \in L^2(\mathbb{Z}^2)$. For $i = j$ we additionally assume $E\epsilon_i^4(0) < \infty$. Suppose for some $g \in L^2(\mathbb{R}^2)$ we have that

$$\lim_{\alpha_1, \alpha_2 \to \infty} \|\tilde{h}^{(\alpha_1, \alpha_2)} - g\| = 0.$$

Then $Q_{ij}(h_{\alpha_1, \alpha_2}) \to_d I_{ij}(g)$ ($\alpha_1, \alpha_2 \to \infty$), where $I_{ij}(g)$ is defined as in (2.1).

### 2.2. The ‘cross-sectional’ intermediate process.

Let $d\mathcal{M}_\beta \equiv \mathcal{M}_\beta(dx_1, dx_2, dB_1, dB_2)$ denote Poisson random measure on $(\mathbb{R}_+ \times C(\mathbb{R}))^2$ with mean

$$d\mu_\beta \equiv \mu_\beta(dx_1, dx_2, dB_1, dB_2) := \psi(1)^2(x_1x_2)^{\beta-3}dx_1dx_2P_B(dB_1)P_B(dB_2),$$

where $\beta > 0$ is parameter and $P_B$ is the Wiener measure on $C(\mathbb{R})$. Let $d\widetilde{\mathcal{M}}_\beta := d\mathcal{M}_\beta - d\mu_\beta$ be the centered Poisson random measure. We shall often use finiteness of the following integrals:

$$\int_{\mathbb{R}_+^2} \min(1, \frac{1}{x_1x_2(x_1+x_2)})dx_1dx_2 < \infty, \quad \forall 0 < \beta < 3/2,$$

$$\int_{\mathbb{R}_+^2} \min(1, \frac{1}{x_1^2})dx_1dx_2 < \infty, \quad \forall 1 < \beta < 3/2.$$
see Appendix. Let
\[ Y_i(u; x) = \int_{-\infty}^{u} e^{-x(u-s)} dB_i(s), \quad u \in \mathbb{R}, \ x > 0 \] (2.10)
be a family of stationary O-U processes subordinated to \( B_i = \{B_i(s), s \in \mathbb{R}\}, B_i, i \in \mathbb{Z} \) being independent BMs. Let
\[ z(\tau; x_1, x_2) := \int_{0}^{\tau} \prod_{i=1}^{2} Y_i(u; x_i) \, du, \quad \tau \geq 0 \] (2.11)
be a family of integrated products of independent O-U processes indexed by \( x_1, x_2 > 0 \). We use the representation of (2.11)
\[ z(\tau; x_1, x_2) = \int_{\mathbb{R}^2} \left\{ \int_{0}^{\tau} \prod_{i=1}^{2} e^{-x_i(u-s_i)} \chi(u > s_i) \, du \right\} dB_1(s_1)dB_2(s_2) \] (2.12)
as the double Itô-Wiener integral as in (2.1). The ‘cross-sectional’ intermediate process \( Z_\beta \) is defined as stochastic integral w.r.t. the Poisson measure \( \mathcal{M}_\beta \), viz.,
\[ Z_\beta(\tau) := \int_{\mathcal{L}_1} z(\tau; x_1, x_2) \, d\mathcal{M}_\beta + \int_{\mathcal{L}_1} z(\tau; x_1, x_2) \, d\tilde{\mathcal{M}}_\beta, \] (2.13)
where
\[ \mathcal{L}_1 := \{(x_1, x_2, B_1, B_2) \in (\mathbb{R}_+ \times C(\mathbb{R}))^2 : x_1x_2(x_1 + x_2) \leq 1\}, \quad \mathcal{L}_1 := (\mathbb{R}_+ \times C(\mathbb{R}))^2 \setminus \mathcal{L}_1 \] (2.14)
and \( \mu_\beta(\mathcal{L}_1) < \infty \). For \( 1/2 < \beta < 3/2 \) the two integrals in (2.13) can be combined in a single one:
\[ Z_\beta(\tau) = \int_{(\mathbb{R}_+ \times C(\mathbb{R}))^2} z(\tau; x_1, x_2) \, d\tilde{\mathcal{M}}_\beta. \] (2.15)
These and other properties of \( Z_\beta \) are stated in the following proposition whose proof is given in the Appendix. We also refer to [25] and [17] for general properties of stochastic integrals w.r.t. Poisson random measure.

**Proposition 2.2.** (i) The process \( Z_\beta \) in (2.13) is well-defined for any \( 0 < \beta < 3/2 \). It has stationary increments, infinitely divisible finite-dimensional distributions, and the joint ch.f. given by
\[ \text{E} \exp \left\{ i \sum_{j=1}^{m} \theta_j Z_\beta(\tau_j) \right\} = \exp \left\{ \int_{(\mathbb{R}_+ \times C(\mathbb{R}))^2} \left( e^{i \sum_{j=1}^{m} \theta_j z(\tau_j; x_1, x_2)} - 1 \right) d\mu_\beta \right\}, \] (2.16)
where \( \theta_j \in \mathbb{R}, \tau_j \geq 0, j = 1, \ldots, m, m \in \mathbb{N} \). Moreover, the distribution of \( Z_\beta \) is symmetric: \( \{Z_\beta(\tau), \tau \geq 0\} =_{\text{d}} \{\tilde{Z}_\beta(\tau), \tau \geq 0\} \).

(ii) \( \text{E}|Z_\beta(\tau)|^p < \infty \) for \( p < 2\beta \) and \( \text{E}Z_\beta(\tau) = 0 \) for \( 1/2 < \beta < 3/2 \).

(iii) For \( 1/2 < \beta < 3/2 \), \( Z_\beta \) can be defined as in (2.15). Moreover, if \( 1 < \beta < 3/2 \), then \( \text{E}Z_{\beta}^2(\tau) < \infty \) and
\[ \text{E}Z_\beta(\tau_1)Z_\beta(\tau_2) = (\sigma_\infty^2/2) (\tau_1^{2(2-\beta)} + \tau_2^{2(2-\beta)} - |\tau_2 - \tau_1|^{2(2-\beta)}) \] (2.17)
where \( \sigma_\infty^2 := \psi(1)^2 \Gamma(\beta - 1)^2/4(2 - \beta)(3 - 2\beta) \).

(iv) For \( 1/2 < \beta < 3/2 \), the process \( Z_\beta \) has a.s. continuous trajectories.
(v) (Asymptotic self-similarity) As $b \to 0$,

$$b^{\beta-2}Z_{\beta}(br) \to_{\text{f.d.d.}} \sigma_{\infty}B_{2-\beta}(\tau) \quad \text{if } 1 < \beta < 3/2,$$

$$b^{-1}\log b^{-1/2}Z_{\beta}(br) \to_{\text{f.d.d.}} \tau V_{2\beta}, \quad \text{if } 0 < \beta < 1,$$

where $\{B_{2-\beta}(\tau), \tau \geq 0\}$ is an FBM with $E[B_{2-\beta}(\tau)]^2 = \tau^{2(2-\beta)}, \tau \geq 0, 2-\beta \in (1/2, 1)$, $\sigma_{\infty}^2$ is given in (2.17), and $V_{2\beta}$ is a symmetric $(2\beta)$-stable r.v. with c.h.f. $Ee^{i\theta V_{2\beta}} = e^{-c_{\infty}|\theta|^{2\beta}}, \theta \in \mathbb{R}$, $c_{\infty} := \psi(1)^22^{1-2\beta}\Gamma(\beta + (1/2))\Gamma(1-\beta)/\sqrt{\pi}$. For any $0 < \beta < 3/2$, as $b \to \infty$,

$$b^{-1/2}Z_{\beta}(br) \to_{\text{f.d.d.}} \mathcal{A}^{1/2}B(\tau),$$

where $\mathcal{A} > 0$ is a $(2\beta/3)$-stable r.v. with Laplace transform $Ee^{-\mathcal{A}x} = \exp\{-\sigma_0x^{2\beta/3}\}, \sqrt{x} \geq 0, \sigma_0 := \psi(1)^22^{-2\beta/3}\Gamma(1-(2\beta/3))B(\beta/3, \beta/3)/2\beta$, and $\{B(\tau), \tau \geq 0\}$ is a standard Brownian motion, independent of $\mathcal{A}$. Finite-dimensional distributions of the limit process in (2.20) are symmetric $(4\beta/3)$-stable.

2.3. The ‘iso-sectional’ intermediate process. Let $d\mathcal{M}_{\beta}^* \equiv \mathcal{M}_{\beta}^*(dx, dB)$ denote Poisson random measure on $\mathbb{R}_+ \times C(\mathbb{R})$ with mean

$$d\mu_{\beta}^* \equiv \mu_{\beta}^*(dx, dB) := \psi(1)x^{\beta-1}dxP_B(dB),$$

where $0 < \beta < 2$ is parameter and $P_B$ is the Wiener measure on $C(\mathbb{R})$. Let $d\mathcal{N}_{\beta}^* := d\mathcal{M}_{\beta}^* - d\mu_{\beta}^*$ be the centered Poisson random measure. Let $\mathcal{Y}(\cdot; x) \equiv \mathcal{Y}_1(\cdot; x)$ be the family of O-U processes as in (2.10), and

$$z^*(\tau; x) := \int_0^\tau \mathcal{Y}^2(u; x)du, \quad \tau \geq 0, \quad x > 0$$

be integrated squared O-U processes. Note $Ez^*(\tau; x) = \tau E\mathcal{Y}^2(0; x) = \tau \int_{-\infty}^0 e^{2xs}ds = \tau/2x$. We will use the representation

$$z^*(\tau; x) = \int_{\mathbb{R}^2} \left\{ \int_0^\tau e^{-x(u-s_i)}1(u > s_i)du \right\}dB(s_1)dB(s_2) + \tau/2x$$

as the double Itô-Wiener integral. The ‘iso-sectional’ intermediate process $Z_{\beta}^*$ is defined for $\beta \in (0, 2), \beta \neq 1$ as stochastic integral w.r.t. the above Poisson measure, viz.,

$$Z_{\beta}^*(\tau) := \int_{\mathbb{R}_+ \times C(\mathbb{R})} z^*(\tau; x) \begin{cases} d\mathcal{M}_{\beta}^*, & 0 < \beta < 1, \\ d\mathcal{N}_{\beta}^*, & 1 < \beta < 2, \end{cases} \quad \tau \geq 0.$$

Proposition 2.3 stating properties of $Z_{\beta}^*$ is similar to Proposition 2.2.

**Proposition 2.3.** (i) The process $Z_{\beta}^*$ in (2.24) is well-defined for any $0 < \beta < 2, \beta \neq 1$. It has stationary increments, infinitely divisible finite-dimensional distributions, and the joint ch.f. given by

$$E \exp \left\{ i \sum_{j=1}^m \theta_j z_{\beta}^*(\tau_j) \right\} = \exp \left\{ \int_{\mathbb{R}_+ \times C(\mathbb{R})} \left( e^{i \sum_{j=1}^m \theta_j z^*(\tau_j; x)} - 1 - i \sum_{j=1}^m \theta_j z^*(\tau_j; x)1(1 < \beta < 2) \right) \right\}d\mu_{\beta}^*,$$

where $\theta_j \in \mathbb{R}, \tau_j \geq 0, j = 1, \ldots, m, m \in \mathbb{N}$.

(ii) $E|Z_{\beta}^*(\tau)|^p < \infty$ for any $0 < p < \beta < 2, \beta \neq 1$ and $EZ_{\beta}^*(\tau) = 0$ for $1 < \beta < 2$.

(iii) For $1 < \beta < 2$, the process $Z_{\beta}^*$ has a.s. continuous trajectories.
(iv) (Asymptotic self-similarity) For any $0 < \beta < 2$, $\beta \neq 1$,

$$b^{-1} Z_{\beta}(b \tau) \rightarrow_{fdd} \begin{cases} \tau V_{\beta}^* & \text{as } b \to 0, \\ \tau V_{\beta}^+ & \text{as } b \to \infty, \end{cases}$$

where $V_{\beta}^+$, $V_{\beta}^*$ are a completely asymmetric $\beta$-stable r.v.s with ch.f.s $E e^{i\theta V_{\beta}^+} = \exp \{ \psi(1) \int_0^\infty (e^{i\theta/2x} - 1 - i(\theta/2x)) x^{\beta-1} dx \}$, $E e^{i\theta V_{\beta}^*} = \exp \{ \psi(1) \int_0^\infty E(e^{i\theta Z/2x} - 1 - i(\theta Z/2x)) 1(1 < \beta < 2)) x^{\beta-1} dx \}$, $\theta \in \mathbb{R}$ and $Z \sim N(0,1)$.

2.4. Conditional long-run variance of products of RCAR(1) processes. We use some facts in Proposition 2.4, below, about conditional variance of the partial sums process of the product $Y_{ij}(t) := X_i(t)X_j(t)$ of two RCAR(1) processes. Split $Y_{ij}(t) = Y_{ij}^+(t) + Y_{ij}^-(t)$, where $Y_{ij}^+(t) = \sum_{s_1 \wedge s_2 \geq 1} a_{i}^{t-s_1} a_{j}^{t-s_2} 1(t \geq s_1 \vee s_2) \varepsilon_i(s_1) \varepsilon_j(s_2)$, $Y_{ij}^-(t) = \sum_{s_1 \wedge s_2 \leq 0} a_{i}^{t-s_1} a_{j}^{t-s_2} 1(t \geq s_1 \vee s_2) \varepsilon_i(s_1) \varepsilon_j(s_2)$. For $i = j$ we assume additionally that $E \varepsilon_i^4(0) < \infty$.

**Proposition 2.4.** We have

$$\text{var}[\sum_{t=1}^n Y_{ij}(t)|a_i, a_j] \sim \text{var}[\sum_{t=1}^n Y_{ij}^+(t)|a_i, a_j] \sim A_{ij}n, \quad n \to \infty$$

where

$$A_{ij} := \begin{cases} \frac{1+a_i a_j}{(1-a_i^2)(1-a_j^2)(1-a_i a_j)}, & i \neq j, \\
\frac{1+a_i a_j}{1-a_i^2} \left( \frac{2}{(1-a_i^2)^2} + \text{cum}_4 \right), & i = j,
\end{cases}$$

where $\text{cum}_4$ is the 4th cumulant of $\varepsilon_i(0)$. Moreover, for any $n \geq 1$, $i, j \in \mathbb{Z}$, $a_i, a_j \in [0,1)$

$$\text{var}[\sum_{t=1}^n Y_{ij}(t)|a_i, a_j] \leq \frac{C_{ij} n^2}{(1-a_i)(1-a_j)} \min \left(1, \frac{1}{n(2-a_i-a_j)} \right),$$

where $C_{ij} := 4 (i \neq j) =: 2(2 + |\text{cum}_4|)$ ($i = j$).

**Proof.** Let $i \neq j$. We have $\text{E}[Y_{ij}(t)Y_{ij}(s)|a_i, a_j] = \text{E}[X_i(t)X_i(s)|a_i] \text{E}[X_j(t)X_j(s)|a_j] = (a_i a_j)^{|t-s|}/(1-a_i^2)(1-a_j^2)$ and hence

$$J_n(a_i, a_j) := \text{E} \left[ \left( \sum_{t=1}^n Y_{ij}(t) \right)^2 | a_i, a_j \right] = \frac{n}{(1-a_i^2)(1-a_j^2)} \sum_{t=-n}^n (a_i a_j)^{|t|} (1 - \frac{|t|}{n}).$$

Relation (2.30) implies (2.27). It also implies $J_n(a_i, a_j) \leq 2n^2/(1-a_i)(1-a_j)$. Note also $1-a_i a_j \geq (1/2)((1-a_i)+(1-a_j))$. Hence and from (2.30) we obtain $J_n(a_i, a_j) \leq \frac{n}{(1-a_i^2)(1-a_j^2)} \left( 1 + \sum_{t=-n}^\infty (a_i a_j)^t \right) \leq 2n/(1-a_i)(1-a_j)(1-a_i a_j) \leq 4n/(1-a_i)(1-a_j)(2-a_i-a_j)$, proving (2.29). The proof of (2.27)-(2.29) for $i = j$ is similar using $\text{cov}[Y_{ii}(t), Y_{ii}(s)|a_i] = 2(a_i^{|t-s|}/(1-a_i^2))^2 + \text{cum}_4 a_i^2|t-s|/(1-a_i^4)$.

