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Effects of Donor Size and Heavy Doping on Optical, Electrical and Thermoelectric Properties of Various Degenerate Donor-Silicon Systems at Low Temperatures

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Abstract: In various degenerate donor-silicon systems, taking into account the effects of donor size and heavy doping and using an effective autocorrelation function for the potential fluctuations expressed in terms of the Heisenberg uncertainty relation and also an expression for the Gaussian average of $e_k^{-\frac{a}{2}}$, $a \geq 1$ $E_k$ being the kinetic energy of the electron, calculated by the Kane integration method (KIM), we investigated the density of states, the optical absorption coefficient and the electrical conductivity, noting that this average expression calculated by the KIM was found to be equivalent to that obtained by the Feynman path-integral method. Then, those results were expressed in terms of $\frac{E_k}{E}$ for total electron energy $E \geq 0$, vanished at the conduction-band edge: $E = 0$, and for $E \leq 0$ exhibited their exponential tails, going to zero as $E \to -\infty$ and $\infty$, and presenting the maxima, in good accordance with an asymptotic form for exponential conduction-band tail obtained by Halperin and Lax, using the minimum counting methods. Further, in degenerate d-Si systems at low temperatures, using an expression for the average of $E_k^p$, $p \geq 3/2$, calculated using the Fermi-Dirac distribution function, we determined the mobility, electrical conductivity, resistivity, Hall factor, Hall coefficient, Hall mobility, thermal conductivity, diffusion coefficient, absolute thermoelectric power, Thomson coefficient, Peltier coefficient, Seebeck thermoelectric potential, and finally dimensionless figure of merit, which were also compared with experimental and theoretical results, suggesting a satisfactory description given for our obtained results.

Keywords: Donor Size, Heavy Doping, Electrical Conductivity, Hall Effect, Diffusion Coefficient

1. Introduction

Donor (acceptor)-silicon d (a)-Si system at a given temperature $T$, doped with a given d (a)-density $N$, assuming that all the impurities are ionized, is the base material of modern semiconductor devices [1-6]. Then, due to the Fermi-Dirac statistics, there are three cases may be classified as: non-degenerate ($T >> T_D$ and $N<<N_c$), moderately degenerate ($T > T_D$ and $N>>N_c$) and degenerate ($T << T_D$ and $N>>N_c$)-cases, $T_D$ and $N_c$ being respectively the degenerate temperature defined in Eq. (15) and critical impurity density.

In the present paper, the degenerate d-Si system is considered, noting that all the optical, electrical, and thermoelectric properties given in the degenerate a-Si system can also be investigated by a same treatment.

So, in the degenerate d-Si system, a good knowledge of: (i) energy-band structure parameters such as: the reduced band gap [7-12], $N_{c(d)}$ [11, 12] and effective donor ionization energy [13], (ii) exponential conduction-band tails [14-25] and Fermi energy [26-28], and finally (iii) optical [29-48], electrical [49-69], and thermoelectric [58, 61, 68, 69] properties, due to the effects of donor size [11, 12, 42, 43, 53], heavy doping [14, 17-30, 34, 35, 48-68], and low T [49, 58], is thus necessary for designing new devices and also understanding their performance.

In Section 2, we studied the effects of donor size [or compression (dilatation)], temperature, and heavy doping on the energy-band-structure parameters. At $T = 0$ K, with
increasing values of donor radius $r_d$, since the effective dielectric constant $\varepsilon_n(r_d)$ decreases, due to the donor-size effect, the effective donor ionization $E_{\text{do}}(r_d)$, unperturbed intrinsic band gap $E_{\text{go}}(r_d)$, and critical donor density $N_{c(d)}^{(r_d)}$ increase, as seen in Table 1. Then, for a given $r_d$, the effective intrinsic band gap $E_{\text{go}}(r_d,T)$, due to the T-effect, decreases with increasing $T$, as given in Eq. (3). Finally, due to the heavy doping effect (HDE), for a given $r_d$, the effective electron mass $m_{\text{ne}}^{(r_d)}(N,T)$, increases with increasing $N$, as given in Eq. (8), and for given $r_d$ and $T$, the reduced band gap $E_{\text{go}}^{(r_d)}(N,T)$ decreases with increasing $N$, as given in Eq. (10).

In Section 3, the effective autocorrelation function for potential fluctuations, $W_\eta$, was determined and in Eq. (B.6) the appendix B, being a central result of the present paper, as noted in Eq. (20). It was suggested that $W_\eta(\varepsilon \rightarrow \pm \infty) \rightarrow \eta_\eta$, $\varepsilon$ and $\eta_\eta$ being respectively the total electron energy and the energy parameter characteristic of the conduction-band tail states, and $W_\eta(\varepsilon \rightarrow \pm 0) \rightarrow 0$. Therefore, the density of states, the optical absorption coefficient and the electrical conductivity, being proportional to our result (20), vanished at the conduction-band edge $\varepsilon = 0$, as given in Eqs. (23, 26). Those results were also compared with other theoretical results, obtained at $\varepsilon = 0$, in the small time approximation [21, 29, 30] and in the full ground-state case and deep-tail approximation [21], which were found to be constant, being not correct, as discussed also in Eq. (26). Then, for $\varepsilon \leq 0$, their exponential tails were obtained in Figures 1, 4 and 7, in good agreement with the corresponding data given in Ref. 12, and it increases with increasing $r_d$, due to the donor-size effect [12]. Then, for $\varepsilon \leq 0$, the exponential band-tail behaviors were investigated and reported in Table 4, and also in Figures 1 and 2a (b).

In Section 4, we determined the average of $\varepsilon^p$ at low temperature $T$ ($T << T_D$), using the Fermi-Dirac distribution function (FDDF), $(\varepsilon^p)^{\text{FDDF}} \equiv \varepsilon_\varepsilon^p \equiv \varepsilon_{F\varepsilon}^p$, for $p \geq 3/2$, as given in Eq. (34) and Table 3, $\varepsilon_{F\varepsilon}$ being the Fermi energy determined in Eq. (A10) of the Appendix C.

In Section V, we determined the critical donor density, as given in Table 1, suggesting that its numerical results are in good agreement with the corresponding data given in Ref. 12, and it increases with increasing $r_d$, due to the donor-size effect [12]. Then, for $\varepsilon \leq 0$, the exponential band-tail behaviors were investigated and reported in Table 4, and also in Figures 1 and 2a (b).

In Section 6, various optical functions were determined in band-to-band transitions ($\varepsilon \geq 0$) as found in Figs. 3a, 3b, 3c, being compared with other theoretical and experimental works [33-35, 38, 44-48], and also the exponential optical absorption-coefficient tail behaviors were investigated when $\varepsilon \leq 0$, as seen in Table 7, and Figures 4 and 5a (b).

In Section 7, for $\varepsilon \geq 0$, using the functions $G_\eta$ obtained at low $T$, given in Table 3, we determined various electric functions as those given in Tables 10-13, in good accordance with the corresponding experimental results [50, 53, 54, 56-61], and for $\varepsilon \leq 0$, we also studied the exponential conductivity-tail behaviors, as those given in Tables 14 and Figures 7, 8a (b).

In Section 8, for $\varepsilon \geq 0$, using also the function $G_\eta$, we studied various thermoelectric functions, and reported their numerical results in Table 15 and Figures 9a, 9b, 9c, 9d, 9e, and 9f, noting that for $\varepsilon \leq 0$ we could also study the exponential thermoelectric function-tail behaviors by a same treatment, as those obtained in Sections 5-7.

Finally, some concluding remarks were given and discussed in Section 9.

2. Energy-Band-Structure Parameters

Here, we study the effects of donor size, temperature, and heavy doping on the energy-band-structure parameters.

2.1. Donor-Size Effect

In donor-Si systems at $T=0$ K, since the d-radius, $r_d$, in tetrahedral covalent bonds is usually either larger (or smaller) than the Si atom-radius, $r_o$, assuming that in the P-Si system $r_p = r_o = 0.117$ nm, with $nm = 10^{-9} m$, a local mechanical strain (or deformation potential-or-strained energy) is induced, according to a compression (dilation) for $r_d > r_o$ ($r_d < r_o$), respectively, or to the donor size ($r_d$)-effect. In the Appendix A of our recent paper [12], basing on an effective Bohr model, such a compression (dilation) occurring in various donor (d)-Si systems was investigated, suggesting that the effective dielectric constant, $\varepsilon_n(r_d)$, decreases with increasing $r_d$. This donor size ($r_d$)-effect affects the changes in all the energy-band-structure parameters or the electronic properties of various donor-Si systems, expressed in terms of $\varepsilon_n(r_d)$, as those investigated in our recent paper [12], noting that $\varepsilon_n(r_p) = 11.4$. In particular, the changes in the unperturbed intrinsic band gap, $E_{\text{go}}(r_p) = 1170$ meV, effective donor ionization energy, $E_{\text{do}}(r_p) = 33.58$ meV, and critical donor (P)-density, $N_{c(p)} = 3.5 \times 10^{18} \text{ cm}^{-3}$, of the P-Si system at 0 K, are obtained in an effective Bohr model, as [12]

$$E_{\text{go}}(r_d) - E_{\text{go}}(r_p) = E_{\text{do}}(r_d) - E_{\text{do}}(r_p) = E_{\text{do}}(r_p) \times \left[\frac{\varepsilon_n(r_p)}{\varepsilon_n(r_d)}\right]^2 - 1,$$

$E_{\text{do}}(r_d)$ and $N_{c(r_d)}$ increase. Those changes, given in our previous paper [12], are now reported in the following Table 1, in which the numerical results of critical donor density, due to the exponential band tail (EBT)-effect, $N_{c(EBT)}^{(r_d)}$, being obtained in the next Section V, are also included for a comparison. Here, $\delta$ is normally equal to 1, but it will be chosen as: $\delta = \delta_n = 1.0028637416$, so that the obtained
results of $N_{ebt}^{(r_d,H)}$, $\delta_x$ would be more accurate.

Table 1. The following values of $r_d$, $\epsilon_n$, $a_{in}$, $E_{go}(r_d)$, and $N_{(cd)}$, given in our previous paper [12, are now reported in this TABLE, in which we also include the numerical results of $N_{ebt}^{(r_d,H)}$, where $b = 1$ or $\delta_x = 1.0028637416$, calculated using Eqs. (41, 42), and their absolute relative errors defined by:

$$|RE| \equiv 1 - \frac{N_{ebt}^{(r_d,H)}}{N_{(cd)}}.$$ Here, nm $\equiv 10^{-9}$ m.

Moreover, it should be noted that in Table I, $E_{go}$ being the effective intrinsic band gap given in the bulk crystalline Si at room temperature, being equal to: (i) the effective mass $m_e(r_d)$, given in the bulk crystalline Si at room temperature, being size of $N_{(cd)}$, calculated using Eqs. (41, 42), and their absolute relative errors defined by:

$$|RE| \equiv 1 - \frac{N_{ebt}^{(r_d,H)}}{N_{(cd)}}.$$ Here, nm $\equiv 10^{-9}$ m.

Moreover, it should be noted that in Table I, $E_{go}$ being the effective intrinsic band gap given in the bulk crystalline Si at room temperature, being size of $N_{(cd)}$, calculated using Eqs. (41, 42), and their absolute relative errors defined by:

$$|RE| \equiv 1 - \frac{N_{ebt}^{(r_d,H)}}{N_{(cd)}}.$$ Here, nm $\equiv 10^{-9}$ m.

Now, the effective Bohr radius can be defined by

$$a_B(r_d, m^*) \equiv \frac{m_e}{\epsilon_n(r_d)} \times 5.3 \text{ nm.} \quad (1)$$

In Eq. (1), $m^*$ is the effective electron mass given in the Si, being equal to: (i) the effective mass $m_e = 0.3216 \times m_o$, $m_o$ being the free electron mass, defined for the calculation of $m_e^{HDE}(N)$, as defined in next Eq. (8), due to the heavy doping effect (HDE), (ii) the reduced effective mass: $m_e = \frac{m_e}{m_e^*}$, $m_e^* = 0.171 \times m_o$, for the optical absorption-coefficient calculation, where $m_p = 0.3664 \times m_o$ is the effective hole mass in the silicon [12], (iii) $m_n^{HDE}(N)$, given in next Eq. (8), for the determination of the density of states, as given in Section 5, and finally (iv) the conductivity effective mass: $m_{cond} = 0.26 \times m_o$ for the electrical conductivity calculation [6], as used in Section 7.

Then, in the degenerate case ($N > N_{(cd)}$), denoting the Fermi wave number by: $k_F(N) \equiv \frac{(3\pi^2 N/g_{e,c})^{1/3}}{\hbar}$, where $g_c = 3$ is the effective average number of equivalent conduction-band edges [11, 12], the effective Wigner-Seitz radius $r_{sn}$ characteristic of the interactions is defined by

$$\gamma \times r_{sn}(N, r_d, m^*) \equiv \frac{k_F^2}{a_B} < 1,$$ \hspace{1cm} (2)

being proportional to $N^{-1/3}$. Here, $\gamma = (4/9\pi)^{1/3}$, and $k_F^2$ means the averaged distance between ionized donors.

2.2. Temperature Effect

Here, in d-Si systems, being inspired from recent works by Pässler [8, 9], we can propose an accurate expression for the effective intrinsic band gap as a function of $r_d$ and $T$, as

$$E_g(T, r_d) = \frac{E_{go}(r_d) - 0.071 (eV) \times \left[1 + \left(\frac{4T}{440.6913 K} \right)^{2.201} \right]^{\frac{1}{2.201}}}{1 - \frac{1}{2.201}},$$ \hspace{1cm} (3)

For example, in the (P, S)-Si systems, for $0 \leq T(K) \leq 3500$, the absolute maximal relative errors of $E_{go}$ are equal to: 0.22%, 0.15%, respectively, calculated using the accurate complicated results given by Pässler [9].

