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ASYMPTOTIC DISTRIBUTION OF PARAMETERS IN RANDOM MAPS

OLIVIER BODINI[1], JULIEN COURTIEL[2], SERGEY DOVGAL[1,3,4], AND HSIEN-KUEI HWANG[5]

Abstract. We consider random rooted maps without regard to their genus, with fixed large number of edges, and address the problem of limiting distributions for six different parameters: vertices, leaves, loops, root edges, root isthmus, and root vertex degree. Each of these leads to a different limiting distribution, varying from (discrete) geometric and Poisson distributions to different continuous ones: Beta, normal, uniform, and an unusual distribution whose moments are characterised by a recursive triangular array.

1. Introduction

1.1. **Motivation for our work.** Rooted maps form a ubiquitous family of combinatorial objects, of considerable importance in combinatorics, in theoretical physics, and in image processing. They describe the possible ways to embed graphs into compact oriented surfaces [LZ04].

The present paper focuses on asymptotic enumeration of some parameters in rooted maps with no restriction on genus. Strangely, this aspect has received little attention in the literature. There exist certain results on the distributions of patterns in planar maps: see e.g. a study of the number of vertices of given degree [DP13]. One of the examples of non-Gaussian limit law in a combinatorial structure is given in [BFSS01] with Airy distribution (this paper involves planar maps as well). In [BBM17], differential equations for coloured planar maps are considered.

Of closest connection to our study here is the paper by Arquès and Béraud [AB00], which contains several characterisations of the number of rooted maps and their generating functions. In particular, they give an explicit formula for the number of maps, expressed as an infinite sum, from which the asymptotic number of maps with n edges can be deduced (which is $(2n+1)!!$). Recently, Carrance [Car17] obtained the distribution of genus in bipartite random maps. To our knowledge, no other asymptotic distribution properties of maps have been properly stated so far. It is also worth mentioning a paper by Flajolet and Noy [FN00] where patterns in chord diagrams are investigated, and [CY17] with the distribution of the so-called *terminal chords*.

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From a generating function viewpoint, if the genus of the maps is not fixed, then the generating function of rooted maps is non-analytic (namely, convergent only at zero) and satisfies a Riccati differential equation. Such divergent Riccati equations appear frequently in enumerative combinatorics. For example, at least 39 entries in Sloane's OEIS [Slo] were found containing sequences whose generating functions satisfy Riccati equations, including some entries related to the families of indecomposable combinatorial objects, moments of probability distributions, chord diagrams [CY17, CYZ16, FN00], Feynman diagrams [CacLP78], etc. Some of these are closely connected to maps. Indeed, it is known that rooted maps with no genus restriction also encode different combinatorial families such as chord diagrams and Feynman diagrams on the one hand, and different fragments of lambda calculus [BGJ13, ZG15] on the other hand. Thus most asymptotic information obtained on maps can often be transferred to the aforementioned objects and lead to a better understanding of them in their respective domains.

From an asymptotic point of view, the analysis of purely formal power series requires tools rather different from the usual analytic combinatorial techniques such as singularity analysis and saddle-point methods; see [FS09]. As Odlyzko writes in his survey [Odl95]: "*There are few methods for dealing with asymptotics of formal power series, at least when compared to the wealth of techniques available for studying analytic generating functions.*" We show the diverse limit laws mentioned in the Abstract; the approaches we use may also be of potential application to other closely related problems.

1.2. **Definitions.** For a rigorous definition of a rooted combinatorial map we refer, for example, to [LZ04, AB00]. For our purposes in this extended abstract we use a less formal but more intuitive definition.

Definition 1 (Maps)**.** *A* map *is a connected multigraph endowed with a cyclic ordering of consequitive half-edges incident to each vertex. Multiple edges and loops are allowed. Around each vertex, each pair of adjacent half-edges is said to form a* corner*. If there is only one half-edge, there is only one corner. A* rooted map *is a map with a distinguished corner.*

Figure 1 shows some examples of rooted maps. Observe that the first two maps are different since the cyclic ordering is different: in the first map, the pendant edge follows counterclockwise the edge after the root (the node pointed to by an arrow), while in the second map it precedes in counterclockwise order. In contrast, the last two maps are equal: although the leaves are at different positions, one can find an isomorphism between the two maps preserving the vertices, the root and the cyclic orderings around each vertex. The corners of the leftmost map are displayed in Figure 2 (left), showing all the possible rootings of this map.