3 Asymptotic distribution of cross-sectional sample covariances

Theorems 3.1 and 3.2 discuss the asymptotic distribution of partial sums process

$$S_{N,n}^{t,s}(\tau) := \sum_{i=1}^N \sum_{u=1}^n X_i(u)X_{i+s}(u+t), \quad \tau \geq 0, \quad (3.1)$$
where $t$ and $s \in \mathbb{Z}$, $s \neq 0$ are fixed and $N$ and $n$ tend to infinity, possibly at a different rate. The asymptotic behavior of sample covariances $\hat{\gamma}_{N,n}(t,s)$ is discussed in Corollary 3.1. As it turns out, these limit distributions do not depend on $t,s$ which is due to the fact that the sectional processes $\{X_i(t), t \in \mathbb{Z}\}, i \in \mathbb{Z}$ are independent and stationary.

**Theorem 3.1.** Let the mixing distribution satisfy condition (1.2) with $0 < \beta < 3/2$. Let $N,n \to \infty$ so as

$$\lambda_{N,n} := \frac{N^{1/2\beta}}{n} \to \lambda_{\infty} \in [0, \infty].$$

Then the following statements (i)-(iii) hold for $S_{N,n}^{t,s}(\tau), (t,s) \in \mathbb{Z}^2, s \neq 0$ in (3.1) depending on $\lambda_{\infty}$ in (3.2).

(i) Let $\lambda_{\infty} = \infty$. Then

$$n^{-2}\lambda_{N,n}^{-\beta}S_{N,n}^{t,s}(\tau) \to_{fdd} \sigma_{\infty} B_{2-\beta}(\tau), \quad 1 < \beta < 3/2,$$

(ii) Let $\lambda_{\infty} = 0$ and $E|\varepsilon(0)|^{2p} < \infty$ ($\exists p > 1$). Then

$$n^{-2}\lambda_{N,n}^{-3/2}S_{N,n}^{t,s}(\tau) \to_{fdd} A_{1/2} B(\tau),$$

(iii) Let $0 < \lambda_{\infty} < \infty$. Then

$$n^{-2}\lambda_{N,n}^{-3/2}S_{N,n}^{t,s}(\tau) \to_{fdd} \lambda_{\infty}^{1/2} Z_{\beta}(\tau/\lambda_{\infty}),$$

where the limit processes are the same as in (2.18), (2.19).

**Theorem 3.2.** Let the mixing distribution satisfy condition (1.2) with $\beta > 3/2$ and assume $E|\varepsilon(0)|^{2p} < \infty$ for some $p > 1$. Then for any $(t,s) \in \mathbb{Z}^2, s \neq 0$ as $N,n \to \infty$ in arbitrary way,

$$n^{-1/2}N^{-1/2}S_{N,n}^{t,s}(\tau) \to_{fdd} \sigma B(\tau), \quad \sigma^2 := EA_{12},$$

where $A_{12}$ is defined in (2.28).

Note that the asymptotic distribution of sample covariances $\hat{\gamma}_{N,n}(t,s)$ in (1.5) coincides with that of the statistics

$$\tilde{\gamma}_{N,n}(t,s) := (Nu)^{-1}S_{N,n}^{t,s}(1) - (\bar{X}_{N,n})^2.$$

For $s \neq 0$ the limit behavior of the first term on the r.h.s. of (3.8) can be obtained from Theorems 3.1 and 3.2. It turns out that for some values of $\beta$, the second term on the r.h.s. can play the dominating role. The limit behavior of $\bar{X}_{N,n}$ was identified in [17] and is given in the following proposition, with some simplifications.

**Proposition 3.1.** Let the mixing distribution satisfy condition (1.2) with $\beta > 0$. Then:

(i) Let $1 < \beta < 2$ and $N/n^\beta \to \infty$. Then

$$N^{1/2}(\beta - 1)/2 \bar{X}_{N,n} \to _d \tilde{\sigma}_\beta Z,$$

(ii) Let $0 < \beta < 1$. Then

$$n^{-1/2} \lambda_{N,n}^{1/2} \bar{X}_{N,n} \to _d \sigma_{\infty} Z_{\beta}(\tau/\lambda_{\infty}),$$

where $Z_{\beta}$ is the intermediate process in (2.13).
where \( Z \sim \mathcal{N}(0, 1) \) and \( \sigma_2^2 := \psi(1)\Gamma(\beta - 1)/(3 - \beta)(2 - \beta) \).

(ii) Let \( 0 < \beta < 1 \) and \( N/n^\beta \to \infty \). Then

\[
N^{1-1/2\beta} \hat{X}_{N,n} \to_d V_{2\beta},
\]

where \( V_{2\beta} \) is a symmetric \((2\beta)\)-stable r.v. with ch.f. \( \mathbb{E} e^{i\theta V_{2\beta}} = e^{-K_\beta|\theta|^{2\beta}}, K_\beta := \psi(1)4^{-\beta}\Gamma(1 - \beta)/\beta \).

(iii) Let \( 0 < \beta < 2 \) and \( N/n^\beta \to 0 \). Then

\[
N^{1-1/\beta}n^{1/2} \hat{X}_{N,n} \to_d W_{\beta},
\]

where \( W_{\beta} \) is a symmetric \( \beta \)-stable r.v. with ch.f. \( \mathbb{E} e^{i\theta W_{\beta}} = e^{-K_\beta|\theta|^{\beta}}, K_\beta := \psi(1)2^{-\beta/2}\Gamma(1 - \beta/2)/\beta \).

(iv) Let \( \beta > 2 \). Then as \( N, n \to \infty \) in arbitrary way,

\[
N^{1/2}n^{1/2} \hat{X}_{N,n} \to_d \sigma Z,
\]

where \( Z \sim \mathcal{N}(0, 1) \) and \( \sigma^2 := \mathbb{E}(1 - a)^{-2} \).

From Theorems 3.1 and Proposition 3.1 we see that the r.h.s. of (3.8) may exhibit two 'bifurcation points' of the limit behavior, viz., as \( N \sim n^{2\beta} \) and \( N \sim n^\beta \). Depending on the value of \( \beta \) the first or the second term may dominate, and the limit behavior of \( \hat{\gamma}_{N,n}(t,s) \) gets more complicated. The following corollary provides this limit without detailing the 'intermediate' situations and also with exception of some particular values of \( \beta \) where both terms on the r.h.s. may contribute to the limit. Essentially, the corollary follows by comparing the normalizations in Theorems 3.1 and Proposition 3.1.

**Corollary 3.1.** Assume that the mixing distribution satisfies condition (1.2) with \( \beta > 0 \) and \( \mathbb{E} |\varepsilon(0)|^{2p} < \infty \) for some \( p > 1 \) and \( (t,s) \in \mathbb{Z}^2, s \neq 0 \) be fixed albeit arbitrary.

(i) Let \( N/n^{2\beta} \to \infty \) and \( 1 < \beta < 3/2 \). Then

\[
N^{1/2}n^{\beta-1} \hat{\gamma}_{N,n}(t,s) \to_d \sigma_\infty Z,
\]

where \( Z \sim \mathcal{N}(0, 1) \) and \( \sigma_\infty \) is the same as in Theorem 3.1 (i).

(ii) Let \( N/n^{2\beta} \to \infty \) and \( 1/2 < \beta < 1 \). Then

\[
\frac{N^{1-1/2\beta}}{\log^{1/2\beta}(N/n^{1/2\beta})} \hat{\gamma}_{N,n}(t,s) \to_d V_{2\beta},
\]

where \( V_{2\beta} \) is symmetric \((2\beta)\)-stable r.v. defined in Theorem 3.1 (i).

(iii) Let \( N/n^{2\beta} \to \infty \) and \( 0 < \beta < 1/2 \). Then

\[
N^{2-1/\beta} \hat{\gamma}_{N,n}(t,s) \to_d -(V_{2\beta})^2,
\]

where \( V_{2\beta} \) is symmetric \((2\beta)\)-stable r.v. defined in Proposition 3.1 (ii).

(iv) Let \( N/n^{2\beta} \to 0, N/n^\beta \to \infty \) and \( 3/4 < \beta < 3/2 \). Then

\[
N^{1-3/4\beta}n^{1/2} \hat{\gamma}_{N,n}(t,s) \to_d W_{4\beta/3},
\]

where \( W_{4\beta/3} \) is a symmetric \((4\beta/3)\)-stable r.v. with characteristic function \( \mathbb{E} e^{i\theta V_{4\beta/3}} = e^{-(\sigma_0/2^{3\beta/3})|\theta|^{3\beta/3}} \) and \( \sigma_0 \) is the same constant as in Theorem 3.1 (ii).
(v) Let $N/n^{2\beta} \rightarrow 0$, $1/2 < \beta < 3/4$ and $N/n^{2\beta/(4\beta-1)} \rightarrow \infty$. Then the convergence in (3.14) holds.

(vi) Let $N/n^{\beta} \rightarrow \infty$, $1/2 < \beta < 3/4$ and $N/n^{3/(4\beta-1)} \rightarrow 0$. Then the convergence in (3.13) holds.

(vii) Let $N/n^{2\beta} \rightarrow 0$, $N/n^{\beta} \rightarrow \infty$ and $0 < \beta < 1/2$. Then the convergence in (3.13) holds.

(viii) Let $N/n^{\beta} \rightarrow 0$ and $3/4 < \beta < 3/2$. Then the convergence in (3.14) holds.

(ix) Let $N/n^{\beta} \rightarrow 0$, $0 < \beta < 3/4$ and $N/n^{2\beta/(5-4\beta)} \rightarrow \infty$. Then

$$N^{2-2/\beta} \gamma_{N,n}(t,s) \rightarrow_d -(\bar{W}_\beta)^2,$$

where $\bar{W}_\beta$ is a symmetric $\beta$-stable r.v. defined in Proposition 3.1 (iii).

(x) Let $0 < \beta < 3/4$ and $N/n^{2\beta/(5-4\beta)} \rightarrow 0$. Then the convergence in (3.14) holds.

(xi) For $3/2 < \beta < 2$, let $N/n^{\beta} \rightarrow [0,\infty]$ and for $\beta > 2$, let $N,n \rightarrow \infty$ in arbitrary way. Then

$$N^{1/2}n^{1/2} \gamma_{N,n}(t,s) \rightarrow_d N(0,\sigma^2),$$

where $\sigma^2$ is given as in Theorem 3.2.

The proof of Theorem 3.1 in cases (i)-(iii) is given subsections 3.1-3.3. To avoid excessive notation, the discussion is limited to the case $(t,s) = (0,1)$ or the partial sums process $S_{N,n}(\tau) := \sum_{i=1}^{N} \sum_{t=1}^{[nr]} X_i(t)X_{i+1}(t)$. Later on we shall extend them to general case $(t,s), s \neq 0$.

Let us give an outline of the proof of Theorem 3.1. Similarly to [17] we use the method of characteristic function combined with ‘vertical’ Bernstein’s blocks, due to the fact that $S_{N,n}$ is not a sum of row-independent summands as in [17]. Write

$$S_{N,n}(\tau) = S_{N,n,q}(\tau) + S_{N,n,q}^1(\tau) + S_{N,n,q}^2(\tau)$$

(3.17)

where the main term

$$S_{N,n,q}(\tau) := \tilde{\tilde{\gamma}}_q \sum_{k=1}^{\tilde{\tilde{\gamma}}_q} Y_{k,n,q}(\tau), \quad Y_{k,n,q}(\tau) := \sum_{(k-1)q<i<qk} \sum_{t=1}^{[nr]} X_i(t)X_{i+1}(t), \quad 1 \leq k \leq \tilde{\tilde{\gamma}}_q := \lceil N/q \rceil (3.18)$$

is a sum of $\tilde{\tilde{\gamma}}_q$ ‘large’ blocks of size $q-1$ with

$$q \equiv q_{N,n} \rightarrow \infty \quad \text{as} \quad N,n \rightarrow \infty. \quad (3.19)$$

The convergence rate of $q \in \mathbb{N}$ in (3.19) will be slow enough (e.g., $q = O(\log N)$) and specified later on. The two other terms in the decomposition (3.17),

$$S_{N,n,q}^1(\tau) := \sum_{k=1}^{\tilde{\tilde{\gamma}}_q} \sum_{t=1}^{[nr]} X_{kq}(t)X_{kq+1}(t), \quad S_{N,n,q}^2(\tau) := \sum_{q \tilde{\tilde{\gamma}}_q<i<qN} \sum_{t=1}^{[nr]} X_i(t)X_{i+1}(t) \quad (3.20)$$

contain respectively $\tilde{\tilde{\gamma}}_q = o(N)$ and $N-q\tilde{\tilde{\gamma}}_q < q = o(N)$ row sums and will be shown to be negligible. More precisely, we show that in each case (i)-(iii) of Theorem 3.1

$$A_{N,n}^{-1} S_{N,n,q}(\tau) \rightarrow_{fdd} S_\beta(\tau), \quad A_{N,n}^{-1} S_{N,n,q}^1(\tau) = o_p(1), \quad A_{N,n}^{-1} S_{N,n,q}^2(\tau) = o_p(1) \quad (3.21)$$

(3.22)
where $A_{N,n}$ and $S_\beta$ denote the normalization and the limit process, respectively, particularly,

$$
A_{N,n} := n^2 \begin{cases} 
\lambda_{N,n}^\beta, & \lambda_\infty = \infty, \ 1 < \beta < 3/2, \\
\lambda_{N,n}(\log \lambda_{N,n})^{1/2\beta}, & \lambda_\infty = \infty, \ 0 < \beta < 1, \\
\lambda_{N,n}^{3/2}, & \lambda_\infty \in [0,\infty), \ 0 < \beta < 3/2,
\end{cases}
$$

(3.23)

Note that the summands $Y_{k,n,q}, 1 \leq k \leq \tilde{N}_q$ in (3.18) are independent and identically distributed, and the limit $S_\beta(\tau)$ is infinitely divisible in cases (i)-(iii) of Theorem 3.1. Hence use of characteristic functions to prove (3.21) is natural. The proofs are limited to one-dimensional convergence at a given $\tau > 0$ since the convergence of general finite-dimensional distributions follows in a similar way. Accordingly, the proof of (3.21) for fixed $\tau > 0$ reduces to

$$
\Phi_{N,n,q}(\theta) \to \Phi(\theta), \quad \text{as } N, n \to \infty, \lambda_{N,n} \to \lambda_\infty, \ \forall \theta \in \mathbb{R},
$$

(3.24)

where

$$
\Phi_{N,n,q}(\theta) := \tilde{N}_q \mathbb{E}[e^{i\theta Y_{1,n,q}(\tau)} - 1], \quad \Phi(\theta) := \log \mathbb{E}e^{i\theta S_\beta(\tau)},
$$

(3.25)

To prove (3.24) write

$$
A_{N,n}^{-1} Y_{1,n,q}(\tau) = \sum_{i=1}^{q-1} y_i(\tau), \quad \text{where } y_i(\tau) := A_{N,n}^{-1} \sum_{i=1}^{[n\tau]} X_i(t)X_{i+1}(t).
$$

(3.26)

We use the identity:

$$
\prod_{1 \leq i < q}(1 + w_i) - 1 = \sum_{1 \leq i < q} w_i + \sum_{|D|\geq 2} \prod_{i \in D} w_i,
$$

(3.27)

where the sum $\sum_{|D|\geq 2}$ is taken over all subsets $D \subset \{1, \ldots, q - 1\}$ of cardinality $|D| \geq 2$. Applying (3.27) with $w_i = e^{i\theta y_i(\tau)} - 1$ we obtain

$$
\Phi_{N,n,q}(\theta) := \tilde{N}_q (q - 1)[\mathbb{E}e^{i\theta y_1(\tau)} - 1] + \tilde{N}_q \sum_{|D|\geq 2} \mathbb{E}\prod_{i \in D} [e^{i\theta y_i(\tau)} - 1]
$$

(3.28)