2.3. Heavy Doping Effect (HDE)

HDE on $m_n$

Now, using Eq. (2) for $r_{sn}(N, r_d, m^* = m_n)$, the ratio of the inverse effective screening length $k_sn$ to Fermi wave number $k_F$ at 0 K is defined by [12]

$$R_{sn}(N, r_d) \equiv \frac{k_{sn}}{k_F} = \frac{k_{sn}^{-1}}{k_{sn}} = R_{snWS} + \left[R_{snTF} - R_{snWS}\right]e^{-r_{sn}} < 1.$$ \hspace{1cm} (4)

It is noted that, in the very high electron-density limit [or in the Thomas-Fermi (TF)-approximation], $R_{sn}$ is reduced to

$$R_{snTF}(N, r_d) \equiv \frac{k_{snTF}}{k_F} = \frac{k_{sn}}{k_{snTF}} = \sqrt{\frac{4\pi r_{sn}}{\pi}} \ll 1, \quad (5)$$

being proportional to $N^{-1/6}$. It should be noted that the effective screening length $k_{snTF}$ is very larger than the averaged distance between ionized donors $k_F^{-1}$ (i.e., this is the TF-condition given in the very degenerate case, $N \gg$
been chosen so that the absolute maximal relative error which is a very simplified form compared with our previous
Further, the empirical parameter given in Table 1 and the electron effective mass \(|RE_s|\) approximated by \([11, 12]\) due to such the HDE, the effective electron mass can be
correlation energy (CE), determined by as \([11, 12]\)

\[
|RE_s| = \frac{\text{optical band-gap \(N, r_d\)}}{\text{in the very low electron-density limit \(N, r_d\)}}.
\]

Here, the values of effective dielectric constant \(\varepsilon_a(N, r_d)\) are given in Table 1 and the electron effective mass \(m^\text{HDE}(N, r_d)\), due to the heavy doping effect, is determined in Eq. (8).

\[
m^\text{HDE}(N, r_d) = \frac{1 + \frac{4\pi}{\text{vol}(\text{growth})} m_n}{\frac{3N_\text{growth}}{1 + \pi^2 N_\text{growth}}}
\]

HDE on \(\varepsilon_a\)

In the degenerate case, the optical band gap is defined by

\[
\text{BGN}(N, r_d) = C_n \times \left(\frac{N}{10^{18} \text{ cm}^{-3}}\right)^{1/2} \times \left(\frac{11.4}{E_{\text{opt}}(r_d)}\right)^{1/2} \times \left[1 + \left(\frac{m^\text{HDE}(N, r_d)}{m_p}\right)^{1/2}\right],
\]

which is a very simplified form compared with our previous complicated expression for BGN \([12]\).

Here, the values of effective dielectric constant \(\varepsilon_a(N, r_d)\) are given in Table 1 and the electron effective mass \(m^\text{HDE}(N, r_d)\), due to the heavy doping effect, is determined in Eq. (8).

Further, the empirical parameter \(C_n = 0.5 \times 10^{-3} \text{ (eV)}\) has been chosen so that the absolute maximal relative error \(\text{RE}\) of our result \((9)\), calculated using the optical band-gap \(\varepsilon_a\) for P-Si systems at 20 K obtained by Wagner, \([7]\)

are found to be minimized.

In a degenerate P-Si system, with use of the next Eq. (43), obtained for the definition of effective density of free electrons given in the conduction band, \(N^* \approx N - N_c(p)\), where the value of \(N_c(p)\) is given in Table I, our present results of \(E_{\text{opt}}(N^*, T = 20 \text{ K}, r_p)\), computed using Eqs. (9, 11), and their absolute relative errors \(\text{RE}\), calculated using the \(E_{\text{opt}}\) data at 20 K \([12]\), are obtained and reported in Table 2, in which our previous accurate \(E_{\text{opt}}(N^*)\)-results and their \(\text{RE}\) are also included \([12]\), for a comparison.

<table>
<thead>
<tr>
<th>(N (10^{18} \text{ cm}^{-3}))</th>
<th>4</th>
<th>8.5</th>
<th>15</th>
<th>50</th>
<th>80</th>
<th>150</th>
</tr>
</thead>
<tbody>
<tr>
<td>(E_{\text{opt}})-data</td>
<td>1.138</td>
<td>1.133</td>
<td>1.129</td>
<td>1.131</td>
<td>1.132</td>
<td>1.133</td>
</tr>
<tr>
<td>(E_{\text{opt}}(N^*)) ([12])</td>
<td>1.149</td>
<td>1.138</td>
<td>1.134</td>
<td>1.126</td>
<td>1.123</td>
<td>1.119</td>
</tr>
<tr>
<td>(\text{RE}) (%)</td>
<td>0.9</td>
<td>0.5</td>
<td>0.5</td>
<td>0.4</td>
<td>0.8</td>
<td>1.2</td>
</tr>
<tr>
<td>(E_{\text{opt}}(N^*), \text{Eq. (9)})</td>
<td>1.160</td>
<td>1.147</td>
<td>1.139</td>
<td>1.123</td>
<td>1.118</td>
<td>1.113</td>
</tr>
<tr>
<td>(\text{RE}) (%)</td>
<td>1.9</td>
<td>1.3</td>
<td>0.9</td>
<td>0.7</td>
<td>1.2</td>
<td>1.7</td>
</tr>
</tbody>
</table>

The underlined \(\text{RE}\)-value is the maximal one.

This table indicates that the maximal value of \(\text{RE}\), obtained from our present \(E_{\text{opt}}(N^*)\)-result, is found to be equal to 1.7%, which can be compared with that equal to 1.2% obtained from our previous \(E_{\text{opt}}(N^*)\)-result \([12]\).

HDE conditions

Finally, in degenerate d-Si systems, the energy parameter \(\eta_n\), being characteristic of the exponential conduction-band tail, is determined in Eq. (B.4) of the Appendix B as

\[
\eta_n(N, r_d) = \frac{\sqrt{2\pi N}}{q^2 k_{\text{F}1/2}^n\eta},
\]

where \(k_{\text{F}1/2}^n\) is determined in Eq. (4). Moreover, in highly degenerate case \((N \gg N_c(d))\) or in the Thomas-Fermi approximation, \(k_{\text{F}1/2}^n \approx k_{\text{F}1/2}^n\eta\) determined in Eq. (5), \(\eta_n\) is found to be proportional to \(N^{1/2}\).

Then, from Eq. (12) and next Eq. (15), we can obtain another heavy doping condition as

\[
\frac{\eta_n}{\varepsilon_{\text{Fno}}} < 1,
\]

being proportional to \(N^{-1/4}\) in this highly degenerate case.

In summary, in the highly degenerate case \((N \gg N_c(d))\) and from Eqs. (2, 4, 13), one has

\[
\frac{k_{\text{F}1/2}^n}{\varepsilon_{\text{Fno}}} < \frac{\eta_n}{\varepsilon_{\text{Fno}}} < \frac{k_{\text{F}1/2}^n}{\varepsilon_{\text{Fno}}} < 1,
\]

where \(\varepsilon_{\text{Fno}}\) is the Fermi energy at 0 K, defined by

\[
\varepsilon_{\text{Fno}}(N) = \frac{k^2 x k_{\text{F}1/2}^n(N)}{2 n m^*}.
\]

In Eq. (15), \(m^*\) is the electron effective mass, defined in Eq. (1), and in this highly degenerate case one has a low \(T\)-condition as: \(T \ll T_D \equiv \varepsilon_{\text{Fno}}(N)/k_B T_D\) and \(k_B\) being the
degeneracy temperature and the Boltzmann’s constant, respectively.

3. Effective Autocorrelation Function and its Applications

In the degenerate d-Si systems, the total screened Coulomb impurity potential energy due to the attractive interaction between an electron charge, \(-q\), at position \(\mathbf{r}\) and an ionized donor charge: \(+q\) at position \(\mathbf{R}_i\) randomly distributed throughout the Si crystal, is defined by

\[
V(\mathbf{r}) \equiv \sum_{i=1}^{N} v_i(\mathbf{r}) + V_o,
\]

where \(N\) is the total number of ionized donors, \(V_o\) is a constant potential energy, and \(v_i(\mathbf{r})\) is a screened Coulomb potential energy for each d-Si system, defined as

\[
v_i(\mathbf{r}) \equiv - \frac{q^2 \times e^{-|x_{sn}(s)| \times |\mathbf{r} - \mathbf{R}_i|}}{e^{|\mathbf{r} - \mathbf{R}_i|}}.
\]

Further, using a Fourier transform, the \(v_i\)-representation in wave vector \(\mathbf{k}\)-space is given by

\[
v_i(\mathbf{k}) \equiv - \frac{q^2}{\varepsilon_{sn}^2} \times \frac{4\pi}{k^2 + k_{\mathbf{k}}^2}
\]

where \(\Omega\) is the total Si-crystal volume and \(k_{\mathbf{k}}\) is the inverse screening length determined in Eq. (4). Moreover, in Eqs. (16, 17), \(V_o\) is defined as a constant so that \(\langle V(\mathbf{r}) \rangle = 0\), reflecting a charge neutrality, where the notation \(\langle \ldots \rangle\) denotes the configuration average \([25, 58]\). In fact, from Eq. (17), one has

\[
\langle V(\mathbf{r}) \rangle = 0.
\]

Therefore, the effective autocorrelation function for potential fluctuations can thus be defined by \([25, 58]\)

\[
W_n(\mathbf{r} - \mathbf{r}') \equiv \langle V(\mathbf{r})V(\mathbf{r}') \rangle \equiv \langle V(\mathbf{r}) \times \langle V(\mathbf{r}') \rangle \rangle ,
\]

where \(\langle V(\mathbf{r})V(\mathbf{r}') \rangle\) denotes the effective second-order cumulant, and \(\mathbf{r}(t)\) and \(\mathbf{r}'(t')\) are the electron positions at the times \(t\) and \(t'\), noting that the cumulant is just the average potential energy, which may be absorbed by a redefinition of the zero energy. Then, the expression for \(W_n\) is determined in Eq. (B.6) of the Appendix B, as

\[
W_n(\mathbf{r},N) \equiv \langle V(\mathbf{r}) \rangle = \eta_n^2 \times \exp \left( -\frac{\mathcal{H}_n \times 2\eta_n}{2\sqrt{W_n}} \right).
\]

Here, \(H_{sn}(N)\) is given in Eq. (4), \(n_n\) is determined in Eq. (12), the constant \(\mathcal{H}_n\) will be chosen in Section V as: \(\mathcal{H}_n = 5.4370\), such that the determination of the density of electrons localized in the conduction-band tail would be accurate, and finally \(\eta_n \equiv -\sqrt{\frac{\mathcal{E}}{\mathcal{E}_{F_{no}}}}\), where \(\mathcal{E}\) is the total electron energy and \(\mathcal{E}_{F_{no}}\) is the Fermi energy at 0 K, determined in Eq. (15).

Now, we calculate the ensemble average of the function:

\[
\langle (E - V)\alpha^{-2} \rangle_{KIM} \equiv \langle E_{\alpha}^{-2} \rangle_{KIM} = \int_{-\infty}^{\infty} (E - V)\alpha^{-2} \times P(V) dV, \text{ for } a \geq 1.
\]

Then, by variable changes: \(s = (E - V)\sqrt{W_n}\) and \(x = -E/\sqrt{W_n}\), and using an identity \([15]\):

\[
\int_{-\infty}^{\infty} s^{a-\frac{1}{2}} \times \exp(-x^2 - s^2) ds \equiv \Gamma(a + \frac{1}{2}) \times \exp(x^2/4) \times D_{-a-\frac{1}{2}}(x),
\]

where \(D_{-a-\frac{1}{2}}(x)\) is the parabolic cylinder function, \(\Gamma(a + \frac{1}{2})\) is the Gamma function, one thus finds

\[
\langle E_{\alpha}^{-2} \rangle_{KIM} = \frac{\exp(-x^2/4) \times W_n^{\frac{8a-1}{2\sqrt{2}}} \times \Gamma(a + \frac{1}{2}) \times D_{-a-\frac{1}{2}}(x)}{\sqrt{2\pi}}.
\]

This result \((20)\) will used to study the optical, electrical, and thermoelectric properties of various degenerate d-Si systems, depending on \(W_n\) defined in Eq. (19) and the variable \(x\), expressed also in terms of \(W_n\), as

\[
W_n(\mathbf{r},N) \equiv \langle V(\mathbf{r}) \rangle = \eta_n^2 \times \exp \left( -\frac{\mathcal{H}_n \times 2\eta_n}{2\sqrt{W_n}} \right).
\]
\[
x = -\frac{E}{\sqrt{W_n}} \equiv A_n \times v_n \times \exp \left( \frac{E_{\text{Fermi}} \times \eta_n}{4 \times \sqrt{W_n}} \right),
A_n \equiv \frac{E_{\text{Fermi}}}{\eta_n}, v_n \equiv -\frac{E}{E_{\text{Fermi}}},
\]

where \(E_{\text{Fermi}}\) and \(\eta_n\) are determined in Eqs. (15, 12), respectively. Therefore, the effective autocorrelation function for potential fluctuations \(W_n\), defined in Eq. (19), is thus a central result of the present paper.

### 3.2. Feynman Path-Integral Method

In the Feynman path-integral method (FPIM), the ensemble average of \((-V)^{a-1}\) is defined by

\[
\langle (-V)^{a-1} \rangle_{\text{FPIM}} \equiv \langle \langle (-V)^{a-1} \rangle_{\text{KIM}} \equiv \frac{\hbar}{2^{3/2} \times \sqrt{2\pi}} \times \Gamma(a + \frac{1}{2}) \times \int_{-\infty}^{\infty} (it)^{-a-1} \times \exp \left( -\frac{i}{\hbar} \left( \frac{\eta_n}{W_n} \right)^2 \right) dt, i^2 = -1,
\]

noting that as \(a=1\), \((it)^{-3/2} \times \exp(-\frac{\eta_n}{W_n})\) is found to be proportional to the averaged Feynman propagator given the dense donors \([16]\).

Then, by variable changes: \(t = \frac{\hbar}{V W_n}\) and \(x = -\frac{E}{\sqrt{W_n}}\) and then using an identity \([15]\):

\[
\int_{-\infty}^{\infty} (s)^{-a-1} \times \exp \left( isx - \frac{x^2}{2} \right) ds \equiv 2^{3/2} \times \Gamma(3/2) \times \exp(-x^2/4) \times D_{-\alpha-\frac{1}{2}}(x),
\]

one finally obtains: \(\langle (-V)^{a-1} \rangle_{\text{FPIM}} \equiv \langle (-V)^{a-1} \rangle_{\text{KIM}}\), where \(\langle (-V)^{a-1} \rangle_{\text{KIM}}\) is determined in Eq. (20).