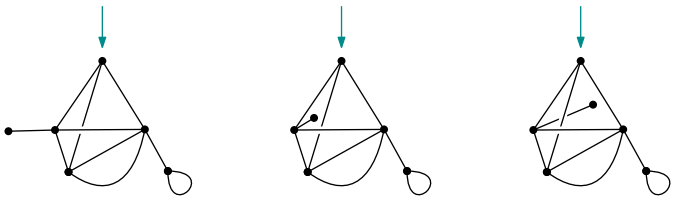

Figure 1. Three rooted maps. Each root is marked by an arrow. The two last maps are equal.

Definition 2 (Map features)**.** *A* face *can be obtained by starting at some corner, moving along an incident half-edge, then switching to the next clockwise half-edge*

and repeating the procedure until the starting corner is met. A loop *is an edge that connects the same vertex. An* isthmus *is an edge such that the deletion of this edge increases the number of connected components of the underlying graph. The* degree *of a vertex is the number of half-edges incident to this vertex.*

These definitions are illustrated in Figure 2 (right).

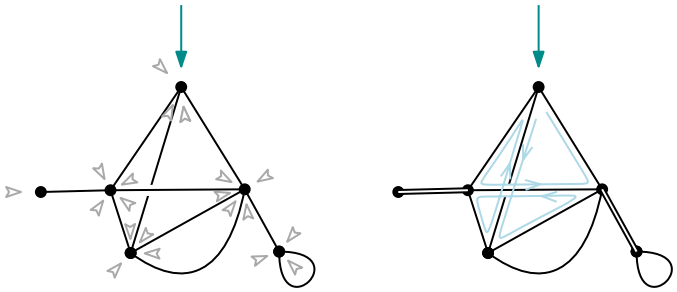

Figure 2. Left. The triangles point at every corner of the map. *Right.* The light-blue line marks the contour of one face of the map. The overlined edges are the isthmuses of the map. The only loop of the map is adjacent to the rightmost isthmus. The vertex incident to this loop has degree 3.

Arquès and Béraud [AB00] prove that the generating function of maps $M(z) := \sum_{n\geqslant 0} m_n z^n$, where m_n enumerates the number of maps with n edges, satisfies the Riccati equation

$$M(z) = 1 + zM(z) + zM(z)^2 + 2z^2\partial_z M(z), \tag{1}$$

a typical Riccati equation whose first few Taylor coefficients read $M(z) = 1 + 2z + 20z^2 + 444z^3 + 16944z^4 + \cdots$.

Table 1. Our results: limit laws for several parameters.

GF	Pattern	Differential equation	Limit law
M	maps (edges)	$M = 1 + zM + zM^2 + 2z^2\partial_z M$	
L	leaves	not discussed	Poisson(1)
X	vertices	$X = v + zX + zX^2 + 2z^2\partial_z X$	$\mathcal{N}(\ln n, \ln n)$
C	root isthmic parts	$C = 1 + zC + vzC\vert_{v=1}C + 2z^2\partial_z C$	$1 + \text{Geometric}(\frac{1}{2})$
E	root edges	$E = 1 + vzE + vzE\vert_{v=1}E + 2vz^2\partial_z E$	$\text{Beta}(1, \frac{1}{2})$
D	root degree	$D = 1 + v^2zD + vzD\vert_{v=1}D + 2vz^2\partial_z D - v^2(1-v)z\partial_v D$	Uniform$[0, 2]$
Y	loops	$Y = v + vzY + vzY\vert_{v=1}Y + 2vz^2\partial_z Y + v^2z(vw-1)\partial_v Y$	A new law*

1.3. **Results and methods.** We address in this paper the analysis of the extended equations of (1) for bivariate (and in one case, trivariate) generating functions $M(z, v) := \sum_{n,k\geqslant 0} m_{n,k} z^n v^k$, where $m_{n,k}$ stands for the number of maps with n edges and the value of the shape parameter equal to k. We obtain limit laws for the distributions of six different parameters (see Figures 3 to 5).

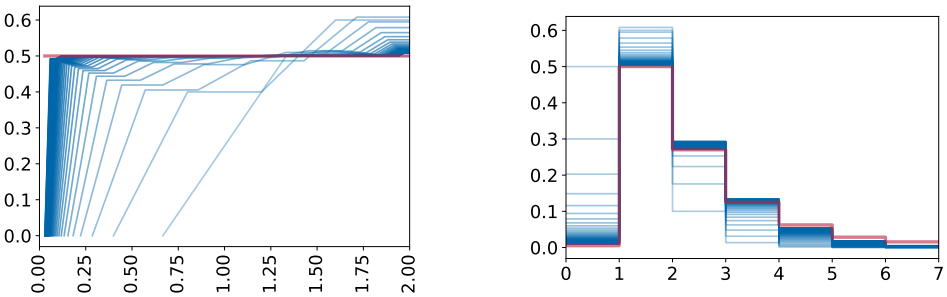

Figure 3. Left. Root vertex degree. *Right.* Number of root isthmic parts.