Thus, since $\tilde{N}_q (q - 1)/N \to 1$, (3.24) follows from

$$
N \mathbb{E}[e^{i\theta y_1(\tau)} - 1] \to \Phi(\theta),
$$

(3.29)

$$
N \sum_{|D|\geq 2} \mathbb{E}\prod_{i \in D} [e^{i\theta y_i(\tau)} - 1] \to 0.
$$

(3.30)

Let us explain the main idea of the proof of (3.29). Assuming $\phi(x) = (1 - x)^{\beta - 1}$ in (1.2) the l.h.s. of (3.29) can be written as

$$
N[\mathbb{E}[e^{i\theta y_1(\tau)} - 1]] = N \int_{[0,1]^2} \mathbb{E}[e^{i\theta y_1(\tau)} - 1] a_i = 1 - z_i, i = 1, 2] (z_1 z_2)^{\beta - 1} dz_1 dz_2
$$

$$
= \frac{N}{B_{N,n}^{2\beta}} \int_{[0,B_{N,n}]^2} \mathbb{E}[e^{i\theta z_{N,n}(\tau;x_1,x_2)} - 1] (x_1 x_2)^{\beta - 1} dx_1 dx_2
$$

(3.31)

where

$$
z_{N,n}(\tau;x_1,x_2) := A_{N,n}^{-1} \sum_{s_1,s_2\in\mathbb{Z}} \varepsilon_1(s_1)\varepsilon_2(s_2) \prod_{t=1}^{[n\tau]} \prod_{i=1}^{2} \left(1 - \frac{x_i}{B_{N,n}}\right)^{-s_i} 1(t \geq s_i)
$$

(3.32)
and $B \rightarrow \infty$ is a scaling factor of the autoregressive coefficient. In cases (ii) and (iii) of Theorem 3.1 (proof of (3.5) and (3.6)) we choose this scaling factor $B = N^{1/2\beta}$ so that $\frac{N}{B} = 1$ and prove that the integral in (3.31) converges to $\int_{\mathbb{R}^+} E[e^{iθz}(r,x_1,x_2) - 1](x_1x_2)^{-1}dx_1dx_2 = Φ(θ)$ where $z(τ;x_1,x_2)$ is a random process and $Φ(θ)$ is the required limit in (3.24). A similar scaling $B_n = (N \log λ_n)1/2\beta$ applies in the case $λ = \infty, 0 < β < 1$ (proof of (3.4)) although in this case the factor $N/B^{2β}$ is $1/2\log λ_n$ in front of the integral in (3.31) does not trivialize and the proof of the limit in (3.24) is more delicate. On the other hand, in the case of the Gaussian limit (3.3), the choice $B = n$ leads to $N/B^{2β} = λ_n \to \infty$ and (3.31) tends to $(1/2)|θ|^2 \int_{\mathbb{R}^+} Eε^2(τ;x_1,x_2)(x_1x_2)^{-1}dx_1dx_2 = Φ(θ)$ with $z(τ;x_1,x_2)$ defined in (2.11) as shown in subsection 3.3 below.

To summarize the above discussion: in each case (i)-(iii) of Theorem 3.1, to prove the limit (3.21) of the main term, it suffices to verify relations (3.29) and (3.30). The proof of the first relation in (3.22) is very similar to (3.21) since $S_{1,n,s}^2(τ)$ is also a sum of i.i.d. r.v.s and the argument of (3.21) applies with small changes. The second relation in (3.22) seems even simpler. In the proofs we repeatedly use the following inequalities:

$$|e^{iz} - 1| ≤ 2 ∧ |z|, \quad |e^{iz} - 1 - iz| ≤ (2|z|) ∧ (z^2/2), \quad z ∈ \mathbb{R}. (3.33)$$

### 3.1 Proof of Theorem 3.1: case (iii) $0 < λ_∞ < ∞$

**Proof of (3.29).** For notational brevity, we assume $λ_n = λ_∞ = 1$ since the general case as in (3.2) requires unsubstantial changes. Recall from (2.16) that $Φ(θ) = \int_{\mathbb{R}^+} E[e^{iθz}(r;x_1,x_2) - 1](x_1x_2)^{-1}dx_1dx_2$ where $z(τ;x_1,x_2)$ is the double Itô-Wiener integral in (2.11). Also recall the representation (3.31), (3.32) where $A_n = n^2, B_n = n$ and $z_n(τ;x_1,x_2) = Q_12(h_n( ; ; τ;x_1,x_2))$ is a quadratic form as in (2.3) with coefficients

$$h_n(s_1,s_2;τ;x_1,x_2) := n^{-2} \sum_{t=1}^{[nr]} \prod_{i=1}^{2} (1 - \frac{x_i}{n})^{-s_i} 1(t ≥ s_i), \quad s_1,s_2 ∈ \mathbb{Z}. (3.34)$$

By Proposition 2.1, with $α_1 = α_2 = n$, the point-wise convergence

$$E[e^{iθz_n;τ;x_1,x_2} - 1] = E[e^{iθQ_12(h_n( ; ; τ;x_1,x_2)) - 1}] \to E[e^{iθz;τ;x_1,x_2} - 1] (3.35)$$

for any fixed $x_1,x_2 ∈ \mathbb{R}$ follows from $L_2$-convergence of the kernels:

$$\|\bar{h}_n( ; ; τ;x_1,x_2) - h( ; ; τ;x_1,x_2)\| \to 0 (3.36)$$

where

$$\bar{h}_n(s_1,s_2;τ;x_1,x_2) := nh_n([ns_1],[ns_2];τ;x_1,x_2) = \frac{1}{n} \sum_{t=1}^{[ns]} \prod_{i=1}^{2} (1 - \frac{x_i}{n})^{t-[ns_i]} 1(t ≥ [ns_i])$$

$$\rightarrow \int_{0}^{τ} \prod_{i=1}^{2} e^{-x_i(t-s_i)} 1(t > s_i)dt =: h(s_1,s_2;τ;x_1,x_2) (3.37)$$

point-wise for any $x_i > 0, s_i ∈ \mathbb{R}, s_i ≠ 0, i = 1, 2, τ > 0$ fixed. We also use the dominating bound

$$|\bar{h}_n(s_1,s_2;τ;x_1,x_2)| ≤ Ch(s_1,s_2;2τ;x_1,x_2), \quad s_1,s_2 ∈ \mathbb{R}, \quad 0 < x_1,x_2 < n, (3.38)$$

where
with $C > 0$ independent of $s_i, x_i, i = 1, 2$ which follows from the definition of $\tilde{h}_n(\cdot; \tau; x_1, x_2)$ and the inequality $1 - x \leq e^{-x}, x > 0$. Since $h(\cdot; 2\tau; x_1, x_2) \in L^2(\mathbb{R}^2)$, (3.37), (3.38) and the DCT imply (3.36) and (3.35).

It remains to show the convergence of the corresponding integrals, viz.,

$$
\int_{(0,n]^2} E[e^{i\theta N,n(\tau;x_1,x_2)} - 1] (x_1,x_2)^\beta - 1 \, dx_1 dx_2 \to \int_{\mathbb{R}^2} E[e^{i\theta (\cdot;x_1,x_2)} - 1] (x_1,x_2)^\beta - 1 \, dx_1 dx_2 = \Phi(\theta). \tag{3.39}
$$

From (3.31) and $E_{N,n}(\tau;x_1,x_2) = 0$ we obtain

$$
|E[e^{i\theta N,n(\tau;x_1,x_2)} - 1]| \leq C \left\{ \begin{array}{ll}
1, & x_1x_2(x_1 + x_2) \leq 1, \\
E_{N,n}^2(\tau;x_1,x_2), & x_1x_2(x_1 + x_2) > 1,
\end{array} \right. \tag{3.40}
$$

where

$$
E_{N,n}^2(\tau;x_1,x_2) = \mathbb{A}_{N,n}^2 \mathbb{E}\left[ \sum_{t=1}^{[\tau]} Y_{12}(t)^2 | a_i = 1 - \frac{x_i}{\overline{N,n}}, i = 1, 2 \right] \\
= n^{-4} \mathbb{E}\left[ \sum_{t=1}^{[\tau]} Y_{12}(t)^2 | a_i = 1 - \frac{x_i}{n}, i = 1, 2 \right] \\
\leq \frac{C}{n^3(\frac{x_1}{n})^2(\frac{x_2}{n})^2} \min \left( n, \frac{1}{x_1 x_2} \right) = \frac{C}{x_1 x_2} \min \left( 1, \frac{1}{x_1 + x_2} \right), \tag{3.41}
$$

see (3.32) and the bound in (2.29). In view of inequality (2.8), the DCT applies, proving (3.39) and (3.29).

Proof of (3.30). Choose $q = q_{N,n} = [\log n]$. Let $J_q(\theta)$ denote the l.h.s. of (3.30). Using the identity $\sum_{D \subset \{1,\ldots,q-1\}, |D| \geq 2} \prod_{i \in D} w_i = \sum_{1 \leq i < j < q} w_i w_j \prod_{1 < k < j} (1 + w_k)$ with $w_i = e^{i\theta y_i(\tau)} - 1$, see (3.27), we can rewrite $J_q(\theta) = \sum_{1 \leq i < j < q} T_{ij}(\theta)$, where

$$
T_{ij}(\theta) := NE\left[ (e^{i\theta y_i(\tau)} - 1)(e^{i\theta y_j(\tau)} - 1) \exp \left\{ i\theta \sum_{1 \leq k < j} y_k(\tau) \right\} (1(a_i < a_{j+1}) + 1(a_i > a_{j+1})) \right] \\
= T'_{ij}(\theta) + T''_{ij}(\theta).
$$

Since $|J_q(\theta)| \leq q^2 \max_{1 \leq i < j < q} |T_{ij}(\theta)| \leq (\log n)^2 \max_{1 \leq i < j < q} |T_{ij}(\theta)|$, (3.30) follows from

$$
|T_{ij}(\theta)| \leq Cn^{-\delta}, \quad \forall \ 1 \leq i < j, \tag{3.43}
$$

with $C, \delta > 0$ independent of $n$. Using $E[y_i(\tau) a_k, \varepsilon_j(k), k, j \in \mathbb{Z}, j \neq i] = 0$ and (3.41) we obtain

$$
|T'_{ij}(\theta)| \leq C \mathbb{E}\left[ \min \left( 1, E[y_i^2(\tau) a_k, k \in \mathbb{Z}] \right) 1(a_i < a_{j+1}) \right] \leq Cn^{-\beta} \int_{(0,n]^3} \min \left\{ 1, \frac{1}{x_1 x_2 (x_1 + x_2 + 1)} \right\} x_1 x_2 (x_1 + x_2 + 1)^{\beta - 1} 1(x_1 + 1 < x_i) dx_1 dx_2 dx_{j+1} \\
= Cn^{-\beta} \int_{(0,n]^2} \min \left\{ 1, \frac{1}{x_1 x_2 (x_1 + x_2)} \right\} x_1 x_2 (x_1 + x_2)^{\beta - 1} 1(x_1 + 1 < x_i) dx_1 dx_2 = Cn^{-\beta} (T'_n + T'_n),
$$

where $T'_n := \int_{0 < x_1 < x_2 < n} \min \left\{ 1, \frac{1}{x_1 x_2 (x_1 + x_2)} \right\} x_1 x_2 (x_1 + x_2)^{\beta - 1} 1(x_1 + 1 < x_i) dx_1 dx_2$, $T''_n := \int_{0 < x_1 < x_2 < n} \min \left\{ 1, \frac{1}{x_1 x_2 (x_1 + x_2)} \right\} x_1 x_2 (x_1 + x_2)^{\beta - 1} 1(x_1 + 1 < x_i) dx_1 dx_2$.

Next,

$$
T'_n \leq C \int_{0}^{1} x_1^{2\beta - 1} dx_1 \int_{x_1}^{1} x_2^{\beta - 1} dx_2 + x_1 \int_{1}^{n} x_1^{\beta - 1} \sqrt{x_1} \, dx_1 \int_{x_1}^{n} x_2^{\beta - 3} dx_2 + \int_{1}^{n} x_1^{\beta - 2} dx_1 \int_{x_1}^{n} x_2^{\beta - 3} dx_2 \\
\leq C \left[ \int_{0}^{1} x_1^{3\beta/2 - 1} dx_1 + \int_{1}^{n} x_1^{3\beta - 4} dx_1 \right] \leq Cn^{3\beta - 1 - \nu} (1 + \log n).\tag{3.44}
$$
implying the point-wise convergence

3.2 Proof of Theorem 3.1 (ii): case

Let us prove the (conditional) CLT:

Similarly, 

\[ T_n'' = \int_0^1 x_1^{2\beta - 1}dx_1 \int_0^x x_2^{2\beta - 1}dx_2 + \int_1^1 x_1^{2\beta - 1}dx_1 \int_0^{x_1^{2\beta - 1}} x_2^{2\beta - 1}dx_2 + \int_1^1 x_1^{2\beta - 3}dx_1 \int_0^x x_2^{2\beta - 2}dx_2 \]

\[ \leq C((\log n)1(\beta < 1) + (\log n)^21(\beta = 1) + n^{3(\beta - 1)}1(\beta > 1)). \]

Whence, the bound in (3.43) follows for \( T''_ij(\theta) \) with any 0 < \( \delta < \beta \wedge (3 - 2\beta) \), for 0 < \( \beta < 3/2 \). Since \( |T''_ij(\theta)| \leq \text{CNE}[\min(1, E[y_j^2(\tau)|a_k, k \in \mathbb{Z}])1(a_{j+1} < a_j)] \) can be symmetrically handled, this proves (3.43) and (3.30).

Proof of (3.22). Since \( A_{N,n}^{-1}S_{N,n,q}(\tau) = \sum_{k=1}^{\hat{N}_q} y_{kq}(\tau) \) is a sum of \( \hat{N}_q \) i.i.d.r.v.s \( y_{kq}(\tau) \), \( k = 1, \ldots, \hat{N}_q \) so the first relation in (3.22) follows from

\[ \hat{N}_q E[e^{i\theta y_1(\tau)} - 1] \to 0, \quad \forall \theta \in \mathbb{R}. \] (3.45)

Clearly, (3.45) is a direct consequence of (3.29) and the fact that \( \hat{N}_q/N \to 0 \).

Consider the second relation in (3.22). Let \( L_q := N - q\hat{N}_q \) be the number of summands in \( S_{N,n,q}(\tau) \). Then \( A_{N,n}^{-1}S_{N,n,q}(\tau) \equiv \text{fdd} \sum_{i=1}^{L_q} y_i(\tau) \) and

\[ E[e^{i\theta A_{N,n}^{-1}S_{N,n,q}(\tau)} - 1] = L_q E[e^{i\theta y_1(\tau)} - 1] + \sum_{|D| \geq 2} E \prod_{i \in D} [e^{i\theta y_i(\tau)} - 1], \] (3.46)

where the last sum is taken over all \( D \subset \{1, \ldots, L_q\}, |D| \geq 2 \). Since \( L_q < q = o(N) \) from (3.29), (3.30) we infer that the r.h.s. of (3.46) vanishes, proving (3.22), and thus completing the proof of Theorem 3.1, case (iii).