In the following, with use of asymptotic forms for \(D_{-\alpha-\frac{1}{2}}(x)\) \([15]\), those given for \(\langle (-V)^{a-1} \rangle_{\text{KIM}}\) will be obtained in the two cases: \(\eta \geq 0\) and \(\eta \leq 0\).

#### \(\eta \geq 0\)-case

As \(\eta \to +\infty\), from Eq. (21), one has: \(v_n \to -\infty\) and \(x \to -\infty\). In this case, one gets \([15]\):

\[
D_{-\alpha-\frac{1}{2}}(x \to -\infty) \approx \sqrt{\frac{\eta_n}{\pi}} \times e^{\frac{x^2}{4a}} \times (-x)^{a-\frac{1}{2}}.
\]

Therefore, Eq. (20) becomes

\[
\langle (-V)^{a-1} \rangle_{\text{KIM}} \approx \langle -V \rangle^{a-\frac{1}{2}}.
\]

Further, as \(\eta \to +0\), from Eq. (21), one has: \(v_n \to 0\) and \(x \to -\infty\). Therefore, one gets \([15]\):

\[
D_{-\alpha-\frac{1}{2}}(x \to -\infty) \approx \beta(a) \times \exp\left( \left( \sqrt{a} + \frac{1}{16a^2} \right) x - \frac{x^2}{16a^2} + \frac{x^3}{24a^3} \right) \to 0, \beta(a) = \frac{\sqrt{\pi}}{2^{3/4} \times \Gamma(\frac{3}{4})}.
\]

Thus, as \(\eta \to +0\), from Eq. (20), one gets:

\[
\langle (-V)^{a-1} \rangle_{\text{KIM}} \to 0,
\]

being in good agreement with our result obtained in Eq. (A3) of the Appendix A.

In summary, from Eqs. (22, 23) and for \(\eta \geq 0\), the expression of \(\langle (-V)^{a-1} \rangle_{\text{KIM}}\) can be approximated by:

\[
\langle (-V)^{a-1} \rangle_{\text{KIM}} \equiv \langle -V \rangle^{a-\frac{1}{2}}.
\]

#### \(\eta \leq 0\)-case

Here, from Eqs. (19, 21), Eq. (20) can be rewritten as

\[
\langle (-V)^{a-1} \rangle_{\text{KIM}} = \frac{\exp(-x^2/4) \times W_n}{\sqrt{2\pi}} \times \Gamma(a + \frac{1}{2}) \times D_{-\alpha-\frac{1}{2}}(x),
\]

\[
= \frac{\exp(-x^2/4) \times W_n}{\sqrt{2\pi}} \times \exp \left( -\frac{3\eta_n}{8 \times \sqrt{W_n}} \right) \times \Gamma(a + \frac{1}{2}) \times D_{-\alpha-\frac{1}{2}}(x).
\]

As \(\eta \to -0\), from Eq. (21), one has: \(v_n \to +0\) and \(x \to +\infty\). Thus, one obtains, for any \(a \geq 1\), \([15]\)
\[ D_{-a}^{-1}(x \to \infty) = \beta(a) \times \exp \left[ -\left( \sqrt{a} + \frac{1}{16a^2} \right)x - \frac{x^2}{16a} - \frac{x^3}{24a^2} \right] \to 0, \beta(a) = \frac{\sqrt{\pi}}{2a^{3/4} \Gamma(3/4)}, \] noting that
\[ \beta(1) = \frac{\sqrt{\pi}}{25 \sqrt{\Gamma(5/4)}} \text{ and } \beta(5/2) = \frac{\sqrt{\pi}}{25/2}. \]

Then, putting \( f(a) \equiv \frac{\eta_n}{\sqrt{2\pi}} \times \Gamma(a + \frac{1}{2}) \times \beta(a) \), Eq. (25) yields
\[ H_n(v_n \to 0, r_d, a) = \frac{(a_f)^{(a-1)/2} \text{KIM}}{f(a)} = \exp \left[ -\frac{\Delta_n R_{an} x^{(2a-1)}}{8x^2|v_n|} - \left( \sqrt{a} + \frac{1}{16a^2} \right)x - \frac{x^2}{16a^2} - \frac{x^3}{24a^2} \right] \to 0, \] which is in good accordance with that given in Eq. (A3) of the Appendix A. In particular, as \( v_n \to 0 \), the first term of \( \ln H_n(v_n \to 0, r_d, a = 1) \) given in Eq. (26), \( -\frac{\Delta_n R_{an}}{8x|v_n|} \), can be compared with the third one given in Eq. (A3), \( -4Bc^2R_{an} \times |v_n|^{-\frac{3}{2}} \). Moreover, as noted in Eqs. (B.5, B.6) of the Appendix B, when the small time approximation (STA) is used: \( \Delta r \approx 0 \) \([17, 21, 29, 30]\), \( \mathcal{W}_n(v_n, N) \equiv \eta_n^H \). Therefore, Eq. (25) now yields \([21, 29, 30]\):
\[ H_n(\text{STA})(v_n \to 0, r_d, a) = \frac{(a_f)^{(a-1)/2} \text{KIM}}{f(a)} = \exp \left( \left( \sqrt{a} + \frac{1}{16a^2} \right)x - \frac{x^2}{16a^2} \right) \times \frac{\mathcal{E}}{\eta_n^H}, \] being equal to 1 for \( \mathcal{E} = -0 \), which is not correct, since we must have: \( H_n(\mathcal{E} \to -0, r_d, a) \to 0 \), as obtained in Eq. (26), due correctly to the Heisenberg uncertainty relation given in Eq. (B.5): \( \Delta r > 0 \) as \( \mathcal{E} \to -0 \). Finally, we also remark that, in the full ground-state case and deep-tail approximation, the exponential conduction- band tail, obtained by Sa-yakamit et al. \([23]\), was also equal to a constant at \( \mathcal{E} = -0 \), being not correct.

Further, from Eq. (21), as \( \mathcal{E} \to -\infty \), one has: \( v_n \to +\infty \) and \( x \to \infty \). Thus, one gets \([15]\):
\[ D_{-a}^{-1}(x \to \infty) \approx x^{-a-\frac{3}{2}} \times e^{-x^2/4} \to 0. \] Therefore, Eq. (25) yields
\[ K_n(v_n \to +\infty, r_d, a) = \frac{(a_f)^{(a-1)/2} \text{KIM}}{f(a)} = \frac{1}{\beta(a)} \times \exp \left( -\frac{(A_n \times v_n)^2}{2} \right) \times (A_n \times v_n)^{-a-\frac{1}{2}} \to 0, \]
being in perfect agreement with a well-known semi-classical Kane’s result \([14]\).

It should be noted that, as \( \mathcal{E} \leq 0 \), the ratios (26) and (27) can be taken in an approximate form as
\[ F_n(v_n, r_d, a) = K_n(v_n, r_d, a) + [H_n(v_n, r_d, a) - K_n(v_n, r_d, a)] \times \exp[-c_1 \times (A_n v_n)^2], \] so that: \( F_n(v_n, r_d, a) \to H_n(v_n, r_d, a) \) for \( 0 \leq v_n \leq 16 \), and \( F_n(v_n, r_d, a) \to K_n(v_n, r_d, a) \) for \( v_n \geq 16 \). For that, in next sections V and VI, the constants \( c_1 \) and \( c_2 \) may be respectively chosen as: \( c_1 = 10^{-40} \) and \( c_2 = 80 \) when \( a = 1 \), being used to the study of reduced density of exponential conduction-band-tail states, and \( c_1 = 20^{-150} \) and \( c_2 = 300 \) when \( a = 5/2 \), for the study of reduced optical absorption coefficient and exponential tails of electrical conductivity.

Here, one remarks that, from Eqs. (26-28) and for a given value of \( a \), since, as \( v_n \to +0 \) and \( +\infty \), \( F_n(v_n, r_d, a) \to 0 \), the maximum of this function thus exists, occurring at \( v_n = v_n(n) \). Hence, in various degenerate d-Si systems, in which \( N = 5 \times 10^{20} \text{cm}^{-3} \) and \( T=0 \text{K} \), for example, using Eq. (28), we can study the behaviors of the function \( \ln[F_n(v_n, r_d, a)] < 0 \), for given \( a \), which can take its approximate form determined by: \( \Delta F(v_n, r_d, v_1, v_2, z, f) < 0 \), obtained in small \( v_n \)-intervals: \( v_1 \leq v_n \leq v_2 \), as
\[ \Delta F(v_n, r_d, v_1, v_2, z, f) = e(r_d, v_1, v_2, z) \times v_n^k + f. \] Here, \( e(r_d, v_1, v_2, z) \) is the slope of this AF-curve, defined by
\[ e(r_d, v_1, v_2, z) \equiv \frac{\ln \mathcal{F}_n(v_1, v_2, r_d, a) - \ln \mathcal{F}_n(v_1, r_d, a)}{v_2^k - v_1^k}, \] which is negative for \( v_n > v_n(n) \) and positive for \( v_n < v_n(n) \), noting that for \( v_n > v_n(n) \) or in particular \( v_n \to +\infty \), from our above results (27, 28), one has: \( z=2 \). So, for \( v_n > v_n(n) \) the values of exponent \( z=1 \), \( 1/2 \), \( 1/3 \) and \( v_n < v_n(n) \) those of exponent \( z=2 \), 1, for \( 1/2 \), \( 1/3 \) could be considered in next Sections V-VII. Moreover, as \( v_n > v_n(n) \), according to \( e(r_d, v_1, v_2, z = 1) < 0 \), the energy parameter characteristic of \((-|e| \times v_0)\)-linear exponential tail states of the function \( F_n(v_n, r_d) \) can be defined by
\[ E_{no}(N, r_d, v_1, v_2) \equiv \frac{\mathcal{E}_{no}(N)}{|e(r_d, v_1, v_2, z = 1)|}, \] as observed in next Figs. 2, 5, and 8.

It should be noted that the important results (20) obtained for any \( \mathcal{E}-values \), (24) for \( \mathcal{E} \geq 0 \), and (28-31) for \( \mathcal{E} \leq 0 \), can be used to determine the density of states and the optical,
electrical and thermoelectric functions in Sections V-VIII, respectively.

4. Low Temperature Effect, Due to the Fermi-Dirac Distribution Function

The Fermi-Dirac distribution function (FDDF) is given by

\[ \langle \mathbb{E}^p \rangle_{\text{FDDF}} \equiv G_p(\mathbb{E}_{\text{Fn}}) \times \mathbb{E}_{\text{Fn}}^p \equiv \int_{-\infty}^{0} \mathbb{E}^p \times \left( -\frac{\partial f}{\partial \mathbb{E}} \right) \mathbb{E} \, d\mathbb{E}, \quad -\frac{\partial f}{\partial \mathbb{E}} \equiv \frac{1}{k_B T} \times \frac{e^\gamma}{(1 + e^{y})^2}. \tag{32} \]

Further, one remarks that, at 0 K, \( -\frac{\partial f}{\partial \mathbb{E}} = \delta(\mathbb{E} - \mathbb{E}_{\text{FNo}}), \delta(\mathbb{E} - \mathbb{E}_{\text{FNo}}) \) being the Dirac delta \((\delta)\)-function and \( \mathbb{E}_{\text{FNo}} \) is the Fermi energy at \( T=0 \) K defined in Eq. (15). Therefore, \( G_p(\mathbb{E}_{\text{FNo}}) = 1 \).

Then, at low \( T \), by a variable change \( y \equiv (\mathbb{E} - \mathbb{E}_{\text{F}})/(k_B T), \) Eq. (32) yields

\[ G_p(\mathbb{E}_{\text{Fn}}) \equiv 1 + \mathbb{E}_{\text{Fn}}^{-p} \times \int_{-\infty}^{0} \frac{e^\gamma}{(1 + e^y)^2} \, \mathbb{E} \, dy = 1 + \sum_{\mu=1,2,...}^{p} C^\beta_{\mu} \times (k_B T)^\beta \times \mathbb{E}_{\text{Fn}}^{-\beta} \times I_\beta, \]

where \( C^\beta_{\mu} \equiv p(p - 1)...(p - \beta + 1)/\beta! \) and the integral \( I_\beta \) is given by

\[ I_\beta = \int_{-\infty}^{0} \left( \frac{e^y}{(1 + e^y)^2} \right)^\beta \, dy = \int_{-\infty}^{0} \frac{y^\beta}{(1 + e^y)^{2\beta}} \, dy, \tag{33} \]

vanishing for old values of \( \beta \). Then, for even values of \( \beta = 2n \), with \( n=1,2,... \), one obtains

\[ I_{2n} = 2 \int_{0}^{\infty} \frac{y^{2n} e^y}{(1 + e^y)^{2\beta}} \, dy. \tag{34} \]

Now, using an identity [15]: \( (1 + e^y)^{-\beta} \equiv \sum_{s=0}^{\infty} (-1)^s s! \times e^y (s-1) \), a variable change: \( s y = -t \), the Gamma function: \( \int_{0}^{\infty} t^{2n} e^{-t} \, dt \equiv \Gamma(2n) = (2n)! \), and also the definition of the Riemann’s zeta function [15]: \( \zeta(2n) \equiv (2^{2n-1} \pi^{2n} |B_{2n}|/(2n)! \), \( B_{2n} \) being the Bernoulli numbers, one finally gets: \( I_{2n} = (2^{2n-2} \times \pi^{2n} \times |B_{2n}| \). So, from Eq. (32), we get in the degenerate case the following ratio:

\[ G_p(\mathbb{E}_{\text{Fn}}) \equiv \left( \frac{\mathbb{E}^p \text{دم}}{\mathbb{E}_{\text{Fn}}} \right)_{\text{FDDF}} = 1 + \sum_{n=1}^{p} C_{n} \times (k_B T)^n \times (|B_{2n}| \times \mathbb{E}_{\text{Fn}}^{-2n} \times |B_{2n}| \times \mathbb{E}_{\text{Fn}}^p \equiv G_p(y), y = \frac{\pi k_B T}{\mathbb{E}_{\text{Fn}}} \approx \frac{\pi k_B T}{\mathbb{E}_{\text{FNo}}} \tag{35} \]

It should be noted that our previous expression for \( G_p(x) \) [58] can now be corrected, replacing \( \beta \) by \( 2n \) and the Bernoulli numbers \( B_{n/2} \) by \( |B_{2n}| \). Further, Jaffe [49] proposed the following result:

\[ G_p(\text{Jaffe})(\mathbb{E}_{\text{Fn}}, T) = 1 + \mathbb{E}_{\text{Fn}}^p \times \sum_{n=1}^{p} (-1)^{2n+1} \times c_{2n} \times (k_B T)^{2n} \times (\frac{2^{2n} \mathbb{E}_{\text{Fn}}^p \mathbb{E}_{\text{Fn}}^p}{(2n)!} \), \tag{35} \]

where \( c_{2n} = \left( \frac{(2 - 2^{2n}) \times \pi^{2n} \times |B_{2n}|}{(2n)!} \right) \). Now, using an identity:

\( \left( \frac{d^{2n} \mathbb{E}^p \mathbb{E}_{\text{Fn}}}{d\mathbb{E}_{\text{Fn}}^{2n}} \right)_{\mathbb{E}=\mathbb{E}_{\text{Fn}}} \equiv p(p-1)...(p-2n+1) \times \mathbb{E}_{\text{Fn}}^{p-2n}, \)

Eq. (35) is found to be identical to our above result (34), which is a more practical result. Then, some usual results of \( G_p(y) \) are given in Table 3.