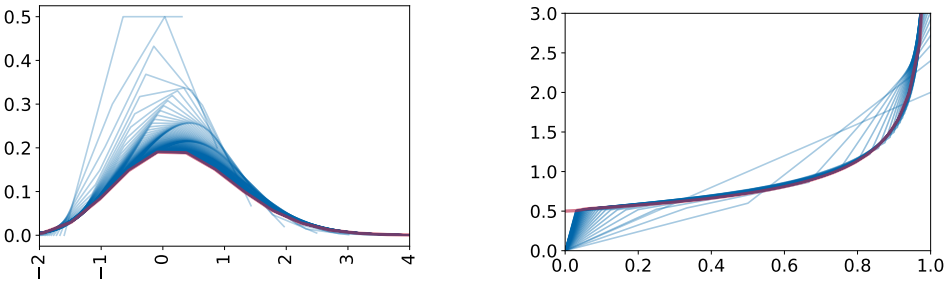

Figure 4. Left. Number of vertices. *Right.* Number of root edges.

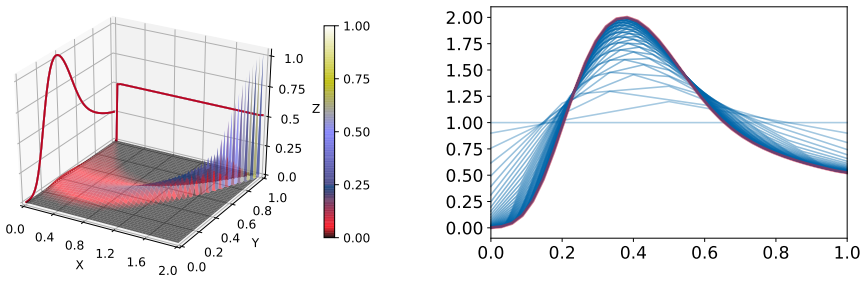

Figure 5. Left. Joint distribution of root vertex degree and the number of loops. *Right.* Number of loops.

We collect the patterns studied here in Table 1 for comparison. Note that some of the limit laws are discrete (Poisson and Geometric), one of the laws is Gaussian with a logarithmic mean, and the other laws are continuous. For the last three lines of the table, the limit law is obtained after dividing the random variable by n, the total number of edges. The distribution of the number of loops follows a rather unusual limit law in the sense that we can only characterise the limit law by its moment sequence, described by a recurrence. The corresponding probability density function of this law remains unknown and does not have an explicit expression at this stage (see Figure 5). On the other hand, by the bijection from [CYZ16] and a known property of chord diagrams in [FN00], it is possible to deduce the limit laws for the number of leaves.

The technique we use several times in our proofs is to linearise the differential equations satisfied by the generating functions, by choosing a suitable transformation. When such a technique fails, we rely then on the method of moments, which consists in computing all higher derivatives of $M(z,v)$ at $v=1$. Then we asymptotically examine the ratios $[z^n]\partial_v^k M(z,v)|_{v=1}/[z^n]M(z,1)$ which correspond to the moments of random variable. Such a procedure also linearises to some extent the more complicated bivariate nature of the differential equations and facilitates the resolution complexity of the asymptotic problem.

1.4. **Structure of the Paper.** In Section 2 we obtain differential equations for the corresponding generating functions. Then in Section 3 we analyse the differential equations described in the previous section and explain how to obtain the limit laws for five different parameters of maps. Finally, we present the combinatorial approaches which supplement our techniques in Section 4, and analyse the sixth parameter – the number of leaves – along with two additional parameters of the dual nature: root face degree and the number of trivial loops.

2. Differential equations for maps

In this section, we describe the differential equations satisfied by the generating functions with the additional variables counting the desired shape patterns, and explain their origin.

2.1. **Univariate generating function of maps.** Since the Riccati equation (1) lies at the basis of all other extended equations in Table 1, we give a quick proof via the recurrence satisfied by m_n, the number of maps with n edges (see Figure 6):

$$m_n = \mathbf{1}_{[n=0]} + \sum_{0\leqslant k<n} m_k m_{n-1-k} + (2n-1)m_{n-1}, \tag{2}$$

which then implies the Riccati equation (1).