3.2 Proof of Theorem 3.1 (ii): case \( \lambda_\infty = 0 \), or \( N = o(n^{2\beta}) \).

Proof of (3.29). Note the log-ch.f. of the r.h.s. in (3.5) can be written as

\[ \Phi(\theta) = \log E[e^{i\theta A_{1/2}B(\tau)}] = -\sigma_0(\theta^2/2)^{2\beta/3} \]

\[ = -\psi(1)^2 \int_{\mathbb{R}^2} \left(1 - \exp \left\{-\frac{\theta^2}{4x_1x_2(x_1+x_2)}\right\}\right) (x_1x_2)^{-1}dx_1dx_2, \] (3.47)

with \( \sigma_0 > 0 \) given by the integral

\[ \sigma_0 := \psi(1)^22^{-2\beta/3} \int_{\mathbb{R}^2} \left(1 - \exp \left\{-\frac{1}{x_1x_2(x_1+x_2)}\right\}\right) (x_1x_2)^{-1}dx_1dx_2. \] (3.48)

Relation (3.47) follows by change of variable \( x_i \to (\theta^2/4)^{1/3}x_i, i = 1, 2 \). The convergence of the integral in (3.48) follows from (2.8). The explicit value of \( \sigma_0 \) in (3.48) is given in Proposition 2.2 (iv) and computed in the Appendix. Recall the representation in (3.31) where \( B_{N,n} = N^{1/2\beta}, N/B_{N,n}^{2\beta} = 1 \) and

\[ z_{N,n}(\tau; x_1, x_2) := N^{-3/4\beta}n^{-1/2} \sum_{s_1, s_2 \in \mathbb{Z}} \varepsilon_1(s_1)\varepsilon_2(s_2) \prod_{t=1}^{[\tau]} (1 - \frac{x_t}{N^{1/2\beta}})^{t-s_1}1(t \geq s_i). \] (3.49)

Let us prove the (conditional) CLT:

\[ z_{N,n}(\tau; x_1, x_2) \to \text{fdd} \ (2x_1x_2(x_1 + x_2))^{-1/2}B(\tau) \] (3.50)

implying the point-wise convergence

\[ E[1 - e^{i\theta z_{N,n}(\tau; x_1, x_2)}] \to 1 - e^{-\theta^2/4x_1x_2(x_1 + x_2)} \] (3.51)
Arguing as in the proof of (2.29) it is easy to show that
\[ \frac{N}{N^{3/2}} \sum_{s=0}^{N-1} f(s,k) \xi_1(s) \xi_2(s) \to \mathcal{N}(0,1/2x_1x_2(x_1+x_2)) \]  
(3.52)

Towards this aim, write \( z_{N,n}^+(x_1, x_2) \) as a sum of zero-mean square-integrable martingale difference array
\[
z_{N,n}^+(x_1, x_2) = \sum_{k=1}^n Z_k, \quad Z_k := \varepsilon_1(k) \sum_{s=1}^{k-1} f(k, s) \varepsilon_2(s) + \varepsilon_2(k) \sum_{s=1}^{k-1} f(s, k) \varepsilon_1(s) + \varepsilon_1(k) \varepsilon_2(k) f(k, k)
\]
with respect to the filtration \( \mathcal{F}_k \) generated by \( \{\varepsilon_i(s), 1 \leq s \leq k, i = 1, 2\} \), \( 0 \leq k \leq n \), where
\[
f(s_1, s_2) := N^{-3/4} n^{-1/2} \sum_{s=1}^n 1(1 - \frac{s}{n})^{t-s_i} 1(t \geq s_i), \quad 1 \leq s_1, s_2 \leq n
\]
Accordingly, the second convergence in (3.52) follows from
\[ \sum_{k=1}^n \mathbb{E}[Z_k^2 | \mathcal{F}_{k-1}] \to \mathbb{E} \frac{1}{2x_1x_2(x_1+x_2)} \quad \text{and} \quad \sum_{k=1}^n \mathbb{E}[Z_k^2 1(\{|Z_k| > \epsilon\})] \to 0 \quad \text{for any } \epsilon > 0. \]  
(3.53)

Note the conditional variance \( v_k^2 := \mathbb{E}[Z_k^2 | \mathcal{F}_{k-1}] = (\sum_{s=1}^{k-1} f(k, s) \xi_2(s))^2 + (\sum_{s=1}^{k-1} f(s, k) \xi_1(s))^2 + f^2(k, k) \), where
\[ \sum_{k=1}^n \mathbb{E}Z_k^2 = \sum_{k=1}^n \mathbb{E}v_k^2 = \sum_{s_1, s_2 = 1}^{n} f^2(s_1, s_2) = \mathbb{E}(z_{N,n}^+(x_1, x_2)^2) \to \frac{1}{2x_1x_2(x_1+x_2)} \]  
(3.54)

is a direct consequence of the asymptotics in (2.27) where \( a_i = 1 - x_i/N^{1/2\beta}, a_j = 1 - x_j/N^{1/2\beta} \). Therefore the first relation in (3.53) follows from (3.54) and
\[ R_n := \sum_{k=1}^n (v_k^2 - \mathbb{E}v_k^2) = o_p(1). \]  
(3.55)

To show (3.55) we split \( R_n = R_n' + R_n'' \) into the sum of ‘diagonal’ and ‘off-diagonal’ parts, viz.,
\[
R_n' := \sum_{i=1}^2 \sum_{1 \leq s < n} c_i(s) (\varepsilon_2^2(s) - 1), \quad R_n'' := \sum_{i=1}^2 \sum_{1 \leq s_1, s_2 < n, s_1 \neq s_2} c_i(s_1, s_2) \varepsilon_i(s_1) \varepsilon_i(s_2)
\]
where \( c_1(s) := \sum_{s < k \leq n} f^2(s, k), \quad c_2(s) := \sum_{s < k \leq n} f^2(k, s), \quad c_1(s_1, s_2) := \sum_{s_1, s_2 < k \leq n} f(s_1, k) f(s_2, k), \quad c_2(s_1, s_2) := \sum_{s_1 < s_2 < k \leq n} f(k, s_1) f(k, s_2). \) Using the elementary bound for \( 1 \leq s_1, s_2 \leq n, \sum_{i=1}^n \sum_{s_i=1}^{s_i \leq x_i} x_i^{a_2} (1 \leq s_i \leq s_2) \leq (a_2^{s_2-s_1} + a_1^{s_1-s_2}) S(a_1, a_2) \), we obtain
\[
|c_i(s)| \leq C n^{-1} x_i^{a_1} (x_1 + x_2)^{-2}, \quad \sum_{s_1, s_2 = 1}^{n} c_2^2(s_1, s_2) \leq C \lambda_{N,n} x_i^{-3} (x_1 + x_2)^{-4}, \quad i = 1, 2. \]  
(3.56)
By (3.56), for $1 < p < 2$ and $x_1, x_2 > 0$ fixed

$$
E|R_n^p|^p \leq C \sum_{i=1}^{n-1} \sum_{s=1}^{n-1} |G_i(s)|^p \leq Cn^{-(p-1)} = o(1),
$$

(3.57)

$$
E|R_n^p|^2 \leq \sum_{i=1}^{n-1} \sum_{s=1}^{n-1} c_i^2(s_1, s_2) \leq C\lambda_{N,n} = o(1),
$$

(3.58)

proving (3.55) and the first relation in (3.53). The proof of the second relation in (3.53) is similar since it reduces to $T_n := \sum_{k=1}^{n} E[|Z_k|^{2p}] = o(1)$ for the same $1 < p < 2$, where $E|Z_k|^{2p} \leq C(\sum_{s=1}^{k-1} f(k, s) \varepsilon_1(s))^{2p} + E|\sum_{s=1}^{k-1} f(s, k) \varepsilon_1(s)|^{2p} \leq C((\sum_{s=1}^{k-1} f(k, s)^p) + (\sum_{s=1}^{k-1} f(k, k)^p + |f(k, k)|^{2p})$ by Rosenthal's inequality, see e.g. ([6], Lemma 2.5.2), and the sum $T_n = O(n^{-(p-1)}) = o(1)$ similarly to (3.57). This proves (3.53), (3.52), and the pointwise convergence in (3.51).

Now we return to the proof of (3.29), whose both sides are written as respective integrals (3.31) and (3.47). Due to the convergence of the integrands (see (3.51)), it suffices to justify the passage to the limit using a DCT argument. The dominating function independent of $N, n$ is obtained from (3.31) and $Ez_{N,n}(\tau; x_1, x_2) = 0$ and from (3.40), (3.41), (2.8) similarly as in the case $\lambda_\infty \in (0, \infty)$ above. This proves (3.29).

Proofs of (3.30) and (3.22) are completely analogous as in the case $\lambda_\infty \in (0, \infty)$ above except that we now choose $q = \log N$ and that $n$ in (3.43) and elsewhere in the proof of (3.30) and (3.22), case $\lambda_\infty \in (0, \infty)$, must be replaced by $N^{1/2\beta}$. This ends the proof of Theorem 3.1, case (ii).

### 3.3 Proof of Theorem 3.1 (i): case $\lambda_\infty = \infty$, or $n = o(N^{1/2\beta})$

**Case $1 < \beta < 3/2$. Proof of (3.29).** In this case, $\Phi(\theta) := -\sigma_\infty^2 \tau^{2(2-\beta)} \theta^2/2, B_{N,n} = n$ and $A_{N,n} = n^2 \lambda_{N,n}^2 = n^{2-\beta} N^{1/2}$. Rewrite the l.h.s. of (3.29) as

$$
N[\hat{\varepsilon}_{N,n}(\tau) - 1] = \int_0^\tau E\Lambda_{N,n}(\theta; \tau; x_1, x_2)(x_1 x_2)^{\beta-1}dx_1dx_2, \\
\Lambda_{N,n}(\theta; \tau; x_1, x_2) := \Lambda_{N,n}^\beta(\theta, \tau; x_1, x_2) - i\theta \lambda_{N,n}^{\beta} \hat{z}_{N,n}(\tau; x_1, x_2)
$$

(3.59)

and where $\hat{z}_{N,n}(\tau; x_1, x_2)$ is defined as in (3.32) with $A_{N,n}$ replaced by $\hat{A}_{N,n} := n^2 = A_{N,n}/\lambda_{N,n}^\beta$. As shown in the proof of Case (iii) (the ‘intermediate limit’), for any $x_1, x_2 > 0$

$$
\hat{z}_{N,n}(\tau; x_1, x_2) \rightarrow z(\tau; x_1, x_2) \quad \text{and} \quad E\hat{z}_{N,n}^2(\tau; x_1, x_2) \rightarrow Ez^2(\tau; x_1, x_2)
$$

(3.60)

see (3.35), where $z(\tau; x_1, x_2)$ is defined in (2.11) and the last expectation in (3.60) is given in (A.2). Then using Skorohod’s representation we extend (3.60) to $\hat{z}_{N,n}(\tau; x_1, x_2) \rightarrow z(\tau; x_1, x_2)$ a.s. implying also $\Lambda_{N,n}(\tau; x_1, x_2) \rightarrow -((\beta^2/2))z^2(\tau; x_1, x_2)$ a.s. Since $|\Lambda_{N,n}(\theta; \tau; x_1, x_2)| \leq Cz^2(\tau; x_1, x_2)$ and (3.60) holds, by Pratt’s lemma we obtain

$$
G_{N,n}(\tau; x_1, x_2) \rightarrow -(1/2)Ez^2(\tau; x_1, x_2), \quad \forall (x_1, x_2) \in \mathbb{R}_+^2.
$$

(3.61)

Relation (3.29) follows from (3.59), (3.61) and the DCT, using the dominating bound

$$
|G_{N,n}(\tau; x_1, x_2)| \leq CE\hat{z}_{N,n}^2(\tau; x_1, x_2) \leq \frac{C}{x_1 x_2} \min\{1, \frac{1}{x_1 + x_2}\} =: \bar{G}(x_1, x_2),
$$

(3.62)

see (3.41), and integrability of $\bar{G}$, see (2.9).
Proof of (3.30) is similar to that in case (iii) \(0 < \lambda_\infty < \infty\) above with \(q = \lfloor \log n \rfloor\). It suffices to check the bound (3.43) for \(T_{ij}(\theta) = T'_{ij}(\theta) + T''_{ij}(\theta)\) given in (3.42). By the same argument as in (3.44), we obtain
\[
|T_{ij}(\theta)| \leq C N^{-\beta} \int_{[0,n]^3} \frac{1}{x_1x_2} \min \left\{ 1, \frac{1}{x_1 + x_2} \right\} (x_1x_2x_3)^{\beta-1} I(x_3 < x_1) dx_1 dx_2 dx_3 \leq C n^{-\beta} (T' + T''),
\]
where
\[
T'_n := \int_0^n \min \left\{ 1, \frac{1}{x_1} \right\} x_1^{\beta-2} dx_1 \int_0^{x_1} x_2^{\beta-2} dx_2 = C \left( \int_0^1 x_1^{3\beta-3} dx_1 + \int_1^n x_1^{3\beta-4} dx_1 \right) \leq C N^{\beta-3}
\]
and
\[
T''_n := \int_0^n \min \left\{ 1, \frac{1}{x_2} \right\} x_2^{\beta-2} dx_2 \int_0^{x_2} x_1^{\beta-2} dx_1 = C \left( \int_0^1 x_2^{3\beta-3} dx_2 + \int_1^n x_2^{3\beta-4} dx_2 \right) \leq C N^{\beta-3}.
\]
Then \(|T''_{ij}(\theta)| \leq C N^{-\beta}[y_i^2(\tau) 1(a_i > a_{j+1})]|\) can be handled in the same way. Whence, the bound in (3.43) follows with any \(0 < \delta < 3 - 2\beta\), for \(1 < \beta < 3/2\). This proves (3.30). Proof of (3.22) using \(\tilde{N}_q/N \to 0\) and \(L_q = N - q\tilde{N}_q < q = o(N)\) is completely analogous to that in case (iii) \(0 < \lambda_\infty < \infty\). This completes the proof of Theorem 3.1, case (i) for \(1 < \beta < 3/2\).

Case \(0 < \beta < 1\). Proof of (3.29). In the rest of this proof, write \(\lambda \equiv \lambda_{N,n} = N^{1/2\beta}/n \to \infty\) for brevity. Also denote \(\lambda' := \lambda \log \lambda \)^{1/2\beta}, \(\log \lambda'/\log \lambda \to 1\). Let \(B_{N,n} := \lambda' n\) then
\[
z_{N,n}(\tau; x_1, x_2) := \frac{1}{\lambda' n^2} \sum_{s_1, s_2 \in \mathbb{Z}} \epsilon_1(s_1) \epsilon_2(s_2) \sum_{t=1}^{[n\tau]} \prod_{i=1}^{t} \left( 1 - \frac{x_i}{\lambda' n} \right)^{t-s_i} I(t \geq s_i).
\]
(3.63)

Split the r.h.s. of (3.29) as follows:
\[
\mathbb{E}[e^{\frac{1}{2} y_{N,n}(\tau)} - 1] = \frac{1}{\log \lambda} \int_{[0,\lambda']^2} \left( I(1 < x_1 + x_2 < \lambda) + I(x_1 + x_2 > \lambda) + I(x_1 + x_2 < 1) \right) \times \mathbb{E}[e^{\frac{1}{2} y_{N,n}(\tau; x_1, x_2)} - 1] (x_1x_2)^{\beta-1} dx_1 dx_2 =: \sum_{i=1}^{3} L_i.
\]
Here, \(L_1\) is the main term and \(L_i, i = 2, 3\) are remainders. Indeed, \(|L_3| = O(1/\log \lambda) = o(1)\). To estimate \(L_2\) we need the bound
\[
\mathbb{E}[y_{N,n}(\tau; x_1, x_2)] \leq \frac{C}{x_1x_2} \min \left( 1, \frac{\lambda'}{x_1 + x_2} \right),
\]
(3.64)
which follows from (2.29) similarly to (3.41). Using (3.64) we obtain
\[
|L_2| \leq \frac{C}{\log \lambda} \int_{x_1 + x_2 > \lambda} \min \left( 1, \frac{\lambda'}{x_1x_2(x_1 + x_2)} \right) (x_1x_2)^{\beta-1} dx_1 dx_2 = \frac{C}{\log \lambda} (J'_{\lambda} + J''_{\lambda}),
\]
(3.65)
where, by change of variables: \(x_1 + x_2 = y, x_1 = yz,\)
\[
J'_{\lambda} := \int_{x_1 + x_2 > \lambda} I(x_1x_2(x_1 + x_2) < \lambda') (x_1x_2)^{\beta-1} dx_1 dx_2
\]
\[
= \int_{\lambda}^\infty \int_0^1 \left( y^3 z(1 - z) < \lambda' \right) y^{2\beta-1} (z(1 - z))^{\beta-1} dz dy
\]
\[
\leq C \int_\lambda^\infty y^{2\beta-1} dy \int_0^{1/2} z^{\beta-1} I(y^3 z < 2\lambda') dz \leq C(\lambda')^\beta \int_\lambda^\infty y^{-\beta-1} dy = C(\log \lambda)^{1/2}
\]
since $0 < \beta < 1$. Similarly,

$$J''_\lambda := \lambda' \int_{x_1 + x_2 > \lambda} 1(x_1 x_2(x_1 + x_2) > \lambda')(x_1 + x_2)^{-1}(x_1 x_2)^{\beta - 2}dx_1 dx_2$$

$$\le C \lambda' \int_{\lambda}^\infty y^{2\beta - 4} dy \int_0^{1/2} z^{\beta - 2} 1(y^3 z > \lambda') dz \le C(\log \lambda)^{1/2}.$$  

This proves $|L_2| = O(1/\log \lambda) = o(1)$.