<table>
<thead>
<tr>
<th>( 2n )</th>
<th>2</th>
<th>4</th>
<th>6</th>
<th>8</th>
<th>10</th>
<th>12</th>
</tr>
</thead>
<tbody>
<tr>
<td>(</td>
<td>B_{2n}</td>
<td>)</td>
<td>1/6</td>
<td>-1/30</td>
<td>1/42</td>
<td>-1/30</td>
</tr>
</tbody>
</table>

\( G_{2}(y) = 1 + \frac{y^{2}}{3} + \frac{2y^{4}}{440} \), \( G_{4}(y) = 1 + \frac{y^{4}}{3} + \frac{5y^{6}}{858} \), \( G_{6}(y) = 1 + y^{6}, G_{8}(y) = 1 + \frac{5y^{8}}{24} + \frac{49y^{10}}{384} \), \( G_{10}(y) = 1 + 2y^{10}, G_{12}(y) = 1 + \frac{25y^{12}}{48} + \frac{147y^{14}}{128} \)...

These functions \( G_p(y) \) obtained in Table 3 will be applied to determine the electrical and thermoelectric properties of the degenerate d-Si systems, being given in Sections 7 and 8, respectively.
5. Determination of Critical Donor Density

In degenerate d-Si systems at \( T=0 \) K, due to the heavy doping effect (HDE), using Eq. (20) for \( a=1 \), \( \langle E_k \rangle_{\text{KIM}} \), the density of states \( D(E) \) is given by:

\[
\langle D(E) \rangle_{\text{KIM}} = \frac{8e^2}{\pi n} \left( \frac{2m^{\text{HDE}} }{\hbar^2} \right)^{3/2} \times \langle E_k \rangle_{\text{KIM}} = \frac{8e^2}{\pi n} \left( \frac{2m^{\text{HDE}} }{\hbar^2} \right)^{3/2} \times \exp \left( -\frac{\sqrt{x^2}}{\sqrt{2\pi}} \right) \times \Gamma \left( \frac{3}{2} \right) \times D \left( \frac{x}{2} \right) = D(E),
\]

where \( m^{\text{HDE}} \) is the electron effective mass, due to the HDE, determined in Eq. (8), and the variable \( x \) is defined in Eq. (21), as

\[
x = \frac{\sqrt{n}}{\sqrt{2\pi}} \equiv A_n \times n \times \exp \left( \frac{3n_x \times R_n}{4 \times \sqrt{|v_n|}} \right), \quad A_n \equiv \frac{E_{F_n}}{n}, \quad v_n(E,N) \equiv \frac{-E}{E_{F_n}}.
\]

Here, \( E_{F_n} \) is determined in Eq. (15) for \( m^* = m_n^{\text{HDE}} \), \( m_n^{\text{HDE}} \) being the electron effective mass due to the HDE and determined in Eq. (8), and the value of Heisenberg empirical parameter \( \beta \) was defined in the Appendix B and proposed here as: \( \beta = 5.4370 \), so that the following determination of the critical density of electrons localized in the exponential conduction-band tail would be accurate. Further, from Eq. (24), one also has

\[
D(E \geq 0) = \frac{8e^2}{\pi n} \left( \frac{2m^{\text{HDE}} }{\hbar^2} \right)^{3/2} \times \sqrt{E}.
\]

Going back to the functions: \( H_n, \ K_n \) and \( F_n \), given respectively in Eqs. (26-28), in which the factor \( \frac{1}{D(E)} \) is now replaced by:

\[
\frac{1}{D(E)_{\text{KIM}}} = F_n(v_n, r_d, a = 1), \quad D_0 = \frac{8e^2}{\pi n} \left( \frac{2m^{\text{HDE}} }{\hbar^2} \right)^{3/2} \times \sqrt{v_n} \times \beta(a = 1) = \frac{\sqrt{\pi}}{2 \times \Gamma(5/4)},
\]

where the reduced density of exponential-tail states: \( F_n(v_n, r_d, a = 1) \equiv F_n(1)v_n, \ r_d \), for a simplicity of presentation, is determined in Eq. (28). Then, in d-Si systems at \( 0 \) K and for \( N = 5 \times 10^{18} \text{ cm}^{-3} \), our results of the functions \( F_n(1)v_n, r_d \) obtained for each \( r_d \)-value, are plotted as functions of \( v_n \) in Figure 1.

Figure 1. Our results of \( F_n(1) \) increase with increasing \( r_d \) for a given \( v_n \), due to the donor-size effect, and present the maxima at \( v_n = v_n(M) \) and go to zero as \( v_n \rightarrow 0 \) and \( \infty \).
Figure 1 shows that:

(i) our results of $F_{(1)}$ increase with increasing $r_d$ for a given $v_n$, due to the donor-size effect, and

(ii) present the maxima at $v_n = v_{n(M)}$ and go to zero as $v_n \rightarrow 0$ and $\infty$, being found to be in good agreement with theoretical results obtained by Lifshitz [18], Friedberg and Luttinger [20], our results given in Eq. (A.3) of the Appendix A, and in particular with an asymptotic form for exponential

Table 4. In the d-Si systems at $T=0 \text{K}$ and for $n = 5 \times 10^{29} \text{cm}^{-3}$, using the reduced density of state of $F_{(1)}(v_n, r_d, a = 1)$, determined in Eq. (38), the numerical results of $\ln[F_{(1)}(v_n, r_d, a = 1)]$, and its approximate form obtained for $a=1$, $AF_{(1)}(v_n, r_d, v_2, z, f) = e^{(r_d, v_1, v_2, z, f)}$, determined in Eqs. (29, 30) for small $v_n$-intervals: $v_1 \leq v_n \leq v_2$, and also those of absolute relative errors defined by: $|RE| = 1 - \frac{AF_{(1)}(v_n, r_d, v_2, z, f)}{F_{(1)}(v_n, r_d, a = 1)}$ are tabulated below.

<table>
<thead>
<tr>
<th>Donor</th>
<th>$v_{n(M)}$</th>
<th>$v_n$</th>
<th>$P$</th>
<th>$C$</th>
<th>$Tl$</th>
<th>$Te$</th>
<th>$Sb$</th>
<th>$S$</th>
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<tbody>
<tr>
<td></td>
<td>$0.19556$</td>
<td>$0.20004$</td>
<td>$0.24824$</td>
<td>$0.25505$</td>
<td>$0.35282$</td>
<td>$0.45819$</td>
<td>$0.53142$</td>
<td>$0.53891$</td>
</tr>
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<td>$P$</td>
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</table>

The underlined $|RE|$-value is the maximal one for each donor-Si system.

Table 4 suggests that: (i) our results of $AF_{(1)}(v_n, r_d, v_2, z, f)$ given for $v_n = v_{n(M)}$ agree with the Urbach law obtained from linear exponential conduction-band-tail behaviors by some workers [21, 29, 30, 37], (ii) ours for $(z=1/2)$-exponent and $v_n > v_{n(M)}$ agree with other theoretical results [19, 37], and finally (iii) ours for $(z=1/3, 1/4)$-exponents when $v_n > v_{n(M)}$, and for $(z=1/4, 1/3, 1/2, 1/4)$-exponents when $v_n < v_{n(M)}$ may thus be new.

Finally, our numerical results of energy parameter, $\ln[n_{n0}(N; r_d, a = 1)]$, obtained in the small interval: $1.2 \leq v_n \leq 1.25$, using Eq. (31), are plotted as functions of $N$ in Figures 2a and 2b, indicating that, for a given $N$, $\ln[n_{n0}]$ increases with increasing $r_d$-values, due to the donor-size effect.
Figures 2. Our results of energy parameter $E_{\alpha 0}(N; r_{d}, a = 1)$, are plotted as functions of $N$, indicating that, for a given $N$, $E_{\alpha 0}$ increases with increasing $r_{d}$-values, due to the donor-size effect.

Now, for $E \leq 0$, using Eqs. (27, 28) for the functions $K_{n}$ and $F_{n}$ as $a=1$, the density of electrons localized in the exponential band tail (EBT) is given by

$$N_{d}^{EBT}(N; r_{d}) = \int_{-\infty}^{0} D(E \leq 0) \, dE,$$

where $D(E \leq 0)$ is determined in Eq. (38).
Then, by a variable change: \( \nu_n \equiv \frac{-E}{k_F n_0} \), Eq. (39) yields

\[
N_{d,\delta}^{\text{EBT}}(\nu_n, r_d) = \frac{e^c x (n^\text{HDE})^{3/2}}{2\pi^2 h^3} \left[ \frac{n_F}{\nu_n E_{\text{Fm}}(\nu_n)} \right] \times \delta \times \left\{ \int_0^{\nu_n} \beta(a = 1) \times F_n(\nu_n, r_d, a = 1) \, d\nu_n + 1 \right\},
\]

where

\[
I_n \equiv \int_0^{\nu_n} \beta(a = 1) \times K_n(\nu_n, r_d, a = 1) \, d\nu_n = \int_0^{\nu_n} e^{-\beta(a = 1) \times K_n(\nu_n, r_d, a = 1)} \times (A_n \nu_n)^{-3/2} \, d\nu_n.
\]

Here, \( \beta(a = 1) = \frac{\sqrt{b}}{2\pi} \) and \( \delta \) is normally equal to 1, but it can be an empirical parameter, being chosen as: \( \delta = \delta_o = 1.0028637416 \) such that the obtained values of \( N_n^{\text{EBT}} \) would be accurate.

Then, by another variable change: \( t = \left[ A_n \nu_n / \sqrt{2} \right]^2 \), the integral \( I_n \) yields [15]

\[
I_n = \frac{1}{2^{7/2} A_n} \int_{y_n}^{\nu_n} \int_0^{t(b-1)} e^{-t} \, dt \equiv \int_{y_n}^{\nu_n} \frac{\Gamma(b, y_n)}{2^{7/2} A_n}
\]

where \( b = -1/4 \), \( y_n = \left[ 16 A_n / \sqrt{2} \right]^2 \), with \( A_n \) being defined in Eq. (38), and \( \Gamma(b, y_n) \) is the incomplete Gamma function, defined by [15]

\[
\Gamma(b, y_n) = y_n^{b-1} \times e^{-y_n} \left[ 1 + \sum_{j=1}^{16} \frac{(b-1)(b-2)\ldots(b-j)}{y_n^j} \right].
\]

Finally, Eq. (40) now yields

\[
N_{d,\delta}^{\text{EBT}}(\nu_n, r_d) = \frac{e^c x (n^\text{HDE})^{3/2}}{2\pi^2 h^3} \left[ \frac{n_F}{\nu_n E_{\text{Fm}}(\nu_n)} \right] \times \delta \times \left\{ \int_0^{\nu_n} \beta(a = 1) \times F_n(\nu_n, r_d, a = 1) \, d\nu_n + \int_{y_n}^{\nu_n} \frac{\Gamma(b, y_n)}{2^{7/2} A_n} \right\},
\]

being the density of electrons localized in the exponential conduction-band tail.

Hence, in the degenerate d-Si system, replacing \( N \), given in the parabolic conduction band of an effective electron gas, by the effective density of free electrons defined here by: \( N^* = N - N_{d,\delta}^{\text{EBT}} \geq 0 \). So, in this system, the Fermi energy given in Eq. (15) is now rewritten as

\[
E_{\text{Fm}}(N^*) = \frac{\pi^2 k_F^2 (N^*)^2}{2m_0^2 |\mu|} N \geq N_{d,\delta}^{\text{EBT}} - N_{c(d)},
\]

(42)

where the Fermi wave number \( k_F \) and \( m_0^{\text{HDE}} \) are respectively determined in Eqs. (2, 8). One notes here that \( E_{\text{Fm}}(N^*) \) vanishes at \( N^* = 0 \), or at the critical donor density defined by: \( N = N_{c(d),\delta} \equiv N_{d,\delta}^{\text{EBT}}(N = N_{c(d),\delta} \geq 0) \), at which the metal-insulator transition thus occurs. Then, the numerical results of \( N_{c(d),\delta} \), for \( \delta = 1 \) and \( \delta_o = 1.0028637416 \), and their absolute relative errors \( |\text{RE}| \), calculated using the given data in Table 1, are obtained and also reported in this Table 1, indicating that those of \( N_{c(d),\delta}^{\text{EBT}} \) and \( N_{c(d),\delta} \) are obtained respectively with accuracies of the orders of \( 1.1 \times 10^{-4} \) and \( 3 \times 10^{-3} \). Hence, these results of \( N_{c(d),\delta}^{\text{EBT}} \) thus confirm our above choice of the Heisenberg parameter value: \( \mathcal{H}_n = 5.4370 \), as that proposed in Eq. (49) and also in the Appendix B. Furthermore, our numerical calculation indicates that, in all the d-Si systems for \( N \geq 1.15 \times N_{c(d)} \), if defining the absolute relative deviations between \( N - N_{c(d),\delta}^{\text{EBT}} \) and \( N - N_{c(d)} \) by: \( |\text{RD}| \equiv \left[ 1 - \frac{N_{c(d),\delta}^{\text{EBT}}}{N_{c(d)}} \right] \), the maximal \( |\text{RD}| \) values, which occur at \( N = 1.15 \times N_{c(d)} \), are approximately equal to 3.2%. So, \( N^* \)

given in the parabolic conduction band of the degenerate d-Si systems can be approximated by [22]

\[
N^* \equiv N - N_{c(d),\delta}^{\text{EBT}} \approx N - N_{c(d)}.
\]

Here, this notion of effective density of free electrons \( N^* \) defined by Eq. (43) should be equivalent to that of \( (N_{d} - \rho_n) \) given in the n-type compensated Si, in which \( N_{d} \) is the total density of donors (or majority electrons) and \( \rho_n \) is the total density of acceptors (or minority holes), assuming that all the impurities are ionized [22]. Finally, in degenerate d-Si systems, in which \( N > N_{c(d)} \) and \( T \leq 77 \text{K} \) or \( T \ll T_D \), \( T_D \) being the degeneracy temperature defined in Eq. (15), this result (43) will be used in all the following Sections.