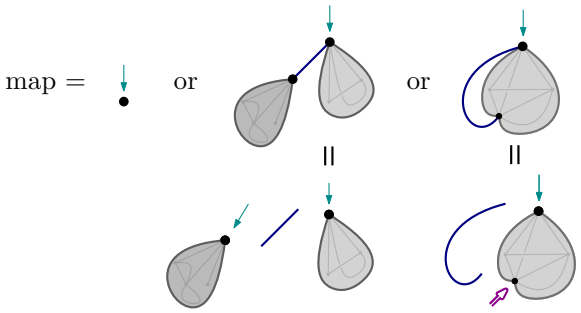

Figure 6. A symbolic construction of rooted maps.

There is only one map with 0 edges, so $m_0 = 1$. Next, a map with n edges can be formed either by connecting the roots of two maps (with k and $n-k-1$ edges respectively) with an isthmus, or by adding an edge to a map with $n-1$ edges, connecting the root and a corner. The number of possible ways to insert an edge in this way is equal to $2n-1$, because there are $2n-2$ corners in a map of size $n-1$, and there are 2 possible ways to insert a new edge at the root corner (either before, or after the root). This proves (2).

2.2. **Vertices.** Consider now the bivariate generating function

$$X(z,v) = \sum_{n,k\geqslant 0} x_{n,k} z^n v^k,$$

where $x_{n,k}$ is equal to the number of rooted maps with n edges and k vertices. Arquès and Béraud [AB00] showed that

$$X(z,v) = v + zX(z,v) + zX(z,v)^2 + 2z^2\partial_z X(z,v). \tag{3}$$

This recurrence can be obtained from (2) by noticing that no new vertex is created when we connect two maps with an isthmus, nor when we add a new root edge to a map. Besides, $X(z,v)$ satisfies another functional equation (see [AB00])

$$X(z,v) = v + zX(z,v)X(z,v+1).$$

2.3. **Root isthmic parts.** In this section we count *root isthmic parts* which are the number of isthmic constructions used at the root vertex. Note that isthmic part may not necessarily be a bridge because the additional edge constructor may induce additional connections.

We show that the bivariate generating function $C(z,v) = \sum_{n,k\geqslant 0} c_{n,k} z^n v^k$, where $c_{n,k}$ enumerates the number of maps with n edges and k root isthmic parts

$$C(z,v) = 1 + zC(z,v) + vzC(z,v)C(z,1) + 2z^2\partial_z C(z,v). \tag{4}$$

In Figure 6, the number of root isthmic parts only changes whenever two maps are connected by an isthmus. This yields $vzC(z,v)C(z,1)$ instead of zC^2.

2.4. **Root edges.** Similarly, consider $E(z,v) = \sum_{n,k\geqslant 0} e_{n,k} z^n v^k$, where $e_{n,k}$ counts the number of rooted maps with n edges and k root edges. We show that $E(z,v)$ satisfies

$$E = 1 + vzE + vzE|_{v=1}E + 2vz^2\partial_z E. \tag{5}$$

This again results from the recurrence (2) and from Figure 6: the non-root edges come from the bottom map in the isthmic construction, hence the summand $vzE(z,v)E(z,1)$.

2.5. **Root degree.** Consider the degree of the root vertex. Note that this may be different from the number of root edges because for the root degree, each loop edge is counted twice, therefore the degree of the root vertex varies from 0 to $2n$. By duality, the distribution of the root face degree is the same as the distribution of the root vertex degree.

Let $D(z,v) = \sum_{n,k\geqslant 0} d_{n,k} z^n v^k$ denote the bivariate generating function for maps with variable v marking root degree. Then

$$D = 1 + v^2 zD + vzD|_{v=1}D + 2vz^2\partial_z D - v^2(1-v)z\partial_v D. \tag{6}$$

In this case, the original construction in Figure 6 is not enough, so we need to consider special cases in Figure 7. When an additional edge becomes a loop, it increases the degree of the root vertex by 2. Otherwise, the root degree is just increased by 1. The number of such special corners is equal to root vertex degree plus one. Suppose that a map has n edges and k root edges. Then, the number of root corners is $k+1$. If $m_n = \sum_k d_{n,k}$ is the total number of maps with n edges, then for the numbers $d_{n,k}$ we can write a reccurence from the symbolic construction

$$d_{n,k} = \mathbf{1}_{[n=0]} + \sum_{j=0}^{n-1} m_j d_{n-j-1,k-1} + (2n-1-(k+1))m_{n-1,k-1} + (k+1)m_{n-1,k-2}.$$

After quick calculations, the desired differential equation is derived. Note that this equation is now partial: the generating function is differentiated by two different variables, which adds an extra difficulty to the analysis.