Consider the main term $L_1$. Although $Ee^{i\theta z_{N,n}(\tau;x_1,x_2)}$ and hence the integrand in $L_1$ point-wise converge for any $(x_1, x_2) \in \mathbb{R}_+^2$, see below, this fact is not very useful since the contribution to the limit of $L_1$ from bounded $x_i$'s is negligible due to the presence of the factor $1/\log \lambda \to 0$ in front of this integral. It turns out that the main (non-negligible) contribution to this integral comes from unbounded $x_1, x_2$ with $x_1/x_2 + x_2/x_1 \to \infty$ and $x_1 x_2 \to z \in \mathbb{R}_+$. To see this, by change of variables: $y = x_1 + x_2, x_1 = yw$ and then $w = z/y^2$ we rewrite

$$L_1 = \frac{1}{\log \lambda} \int_1^\lambda V_{N,n}(\theta; y) \frac{dy}{y},$$  

where

$$V_{N,n}(\theta; y) := 2 \int_0^{y^{2/2}} E[\exp \{ i\theta z_{N,n}(\tau; \frac{z}{y}, y(1 - \frac{z}{y^2}) \} - 1] z^{\beta - 1}(1 - \frac{z}{y^2})^{\beta - 1} dz.$$  

In view of $L_i = o(1), i = 2, 3$ relation (3.29) follows from representation (3.66) and the existence of the limit:

$$\lim_{y \to \infty, y = O(\lambda)} V_{N,n}(\theta; y) = V(\theta) := -k_\infty |\theta|^{2\beta} z^{2\beta},$$  

where the constant $k_\infty > 0$ is defined below in (3.71). More precisely, (3.68) says that for any $\epsilon > 0$ there exists $K > 0$ such that for any $N, n, y \ge K$ satisfying $y \le \lambda, \lambda \ge K$

$$|V_{N,n}(\theta; y) - V(\theta)| < \epsilon.$$  

To show that (3.69) implies $L_1 \to V(\theta)$ it suffices to split $L_1 - V(\theta) = (\log \lambda)^{-1} \int_K^\lambda (V_{N,n}(\theta; y) - V(\theta)) \frac{dy}{y}$ and use (3.69) together with the fact that $|V_{N,n}(\theta; y)| \le C$ is bounded uniformly in $N,n,y$.

To prove (3.69), rewrite $V(\theta)$ of (3.68) as the integral

$$V(\theta) = 2 \int_0^\infty z^{\beta - 1} E(e^{i\theta \tau Z_1 Z_2/2\sqrt{\pi} - 1}) dz = -2E \int_0^\infty z^{\beta - 1}(1 - e^{-\theta^2 \tau^2 Z_1^2/8\tau}) dz = -k_\infty |\theta|^{2\beta} z^{2\beta}$$  

with $Z_1, Z_2 \sim N(0,1)$ independent normals and

$$k_\infty = 2E \int_0^\infty z^{\beta - 1}(1 - e^{-Z_1^2/8\tau}) dz = 2^{1 - 3\beta} E|Z_1|^{2\beta} \int_0^\infty z^{\beta - 1}(1 - e^{-1/z}) dz$$

$$= 2^{1 - 2\beta} \Gamma(\beta + \frac{1}{2}) \Gamma(1 - \beta)/\sqrt{\pi}.$$  

Let $\Lambda_{N,n}(z; y) := E[\exp \{ i\theta z_{N,n}(\tau; \frac{z}{y}, y(1 - \frac{z}{y^2}) \} - 1], \Lambda(z) := E[e^{i\theta \tau Z_1 Z_2/2\sqrt{\pi} - 1}$ denote the corresponding expectations in (3.67), (3.70). Clearly, (3.69) follows from

$$\lim_{y \to \infty, y = O(\lambda)} \Lambda_{N,n}(z; y) = \Lambda(z), \quad \forall \ z > 0$$  

and

$$|\Lambda_{N,n}(z; y)| \le C(1 \wedge (1/z)), \quad \forall \ 0 < y < \lambda, \ 0 < z < y^2/2.$$  

(3.73)
The dominating bound in (3.73) is a consequence of (3.64). To show (3.72) use Proposition 2.1 by writing $z_{N,n}(\tau; z/y, y') := y(1 - z/y^2)$ in (3.67) as the quadratic form: $z_{N,n}(\tau; z/y, y') = Q_{12}(h_{\alpha_1, \alpha_2}(\cdot; \tau; z))$ with

$$h_{\alpha_1, \alpha_2}(s_1, s_2; \tau; z) := \sqrt{z/y} \frac{1}{\sqrt{\alpha_1 \alpha_2 n}} \sum_{t=1}^{[\tau]} \prod_{i=1}^{2} \left(1 - \frac{1}{\alpha_i}\right)^{t-s_i} \mathbf{1}(t \geq s_i), \quad s_1, s_2 \in \mathbb{Z},$$

$$\alpha_1 := \lambda'ny/z, \quad \alpha_2 := \lambda'n/y'.$$

If

$$n, \alpha_1, \alpha_2, y, y' \to \infty$$

so that $y/y' \to 1$ and $n = o(\alpha_i), \; i = 1, 2,$

then

$$\tilde{h}_{\alpha_1, \alpha_2}(s_1, s_2; \tau; z) := \sqrt{\alpha_1 \alpha_2} h_{\alpha_1, \alpha_2}([\alpha_1 s_1], [\alpha_2 s_2]; \tau; z)$$

$$= \sqrt{z/y} \frac{1}{\sqrt{\alpha_1 \alpha_2 n}} \sum_{t=1}^{[\tau]} \prod_{i=1}^{2} \left(1 - \frac{1}{\alpha_i}\right)^{t-[\alpha_i s_i]} \mathbf{1}(t \geq [\alpha_i s_i])$$

$$\to \frac{\tau}{\sqrt{z}} \prod_{i=1}^{2} \mathbf{1}(s_i < 0) = h(s_1, s_2; \tau; z)$$

point-wise for any $\tau > 0, z > 0, s_i \in \mathbb{R}, s_i \neq 0, i = 1, 2$ fixed. Moreover, under the same conditions (3.75), $\|\tilde{h}_{\alpha_1, \alpha_2}(\cdot; \tau; z) - h(\cdot; \tau; z)\| \to 0$, implying the convergence $Q_{12}(h_{\alpha_1, \alpha_2}(\cdot; \tau; z)) \to d_1 I_{12}(h(\cdot; \tau; z)) = d_1 \tau Z_1 Z_2/2\sqrt{\pi}, Z_i \sim N(0,1), \; i = 1, 2$ by Proposition 2.1. Conditions on $N, n, y, y', \lambda$ in (3.75) are obviously satisfied due to $y, y' = O(\lambda) = o(\lambda')$. This proves (3.72) and (3.68), thereby completing the proof of (3.29).

Proof of (3.30). For $T_{ij}(\theta)$ defined by (3.42) let us prove (3.43). Denote $N'_{\lambda} := (N \log \lambda)^{1/2\beta}$. Similarly to (3.44) we have that

$$|T_{ij}(\theta)| \leq \frac{C}{N^{1/2}(\log \lambda)^{3/2}} \int_{(0,N'_{\lambda})^3} \min \left\{1, E z_{n,n}^2(\tau; x_1, x_2)\right\}(x_1 x_2 x_3)^{\beta-1} \mathbf{1}(x_3 < x_1) dx_1 dx_2 dx_3$$

with $z_{n,n}(\tau; x_1, x_2)$ defined by (3.63). Whence using (3.64) similarly as in the proof of case (i) we obtain

$$|T_{ij}(\theta)| \leq \frac{C}{N^{1/2}(\log \lambda)^{3/2}} \int_{(0,N'_{\lambda})^2} \min \left\{1, \frac{C}{x_1 x_2} \min \left\{1, \frac{\lambda'}{x_1 + x_2}\right\}\right\} x_1^{2\beta-1} x_2^{\beta-1} dx_1 dx_2$$

$$= \frac{C}{N^{1/2}(\log \lambda)^{3/2}} \sum_{i=1}^{3} T_{\lambda,i},$$

where

$$T_{\lambda,1} := \int_{(0,N'_{\lambda})^2} \mathbf{1}(x_1 + x_2 < \lambda') \min \left\{1, \frac{1}{x_1 x_2}\right\} x_1^{2\beta-1} x_2^{\beta-1} dx_1 dx_2,$$

$$T_{\lambda,2} := \int_{(0,N'_{\lambda})^2} \mathbf{1}(x_1 x_2(x_1 + x_2) < \lambda', x_1 + x_2 > \lambda') x_1^{2\beta-1} x_2^{\beta-1} dx_1 dx_2,$$

$$T_{\lambda,3} := \lambda' \int_{(0,N'_{\lambda})^2} \mathbf{1}(x_1 x_2(x_1 + x_2) > \lambda', x_1 + x_2 > \lambda') x_1^{2\beta-2} x_2^{\beta-2} dx_1 dx_2/(x_1 + x_2).$$

By changing variables $x_1, x_2$ as in (3.66)-(3.67) we get $T_{\lambda,1} \leq C \int_{0}^{\lambda'} y^{\beta-1} dy \leq C(\lambda')^{\beta}$. Also, similarly to the estimation of $J'_\lambda, J''_\lambda$ following (3.65) we obtain $T_{\lambda,2} + T_{\lambda,3} \leq C(\lambda')^{\beta} \int_{\lambda'}^{2N'_{\lambda} / \lambda} y^{\beta-1} dy \leq C(\lambda')^{\beta} \log(2N'_{\lambda}/\lambda')$. Hence, we conclude that

$$|T_{ij}(\theta)| \leq \frac{C(\lambda')^{\beta} \log(N'_{\lambda}/\lambda')}{N^{1/2}(\log \lambda)^{3/2}} \leq \frac{C \log n}{n^{\beta} \log \lambda},$$

20
proving (3.43) with any \(0 < \delta < \beta\). This proves (3.30). We omit the proof of (3.22) which seems completely similar to that in case (iii) and elsewhere. This completes the proof of Theorem 3.1 for \((t,s) = (0,1)\).

Proof of Theorem 3.1 in the general case \((t,s) \in \mathbb{Z}^2, s \geq 1\). Similarly to (3.17) we decompose \(S_{N,n}^{t,s}(\tau)\) in (3.1) as

\[
S_{N,n}^{t,s}(\tau) = S_{N,n}^{t,s}(\tau) + S_{N,n}^{t,s\uparrow}(\tau) + S_{N,n}^{t,s\downarrow}(\tau)
\]

where the main term

\[
S_{N,n}^{t,s}(\tau) := \sum_{k=1}^{\tilde{N}_q} \sum_{q-k-s \leq i \leq q} X_i(u)X_{i+s}(u + t)
\]

is a sum of independent \(\tilde{N}_q = \left[\frac{N}{q}\right]\) blocks of size \(q - s = q_N, n - s \to \infty\), and

\[
S_{N,n}^{t,s\uparrow}(\tau) := \sum_{k=1}^{\tilde{N}_q} \sum_{q-k-s \leq i \leq q} X_i(u)X_{i+s}(u + t), \quad S_{N,n}^{t,s\downarrow}(\tau) := \sum_{q\tilde{N}_q \leq i \leq N} \sum_{u=1}^{\lceil \nu r \rceil} X_i(u)X_{i+s}(u + t)
\]

are remainder terms. The proof of (3.29)-(3.30) for \(A_{N,n}^{-1}Y_{N,n}^{t,s}(\tau) = \sum_{i=1}^{\lceil \nu r \rceil} y_i^{t,s}(\tau), \ y_i^{t,s}(\tau) := A_{N,n}^{-1} \sum_{u=1}^{\lceil \nu r \rceil} X_i(u)X_{i+s}(u + t)\) is completely analogous since the distribution of \(y_i^{t,s}(\tau)\) does not depend on \(t\) and \(s \neq 0\).

3.4 Proof of Theorem 3.2

The proof uses the following result of [20].

Lemma 3.1. ([20], Lemma 7.1) Let \(\{\xi_{ni}, 1 \leq i \leq N_n\}, n \geq 1\), be a triangular array of \(m\)-dependent r.v.s with zero mean and finite variance. Assume that: (L1) \(\xi_{ni}, 1 \leq i \leq N_n\), are identically distributed for any \(n \geq 1\), (L2) \(\xi_{n1} \to_d \xi\), \(E\xi_{n1}^2 \to E\xi^2 < \infty\), for some r.v. \(\xi\) and (L3) \(\text{var} \left( \sum_{i=1}^{N_n} \xi_{ni} \right) \sim \sigma^2 N_n, \sigma^2 > 0\). Then \(N_n^{-1/2} \sum_{i=1}^{N_n} \xi_{ni} \to_d N(0, \sigma^2)\).

For notational simplicity, we consider only one-dimensional convergence at \(\tau > 0\). Let \((Nn)^{-1/2} S_{N,n}^{t,s}(\tau) = N^{-1/2} \sum_{i=1}^{N} \xi_{ni}\), where \(\xi_{ni} := n^{-1/2} \sum_{u=1}^{\lceil \nu r \rceil} X_i(u)X_{i+s}(u + t)\), \(1 \leq i \leq N\) are \(|s|\)-dependent, identically distributed random variables with zero mean and finite variance. Since \(\xi_{ni}, 1 \leq i \leq N\) are uncorrelated, it follows that \(E(\sum_{i=1}^{N} \xi_{ni})^2 = NE\xi_{n1}^2\), where \(\xi_{n1} = \tau A_{12} \xi_1\), and so \(E\xi_{n1}^2 \sim \sigma^2\), where \(\sigma^2 := E\xi_{12} < \infty\). It remains to show that \(\xi_n \to_d \tau A_{12} B(\tau)\), where \(A_{12}\) is independent of \(B(\tau)\). This follows from the martingale CLT similarly to (3.50). By the lemma above, we conclude that \((Nn)^{-1/2} S_{N,n}^{t,s}(\tau) \to_d \sigma B(\tau)\). Theorem 3.2 is proved.

4 Asymptotic distribution of temporal (iso-sectional) sample covariances

The limit distribution of iso-sectional sample covariances \(\tilde{Y}_{N,n}(t,0)\) in (1.5) and the corresponding partial sums process \(S_{N,n}^{t,0}(\tau)\) of (3.1) is obtained similarly as in the cross-sectional case, with certain differences which are discussed below. Since the conditional expectation \(E[S_{N,n}^{t,0}(\tau)|a_1, \cdots, a_N] =: T_{N,n}(\tau) \neq 0\), a natural decomposition is

\[
S_{N,n}^{t,0}(\tau) = \tilde{S}_{N,n}^{t,0}(\tau) + T_{N,n}(\tau)
\]
where $S_{N,n,\tau}^t(\tau) := S_{N,n,n}^t(\tau) - T_{N,n,\tau}^t(\tau)$ is the conditionally centered term with $E[S_{N,n,\tau}^t(\tau)|a_1, \cdots, a_N] = 0$, and

$$T_{N,n,\tau}^t(\tau) := [\alpha] \sum_{i=1}^N a_i^t/(1 - a_i^2), \quad t \geq 0 \quad (4.2)$$

is proportional to a sum of i.i.d.r.\'s $a_i^t/(1 - a_i^2), 1 \leq i \leq N$ with regularly decaying tail distribution function

$$P(a_i^t/(1 - a_i^2) > x) \sim P(a > 1 - \frac{1}{2x}) \sim c_\alpha x^{-\beta}, \quad x \to \infty, \quad c_\alpha := \psi(1)/2^{\beta},$$

see condition (1.2). Accordingly, the limit distribution of appropriately normalized and centered term $T_{N,n,\tau}^t(\tau)$ does not depend on $t$ and can be found from the classical CLT and turns out to be a $(\beta \wedge 2)$-stable line, under normalization $nN^{1/(\beta \wedge 2)} (\beta \neq 2)$. The other term, $S_{N,n,\tau}^t(\tau)$, in (4.1), is a sum of mutually independent partial sums processes $Y_{i,n}^t(\tau) := \sum_{u=1}^{[\alpha]} (X_i(u)X_i(u + t) - E[X_i(u)X_i(u + t)|a_i]), 1 \leq i \leq N$ with conditional variance

$$\text{var}[Y_{i,n}^t(1)|a_i] \sim nA_{i}$, \quad n \to \infty, \quad \text{where} \quad A_{i}^{t,0} := \frac{1+a_i^t}{1-a_i^2} \left(\frac{1+a_i^t}{1-a_i^2} + \frac{a_i^t(2t+\text{cumn}_i)}{1-a_i^2}\right).$$

The proof of the last fact follows similarly to that of (2.28) and is omitted. As $a \uparrow 1, A_{i}^{t,0} \sim 1/2(1 - a_i)^3$ and the limit distribution of $S_{N,n,\tau}^t(\tau)$ can be shown to exhibit a trichotomy on the interval $0 < \beta < 3$ depending on the limit $\lambda_\infty^*$ in (4.3). It turns out that for $\beta > 2$ the asymptotically Gaussian term $T_{N,n,\tau}^t(\tau)$ dominates $S_{N,n,\tau}^t(\tau)$ in all cases of $\lambda_\infty^*$, while in the interval $0 < \beta < 2 T_{N,n,\tau}^t(\tau)$ and $S_{N,n,\tau}^t(\tau)$ have the same convergence rate. Somewhat surprisingly, the limit distribution of $S_{N,n,\tau}^t(\tau)$ is a $\beta$-stable line in both cases $\lambda_\infty^* = \infty$ and $\lambda_\infty^* = 0$ with different scale parameters of the random slope coefficient of this line.