6. Optical Properties

The problem of exponential optical absorption-coefficient tails has by now a rather long history. We will limit our study here to the degenerate d-Si systems, although the band structure of random alloys and amorphous materials is a problem with many common features [41].

Optical properties of any medium can be described by the complex refraction index \( N \) and the complex dielectric function \( \varepsilon \), defined by: \( \varepsilon \equiv n - i \varepsilon \) and \( \varepsilon \equiv 1 - i \varepsilon_2 \), where \( i^2 = -1 \) and \( \varepsilon \equiv \varepsilon^2 \), and by the optical absorption coefficient \( \alpha \), which is related to the imaginary part of \( \varepsilon_2 \), the refraction index \( n \), the extinction coefficient \( k \), and the conductivity \( \sigma_0 \), due to the electro-optical effect, as [29-48]

\[
\alpha(E) = \frac{4\pi n_0 \varepsilon_0 E}{c} \times n(E) = \frac{4\pi n_0 \varepsilon_0 E}{c} = \frac{4\pi n_0 \varepsilon_0 E}{c} \times n(E) \times \varepsilon_0.
\]

(44)
One remarks that the real part of $\varepsilon$ is defined by

$$
\varepsilon_1(E) \equiv n(E)^2 - \kappa(E)^2,
$$

and the normal-incidence reflectance $R(E)$, by

$$
R(E) = \frac{[n(E) - 1]^2 + \kappa(E)^2}{[n(E) + 1]^2 + \kappa(E)^2},
$$

which are the optical dispersion relations since in general the values of those optical functions are expressed as functions of the multi-photon energy [46], $E \equiv \hbar \omega_0, 2\hbar \omega_0, 3\hbar \omega_0, 4\hbar \omega_0, \ldots$. In the present work, we only focus our attention to the case of photon energy $E \equiv \hbar \omega_0$. Here, $h$, $v$, $\omega$, $\varepsilon_1$, and $\varepsilon_2$ respectively represent the electron charge, Dirac's constant, matrix elements of the velocity operator between valence-and-conduction bands in n-type semiconductors, photon frequency, permittivity of free space, velocity of light, and joint density of states (JDOS). One remarks here that: (i) if some optical functions are known such as: $(J, n, |v|^2)$, $(n, \kappa)$, or $(\varepsilon_1, \varepsilon_2)$, then, all other ones are determined, and (ii) in n-type semiconductors, all the optical functions will be expressed in terms of the total energy of the electron, defined by:

$$
E \equiv E - E_{\text{gn}},
$$

where the band gap given in the degenerate d-Si systems, $E_{\text{gn}}$, can be equal to: $E_{\text{gr}}$, $E_{\text{gn1}}$, and $E_{\text{gn2}}$, defined in Eqs. (3, 9, 10), respectively.

Further, for any E or $\varepsilon$, using Eq. (20), Eq. (48) becomes

$$
\alpha(E, a) \equiv q^2 |v(E)|^2 \pi^2 m^2 / 4 \times \frac{m}{\hbar^2} \times \frac{2a-1}{\tau^2} \times D_{\alpha}^{-a-1} \times \Gamma(a + \frac{3}{2}) \times \tau^2 \cdot \frac{1}{\pi^2} \times \frac{\sqrt{2a-1}}{2a-1} \times \frac{n(E)^2}{\hbar^2},
$$

proposed in his FB-method (FB-M) a familiar four-term expression for extinction coefficient, $\kappa(E, 5/2)$, expressed in terms of $(E - E_{\text{gn}})^2$ for both direct-and-indirect band-gap semiconductors, being thus correct only in indirect band-gap ones. Further, their result is not correct when $E \rightarrow \infty$ since one must have: $\kappa(E \rightarrow \infty) \rightarrow 0$ [36, 41]. Furthermore, in the d-Si systems, from Eqs. (44, 48) one can determine the extinction coefficient $\kappa$, obtained for $a=5/2$, as

$$
\kappa(E \geq E_{\text{gn}}) \equiv \frac{q^2 m^2 / 4 \pi^2 \hbar^2}{\sqrt{2a-1} \times \pi^2 n(E)^2 \hbar^2} \times |v(E)|^2 \times \left(\frac{E - E_{\text{gn}}}{\hbar^2}\right)^2.
$$

We now propose an improved FB-M (IFB-M).

First, if putting $F(E) \equiv \sum_{i=1}^{4} \frac{A_i}{E - B_i \cdot \varepsilon_{\text{gr}}^2},$ where the values of empirical parameters: $A_i, B_i, C_i, C_i$, are given in the FB-M for the Si [40], and simply replacing the band-gap energy $E_{\text{g}} = 1.06 \text{ eV}$ [40] by $E_{\text{gr}}$, which can be equal to: $E_{\text{gr}}$, $E_{\text{gr}1}$, and $E_{\text{gr}2}$, which are determined respectively in Eqs. (3, 10, 11), we can now propose, as that done by O’Leary et al. for very large values of $E$, [39]
Our IFB-M, we can further propose functions can thus be determined. So, at 298 K and photon energy $E$ (eV) are also evaluated and tabulated in this Table, in which the corresponding Studna [33], are tabulated in the Table 6.

Obtained from the FB-M, and the absolute errors of those, calculated using the optical-function data obtained by Aspnes and "%

Secondly, by putting

$$\kappa_{\text{IFB-M}}(E \geq \varepsilon_{\text{gn}}) = f(E) \times \left(\frac{E - \varepsilon_{\text{gn}}}{E^{2}}\right)^{2}, \text{for } E \geq 6 \text{ eV},$$

$$= f(E) \times \frac{(E - \varepsilon_{\text{gn}})^{2}}{E^{3}}, \text{for } \varepsilon_{\text{gn}} \leq E \leq 6 \text{ eV},$$

so that $\kappa_{\text{IFB-M}}(E \rightarrow \infty)$ goes to 0 as $E^{-2}$, in good accordance with both experimental [36] and theoretical [41] results.

For example, in intrinsic d-Si systems at 298 K, in which we now replace $\varepsilon_{g} = 1.06$ eV by $\varepsilon_{\text{gn}}$ for our IFB-M, we can further propose

$$n_{\text{IFB-M}}(E) = n_{\infty} + n_{0} \times \Delta n, \text{for } E \geq 6 \text{ eV}, n_{\infty} \equiv \sqrt{\varepsilon_{n}}$$

$$= 1.93 + \Delta n, \text{for } \varepsilon_{\text{gn}} \leq E \leq 6 \text{ eV},$$

so that $n_{\text{IFB-M}}(E \rightarrow \infty) \rightarrow \sqrt{\varepsilon_{n}}$, where the values of $\varepsilon_{n}$ are given in Table I, giving a correct asymptotic behavior of $n_{\text{IFB-M}}(E)$. Here, $n_{0}$ is the factor to be determined so that the function $n_{\text{IFB-M}}(E)$ for $E \geq 6$ eV is continuous at $E = 6$ eV, depending on $T$, $r_{d}$, and $N$.

For example, in intrinsic d-Si systems at 298 K, in which $\varepsilon_{\text{gn}} = \varepsilon_{\text{gl}}(T = 298 \text{ K}, r_{d}) = 1.125$ eV is determined in Eq. (3), the values of $n_{0}(r_{d})$ are evaluated and tabulated in Table 5.

### Table 5. Intrinsic donor-Si systems at 298 K, the numerical results of Factor $n_{0}(r_{d})$, being due to the donor-size effect and expressed as functions of donor-radius $r_{d}$, are determined so that the function $n_{\text{IFB-M}}(E)$ given in Eq. (52) for $E \geq 6$ eV is continuous at $E = 6$ eV.

<table>
<thead>
<tr>
<th>Donor</th>
<th>Sb</th>
<th>P</th>
<th>As</th>
<th>Bi</th>
<th>Ti</th>
<th>Te</th>
<th>Se</th>
<th>S</th>
</tr>
</thead>
<tbody>
<tr>
<td>$n_{0}(r_{d})$</td>
<td>2.571289</td>
<td>2.482479</td>
<td>2.032121</td>
<td>1.944450</td>
<td>1.290620</td>
<td>0.805033</td>
<td>0.430750</td>
<td>0.384780</td>
</tr>
</tbody>
</table>

As noted in Eqs. (44-46), if from Eqs. (51, 52) the values of $\kappa_{\text{IFB-M}}(E)$ and $n_{\text{IFB-M}}(E)$ are evaluated, all other optical functions can thus be determined. So, at 298 K and $1.5 \leq E(\text{eV}) \leq 6$, in the intrinsic P-Si systems, in which $\varepsilon_{\text{gn}} = \varepsilon_{\text{gl}}(T = 298 \text{ K}, r_{d}) = 1.125$ eV is evaluated using Eq. (3), our results of all the optical functions and the corresponding ones obtained from the FB-M, and the absolute errors of those, calculated using the optical-function data obtained by Aspnes and Studna [33], are tabulated in the Table 6.

### Table 6. Intrinsic P-Si systems at 298 K and for $1.5 \leq E(\text{eV}) \leq 6$, our numerical results of all the optical functions (OF) are calculated, using Eqs. (44-46, 51, 52) obtained in our IFB-M, and using the OF-data obtained by Aspnes and Studna [33], their absolute maximal relative errors ($\text{MREs}$) determined at the photon energy $E$ (eV) are also evaluated and tabulated in this Table, in which the corresponding $\text{MREs}$ obtained in FB-M are also included.

<table>
<thead>
<tr>
<th>MRE</th>
<th>$E$ (eV)</th>
<th>$\varepsilon_{1}\times\text{MRE}$</th>
<th>$\varepsilon_{2}\times\text{MRE}$</th>
<th>$n-\text{MRE}$</th>
<th>$\kappa-\text{MRE}$</th>
<th>$R-\text{MRE}$</th>
<th>$\alpha-\text{MRE}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>FB-M</td>
<td>1.5</td>
<td>0.57</td>
<td>0.58</td>
<td>0.58</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>(\varepsilon_{0} = 1.95, \varepsilon_{g} = 1.06 \text{ eV})</td>
<td>2.2</td>
<td>0.66</td>
<td>0.06</td>
<td>0.07</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>4.2</td>
<td>0.40</td>
<td>0.41</td>
<td>0.40</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>5.0</td>
<td>0.30</td>
<td>0.08</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Our IFB-M</td>
<td>1.5</td>
<td>0.30</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>(\varepsilon_{0} = \sqrt{14A_{i}\varepsilon_{gl}} = 1.125 \text{ eV})</td>
<td>2.6</td>
<td>0.30</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>4.3</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>5.0</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

The underlined $\text{MRE}$-value is the maximal one for each optical function.

Further, our numerical calculation indicates that, for a given $E$, since $\kappa_{\text{IFB-M}}(E)$ given in Eq. (51) is expressed in terms of $(E - \varepsilon_{\text{gn}})^{2}$, if $\varepsilon_{\text{gn}}$ increases (decreases), then other functions such as: $(E - \varepsilon_{\text{gn}})^{2}$, $\kappa_{\text{IFB-M}}(E)$, $\varepsilon_{2}(\text{IFB-M})$, and $\varepsilon_{2}(\text{IFB-M})$ decrease (increase), respectively. This useful remark will be used in our IFB-M to explain all the following results.

In the intrinsic P-Si system, $\varepsilon_{\text{gn}} = \varepsilon_{\text{gl}}(T, r_{p})$, calculated from Eq. (3), decreases with increasing $T$. So, for a given $E$, our results of $(E - \varepsilon_{\text{gn}})^{2}$ and $\varepsilon_{2}(\text{IFB-M})$, obtained in absolute values, increase with increasing $T$, in good
accordance with experimental results [32, 38, 44, 48], as observed in the following Figure 3a.

Figure 3a. In the intrinsic P-Si system, our results of $\epsilon_{2}(\mathrm{fB-M})(E)$, obtained in absolute values, increase with increasing $T$.

In intrinsic donor-Si systems, $E_{\text{g}} \equiv E_{\text{g}}(r_d, T = 298 \, \text{K})$, calculated from Eq. (3), increases with increasing $r_d$, as seen in Table 1. Thus, for a given $E$, our results of $[E - E_{\text{g}}(r_d, T = 298 \, \text{K})]^2$ and $E_{1(2)}(\mathrm{fB-M})(E)$, in absolute values, decrease with increasing $r_d$, as obtained in the following Figure 3b, in which we also observe the correct asymptotic results: $E_{1(2)}(\mathrm{fB-M})(E \to \infty) \to \epsilon_n$, being identical to the values of $\epsilon_n$ given in Table 1, and $E_{2}(\mathrm{fB-M})(E \to \infty) \to 0$.

Figure 3b. In intrinsic donor-Si systems, our results of $\epsilon_{2}(\mathrm{fB-M})(E)$, in absolute values, decrease with increasing $r_d$.