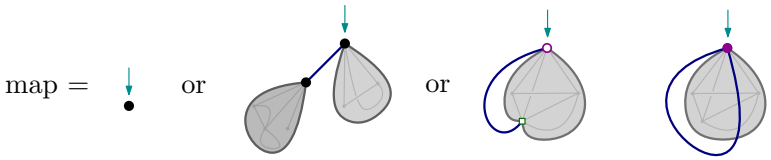

Figure 7. Symbolic method to count root degree and loops in rooted maps.

2.6. **Loops.** Finally, we look at the number of loops whose enumeration necessitates the consideration of the joint distribution of the number of loops and the number of root edges. So we consider the trivariate generating function $Y(z,v,w) = \sum_{n,k,m} y_{n,k,m} z^n v^k w^m$, where $y_{n,k,m}$ denotes the number of rooted maps with n edges, root degree equal to k, and m loops. We show that $Y(z,v,w)$ satisfies a partial differential equation

$$Y = 1 + zvY + zvY|_{v=1}Y + 2z^2 v\partial_z Y + zv^2(vw-1)\partial_v Y. \tag{7}$$

As in the previous subsection, in the symbolic construction of Figure 7, a new edge becomes a loop only if it is attached to one of the corners incident to the root vertex. The differential equation (7) is then a modification of (6) with an additional marking variable attached in the loop case.

Note that Equation (7) is *catalytic* with respect to the variable v, i.e. putting $v = 1$ introduces a new unknown object $\partial_v Y|_{v=1}$ entering the differential equation. One of the strategies for dealing with catalytic equations was developed by Bousquet-Mélou and Jehanne [BMJ06], generalising the so-called kernel method and quadratic method. However, their method does not work in our case because our equation is differentially algebraic.

3. Limit laws

This section describes the techniques we use to establish the limit laws.

From now on, by a random map (with n edges) we assume that all rooted map with n edges are equally likely. For notational convention, we use $X' = \partial_z X$ to denote derivative with respect to z. Due to space limit, we give only the sketches of the proofs.

3.1. **Transformation into a linear differential equation.** For most of the equations in the previous section, it turns out that a transformation similar to that used for Riccati equations largely simplifies the resolution and leads to solvable recurrences, which are then suitable for our asymptotic purposes. We begin by solving the standard Riccati equation (1) and see how a similar idea extends to other differential equations.

Proposition 3. *The number m_n of maps with n edges satisfies*

$$m_n = \phi_n \left(2n - 1 + O\big(n^{-1}\big)\right), \quad \textit{where} \quad \phi_n = \frac{(2n)!}{2^n n!} = (2n-1)!!.$$

Proof. We solve the Riccati equation (1) by considering the transformation

$$M(z) = 1 + \frac{2z\phi'(z)}{\phi(z)}, \tag{8}$$

with $\phi(0) = 1$. Substituting this form into the equation (2), we get the second-order differential equation $2z^2\phi'' + (5z-1)\phi' + \phi = 0$. From there, the coefficients ϕ_n must satisfy the recurrence $\phi_{n+1} = (2n+1)\phi_n$, which implies the double factorial form of ϕ_n.

Moreover, by extracting the coefficient of z^n in Equation (8), we obtain a relation between the coefficients m_k and ϕ_ℓ. By the inequality $m_n \geqslant (2n-1)m_{n-1}$ (see Equation (2)), we obtain the asymptotic relation $m_n = \phi_n\left(2n - 1 + O\left(n^{-1}\right)\right)$. □

Theorem 4. *Let X_n denote the number of vertices in a random rooted map with n edges. Then X_n follows a central limit theorem with logarithmic mean and logarithmic variance:*

$$\frac{X_n - \mathbb{E}(X_n)}{\sqrt{\mathbb{V}(X_n)}} \xrightarrow{d} \mathcal{N}(0,1), \quad \mathbb{E}(X_n) \sim \ln n, \quad \mathbb{V}(X_n) \sim \ln n. \tag{9}$$