Rigorous description of the above limit results is given in the following Theorems 4.1 and 4.2. The proofs of these theorems are similar and actually simpler than the corresponding Theorems 3.1 and 3.2 dealing with non-horizontal sample covariances, due to the fact that $S_{N,n,\tau}^t(\tau)$ is a sum of row-independent summands contrary to $S_{N,n,\tau}^{t,s}(\tau), s \neq 0$. Because of this, we omit some details of the proof of Theorems 4.1 and 4.2. We also omit the more delicate cases $\beta = 1$ and $\beta = 2$ where the limit results may require a change of normalization or additional centering.

**Theorem 4.1.** Let the mixing distribution satisfy condition (1.2) with $0 < \beta < 2, \beta \neq 1$. Let $N, n \to \infty$ so that

$$\lambda_\infty^* := \frac{N^{1/\beta}}{n} \to \lambda_\infty^* \in [0, \infty]. \quad (4.3)$$

In addition, assume $Ez^4(0) < \infty$. Then the following statements (i)-(iii) hold for $S_{N,n,\tau}^t(\tau), t \in \mathbb{Z}$ in (3.1) depending on $\lambda_\infty^*$ in (4.3).

(i) Let $\lambda_\infty^* = \infty$. Then

$$n^{-1}N^{-1/\beta}(S_{N,n,\tau}^t(\tau) - E S_{N,n,\tau}^t(\tau)1(1 < \beta < 2)) \to \text{fdd} \tau V_\beta^*, \quad (4.4)$$

where $V_\beta^*$ is a completely asymmetric $\beta$-stable r.v. with characteristic function in (4.7) below.

(ii) Let $\lambda_\infty^* = 0$. Then

$$n^{-1}N^{-1/\beta}(S_{N,n,\tau}^t(\tau) - E S_{N,n,\tau}^t(\tau)1(1 < \beta < 2)) \to \text{fdd} \tau V_\beta^+ \quad (4.5)$$

where $V_\beta^+$ is a completely asymmetric $\beta$-stable r.v. with characteristic function in (4.7) below.
(iii) Let $0 < \lambda_\infty < \infty$. Then

\[ n^{-1}N^{-1/\beta}(S_{N,n}^{t,0}(\tau) - ES_{N,n}^{t,0}(\tau)1(1 < \beta < 2)) \xrightarrow{\text{fdd}} \lambda_\infty Z_\beta^*(\tau/\lambda_\infty), \]  

(4.6)

where $Z_\beta^*$ is the ‘diagonal intermediate’ process in (2.24).

**Remark 4.1.** The r.v.s $V_\beta^*$ and $V_\beta^+$ in (4.4) and (4.5) have respective stochastic integral representations

\[ V_\beta^* = \int_{\mathbb{R} \times C([\mathbb{R}])} \left\{ \int_{-\infty}^{0} e^{xs}dB(s) \right\}^2 d(M_\beta^* - EM_\beta^*1(1 < \beta < 2)), \]

\[ V_\beta^+ = \int_{\mathbb{R} \times C([\mathbb{R}])} (2x)^{-1}d(M_\beta^* - EM_\beta^*1(1 < \beta < 2)) \]

w.r.t. Poisson random measure $M_\beta^*$ in (2.21). Note $\int_{-\infty}^{0} e^{xs}dB(s) =_{\text{law}} Z/\sqrt{x}, Z \sim N(0,1)$. The fact that both $V_\beta^*$ and $V_\beta^+$ have $\beta$-stable distribution follows from their characteristic functions:

\[ E\exp\{i\theta V_\beta^*\} = \exp\left\{\psi(1) \int_{0}^{\infty} (e^{i\theta Z^2/2x} - 1 - i(\theta Z/2x)1(1 < \beta < 2))x^{\beta-1}dx\right\} = e^{-c_\beta^*(\text{sign}(\theta))|\theta|^{\beta}}, \quad \theta \in \mathbb{R}, \]

\[ E\exp\{i\theta V_\beta^+\} = \exp\left\{\psi(1) \int_{0}^{\infty} (e^{i\theta Z^2/2x} - 1 - i(\theta Z/2x)1(1 < \beta < 2))x^{\beta-1}dx\right\} = e^{-c_\beta^+(\text{sign}(\theta))|\theta|^{\beta}}, \quad \theta \in \mathbb{R}, \]  

(4.7)

where

\[ c_\beta^*(\pm) := -\psi(1) \int_{0}^{\infty} E\left(e^{\pm iZ^2/2x} - 1 - (\pm iZ^2/2x)1(1 < \beta < 2))\right)x^{\beta-1}dx, \]

\[ c_\beta^+(\pm) := -\psi(1) \int_{0}^{\infty} (e^{\pm iZ^2/2x} - 1 - (\pm i/2x)1(1 < \beta < 2))x^{\beta-1}dx. \]

Note $c_\beta^*(\pm) = E|Z|^{2\beta}c_\beta^+(\pm)$ where $E|Z|^{2\beta} = 2^\beta \Gamma(\beta + 1/2)/\sqrt{\pi} \neq 1$ unless $\beta = 1$, implying that $V_\beta^*$ and $V_\beta^+$ have different distributions.

**Theorem 4.2.** Let the mixing distribution satisfy condition (1.2) with $\beta > 2$. In addition, assume $E\varepsilon^4(0) < \infty$. Then for any $t \in \mathbb{Z}$ as $N, n \to \infty$ in arbitrary way,

\[ n^{-1}N^{-1/\beta}(S_{N,n}^{t,0}(\tau) - ES_{N,n}^{t,0}(\tau)) \xrightarrow{\text{fdd}} \tau \sigma_\tau^* Z, \]

(4.8)

where $Z \sim N(0,1)$ and $(\sigma_\tau)^2 := \text{var}(a|t|/(1 - a^2))$.

**Remark 4.2.** If $\beta < 1$, then $\gamma(t,0)$ is undefined for any $t \in \mathbb{Z}$. By abuse of notation, we set $\gamma(t,0)1(1 < \beta < 2) := 0$ if $\beta < 1$ and $\gamma(t,0)$ if $\beta > 1$.

**Corollary 4.1.** (i) Let the conditions of Theorem 4.1 (i) be satisfied. Then for any $t \in \mathbb{Z}$

\[ N^{1-1/\beta}(\delta_{N,n}(t,0) - \gamma(t,0)1(1 < \beta < 2)) \xrightarrow{\text{d}} V_\beta^*. \]

(ii) Let the conditions of Theorem 4.1 (ii) be satisfied. Then for any $t \in \mathbb{Z}$

\[ N^{1-1/\beta}(\delta_{N,n}(t,0) - \gamma(t,0)1(1 < \beta < 2)) \xrightarrow{\text{d}} V_\beta^+. \]

(iii) Let the conditions of Theorem 4.1 (iii) be satisfied. Then for any $t \in \mathbb{Z}$

\[ N^{1-1/\beta}(\delta_{N,n}(t,0) - \gamma(t,0)1(1 < \beta < 2)) \xrightarrow{\text{d}} \lambda_\infty^* Z_\beta^*(1/\lambda_\infty^*). \]

(iv) Let the conditions of Theorem 4.2 be satisfied. Then for any $t \in \mathbb{Z}$

\[ N^{1/2}(\delta_{N,n}(t,0) - \gamma(t,0)) \xrightarrow{\text{d}} \sigma_\tau^* Z, \quad Z \sim N(0,1). \]
Proof of Theorem 4.1. Let \( t \geq 0 \) and
\[
y^{t,0}_\tau := \frac{1}{nN^{1/\beta}} \sum_{u=1}^{[nt]} (X(u)X(u+t) - EX(u)X(u+t)\mathbb{1}(1 < \beta < 2)).
\] (4.9)

It suffices to prove that
\[
\Phi^{t,0}_{N,n}(\theta) \to \Phi(\theta), \quad \text{as } N, n \to \infty, \lambda^*_{N,n} \to \lambda^*_\infty, \quad \forall \theta \in \mathbb{R},
\] (4.10)

where using \( 

\text{E}y^{t,0}_\tau \mathbb{1}(1 < \beta < 2) = 0 \)
\[
\Phi^{t,0}_{N,n}(\theta) := \text{NE}[e^{i\theta y^{t,0}_\tau} - 1 - i\theta y^{t,0}_\tau \mathbb{1}(1 < \beta < 2)], \quad \Phi(\theta) := \log \text{E}e^{i\Theta^*_{\beta}(\tau)},
\] (4.11)

and \( \Theta^*_{\beta}(\tau) \) denotes the limit process in (4.4)-(4.6). Similarly to (3.31)
\[
\Phi^{t,0}_{N,n}(\theta) = \psi(1) \int_{(0,1/N^{1/\beta}]} E[e^{i\theta z^{t,0}_{N,n}(\tau;x)} - 1 - i\theta z^{t,0}_{N,n}(\tau;x) \mathbb{1}(1 < \beta < 2)]x^{\beta-1}dx
\] (4.12)

where \( z^{t,0}_{N,n}(\tau;x) := y^{t,0}_\tau |_{a=1-x/N^{1/\beta}} \). Next we decompose \( y^{t,0}_\tau = y^*(\tau) + y^+(\tau) \) where
\[
y^*(\tau) := \frac{1}{nN^{1/\beta}} \sum_{u=1}^{[nt]} (X(u)X(u+t) - EX(u)X(u+t)|a]),
\]
y^+(\tau) := \frac{[nt]}{nN^{1/\beta}} (E[X(0)X(t)|a] - E[X(0)X(t)] \mathbb{1}(1 < \beta < 2))) = \frac{[nt]}{nN^{1/\beta}} \left( \frac{a^1}{1-a^2} - \frac{E(a^1(1 < \beta < 2))}{1-a^2} \right).

Accordingly, we decompose \( z^{t,0}_{N,n}(\tau;x) = z^{*}_{N,n}(\tau;x) + z^{+}_{N,n}(\tau;x) \) where
\[
z^{*}_{N,n}(\tau;x) := \frac{1}{nN^{1/\beta}} \sum_{s_1, s_2 \in Z} \varepsilon(s_1)\varepsilon(s_2) \sum_{u=1}^{[nt]} (1 - \frac{x}{N^{1/\beta}})^{2u+t-s_1-s_2} \mathbb{1}(u \geq s_1, u+t \geq s_2),
\] (4.13)
\[
z^{+}_{N,n}(\tau;x) := \frac{[nt]}{nN^{1/\beta}} \left( \frac{1 - \frac{x}{N^{1/\beta}}}{1 - (1 - \frac{x}{N^{1/\beta}})^2} - E[a^1(1 < \beta < 2)] \right)
\]
where \( \varepsilon(s_1)\varepsilon(s_2) := \varepsilon(s_1)\varepsilon(s_2) - E\varepsilon(s_1)\varepsilon(s_2) \).

Proof of (4.10), case \( 0 < \lambda^*_\infty < \infty \). We have
\[
\Phi^*(\theta) = \psi(1) \int_0^\infty E[e^{i\theta \lambda^*_\infty z^*(x/\lambda^*_\infty ; x)} - 1 - i\theta \lambda^*_\infty z^*(x/\lambda^*_\infty ; x) \mathbb{1}(1 < \beta < 2)]x^{\beta-1}dx,
\] (4.14)

where the last expectation is taken w.r.t. the Wiener measure \( P_B \). Similarly as in the proof of (3.29) we prove the point-wise convergence of the integrands in (4.12) and (4.14): for any \( x > 0 \)
\[
A^{t,0}_{N,n}(\theta;x) := E[e^{i\theta z^{t,0}_{N,n}(\tau;x)} - 1 - i\theta z^{t,0}_{N,n}(\tau;x) \mathbb{1}(1 < \beta < 2)]
\] (4.15)
\[
\rightarrow E[e^{i\theta \lambda^*_\infty z^*(x/\lambda^*_\infty ; x)} - 1 - i\theta \lambda^*_\infty z^*(x/\lambda^*_\infty ; x) \mathbb{1}(1 < \beta < 2)].
\]

The proof of (4.15) using Prop. 2.1 is very similar to that of (3.35) and we omit the details. Using (4.15) and the DCT we can prove the convergence of integrals, or (4.10). The application of the DCT is guaranteed by the dominating bound
\[
|A^{t,0}_{N,n}(\theta;x)| \leq C(1 \wedge 1/x)\{1(0 < \beta < 1) + (1/x)1(1 < \beta < 2)\},
\] (4.16)
which is a consequence of $|z_{N,n}^+(\tau;x)| \leq C/x, E(z_{N,n}^+(\tau;x))^2 \leq Cx^{-2}$, see (2.29). Particularly, for $0 < \beta < 1$ we get $|\Lambda_{N,n}^{t,0}(\theta;x)| \leq 2 (0 < x \leq 1), \leq E(|z_{N,n}^+(\tau;x)| + |z_{N,n}^+(\tau;x)|) \leq C(\sqrt{E}|z_{N,n}^+(\tau;x)|^2 + (1/x)) \leq C/x$ and (4.16) follows. For $1 < \beta < 2$ (4.16) follows similarly. This proves (4.10) for $0 < \lambda_*^\infty < \infty$.

Proof of (4.10), case $\lambda_*^\infty = 0$. In this case

$$\Phi^*(\theta) = \psi(1) \int_{\mathbb{R}^+} \left[ e^{i\theta(x/2x)} - 1 - i\theta(x/2x) \right] (1 < \beta < 2) x^{\beta-1} dx,$$

see (4.7). From (2.29) we have $E(z_{N,n}^+(\tau;x))^2 \leq Cx^{-2} \min(1, \lambda_*^\infty/x) \to 0$ and hence

$$\Lambda_{N,n}^{t,0}(\theta;x) \to e^{i\theta(x/2x)} - 1 - i\theta(x/2x) (1 < \beta < 2)$$

for any $x > 0$ similarly as in (4.15). Finally, the use of the dominating bound in (4.16) which is also valid in this case completes the proof of (4.10) for $\lambda_*^\infty = 0$. Proof of (4.10), case $\lambda_*^\infty = \infty$. In this case,

$$\Phi^*(\theta) = \psi(1) \int_{\mathbb{R}^+} E\left[ e^{i\theta(xZ^2/2x)} - 1 - i\theta(xZ^2/2x) \right] (1 < \beta < 2) x^{\beta-1} dx,$$

(4.17) see (4.7). Write $z_{N,n}^+(\tau;x)$ in (4.13) as quadratic form: $z_{N,n}^+(\tau;x) = Q_{11}(h(\tau;x; \cdot))$ in (2.3) and apply Proposition 2.1 with $a_1 = a_2 = \alpha := N^{1/\beta}$. Note $\tilde{h}((1,1)) = (1 - x/N^{1/\beta})u - N^{1/\beta}s1(u \geq N^{1/\beta}s1, u + t \geq N^{1/\beta}s2) \to g(s1, s2) := \tau e^{\tau(s1+s2)}1(s1 \vee s2 \leq 0)$ point-wise a.e. in $(s1, s2) \in \mathbb{R}^2$ and also in $L^2(\mathbb{R}^2)$. Then conclude $z_{N,n}^+(\tau;x) \to g_{11} = \int_{\mathbb{R}^2} g(s1, s2) dB(s1)dB(s2)$ is $\tau \{ (\int_0^\infty e^{\tau} dB(s))^2 - E(\int_0^\infty e^{\tau} dB(s))^2 \} = \tau (Z^2 - 1)/2x$ for any $x > 0$, where $Z \sim N(0,1)$. On the other hand, $z_{N,n}^+(\tau;x) \to \tau/2x$ and therefore

$$\Lambda_{N,n}^{t,0}(\theta;x) \to E\left[ e^{i\theta(xZ^2/2x)} - 1 - i\theta(xZ^2/2x) \right] (1 < \beta < 2)$$

for any $x > 0$, proving the point-wise convergence of the integrands in (4.12) and (4.17). The remaining details are similar as in the previous cases and omitted. This ends the proof of Theorem 4.1.