In degenerate P-Si systems at $T=4.2$ K, in which $E_{\text{g}} \equiv E_{\text{g}}(N)$, being the optical band gap determined in Eq. (9), increases with increasing $N$, due to the heavy-doping effect.

So, for a given $E$, the absolute values of $[E - E_{\text{g}}(N)]^2$ and $E_{1(2)}(\mathrm{fB-M})(E)$ decrease with increasing $N$, in good accordance with experiments by Aspnes et al. [34], and Vina and Cardona [35], as seen in the following Figure 3c.

Figure 3c. In degenerate P-Si systems, our results of $\epsilon_{1(2)}(\mathrm{fB-M})(E)$, in absolute values, decrease with increasing $N$.

Finally, in degenerate P-Si systems, in which $E_{\text{g}} \equiv E_{\text{g}}(N)$, being the reduced band gap determined in Eq. (10), decreases with increasing $N$, due to the heavy-doping effect. Consequently, for a given $E$, the absolute values of $[E - E_{\text{g}}(N)]^2$ and $E_{1(2)}(\mathrm{fB-M})(E)$ increase with increasing $N$.

Now, identifying our above results (50, 51) and using Eq. (52), we can propose an useful expression for $|\nu(E)|^2$ as

$$|\nu(E)|^2 = \frac{\sqrt{2}\times n^{2}\times A	imes n_{\text{fB-M}}(E)\times r_{n}\times E_{\text{F}}^{3/2}\times f(E)}{q^{2}\times m_{e}^{3/2}} \times \left(\frac{6 \, \text{eV}}{\epsilon}\right)^{3}, \text{for } E \geq 6 \, \text{eV},$$

$$= \frac{\sqrt{2}\times n^{2}\times A	imes n_{\text{fB-M}}(E)\times r_{n}\times E_{\text{F}}^{3/2}\times f(E)}{q^{2}\times m_{e}^{3/2}}, \text{for } E_{\text{g}} \leq E \leq 6 \, \text{eV},$$

$$= \frac{\sqrt{2}\times n^{2}\times A	imes n_{\text{fB-M}}(E_{\text{g}})\times r_{n}\times E_{\text{F}}^{3/2}\times f(E_{\text{g}})}{q^{2}\times m_{e}^{3/2}}, \text{for } E \leq E_{\text{g}} \text{ or for } E \leq 0.$$

(53)

6.2. Behaviors of Optical Functions Obtained for $E \leq E_{\text{g}}$ or $E \leq 0$

Here, going back to the functions: $H_n$, $K_n$, and $F_n$, given respectively in Eqs. (26-28) for $a=5/2$, in which the factor \(\frac{(\tilde{\epsilon}_n)^2}{4/(a=5/2)}\) is now replaced by: \(\frac{(\tilde{\epsilon}_n)^2}{4/(a=5/2)} = \frac{(\tilde{\epsilon}_n)^2}{4/(a=5/2)} = F_n\times N_{\text{g}}\times r_a, a = 5/2 \equiv F_n\times N_{\text{g}}\times r_a\) calculated using Eq. (28), for a simplicity of presentation, $\alpha_n(E = E_{\text{g}})$ being determined from Eqs. (26, 49, 53) as: $\alpha_n = \frac{\epsilon_{\text{g}}}{c\times h\times E_{\text{g}}}$, then our numerical results of reduced optical absorption coefficient given in degenerate d-Si systems at 0 K and for $N = 5 \times 10^{20} \, \text{cm}^{-3}$, $F_n\times N_{\text{g}}\times r_a$ are plotted in Figure 4, as functions of $\nu_n$. 
Figure 4 shows that:

(i) our results of $F_{n}^{(2)}$ increase with increasing $r_d$ for a given $v_n$, due to the donor-size effect, and
(ii) present the maxima at $v_n = v_{n(M)}$ and go to zero as $v_n \to 0$ and $\infty$, being found to be in good agreement with theoretical results obtained by Lifshitz [18], Friedberg and Luttinger [20], our results given in Eq. (A.3) of the Appendix A, and in particular with an asymptotic form for exponential conduction-band tail, obtained for $0 \leq v_n \leq \infty$, by Halperin and Lax [19], using the minimum counting methods.

Then, in degenerate d-Si systems at $0 \text{ K}$ and $N = 5 \times 10^{20} \text{ cm}^{-3}$, our numerical results of $\ln[F_n(v_p/r_d,a = 5/2)] < 0$, which can take its approximate form, obtained in small $v_n$-intervals: $v_1 \leq v_n < v_2$, by: $\ln[F_n(v_n,r_d,a = 5/2)] = A F(v_n,r_d,v_1,v_2,z,f) = (r_d,v_1,v_2,z) + f$, determined in Eqs. (29, 30) for small $v_n$-intervals: $v_1 \leq v_n < v_2$, and those of absolute relative errors defined by: $|RE| = 1 - \frac{[AF(v_n,r_d,v_1,v_2,z,f)]}{[AF(v_n,r_d,v_1,v_2,z,f)_{theoretical}]}$ are calculated and tabulated below.

Table 7. In the d-Si systems at $T=0 \text{ K}$ and for $N = 5 \times 10^{20} \text{ cm}^{-3}$, using the reduced optical absorption coefficient determined in Eq. (28), $F_{n}^{(2)}(v_p/r_d,a = 5/2)$, the numerical results of $\ln[F_n(v_p/r_d,a = 5/2)]$ and its approximate form for $a=5/2$: $A F(v_p,r_d,v_1,v_2,z,f) = (r_d,v_1,v_2,z) + f$, determined in Eqs. (29, 30) for small $v_n$-intervals: $v_1 \leq v_n < v_2$, and those of absolute relative errors defined by: $|RE| = 1 - \frac{[AF(v_n,r_d,v_1,v_2,z,f)]}{[AF(v_n,r_d,v_1,v_2,z,f)_{theoretical}]}$ are calculated and tabulated below.
Table 7 suggests that: (i) our results of \( AF(v_n, r_d, v_f, v_e, z_0, f) \) given for \( z=1 \)-exponent agree with the Urbach law obtained from linear exponential conduction-band tail-behaviors by some workers [21, 29, 30, 37], (ii) ours for \( z=1/2 \)-exponent and \( v_n > v_n(M) \) agree with other theoretical results [19, 37], and finally (iii) ours for \( z=1/3, 1/4 \)-exponents when \( v_n > v_n(M) \), and for \( z=1/4, 1/3, 1/2, 1, 2 \)-exponents when \( v_n < v_n(M) \) may thus be new.

Finally, our numerical results of energy parameter, \( E_{no}(N; r_d, a = 5/2) \), obtained in the small interval: \( 1.2 \leq v_n \leq 1.25 \), using Eq. (31), are plotted as functions of \( N \) in Figures 5a and 5b, indicating that, for a given \( N \), \( E_{no} \) increases with increasing \( r_d \)-values, due to the donor-size effect.

\[ \text{Table 7} \]

<table>
<thead>
<tr>
<th>Donor</th>
<th>Sb</th>
<th>P</th>
<th>As</th>
<th>Bi</th>
<th>Ti</th>
<th>Te</th>
<th>Se</th>
<th>S</th>
</tr>
</thead>
<tbody>
<tr>
<td>( v_n(M) )</td>
<td>0.20000</td>
<td>0.23000</td>
<td>0.25100</td>
<td>0.25000</td>
<td>0.36930</td>
<td>0.50457</td>
<td>0.63809</td>
<td>0.65575</td>
</tr>
<tr>
<td>e</td>
<td>77.110</td>
<td>80.266</td>
<td>90.763</td>
<td>91.957</td>
<td>69.455</td>
<td>46.413</td>
<td>37.693</td>
<td>37.021</td>
</tr>
<tr>
<td>(</td>
<td>RE</td>
<td>)</td>
<td>1.3 \times 10^{-4}</td>
<td>6.7 \times 10^{-4}</td>
<td>1.7 \times 10^{-4}</td>
<td>1.7 \times 10^{-4}</td>
<td>1.5 \times 10^{-4}</td>
<td>1.3 \times 10^{-4}</td>
</tr>
</tbody>
</table>

For \( 0.061 \leq v_n \leq 0.064, AF = (e \times v_n + f) \) is accurate to within \( 5.1 \times 10^{-4} \), where

\[ e = 306.576 | 319.993 | 366.932 | 368.556 | 267.210 | 159.872 | 120.930 | 117.973 |
\[ \|RE\| = 3.9 \times 10^{-4} | 4.3 \times 10^{-4} | 4.9 \times 10^{-4} | 5.1 \times 10^{-4} | 2.3 \times 10^{-4} | 2.8 \times 10^{-4} | 2.4 \times 10^{-4} |

For \( 0.054 \leq v_n \leq 0.056, AF = (e \times v_n^2 + f) \) is accurate to within \( 8.7 \times 10^{-4} \), where

\[ e = 4254.38 | 4420.99 | 5198.02 | 5318.51 | 3719.15 | 2095.41 | 1520.14 | 1476.98 |
\[ \|RE\| = 3.9 \times 10^{-4} | 4.3 \times 10^{-4} | 5.5 \times 10^{-4} | 8.7 \times 10^{-4} | 4.2 \times 10^{-4} | 3.0 \times 10^{-4} | 2.3 \times 10^{-4} | 2.3 \times 10^{-4} |

The underlined \( |RE| \)-value is the maximal one for each donor-Si system.

7. Electrical Properties

Here, \( m^* \equiv m_{\text{cond}} = 0.26 \times m_0 \). Then, the electrical functions, obtained in the two cases: \( \Xi \geq 0 \) and \( \Xi \leq 0 \), will be considered as follows.

7.1. Electrical Functions Obtained as \( \Xi \geq 0 \)

In the effective electron gas at 0 K [66], denoting the relaxation time by \( \tau \), the mobility is defined by

\[ \mu \equiv \frac{\hbar^2}{m_{\text{cond}}}. \]

the conductivity \( \sigma \) (or resistivity \( \rho \equiv 1/\sigma \)), given in the Drude model, by
\[ \sigma \equiv q \times N \times \mu = q^2 \times N \times \frac{r}{m_{\text{cond}}}, \quad (55) \]

the Hall conductivity \( \sigma_H \), by

\[ \sigma_H \equiv -\sigma \times \mu = -q^3 \times N \times \frac{r}{m_{\text{cond}}} < 0, \quad (56) \]

and finally, from Eqs. (55, 56), the Hall coefficient at 0 K, by

\[ R_H(N) \equiv \frac{\sigma_H}{(\sigma)^2} = -\frac{1}{N \times q} \times \left( \frac{r^2}{(\sigma)^2} \right) < 0, \quad (57) \]

This result (57) is not correct for the degenerate donor (d)-Si systems at low temperatures, where \( N \) may be replaced by the total effective density of free electrons given in the conduction band, \( N' = N - N_{c(d)} \), as that given in Eq. (43), in which the values of critical donor density \( N_{c(d)} \) are given in Table I. In those degenerate d-Si systems, the relaxation time can be defined by

\[ \frac{1}{\tau(k)} = N' \times \frac{\hbar k}{m_{\text{cond}}} \times \pi (C \times k)^{-2} \times \prod_{i=1}^{\infty} x_i, \quad (58) \]

where \( \hbar k/(m_{\text{cond}} \times m) \) is the electron velocity, \( C \) is an empirical parameter, \( (C \times k)^{-2} \) is the scattering cross section, and finally the factors \( x_i \) are included to represent the high donor-density conditions when \( k = k_Fn \), as those given in Eq. (14), such that \( \frac{1}{\tau(k_Fn)} < 1 \).

We now propose our present method (PM) to determine all the electrical functions as follows.

The Hall factor 

\[ \mu_{\text{PM}}(N', T, r_d) \equiv \mu_{\text{PM}}(N', T, r_d) \equiv \frac{-G_1(y) + \Delta_{\text{PM}} \times G_1(y)}{\frac{\eta_n}{E_{\text{Fp}\sigma}},} \quad (62) \]

where \( G_1(y) \) is given in Table III, with \( y = \frac{\eta_n}{E_{\text{Fp}\sigma}} \), noting that \( G_1(y) = 1 \),

\[ \sigma_{\text{PM}}(N', T, r_d) \equiv \sigma_{\text{PM}}(N', T, r_d) \equiv \frac{-\Delta_{\text{PM}} \times G_1(y)}{\frac{\eta_n}{E_{\text{Fp}\sigma}}}, \quad (63) \]

Further, the Hall coefficient is defined by

\[ R_{\text{H}(\text{PM})}(N', T, r_d) \equiv \frac{-G_1(y) + \Delta_{\text{PM}} \times G_1(y)}{\frac{\eta_n}{E_{\text{Fp}\sigma}}}, \quad (64) \]

where \( \eta_n = E_{\text{Fp}\sigma} \), noting that \( G_1(y) = 1 \).
found to be proportional to $\frac{E_{F_{\text{Fno}}}}{kT}$. Further, using now the total correction given by:  
$$\Delta_\text{PM} = 0.04 \times \frac{k_f \xi_r}{k_f N} + 0.2 \times \frac{k_f^2}{a_{\text{g}}(r_d, p_{\text{cond}})} - 0.03 \times \frac{n_{\text{n}}}{L_{\text{Fno}}}$$  
which is proportional to $E_{F_{\text{Fno}}}^{1/2}$, 

$$\mu_\text{PM}(N', T, r_d) \approx \frac{(0.85)^2}{\pi} \times \frac{\xi_r}{\hbar} \times \frac{a_{\text{g}}(r_d, r_{\text{cond}})}{k_f} \times \frac{k_f^2}{k_f N} \times \frac{E_{F_{\text{Fno}}}}{n_{\text{n}}}$$

where $r_{\text{cond}}$ is the empirical parameter chosen to minimize the absolute deviations between the numerical results of $\mu_\text{PM}$ and the corresponding $r_{\text{data}}$-data, and the functions $G_2(y)$ and $G_2(y)$ are given in Table III. Then, the expression for electrical conductivity is given by

$$\sigma_\text{PM}(N', T, r_d) = \sigma_\text{g}(E_{F_{\text{Fno}}}) \times \left\{ G_2(y) + \Delta_\text{PM} \times G_2(y) \right\},$$

where $\sigma_\text{g}(E_{F_{\text{Fno}}}) = \frac{(0.85)^2}{\pi} \times \frac{\xi_r}{\hbar} \times \frac{a_{\text{g}}(r_d, r_{\text{cond}})}{k_f} \times \frac{k_f^2}{k_f N} \times \frac{E_{F_{\text{Fno}}}}{n_{\text{n}}}$, being proportional to $E_{F_{\text{Fno}}}^2$.