Proof. By a substitution similar to (8), we define a bivariate generating function $S(z,v) = \sum_{n\geqslant 0} s_n(v) z^n$ such that

$$X(z,v) = v + \frac{2zS'}{S}, \quad S(0) = 1.$$

Substituting this $X(z,v)$ into (3) leads to a linear differential equation

$$4z^3S'' - 2z(1-(3-2v)z)S' + v(1+v)zS = 0.$$

from which one can extract the recurrence

$$s_n(v) = \frac{(2n+v-2)(2n+v-1)}{2n}\, s_{n-1}(v).$$

We then find an explicit expression for $s_n(v)$,

$$s_n(v) = \frac{1}{2^n n!} \cdot \frac{\Gamma(v+2n)}{\Gamma(v)}, \tag{10}$$

from which we can deduce that

$$\mathbb{E}(v^{X_n}) = \frac{2^{v-1}}{\Gamma(v)}\, n^{v-1}(1 + O(n^{-1})),$$

and conclude by applying the Quasi-Powers Theorem [FS09, Hwa98]. □

A finer Poisson($\log n + c$) approximation, for a suitably chosen c, is also possible, which results in a better convergence rate $O(\log n)^{-1}$ instead of $(\log n)^{-\frac{1}{2}}$; see [Hwa99] for details.

Theorem 5. *Let C_n denote the number of root isthmic parts in a random rooted map with n edges. Then,*

$$C_n \xrightarrow{d} 1 + \text{Geometric}\left(\frac{1}{2}\right).$$

Proof. Since $C(1,z) = M(z)$, we use again the substitution (8) and apply it to (4):

$$2z^2(\phi C' + v\phi' C) = (1-(1+v)z)\phi C - \phi.$$

The trick here is to multiply both sides by $\phi(z)^{v-1}$ and set $Q(z,v) = \phi^v C(z,v)$. We then obtain

$$2z^2 Q' = (1-(1+v)z)Q - \phi^v.$$

Set $q_n(v) := [z^n]Q(z,v)$. Translating the differential equation into a recurrence for normalised coefficients $\widehat{q}_n(v) := q_n(v)/\phi_n$, and by using an approximation $[z^n]\phi^v = v\phi_n(1+O(n^{-1}))$, we obtain a recurrence

$$\widehat{q}_n(v) = \frac{2n-1+v}{2n+1}\,\widehat{q}_n(v) + \frac{v}{2n+1} + O(n^{-2})$$

which, by iteration, becomes

$$\widehat{q}_n(v) = \frac{v}{2n+1}\sum_{2\leqslant k\leqslant n}\frac{\gamma_k(v)}{\gamma_n(v)} + \frac{3\gamma_1(v)}{(2n+1)\gamma_n(v)} + O(n^{-1}),$$

where $\gamma_k(v) = \frac{\Gamma(k+\frac{1}{2})}{2\Gamma(k+\frac{v+1}{2})}$. For large k and fixed v, the ratio of Gamma functions can be approximated by

$$\gamma_k(v) = \frac{k^{-v/2}}{2}(1+O(k^{-1})).$$

Finally, we find that the n-th coefficient of Q is proportional to

$$\widehat{q}_n(v) = \frac{v}{2n}\sum_{k=1}^{n}\left(\frac{n}{k}\right)^{v/2} + O(n^{-1/2}) = \frac{v}{2-v} + O(n^{-\frac{1}{2}}).$$

This corresponds to a (shifted by 1) geometric distribution with parameter $\frac{1}{2}$. By the relation $Q(z,v) = \phi^v C(z,v)$, we deduce that the limiting distribution of C_n is also geometric with parameter $\frac{1}{2}$. □

Theorem 6. *Let E_n denote the number of edges incident to the root vertex in a random rooted map with n edges. Then E_n follows asymptotically a Beta distribution:*

$$\frac{E_n}{n} \xrightarrow{d} \mathrm{Beta}\big(1,\tfrac{1}{2}\big),$$

with the density function $\frac{1}{2}(1-t)^{-\frac{1}{2}}$ for $t\in[0,1)$.