Proof of Theorem 4.2. Consider the decomposition in (4.1), where $n^{-1}T_{N,n}^{t,0}(\tau) = (1/n) \sum_{i=1}^N a_i^t/(1 - a_i^2)$ is a sum of i.i.d.r.v.s with finite variance $(\sigma_t^2) = \text{var}(a_i^t/(1 - a_i^2))$ and therefore

$$n^{-1}N^{-1/2} \left( T_{N,n}^{t,0}(\tau) - E T_{N,n}^{t,0}(\tau) \right) \to fdd \tau \sigma_t^* Z$$

holds by the classical CLT as $N, n \to \infty$ in arbitrary way and where $Z \sim N(0,1)$. Hence the statement of the theorem follows from $\tilde{S}_{N,n}^{t,0}(1) = o_P(nN^{1/2})$. By Proposition 2.4 (2.29) we have that $\text{var}(\tilde{S}_{N,n}^{t,0}(1)) = N\text{Var} \left[ \sum_{u=1}^N X(u) X(u+t)|a| \right] \leq C N^{n/2}(1 - a)^{-2} \min(1, (n(1 - a))^{-1})$, where the last expectation vanishes as $n \to \infty$, due to $E(1 - a)^{-2} < \infty$. Theorem 4.2 is proved.

A Appendix

A.1 Proofs of Prop. 2.2 and 2.3

Proof of Prop. 2.2. (i) The existence of $Z_\beta$ follows from

$$J_\beta := \int_{\mathbb{R}^1} |z(\tau; x_1, x_2)|^2 d\mu_\beta < \infty$$

(A.1)
and \( \mu_\beta(L_1) < \infty \). We have \( \mu_\beta(L_1) = \psi(1)^2 \int_{\mathbb{R}_+^2} 1(x_1 x_2(x_1 + x_2) < 1)(x_1 x_2)^{\beta - 1}dx_1 dx_2 \leq C \int_0^\infty x_1^{\beta - 1}dx_1 \int_0^{x_1} 1(x_2 < 1/x_1^2)x_2^{\beta - 1}dx_2 = C(\int_0^1 x_1^{\beta - 1}dx_1 \int_0^{x_1} x_2^{\beta - 1}dx_2 + \int_1^\infty x_1^{\beta - 1}dx_1 \int_0^{1/x_1^2} x_2^{\beta - 1}dx_2) \leq C(\int_0^1 x_1^{\beta - 1}dx_1 + \int_1^{\infty} x_1^{\beta - 1}dx_1) < \infty \) since \( \beta > 0 \).

Consider (A.1). Then

\[
J_\beta = C \int_{\mathbb{R}_+^2} 1(x_1 x_2(x_1 + x_2) > 1)E|z(\tau; x_1, x_2)|^2(x_1 x_2)^{\beta - 1}dx_1 dx_2,
\]

where

\[
E|z(\tau; x_1, x_2)|^2 = \int_{(0, \tau)^2} \prod_{i=1}^2 E|\mathcal{V}_i(u_1; x_i)\mathcal{V}_i(u_2; x_i)|du_1 du_2 = \frac{1}{4x_1 x_2} \int_{(0, \tau)^2} e^{-(x_1 + x_2)|u_1 - u_2|} du_1 du_2 \leq \frac{C \tau^2}{x_1 x_2} \left( 1 \wedge \frac{1}{\tau(x_1 + x_2)} \right). \tag{A.2}
\]

Hence,

\[
J_\beta \leq C \int_{\mathbb{R}_+^2} 1(x_1 x_2(x_1 + x_2) > 1)(x_1 + x_2)^{-1}(x_1 x_2)^{\beta - 2}dx_1 dx_2
\]

\[
\leq C \int_{\mathbb{R}_+^2} 1(x_2 > x_1, x_1 x_2^2 > 1)x_1^{\beta - 2} x_2^{\beta - 3}dx_1 dx_2
\]

\[
= C \left( \int_0^1 x_1^{\beta - 2}dx_1 \int_{x_1^{-1/2}}^\infty x_2^{\beta - 3}dx_2 + \int_1^\infty x_1^{\beta - 2}dx_1 \int_{x_1^{-1/2}}^\infty x_2^{\beta - 3}dx_2 \right) < \infty
\]

if \( 0 < \beta < 3/2 \). The remaining facts in (i) are easy and we omit the details.

(ii) Similarly as in ([17], proof of Prop. 3.1(ii)) it suffices to show for any \( 0 < p < 2\beta \) that

\[
\infty > J_{p, \beta}(\tau) := \left\{ \begin{array}{ll}
\int_{\mathbb{R}_+^2} E|z(\tau; x_1, x_2)|^p(x_1 x_2)^{\beta - 1}dx_1 dx_2, & 0 < p \leq 2, \\
\int_{\mathbb{R}_+^2} E[|z(\tau; x_1, x_2)|^p \vee |z(\tau; x_1, x_2)|^2](x_1 x_2)^{\beta - 1}dx_1 dx_2, & p > 2.
\end{array} \right. \tag{A.3}
\]

Let first \( 0 < p \leq 2 \). Using \( E|z(\tau; x_1, x_2)|^p \leq (E|z(\tau; x_1, x_2)|^2)^{p/2} \) and (A.2), we obtain

\[
J_{p, \beta}(\tau) \leq C \int_{\mathbb{R}_+^2} \left( \int_{(0, \tau)^2} e^{-(x_1 + x_2)|u_1 - u_2|} du_1 du_2 \right)^{p/2}(x_1 x_2)^{\beta - 1 - p/2}dx_1 dx_2 =: C \tau^{2(p - \beta)} I_{p, \beta}, \tag{A.4}
\]

where

\[
I_{p, \beta} \leq \int_{\mathbb{R}_+^2} \left( 1 \wedge \frac{1}{x_1 + x_2} \right)^{p/2}(x_1 x_2)^{\beta - 1 - p/2}dx_1 dx_2
\]

\[
\leq C \int_0^\infty \int_0^{x_1} \left( 1 \wedge \frac{1}{x_1} \right)^{p/2}(x_1 x_2)^{\beta - 1 - p/2}dx_1 dx_2
\]

\[
= C \int_0^\infty \left( 1 \wedge \frac{1}{x_1} \right)^{p/2} x_1^{2\beta - p - 1}dx_1 < \infty \tag{A.5}
\]

if \( p/2 < \beta < 3p/4 \), thus proving (A.3) for \( 0 < p \leq 2 \).

Next for \( 2 < p < 3 \) we need the inequality for double Itô-Wiener integrals: for any \( p \geq 2, g \in L^2(\mathbb{R}^2) \)

\[
E \left[ \int_{\mathbb{R}^2} g(s_1, s_2)dB_1 dB_2 \right]^p \leq C \left( E \left[ \int_{\mathbb{R}^2} g(s_1, s_2)dB_1 dB_2 \right]^2 \right)^{p/2} = C \left( \int_{\mathbb{R}^2} |g(s_1, s_2)|^2ds_1 ds_2 \right)^{p/2}. \tag{A.6}
\]
Indeed, by using Gaussianity and independence of $B_1, B_2$ and Minkowski inequality for $I_2(g) := \int_{\mathbb{R}^2} g(s_1, s_2) dB_1 dB_2$ we obtain

$$
(E|I_2(g)|^p)^{2/p} = (E_{B_1} E_{B_2} [|I_2(g)|^p|B_1|])^{2/p} \leq C (E_{B_1} (E_{B_2} [|I_2(g)|^2|B_1|])^{p/2})^{2/p} \leq C E_{B_2} (E_{B_1} [|I_2(g)|^2|B_2|])^{p/2})^{2/p} \leq C E_{B_2} E_{B_1} [|I_2(g)|^2] = CE|I_2(g)|^2.
$$

Using inequality (A.6) and (A.4), (A.5) we obtain

$$
J_{p,\beta}(\tau) \leq C \left( \int_{\mathbb{R}^2_+} E|z(\tau; x_1, x_2)|^p(x_1 x_2)^{\beta-1}dx_1dx_2 + \int_{\mathbb{R}^2_+} E|z(\tau; x_1, x_2)|^2(x_1 x_2)^{\beta-1}dx_1dx_2 \right)
\leq C (J_{p,\beta}(\tau) + I_{2,\beta}(\tau)) < \infty
$$
if $p/2 < \beta < 3p/4$, thus proving (A.3) and part (ii).

(iii) Follows from stationarity of increments of $Z_{\beta}$ (part (i)) and $J_{2,\beta}(\tau) = \sigma_\infty^2 \tau^{2(2-\beta)}$, where according to (A.2),

$$
\sigma_\infty^2 = \int_{\mathbb{R}^2_+} E z^2(1; x_1, x_2) d\mu_{\beta}
= (\psi(1)/2)^2 \int_{(0,1]^2} du_1 du_2 \left( \int_0^{\infty} e^{-x|u_1-u_2|\beta-2} dx \right)^2
= (\psi(1)/2)^2 \Gamma(\beta-1)^2 \int_{(0,1]^2} |u_1-u_2|^{2(1-\beta)} du_1 du_2
= (\psi(1)/2)^2 \Gamma(\beta-1)^2 / (2-\beta)(3-2\beta).
$$

(iv) Follows from stationarity of increments, $E|Z_{\beta}(\tau)|^p \leq C J_{p,\beta}(\tau)$, $1 < p \leq 2$, where $J_{p,\beta}(\tau)$ is the same as in (A.3), and Kolmogorov’s criterion; c.f ([17], proof of Prop. 3.1(iv)).

(v) The proofs are very similar to those of Theorem 3.1 (i), (ii), hence we omit some details. For notational simplicity, we only prove one-dimensional convergence at $\tau > 0$.

**Proof of (2.18).** As $b \to 0$, consider

$$
\Phi_b(\theta) := \log E \exp(i \theta b^{-2} z_{\beta}(b \tau)) = \psi(1)^2 \int_{\mathbb{R}^2_+} E \Psi(\theta b^{-2} z(b \tau; x_1, x_2))(x_1 x_2)^{\beta-1}dx_1dx_2,
$$

where $\Psi(z) := e^{iz} - 1 - iz, z \in \mathbb{R}$. Since $b^{-2} z(b \tau; x_1, x_2) \to z(\tau; bx_1, bx_2)$, rewrite

$$
\Phi_b(\theta) = \psi(1)^2 b^{-2\beta} \int_{\mathbb{R}^2_+} E \Psi(\theta b^{\beta} z(\tau; x_1, x_2))(x_1 x_2)^{\beta-1}dx_1dx_2,
$$

where $b^{-2\beta} \Psi(\theta b^{\beta} z(\tau; x_1, x_2)) \to - (\theta^2/2) z^2(\tau; x_1, x_2) a.s.$ Note $|b^{-2\beta} \Psi(\theta b^{\beta} z(\tau; x_1, x_2))| \leq (\theta^2/2) z^2(\tau; x_1, x_2)$, where the dominating function satisfies (A.2) and (2.9). Hence, by the dominated convergence theorem,

$$
\Phi_b(\theta) \to - (\theta^2/2) \psi(1)^2 \int_{\mathbb{R}^2_+} E z^2(\tau; x_1, x_2)(x_1 x_2)^{\beta-1}dx_1dx_2 = \log E \{i \theta \sigma_{\infty} B_{2-\beta}(\tau)\},
$$

which finishes the proof.

**Proof of (2.19) follows that of Thm. 3.1 (i) case $0 < \beta < 1$.** As $b \to 0$, consider

$$
\Phi_b(\theta) := \log E e^{i \theta b^{-1}(\log b^{-1})^{-1/2\beta} z_{\beta}(b \tau)} = \frac{\psi(1)^2}{\log b^{-1}} \int_{\mathbb{R}^2_+} E [e^{i \theta z_{\beta}(\tau; x_1, x_2)} - 1](x_1 x_2)^{\beta-1}dx_1dx_2
$$

27
where
\[ z_b(\tau; x_1, x_2) := b^{-1}(\log b^{-1})^{-1/2} \beta \left( b\tau; (\log b^{-1})^{-1/2} \beta x_1, (\log b^{-1})^{-1/2} \beta x_2 \right) \]
satisfies
\[ E|z_b(\tau; x_1, x_2)|^2 \leq \frac{C}{x_1 x_2} \left( 1 \land \frac{b^{-1}(\log b^{-1})^{1/2} \beta}{x_1 + x_2} \right), \quad (A.7) \]
see (A.2). Split
\[ \Phi_b(\theta) = \frac{\psi(1)^2}{\log b^{-1}} \int_{\mathbb{R}_+^2} (1(1 < x_1 + x_2 < b^{-1}) + 1(x_1 + x_2 > b^{-1}) + 1(x_1 + x_2 < 1)) \]
\[ \times E[e^{i\theta z_b(\tau;x_1,x_2)} - 1](x_1 x_2)^{3-1}dx_1 dx_2 =: \sum_{i=1}^{3} L_i. \]
Using (A.7), we can show that \( L_i, i = 2, 3 \) are remainders. By change of variables: \( y = x_1 + x_2, x_1 = yw \) and then \( w = z/y^2 \), we rewrite the main term
\[ L_1 = \frac{1}{\log b^{-1}} \int_{1}^{b^{-1}} V_b(\theta; y) \frac{dy}{y}, \quad V_b(\theta; y) := 2\psi(1)^2 \int_0^{y/2} \Lambda_b(z; y) z^{\beta-1} \left( 1 - \frac{z}{y^2} \right)^{\beta-1} dz \quad (A.8) \]
with \( \Lambda_b(z; y) := E[\exp\{i\theta z_b(\tau; \tilde{z}, y(1 - \frac{z}{y^2}))\} - 1] \), which satisfies \( |\Lambda_b(z; y)| \leq C(1 \land \frac{1}{z}) \) for all \( 0 < \frac{z}{y^2} < \frac{1}{2}, 0 < y < b^{-1} \). Here the dominating bound is a consequence of (A.7). Then
\[ L_1 \rightarrow \log Ee^{i\theta V_2\beta} = 2\psi(1)^2 \int_0^{\infty} \Lambda(z) z^{\beta-1} dz, \quad (A.9) \]
where \( \Lambda(z) := E[e^{i\theta Z_i Z_i/2\sqrt{\beta} - 1} \] with \( Z_i \sim N(0, 1), i = 1, 2 \) being independent r.v.s, follows from
\[ \lim_{y \to \infty, y = O(b^{-1})} \Lambda_b(z; y) = \Lambda(z), \quad \forall z > 0, \quad (A.10) \]
for more details we refer the reader to the proof of Thm. 3.1 (i) case \( 0 < \beta < 1 \). More precisely, (A.10) says that for every \( \epsilon > 0 \) there exists a small \( \delta > 0 \) such that for all \( 0 < b < \delta \), if \( \delta^{-1} < y < b^{-1} \), then \( |\Lambda_b(z; y) - \Lambda(z)| < \epsilon \). To show (A.10), note \( z_b(\tau; \tilde{z}, y(1 - \frac{z}{y^2})) = I_{12}(\tilde{h}_b(\cdot; \tau; z)) \) is a double Itô-Wiener stochastic integral w.r.t. independent standard Brownian motions \( \{ B_i(s), s \in \mathbb{R} \}, i = 1, 2 \) for
\[ h_b(s_1, s_2; \tau; z) := (\log b^{-1})^{-1/2} \beta \int_0^\tau \prod_{i=1}^2 e^{-\frac{b}{2\beta}(bu - s_i)} 1(s_i < bu) du, \quad s_1, s_2 \in \mathbb{R}, \]
\[ \alpha_1 := (\log b^{-1})^{1/2} \beta y/z, \quad \alpha_2 := (\log b^{-1})^{1/2} \beta y', \quad y' := y(1 - \frac{z}{y^2}). \]
We have that \( z_b(\tau; \tilde{z}, y(1 - \frac{z}{y^2})) = d I_{12}(\tilde{h}_b(\cdot; \tau; z)), \) where
\[ \tilde{h}_b(s_1, s_2; \tau; z) := \sqrt{\alpha_1 \alpha_2} h_b(\alpha_1 s_1, \alpha_2 s_2; \tau; z) \]
\[ = \sqrt{\frac{y}{z y'}} \int_0^\tau \prod_{i=1}^2 e^{-\frac{b}{2\beta}(bu - s_i)} 1(s_i < bu) du, \quad s_1, s_2 \in \mathbb{R}. \]
If \( b \to 0, y, y' \to \infty \) so that \( y/y' \to 1 \) and \( b/\alpha_i \to 0, i = 1, 2, \) then \( ||\tilde{h}_b(\cdot; \tau; z) - h(\cdot; \tau; z)|| \to 0 \) with
\[ h(s_1, s_2; \tau; z) := \frac{\tau}{\sqrt{z}} \prod_{i=1}^2 e^{\alpha_i} 1(s_i < 0), \quad s_1, s_2 \in \mathbb{R} \quad (A.11) \]
implies the convergence 

\[ z_b(\tau; \tilde{z}, y(1 - \frac{z}{b})) \to_d I_{12}(h(\cdot; \tau); z) =_d \tau Z_1 Z_2/2\sqrt{z}. \]

Conditions on \( b, y, y' \) are obviously satisfied due to \( y, y' = O(b^{-1}) = o(b^{-1}(\log b^{-1})^{1/2}) \). This proves (A.10) and (A.9), thereby completing the proof of (2.19).