Furthermore, the Hall coefficient is defined by

$$R_{H(\text{PM})}(N', T, r_d) = -\frac{(\sigma_{\text{PM}} \times r_{\text{PM}})}{(\sigma_{\text{PM}})^2} = -\frac{1}{N' \times q} \times r_{\text{H(PM)}} < 0,$$

where the Hall factor is given by

$$r_{\text{H(PM)}}(N', T, r_d) \equiv \frac{(\tau_2)}{(\tau_1)} = \frac{G_2(y) + \Delta_\text{PM}^2 \times G_2(y) + 2 \times \Delta_\text{PM} \times G_2(y)}{[G_2(y) + \Delta_\text{PM}^2 \times G_2(y)]^{1/2}},$$

Furthermore, the Hall mobility yields

$$\mu_{\text{H(PM)}}(N', T, r_d) = \mu_\text{PM} \times r_{\text{H(PM)}}.$$  

Our numerical calculation indicates that in degenerate d-Si systems the $r_{\text{H}}$-behaviors obtained in PVCMM and PM, using Eqs. (65, 70), are almost the same. So, in the PM, our numerical results of Hall factors $r_{\text{H}}$ obtained in various d-Si systems at 77 K, using Eq. (70), are plotted as functions of $N$ in Figures 6a and 6b.

![Image](image)

**Figures 6.** In the PM, our results of Hall factors $r_{\text{H}}$ obtained in various d-Si systems are plotted as functions of $N$, decreasing with increasing $N$, in good agreement with the result obtained in an effective electron gas [66].

Then, in particular, in the As-Si system at $T=10$ K and for $N = 2.7 \times 10^{19}$ cm$^{-3}$, the numerical results of Hall coefficient, $[R_{\text{H}}(N')]$, where $N' = N - N_{\text{c(As)}}$, $N_{\text{c(As)}} = 8.58 \times 10^{18}$ cm$^{-3}$, and Hall mobility, $\mu_{\text{H}}(N')$, obtained using Eqs. (64, 66) for the PVCMM, and Eqs. (69, 71) for the PM, and their absolute relative errors, $|\text{REs}|$, calculated using the corresponding data obtained by Morin and Maita [50], are tabulated in Table 8.

<table>
<thead>
<tr>
<th>$N'(10^{19}$ cm$^{-3}$)</th>
<th>$\mu_{\text{H}} [\text{RE}]$</th>
<th>$R_{\text{H}[\text{RE}]$</th>
<th>PVCMM</th>
<th>$\mu_{\text{H}} [\text{RE}]$</th>
<th>$R_{\text{H}[\text{RE}]$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.79</td>
<td>2.34</td>
<td>0.34</td>
<td>156.5</td>
<td>0.34</td>
<td></td>
</tr>
<tr>
<td>0.17</td>
<td>0.10</td>
<td>0.03</td>
<td>0.01</td>
<td>0.03</td>
<td></td>
</tr>
</tbody>
</table>

Table 8 indicates $|\text{REs}|$ of $\mu_{\text{H}}(N')$ and $|R_{\text{H}}(N')|$ are equal to 17% and 3% obtained for the PM, and 0.6% and 3% for PVCMM, respectively, confirming thus the use of $N'$ for the effective density of free electrons given in the conduction band when $N > N_{\text{c(d)}}$, given in Eq. (43).
In the P (As)-Si systems at $T=4.2$ K, $N_{c(p)} = 3.52 \times 10^{18}$ cm$^{-3}$ and $N_{c(As)} = 8.58 \times 10^{18}$ cm$^{-3}$, as given in Table 1, the numerical results of resistivity $\rho(N^*) = 1/\sigma(N^*)$, $\sigma(N^*)$ being determined in Eq. (63) for the PVCMM and in Eq. (68) for the PM, are tabulated in Table 9, in which their absolute relative errors $|\text{RE}|$, calculated using the data obtained by Chapman et al. [54], are also included, suggesting that the maximal $|\text{RE}|$ of $\rho(N^*)$ are equal to 10% (11%), obtained respectively for the PM (PVCMM).

The underlined $|\text{RE}|$-value is the maximal one.

In the P-Si system at $T=77$ K, the numerical results of conductivity $\sigma(N^*)$, obtained respectively for the PVCMM and PM, are tabulated in this Table 10, in which their absolute relative errors $|\text{RE}|$, calculated using the $\sigma$ data obtained by Finetti and Mazzone [60], are also included. This indicates that the maximal $|\text{RE}|$ of $\sigma(N^*)$ are equal to 12% and 14% for PM and PVCMM, respectively.

<table>
<thead>
<tr>
<th>$N$ (10$^{18}$ cm$^{-3}$)</th>
<th>1.1</th>
<th>1.6</th>
<th>2.7</th>
<th>3.9</th>
<th>5</th>
<th>7</th>
<th>13</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\rho_{\text{PM}}$ (donor)</td>
<td>33 (P)</td>
<td>23 (P)</td>
<td>13 (P)</td>
<td>9.4 (P)</td>
<td>13 (As)</td>
<td>6 (P)</td>
<td>3.8 (P)</td>
</tr>
</tbody>
</table>

In the PM, the results of $\rho$ are accompanied by their $|\text{ARE}|$ as:

- For given $N$, it increases at a given $N$, it increases with increasing $r_d$. That means: $\rho(r_{dB}) < \rho(r_p) < \rho(r_{AS}) < \cdots < \rho(r_{dA}) < \rho(r_{d})$, in good agreement with the observations by Logan et al. [53].

Table 11 indicates that (i) at a given $r_d$, the resistivity decreases with increasing $N$, and (ii) at a given $N$, it increases with increasing $r_d$. That means: $\rho(r_{dB}) < \rho(r_p) < \rho(r_{AS}) < \cdots < \rho(r_{dA}) < \rho(r_{d})$, in good agreement with the observations by Logan et al. [53].

Table 12 suggests that (i) for a given $r_d$, the mobility and the Hall mobility decrease with increasing $N$, (ii) for given $N$, they decrease with increasing $r_d$, since $\mu$ (or $\mu_H$) is proportional to $\sigma \equiv 1/\rho$, where $\rho$ increases with increasing $r_d$, as observed in above Table XI, (iii) for given $N$ and $r_d$, 

<table>
<thead>
<tr>
<th>$N$ (10$^{18}$ cm$^{-3}$)</th>
<th>1.85</th>
<th>5.55</th>
<th>8.65</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\sigma_{\text{abs}}$ (ohm$^{-1}$ cm$^{-1}$)</td>
<td>559</td>
<td>1500</td>
<td>2000</td>
</tr>
</tbody>
</table>

In the PM, the results of $\sigma$ are accompanied by their $|\text{ARE}|$ as:

<table>
<thead>
<tr>
<th>Donor</th>
<th>Sb</th>
<th>P</th>
<th>As</th>
<th>Bi</th>
<th>Ti</th>
<th>Te</th>
<th>Se</th>
<th>S</th>
</tr>
</thead>
<tbody>
<tr>
<td>$N$ (10$^{19}$ cm$^{-3}$)</td>
<td>$\rho$</td>
<td>$\rho$</td>
<td>$\rho$</td>
<td>$\rho$</td>
<td>$\rho$</td>
<td>$\rho$</td>
<td>$\rho$</td>
<td>$\rho$</td>
</tr>
<tr>
<td>3</td>
<td>11.58</td>
<td>12.68</td>
<td>22.37</td>
<td>25.92</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>6</td>
<td>6.40</td>
<td>6.98</td>
<td>11.56</td>
<td>12.97</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>12</td>
<td>4.12</td>
<td>4.49</td>
<td>7.32</td>
<td>8.14</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>40</td>
<td>1.22</td>
<td>1.33</td>
<td>2.18</td>
<td>2.41</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>70</td>
<td>0.74</td>
<td>0.81</td>
<td>1.33</td>
<td>1.48</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>100</td>
<td>0.53</td>
<td>0.58</td>
<td>0.97</td>
<td>1.08</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>$\mu$ (m$^{-1}$ cm$^{-1}$)</th>
<th>$\mu$ (m$^{-1}$ cm$^{-1}$)</th>
<th>$\mu$ (m$^{-1}$ cm$^{-1}$)</th>
</tr>
</thead>
<tbody>
<tr>
<td>3</td>
<td>200 (228)</td>
<td>186 (213)</td>
</tr>
<tr>
<td>6</td>
<td>171 (180)</td>
<td>158 (177)</td>
</tr>
<tr>
<td>12</td>
<td>156 (160)</td>
<td>144 (148)</td>
</tr>
<tr>
<td>40</td>
<td>129 (129)</td>
<td>118 (119)</td>
</tr>
<tr>
<td>70</td>
<td>121 (122)</td>
<td>111 (111)</td>
</tr>
<tr>
<td>100</td>
<td>117 (117)</td>
<td>107 (107)</td>
</tr>
</tbody>
</table>

Table 12. In the degenerate $d$-Si systems at $T=77$ K, the numerical results of mobility $\mu(N^*, T, r_d)$ and Hall mobility $\mu_H(N^*, T, r_d)$, both expressed in (cm$^2$/V sec) and obtained respectively from Eqs. (67, 71), are tabulated here. This indicates that $\mu_H = \mu$ at $N = 10^{21}$ cm$^{-3}$. 

Table 11. In the degenerate $d$-Si systems at $T=77$ K, the numerical results of resistivity $\rho(N^*, T, r_d)$ are expressed in $10^{-4}$ ohm cm. 

The underlined $|\text{RE}|$-value is the maximal one.
\[ \mu_H > \mu, \text{ and finally } \mu_H = \mu \text{ for } N = 10^{21} \text{ cm}^{-3}, \text{ since the Hall factor } r_H \text{ is equal to 1, as that given in the effective electron gas [66].} \]

Now, in degenerate (d)-Si systems at 77 K, from the generalized Einstein relation [62-67], it is interesting to present in following Table 13 our numerical results of diffusion coefficients: \( D(N, T, r_d) \), \( D_a(N, T, r_d) \), and \( D_2(N, T, r_d) \), determined respectively in Eqs. (A15, A16, A17) of the Appendix C, being related to the mobility \( \mu(N, T, r_d) \) given in Eq. (67).

### Table 13

<table>
<thead>
<tr>
<th>Donor</th>
<th>Sb</th>
<th>P</th>
<th>As</th>
<th>Bi</th>
<th>Ti</th>
<th>Te</th>
<th>Se</th>
<th>S</th>
</tr>
</thead>
<tbody>
<tr>
<td>N</td>
<td>( D(D_1) )</td>
<td>( D(D_2) )</td>
<td>( D(D_1) )</td>
<td>( D(D_2) )</td>
<td>( D(D_1) )</td>
<td>( D(D_2) )</td>
<td>( D(D_1) )</td>
<td>( D(D_2) )</td>
</tr>
<tr>
<td>3</td>
<td>5.795 (5.793)</td>
<td>5.296 (5.288)</td>
<td>3.074 (3.069)</td>
<td>2.701 (2.695)</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>6</td>
<td>8.264 (8.266)</td>
<td>7.559 (7.560)</td>
<td>4.539 (4.539)</td>
<td>4.049 (4.049)</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>10</td>
<td>10.873 (10.876)</td>
<td>9.947 (9.949)</td>
<td>6.036 (6.037)</td>
<td>5.415 (5.416)</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>40</td>
<td>23.587 (23.593)</td>
<td>21.523 (21.528)</td>
<td>13.004 (13.007)</td>
<td>12.82 (12.82)</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>70</td>
<td>32.576 (32.583)</td>
<td>29.684 (29.691)</td>
<td>17.809 (17.813)</td>
<td>17.187 (17.187)</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>( N )</td>
<td>( D(D_1) )</td>
<td>( D(D_2) )</td>
<td>( D(D_1) )</td>
<td>( D(D_2) )</td>
<td>( D(D_1) )</td>
<td>( D(D_2) )</td>
<td>( D(D_1) )</td>
<td>( D(D_2) )</td>
</tr>
</tbody>
</table>

Table 13 indicates that: (i) for a given \( r_d \), \( D_1 \) and \( D_2 \) increase with increasing \( N \), (ii) for a given \( N \), since \( D_1 \) and \( D_2 \) increase with increasing \( r_d \), as observed in above Table 11, our results of \( D \) and \( D_1 \) and \( D_2 \) are found to be almost the same, suggesting that the asymptotic behaviors of \( D \) and \( D_1 \) and \( D_2 \) are correct.

### 7.2. Behaviors of Electrical Functions Obtained for \( E \leq 0 \)

First of all, it should be noted from Eqs. (26, 68) that for any \( E \) the conductivity can be rewritten in a general form as

\[
\sigma_{PM}(E, r_d) \equiv \sigma(E, r_d) = \sigma_0(E_{Fno}) \times \left\{ \frac{\langle A^{5/2} \rangle_{KIM}^{\sqrt{\Delta \rho}}}{\eta_n} + \Delta \rho \times \frac{\langle A^{5/2} \rangle_{KIM}^{\sqrt{\Delta \rho}}}{\eta_n} \right\},
\]

(72)

where \( \sigma_0(E_{Fno}) = \frac{(0.85)^2}{\pi} \times \frac{a^2 \times k_{Fno}^2}{\hbar} \times \frac{\Delta \rho \times k_{Fno}^2}{\hbar} \times \frac{k_{Fno}}{\eta_n} \) is proportional to \( E_{Fno}^2 \), and \( \langle A^{5/2} \rangle_{KIM} \) is determined in Eq. (26) for \( a = 5/2 \) and \( a = 2 \), respectively.