Proof. We use again the substitution $E(z,1) = M(z) = 1 + 2z\frac{\phi'}{\phi}$ so that

$$2vz^2(\phi E' + \phi' E) = (1-2vz)\phi E - \phi.$$

If we set $Q(z,v) = \phi(z)E(z,v)$, we then obtain

$$2vz^2Q' = (1-2vz)Q - \phi. \tag{11}$$

This linear differential equation translates into a recurrence for the coefficients $q_n(v)$ of $Q(z,v)$,

$$q_n(v) = 2vnq_{n-1}(v) + \phi_n,$$

which yields the closed-form expression

$$q_n(v) = 2^n n!\sum_{j=0}^{n}\binom{2j}{j}4^{-j}v^{n-j}. \tag{12}$$

Returning to $E(z,v)$, we find that its coefficients asymptotically behave like $q_n(v)$. This implies Beta limit law for random variable E_n/n since $\binom{2j}{j}4^{-j} \sim \frac{1}{\sqrt{\pi j}}$ for large values j. □

Theorem 7. *Let D_n denote the degree of the root vertex in a random rooted map with n edges. Then, D_n divided by the number of edges converges in law to the uniform distribution on $[0,2]$:*

$$\frac{D_n}{n} \xrightarrow{d} \mathrm{Uniform}\,[0,2]\,.$$

Proof. The substitutions

$$D(z,1) = M(z) = 1 + \frac{2z\phi'}{\phi}, \quad \text{and} \quad D(z,v) = \frac{Q(z,v)}{\phi(z)}$$

lead to a partial differential equation

$$Q = 2vz^2\partial_z Q - v^2(1-v)z\partial_v Q + vz(1+v)Q + \phi \tag{13}$$

with the boundary conditions $Q(0,v) = 1$ and $Q(z,0) = \phi(z)$. This in turn yields the recurrence for the coefficients $q_n(v) := [z^n]Q(z,v)$:

$$q_n(v) = v(2n-1+v)q_{n-1} - v^2(1-v)q'_{n-1}(v) + \phi_n$$

with $q_0(v) = 1$. We then get the exact solution $q_n(v) = \phi_n(1 + v + \cdots + v^{2n})$. Accordingly, $d_n(v) := [z^n]D(z,v) \sim q_n(v)$. This implies a uniform limit law for the random variable D_n/n. □

3.2. **Approximation and method of moments.** Unlike all previous proofs, we use the method of moments to establish the limiting distribution of the number of loops. The situation is complicated by the presence of the term involving $\partial_v Y$ in (7), which induces arbitrary order derivatives with respect to v if $v = 1$. We still manage to prove the following theorem.

Theorem 8. *Let Y_n denote the total number of loops in a random rooted map with n edges. Then*

$$\frac{Y_n}{n} \xrightarrow{d} \mathcal{L}, \tag{14}$$

where $\mathcal{L}$ is a continuous law with a computable density on $[0,1]$.

Proof. First, we show by induction that there exist constants $\eta_{k,\ell}$, such that as $n \to \infty$,

$$[z^n]\partial_v^k\partial_w^\ell Y(z,v,w)\big|_{v=w=1} \sim \eta_{k,\ell} n^{k+\ell+1}\phi_n, \quad k,\ell \geqslant 0. \tag{15}$$

For $k = \ell = 0$ the statement clearly holds with $\eta_{0,0} = 2$. For larger k and ℓ Let $y_n^{(k,\ell)}$ be $[z^n]\partial_v^k\partial_w^\ell Y(z,v,w)\big|_{v=w=1}$. By translating (7) into the corresponding recurrence for the coefficients and by collecting the dominant terms (using the induction hypothesis (15)), we obtain

$$y_n^{(k,\ell)} \sim (2n+k)y_{n-1}^{(k,\ell)} + \ell y_{n-1}^{(k+1,\ell-1)} + (2kn-2k)y_{n-1}^{(k-1,\ell)} + \mathbf{1}_{[k=0]}y_{n-1}^{(k,\ell)}.$$

After normalizing by ϕ_n, i.e. by denoting $\widehat{y}_n^{(k,\ell)} := y_n^{(k,\ell)}/\phi_n$, we obtain

$$\widehat{y}_n^{(k,\ell)} \sim \frac{\ell}{2n-1}\widehat{y}_{n-1}^{(k+1,\ell-1)} + \frac{2kn-2k}{2n-1}\widehat{y}_{n-1}^{(k-1,\ell)} + \frac{2n+k+\mathbf{1}_{[k=0]}}{2n-1}\widehat{y}_{n-1}^{(k,\ell)}. \tag{16}$$

After fitting $\widehat{y}_n^{(k,\ell)} \sim \eta_{k,\ell}n^{k+\ell+1} + d_{k,\ell}n^{k+\ell}$, we then deduce the recurrence

$$\eta_{k,\ell} = \frac{1}{k+2\ell+\mathbf{1}_{[k>0]}}(2k\eta_{k-1,\ell} + \ell\eta_{k+1,\ell-1}).$$