**Proof of (2.20)** follows that of Theorem 3.1 (ii). We will prove that as \( b \to \infty \),

\[ \log E[e^{\theta b^{-1/2}Z(br)}] = \psi(1)^2 \int_{\mathbb{R}^2_+} E \left[ \exp \left\{ i\theta b^{-1/2}z(br; x_1, x_2) \right\} - 1 \right] (x_1x_2)^{3/2}dx_1dx_2 \]  

\[ \to \psi(1)^2 \int_{\mathbb{R}^2_+} \left[ \exp \left\{ - \frac{\theta^2 r}{4x_1x_2(x_1 + x_2)} \right\} - 1 \right] (x_1x_2)^{3/2}dx_1dx_2 = \log E[e^{\theta A_{1/2}B(\tau)}]. \]

By (A.2), we have that \( E[\exp \{i\theta b^{-1/2}z(br; x_1, x_2)\} - 1] \leq C \min\{1, (x_1x_2(x_1 + x_2))^{-1}\} \). In view of (2.8), the dominated convergence theorem applies if the integrands on the r.h.s. of (A.12) converge pointwise, i.e. for every \((x_1, x_2) \in \mathbb{R}^2_+\),

\[ b^{-1/2}z(br; x_1, x_2) \to_d \frac{B(\tau)}{\sqrt{2x_1x_2(x_1 + x_2)}}. \]  

(A.13)

To simplify notation, let \( \tau = 1 \) and all \( b \in \mathbb{N} \). Define

\[ z_b^+(x_1, x_2) := \int_0^b \int_0^b f(s_1, s_2)dB_1(s_1)dB_2(s_2), \quad f(s_1, s_2) := b^{-1/2} \int_0^b \prod_{i=1}^2 e^{-x_i(u-s_i)}1(u > s_i)du. \]

and \( z_b^-(x_1, x_2) := b^{-1/2}z(b; x_1, x_2) - z_b^+(x_1, x_2) \). Since \( E(z_b^-(x_1, x_2))^2 = O(b^{-1}) \) implies \( z_b^-(x_1, x_2) = o_p(1) \), we only need to prove that

\[ z_b^+(x_1, x_2) \to_d N\left(0, \frac{1}{2x_1x_2(x_1 + x_2)}\right). \]  

(A.14)

Write \( z_b^+(x_1, x_2) = \sum_{k=1}^b Z_k \) as a sum of a sum of a zero-mean square-integrable martingale difference array

\[ Z_k := \int_{k-1}^k \int_0^{k-1} f(s_1, s_2)dB_1(s_1)dB_2(s_2) + \int_{k-1}^k \int_0^{k-1} f(s_1, s_2)dB_1(s_1)dB_2(s_2) \]

\[ + \int_{k-1}^k \int_{k-1}^k f(s_1, s_2)dB_1(s_1)dB_2(s_2) \]

w.r.t. the filtration \( \mathcal{F}_k \) generated by \( \{B_i(s), 0 \leq s \leq k, i = 1, 2\}, k = 0, \ldots, b \). By the martingale CLT in Hall and Heyde [9], (A.14) then follows from

\[ \sum_{k=1}^b E[Z_k^2|\mathcal{F}_{k-1}] \to_p \frac{1}{2x_1x_2(x_1 + x_2)} \quad \text{and} \quad \sum_{k=1}^b E[Z_k^21(|Z_k| > \epsilon)] \to 0 \quad \text{for any} \ \epsilon > 0. \]  

(A.15)

Since \( \sum_{k=1}^b E[Z_k^2] = \int_0^b \int_0^b f^2(s_1, s_2)ds_1ds_2 = E(z_b^+(x_1, x_2))^2 \to (2x_1x_2(x_1 + x_2))^{-1} \), consider \( R_b := \sum_{k=1}^b (E[Z_k^2|\mathcal{F}_{k-1}] - E[Z_k^2]), \) where

\[ E[Z_k^2|\mathcal{F}_{k-1}] = \int_{k-1}^k \left( \int_0^{k-1} f(s_1, s_2)dB_2(s_2) \right)^2 ds_1 + \int_{k-1}^k \left( \int_0^{k-1} f(s_1, s_2)dB_1(s_1) \right)^2 ds_2 \]

\[ + \int_{k-1}^k \int_{k-1}^k f^2(s_1, s_2)ds_1ds_2. \]

By rewriting \( R_b = \sum_{i=1}^2 \int_0^b \int_0^b c_i(s_1, s_2)dB_1(s_1)dB_2(s_2) \) with \( c_1(s_1, s_2) = \int_{s_1}^b \int_{s_2}^b f(s_1, s)f(s_2, s)ds, \)

\( c_2(s_1, s_2) = \int_{s_1}^b \int_{s_2}^b f(s_1, s)f(s_2, s)ds \) and using the elementary bound:

\[ f(s_1, s_2) \leq Cb^{-1/2}(e^{-x_1(s_2-s_1)}1(s_1 < s_2) + e^{-x_2(s_1-s_2)}1(s_1 \geq s_2)), \quad 0 \leq s_1, s_2 \leq b, \]  

(A.16)
we obtain $E|R_b|^2 = \sum_{i=1}^{2} \int_0^b \int_0^b c_i^2(s_1, s_2)ds_1ds_2 = O(b^{-1}) = o(1)$, which proves $R_b = o_p(1)$ and completes the proof of the first relation in (A.15). Finally, using (A.6), (A.16), we obtain $\sum_{k=1}^{b} |E|Z_k|^4 = O(b^{-1}) = o(1)$, which implies the second relation in (A.15) and completes the proof of (A.14).

Proposition 2.2 is proved.

\[\square\]

Proof of Prop. 2.3. (i) Split $Z_\beta^\tau(\tau) = \tilde{Z}_\beta^\tau(\tau) + \tau V_\beta^+$ with

$$\tilde{Z}_\beta^\tau(\tau) := \int_{R_+ \times C(\mathbb{R})} \left( z^\tau(\tau; x) - \frac{\tau}{2x} \right) d(M^\tau_\beta - EM^\tau_\beta 1(1 < \beta < 2)).$$

$$V_\beta^+ := \int_{R_+ \times C(\mathbb{R})} \frac{1}{2x} d(M^\tau_\beta - EM^\tau_\beta 1(1 < \beta < 2)),$$

where $M^\tau_\beta$ is a Poisson random measure on $R_+ \times C(\mathbb{R})$ with mean $\mu^\tau_\beta = EM^\tau_\beta$ given in (2.21). The existence of $V_\beta^+$ follows from $\int_0^\infty \min(1, \frac{x^{-\beta}}{x}) dx < \infty$ if $\beta \in (0, 1)$ and $\int_0^\infty \min(x^{-1}, x^{-2}) x^{1-\beta} dx < \infty$ if $\beta \in (1, 2)$. The process $\tilde{Z}_\beta^\tau$ is well-defined if

$$J_{p, \beta}^\tau(\tau) := \int_{R_+ \times C(\mathbb{R})} |z^\tau(\tau; x) - \tau/2x|^p d\mu^\tau_\beta = C \int_0^\infty E|z^\tau(\tau; x) - \tau/2x|^p x^{\beta-1} dx < \infty,$$

where $0 < p \leq 1$ for $\beta \in (0, 1)$ and $1 \leq p \leq 2$ for $\beta \in (1, 2)$. We have $E|z^\tau(\tau; x) - \tau/2x|^p \leq (\text{var}(z^\tau(\tau; x)))^{p/2}$, where

$$\text{var}(z^\tau(\tau; x)) = \int_{(0, \tau]^2} \text{cov}(Y^2(u_1; x), Y^2(u_2; x)) du_1 du_2 = 2 \int_{(0, \tau]^2} \int_{\mathbb{R}^2} ds_1 ds_2 e^{-2x(u_1 + u_2 - s_1 - s_2)} 1(s_1 < s_2 < u_1 \land u_2)$$

$$= \frac{1}{2x^2} \int_{(0, \tau]^2} e^{-2x|u_1 - u_2|} du_1 du_2 = \frac{1}{8x^4} \left( 2x^2 - 1 + e^{-2x^\tau} \right) \leq C \frac{\tau^2}{x^2} \left( 1 + \frac{1}{x^\tau} \right),$$

hence, $J_{p, \beta}^\tau(\tau) \leq C \tau^{2p-\beta} < \infty$ for $p < \beta < 3p/2$. This completes the proof of part (i).

(ii) $E|V_\beta^+|^p < \infty$ for $0 < p < \beta$, since $V_\beta^+$ is a $\beta$-stable random variable. Similarly to (A.3), $E|\tilde{Z}_\beta^\tau(\tau)|^p < \infty$ follows from $J_{p, \beta}^\tau(\tau) < \infty$ in (A.17), where $p$ is sufficiently close to $\beta$ and such that $0 < p < \beta < 3p/2$. This proves part (ii).

(iii) Follows from part (ii) by Kolmogorov’s criterion, similarly as in the proof of Proposition 2.2.

(iv) For notational simplicity, we only prove one-dimensional convergence at $\tau > 0$. We have

$$\text{log } E \exp\{i\theta b^{-1} Z_\beta(b\tau)\} = \psi(1) \int_{R_+} \Lambda_b(x) x^{\beta-1} dx,$$

where

$$\Lambda_b(x) := E[ \exp\{i\theta b^{-1} z^\tau(b\tau; x) - 1 - i\theta b^{-1} z^\tau(b\tau; x)\} 1(1 < \beta < 2)].$$

Substituting $E|z^\tau(b\tau; x)| \leq (E|z^\tau(b\tau; x)|^2)^{1/2}$ and $E|z^\tau(b\tau; x)|^2 = \text{var}(z^\tau(b\tau; x)) + (b\tau/2)^2 \leq C(b/x)^2$ by (A.18) into

$$|\Lambda_b(x)| \leq C \begin{cases} \min \{1, b^{-1} E|z^\tau(b\tau; x)|\}, & 0 < \beta < 1, \\ \min \{b^{-1} E|z^\tau(b\tau; x)|, b^{-2} E|z^\tau(b\tau; x)|^2\}, & 1 < \beta < 2, \end{cases}$$

we obtain the bounds: $|\Lambda_b(x)| \leq C \min\{1, x^{-1}\}$ if $0 < \beta < 1$, and $|\Lambda_b(x)| \leq C \min\{x^{-1}, x^{-2}\}$ if $1 < \beta < 2$. The result then follows from the dominated convergence theorem once we show that for all $x \in R_+$,

$$\Lambda_b(x) \rightarrow \begin{cases} \exp\{i\theta \tau/2x\} - 1 - (i\theta \tau/2x) 1(1 < \beta < 2) & \text{as } b \rightarrow \infty, \\ E[\exp\{i\theta Z^2/2x\} - 1 - (i\theta Z^2/2x) 1(1 < \beta < 2)] & \text{as } b \rightarrow 0, \end{cases}$$

(A.19)
where \( Z \sim N(0,1) \). Using (A.18), we get 
\[
E|b^{-1}z^*(b\tau;x) - (\tau/2x)|^2 = b^{-2} \text{var}(z^*(b\tau;x)) \leq Cb^{-1} = o(1)
\]
as \( b \to \infty \), which implies the first convergence in (A.19). To prove the second convergence in (A.19), note 
\[
Z/\sqrt{2x} \overset{d}{\rightarrow} \mathcal{Y}(0; x).
\]
It suffices to show that as \( b \to 0 \),
\[
E|b^{-1}z^*(b\tau;x) - \mathcal{Y}^2(0;x)| = E \int_0^T (\mathcal{Y}^{2}(bu;x) - \mathcal{Y}^{2}(0;x))du \leq \int_0^T E|\mathcal{Y}^2(bu;x) - \mathcal{Y}^2(0;x)|du = o(1).
\]
Factorizing the difference of squares and applying the Cauchy-Schwarz inequality, this follows from
\[
E|\mathcal{Y}(bu;x) - \mathcal{Y}(0;x)|^2 = \int_0^{bu} e^{-2x}ds + \frac{1}{2x}(e^{-xbu} - 1)^2 \leq Cu.
\]
Proposition 2.3 is proved. \( \square \)

**Calculation of the constant \( \sigma_0 \) in Theorem 3.1 (ii).** We have
\[
\sigma_0 \cdot \frac{2^{2\beta/3}}{\psi_1(1)}^2 = \int_{\mathbb{R}^2_+} (1 - \exp\{-u_1 + u_2\}^{-1}(u_1u_2)^{-1}) (u_1u_2)^{\beta-1} du_1 du_2
\]
\[
= \int_{u_2=v_2} \int_{u_1=v_1^{-1/3}} (1 - \exp\{-u_1^{-3}(1 + v_2)^{-1}v_1^{2\beta/3-1}v_2^{\beta-1}\} u_1^{2\beta/3-1}v_2^{\beta-1} du_1 dv_2
\]
\[
= \frac{1}{3} \int_{\mathbb{R}^2_+} \left( \int_{v_2=s^{-1/3}}^{1/(1+v_2)} e^{-v_1 t} dt \right) v_1^{2\beta/3-1}v_2^{\beta-1} dv_1 dv_2
\]
\[
= \frac{\Gamma(1 - \frac{2\beta}{3})}{3} \int_0^\infty v_2^{-1} dv_2 \int_0^1 (1 + v_2)^{-1} t^{2\beta/3-1} dt
\]
\[
= \frac{\Gamma(1 - \frac{2\beta}{3})}{2\beta} \int_0^\infty (1 + v_2)^{-2\beta/3} v_2^{\beta/3-1} dv_2
\]
\[
= \frac{\Gamma(1 - \frac{2\beta}{3})}{2\beta} \int_0^1 s^{2\beta/3} (s^{-1} - 1)^{\beta/3-1} s^{-2} ds
\]
\[
= \frac{\Gamma(1 - \frac{2\beta}{3}) B(\frac{\beta}{3}, \frac{\beta}{3})}{2\beta}.
\]

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**References**