In particular, when $\ell = 0$, we obtain the moments of the random variable E_n, the number of root edges: $\eta_{k,0} = \frac{2^{k+1}}{k+1}$, which coincides with the moments of the uniform random variable Uniform$[0,2]$. Finally, it is not complicated to check that the numbers $\eta_{0,\ell}$ satisfy the condition of Hausdorff moment problem, i.e. $\eta_{0,\ell}$ uniquely determine the limiting random variable defined on the segment $[0,1]$. □

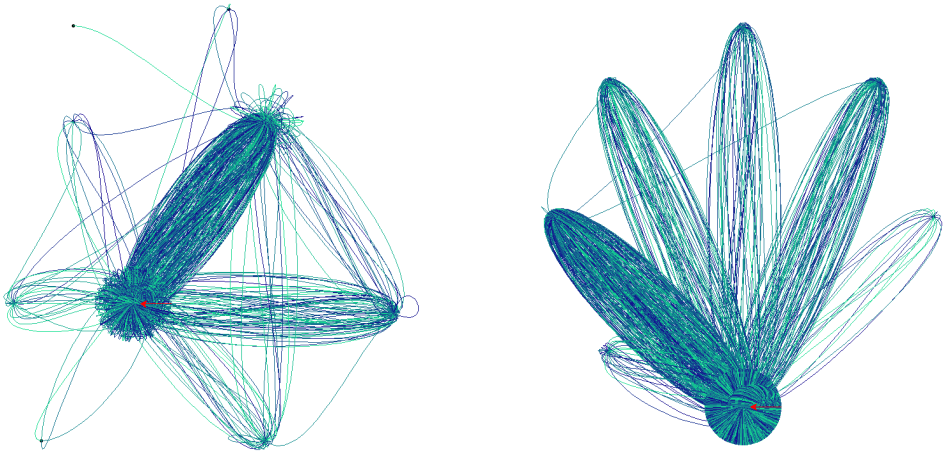

Figure 8. Random rooted maps, respectively with 1000 and 20000 edges.

4. Combinatorial approaches

The answers provided by the previous section, because of their simplicity, raise new questions. They suggest the existence of a combinatorial background which should make the enumeration easier. This section gives new answers by introducing the bijections with indecomposable chord diagrams.

A *chord diagram* with n chords is a set of vertices labelled with numbers $\{1, 2, \dots, 2n\}$ equipped with a perfect matching. A chord diagram is *indecomposable* if it cannot be expressed as a concatenation of two smaller diagrams.

4.1. **Why the root degree follows a uniform law.** In [Cor09], Cori described a bijection between rooted maps and indecomposable diagrams. It turns out that this bijection satisfies the following proposition.

Proposition 9. *There exists a bijection between rooted maps of root degree d with n edges, and indecomposable diagrams with $n+1$ chords such that vertex $k-2$ is matched with vertex* 1.

This proposition proves Theorem 7 in a much simpler way: in a (non necessarily indecomposable) diagram, the label of the vertex matched with 1 exactly follows a uniform law on $\{2, \dots, 2n\}$. But a diagram is almost surely an indecomposable diagram (because their cardinalities are asymptotically the same), so the label of the vertex matched with 1 divided by $2n$ asymptotically obeys a uniform law on $[0, 1]$.

4.2. **Uniform random generation.** One also can use Cori's bijection to write a uniform generator for rooted maps. Uniform sampling a random diagram can be accomplished by adding the chords sequentially. If the sampled diagram is not indecomposable, it is rejected (it happens with probability close to 0). After that, the diagram is transformed into a map thanks to Cori's bijection. Figure 8 shows examples of random maps thus generated.

4.3. **Number of leaves.** To obtain the asymptotic distribution of leaves, we use another bijection coming from [CYZ16]. This bijection sends leaves of a map into the *isolated* chords (that is, edges connecting vertices k and $k+1$) of a indecomposable chord diagram. According to [FN00, Theorem 2], the number of isolated edges in a random chord diagram has a Poisson distribution with parameter 1. We can then deduce the following theorem.

Theorem 10. *The number of leaves in a random map with n edges, follows asymptotically a Poisson law with parameter* 1.

4.4. **Two dual parameters.** We briefly remark that two other parameters, namely *root face degree* and the number of *trivial loops* cannot be easily examined with the method of generating functions because marking them requires additional nested information like the degrees of all the faces. However, such parameters can be easily marked in their corresponding dual maps. Their distributions follow, respectively, uniform and Poisson limit laws.

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