Here, as \( E \leq 0 \), using the functions: \( H_{n}, K_{n} \) and \( F_{n} \), given respectively in Eqs. (26-28) for \( a = 5/2 \) and \( a = 2 \), the conductivity, given in Eq. (72), is now rewritten by

\[
\sigma_{PM}(E, r_d) \equiv \sigma(E, r_d) = \sigma_0(E_{Fno}) \times \left\{ \frac{\langle A^{5/2} \rangle_{KIM}^{\sqrt{\Delta \rho}}}{\eta_n} \times F_n(a = 5/2) + \Delta \rho \times \frac{\langle A^{5/2} \rangle_{KIM}^{\sqrt{\Delta \rho}}}{\eta_n} \times F_n(a = 2) \right\},
\]

(73)

So, our numerical results of exponential tails of the electrical conductivity \( \sigma(E, r_d) \) at 0 K and for \( N = 5 \times 10^{20} \text{ cm}^{-3} \), calculated using Eq. (73), are plotted in Figure 7, as functions of \( v_n \).

Figure 7 shows that:

(i) our results of \( \sigma(E, r_d) \) increase with increasing \( r_d \) for a given \( v_n \), due to the donor-size effect, and

(ii) present the maxima at \( v_n = v_{n}(M) \) and go to zero as \( v_n \rightarrow 0 \) and \( \infty \).

Figure 7. Our results of electrical conductivity \( \sigma(E, r_d) \) increase with increasing \( r_d \) for a given \( v_n \), due to the donor-size effect, and present the maxima at \( v_n = v_{n}(M) \) and go to zero as \( v_n \rightarrow 0 \) and \( \infty \).
Table 14. In the d-Si systems at $T=0$ K and for $N = 5 \times 10^{20}$ cm$^{-3}$, using the expression for electrical conductivity determined in Eq. (73), $\sigma(V_{n}r_{s})$, the numerical results of $\ln(\sigma (V_{n}r_{s}))$ and its approximate form: $A F(V_{n}r_{s}a_{1}, V_{n}r_{s}z, f) = e (V_{n}r_{s}V_{n}z) \times x_{v}^{2} + f$. determined in Eq. (29) for small $v_{n}$-intervals: $v_{x} \leq v_{n} \leq v_{e}$ and those of absolute relative errors: $|RE| \equiv 1 - \frac{A F(V_{n}r_{s}a_{1}, V_{n}r_{s}z, f)}{\ln(\sigma (V_{n}r_{s}))}$ are evaluated and tabulated below.

<table>
<thead>
<tr>
<th>Donor</th>
<th>Sb</th>
<th>P</th>
<th>As</th>
<th>Bi</th>
<th>Ti</th>
<th>Te</th>
<th>Se</th>
<th>S</th>
</tr>
</thead>
<tbody>
<tr>
<td>$v_{n}(M)$</td>
<td>0.24145</td>
<td>0.25005</td>
<td>0.31043</td>
<td>0.32482</td>
<td>0.48207</td>
<td>0.67829</td>
<td>0.86355</td>
<td>0.88790</td>
</tr>
</tbody>
</table>

For $1.37 \leq v_{n} \leq 1.42, AF = (e \times v_{n} + f)$ is accurate to within $1.9 \times 10^{-4}$, where

$\epsilon$

\begin{align*}
- e & 17.734 \\
- f & 9.234 \\
|RE| & 1.2 \times 10^{-4} \\
& 1.2 \times 10^{-4} \\
& 1.0 \times 10^{-4} \\
& 1.2 \times 10^{-4} \\
& 1.9 \times 10^{-3} \\
& 2.4 \times 10^{-4} \\
& 1.2 \times 10^{-4} \\
& 1.2 \times 10^{-4}
\end{align*}

For $1.28 \leq v_{n} \leq 1.30, AF = (e \times v_{n}^{1/2} + f)$ is accurate to within $1.5 \times 10^{-4}$, where

$\epsilon$

\begin{align*}
- e & 38.134 \\
- f & 29.621 \\
|RE| & 4.7 \times 10^{-5} \\
& 2.7 \times 10^{-5} \\
& 1.2 \times 10^{-4} \\
& 4.7 \times 10^{-5} \\
& 1.5 \times 10^{-3} \\
& 6.2 \times 10^{-5} \\
& 3.1 \times 10^{-5} \\
& 1.3 \times 10^{-4}
\end{align*}

For $1.180 \leq v_{n} \leq 1.185, AF = (e \times v_{n}^{1/3} + f)$ is accurate to within $9.4 \times 10^{-4}$, where

$\epsilon$

\begin{align*}
- e & 53.080 \\
- f & 44.192 \\
|RE| & 4.1 \times 10^{-5} \\
& 1.8 \times 10^{-5} \\
& 6.9 \times 10^{-5} \\
& 6.7 \times 10^{-5} \\
& 1.4 \times 10^{-3} \\
& 6.8 \times 10^{-5} \\
& 1.8 \times 10^{-5} \\
& 7.9 \times 10^{-5}
\end{align*}

For $0.110 \leq v_{n} \leq 0.115 < v_{n}(M), AF = (e \times v_{n}^{1/4} + f)$ is accurate to within $2.1 \times 10^{-3}$, where

$\epsilon$

\begin{align*}
- e & 69.838 \\
- f & 60.894 \\
|RE| & 4.1 \times 10^{-5} \\
& 1.8 \times 10^{-5} \\
& 5.3 \times 10^{-4} \\
& 4.9 \times 10^{-4} \\
& 8.2 \times 10^{-4} \\
& 2.1 \times 10^{-3} \\
& 8.3 \times 10^{-4} \\
& 4.3 \times 10^{-4}
\end{align*}

For $0.088 \leq v_{n} \leq 0.090, AF = (e \times v_{n}^{1/3} + f)$ is accurate to within $4.3 \times 10^{-3}$, where

$\epsilon$

\begin{align*}
- e & 37.452 \\
- f & -21.105 \\
|RE| & 9.5 \times 10^{-5} \\
& 1.4 \times 10^{-4} \\
& 9.7 \times 10^{-5} \\
& 1.6 \times 10^{-4} \\
& 3.0 \times 10^{-4} \\
& 4.1 \times 10^{-4} \\
& 4.3 \times 10^{-4} \\
& 9.5 \times 10^{-4}
\end{align*}

For $0.070 \leq v_{n} \leq 0.073, AF = (e \times v_{n}^{1/2} + f)$ is accurate to within $1.6 \times 10^{-3}$, where

$\epsilon$

\begin{align*}
- e & 59.884 \\
- f & -21.865 \\
|RE| & 3.6 \times 10^{-4} \\
& 3.9 \times 10^{-4} \\
& 4.5 \times 10^{-4} \\
& 7.5 \times 10^{-4} \\
& 8.9 \times 10^{-4} \\
& 6.1 \times 10^{-4} \\
& 1.6 \times 10^{-3} \\
& 9.2 \times 10^{-4}
\end{align*}

For $0.061 \leq v_{n} \leq 0.064, AF = (e \times v_{n} + f)$ is accurate to within $1.5 \times 10^{-2}$, where

$\epsilon$

\begin{align*}
- e & 162.658 \\
- f & -17.234 \\
|RE| & 5.9 \times 10^{-4} \\
& 6.7 \times 10^{-4} \\
& 7.3 \times 10^{-4} \\
& 6.9 \times 10^{-4} \\
& 8.3 \times 10^{-4} \\
& 9.3 \times 10^{-4} \\
& 1.1 \times 10^{-3} \\
& 1.5 \times 10^{-3}
\end{align*}

For $0.054 \leq v_{n} \leq 0.056, AF = (e \times x_{v}^{2} + f)$ is accurate to within $9.9 \times 10^{-4}$, where

$\epsilon$

\begin{align*}
- e & 2147.35 \\
- f & -15.033 \\
|RE| & 5.4 \times 10^{-4} \\
& 5.2 \times 10^{-4} \\
& 6.5 \times 10^{-4} \\
& 6.2 \times 10^{-4} \\
& 6.6 \times 10^{-4} \\
& 8.6 \times 10^{-4} \\
& 8.8 \times 10^{-4} \\
& 9.9 \times 10^{-4}
\end{align*}

The underlined $|RE|$-value is the maximal one for each donor-Si system.

Table 14 suggests that: (i) our results of $AF(V_{n}r_{s}a_{1}, V_{n}r_{s}z, f)$ given for ($z=1$)-exponent agree with the Urbach law obtained from linear exponential conduction-band tail-behaviors by some workers [21, 29, 30, 37], (ii) ours for ($z=1/2$)-exponent and $v_{n} > v_{n}(M)$ agree with other theoretical results [19, 37], and finally (iii) ours for ($z=1/3, 1/4$)-exponents when $v_{n} > v_{n}(M)$, and for ($z=1/4, 1/3, 1/2, 1, 2$)-exponents when $v_{n} < v_{n}(M)$ may thus be new.

Finally, our numerical results of energy parameter, $E_{no}(N; r_{s}A)$, obtained in the small interval: $1.37 \leq v_{n} \leq 1.42$, using Eq. (31), are plotted as functions of $N$ in Figures 8a and 8b, indicating that, for a given $N$, $E_{no}$ increases with
increasing \(r_d\)-values, due to the donor-size effect.

\[
\vec{j}(\vec{r}) = L^{(1)} \times \vec{E} + L^{(2)} \times \vec{T} \times \vec{v}(T^{-1}),
\]

(75)

where \(\vec{j}\) is the electric current density, \(\vec{E}\) is the electric field, and \(L^{(i)}\) is the transport coefficient determined in an isotropic system. Now, using the average of \((E^P)_{\text{FDDF}} \equiv \text{G}_D(y) \times E_{\text{Pho}}^P\), where the expressions for \(\text{G}_D(y)\), \(y = \frac{n_k T}{E_{\text{Pho}}} = \frac{n_k T}{E_{\text{Pho}}}\), are determined in Eq. (34) and given in Table III, calculated using the Fermi-Dirac distribution function (FDDF), and using also the expression for electrical conductivity as a function of \(E\), derived from Eq. (68) for \(E \geq 0\), as

\[
\sigma(E, r_d) = \sigma_{\text{G}D}(E_{\text{Pho}}) \left\{ \frac{E^2}{E_{\text{Pho}}} + \Delta_{\text{PM}} \times \frac{E^{y/2}}{E_{\text{Pho}}^{y/2}} \right\},
\]

(76)

the Onsager relations are found to be given as follows.

First, one has [58, 61]

\[
L^{(1)} \equiv (\sigma(E, r_d))_{\text{FDDF}} = \sigma_{\text{G}D}(E_{\text{Pho}}) \left\{ \frac{E^2}{E_{\text{Pho}}} + \Delta_{\text{PM}} \times \frac{E^{y/2}}{E_{\text{Pho}}^{y/2}} \right\},
\]

(77)

which is just the result obtained in Eq. (68).

Then, one gets [58]

\[
L^{(2)} = L^{(3)} = -\frac{1}{q} \times (E \times \sigma(E, r_d))_{\text{FDDF}} = -\frac{\sigma_{\text{G}D}(E_{\text{Pho}})}{q} \times \left\{ \text{G}_D(y) + \Delta_{\text{PM}} \times \text{G}_D(y) \right\}.
\]

(78)

Finally, one obtains [58]

\[
L^{(4)} = \frac{1}{q^2} \times (E^2 \times \sigma(E, r_d))_{\text{FDDF}} = \frac{\sigma_{\text{G}D}(E_{\text{Pho}})}{q^2} \times \left\{ \text{G}_D(y) + \Delta_{\text{PM}} \times \text{G}_D(y) \right\}.
\]

(79)

Now, from Eqs. (77-79), one can define the thermal conductivity by [58, 61]

\[
K_T(N, T, r_d) \equiv \frac{1}{T} \times \left\{ L^{(4)} - \frac{L^{(2)}L^{(3)}}{L^{(5)}} \right\}.
\]

(80)

Some remarks obtained from Eq. (80) are given as follows. (i) First, our numerical calculation indicates that, in the degenerate (P)-Si system, for \(N = 10^{21} \text{ cm}^{-3}\) and at \(T=3 \text{ K}\) and \(300 \text{ K}\), our results of \(K_T\) are equal to \(8 \times 10^{-4}\) and \(0.125 \text{ W/cm.K}\), in good agreement with the experimental results obtained by Slack [68]: \(5 \times 10^{-4}\) and between 0.1 and 0.2 W/cm.K, respectively.

(ii) Second, at \(N = 10^{21} \text{ cm}^{-3}\) and \(T=3 \text{ K}\), the values of relative deviations between our results of \(K_T(N, T, r_d)/[T \times \sigma(N, T, r_d)]\), calculated using Eqs. (68) and (80), and the constant: \(n_k^2 \frac{E_{\text{Pho}}}{q} = 2.443 \times 10^{-8} \text{ W.cm.K}^{-2}\), being obtained from the Wiedemann-Franck law for metals [58, 61], are tabulated in Table 15, indicating that our result (80) well verifies this law, with a precision of the order of \(6.52 \times 10^{-7}\).
The values of the relative deviations (RD) between our results of \( \frac{K_T}{T} \) obtained in various degenerate donor-Si systems, and the constant \( K_T = 2.443 \times 10^{-8} \text{W} K^{-2} \), obtained from the Wiedemann-Franck law for metals, indicating a perfect agreement between those results.

<table>
<thead>
<tr>
<th>Donor</th>
<th>Sb</th>
<th>P</th>
<th>As</th>
<th>Bi</th>
<th>Ti</th>
<th>Te</th>
<th>Se</th>
<th>S</th>
</tr>
</thead>
<tbody>
<tr>
<td>RE</td>
<td>4.72 \times 10^{-7}</td>
<td>-9.40 \times 10^{-8}</td>
<td>-3.51 \times 10^{-8}</td>
<td>3.79 \times 10^{-7}</td>
<td>1.67 \times 10^{-7}</td>
<td>-6.52 \times 10^{-7}</td>
<td>-1.53 \times 10^{-7}</td>
<td>-5.95 \times 10^{-8}</td>
</tr>
</tbody>
</table>

The underlined \(|RE|\) value is the maximal one for each donor-Si system.

(iii) Finally, our numerical calculation shows that, in degenerate (d)-Si systems, for \( N = 10^{21} \text{cm}^{-3} \) and in the temperature range from \( T=3 \) to 300 K, the maximal value of absolute deviations between \( K_T(N^*, T, r_d) \) given in Eq. (80) and its approximate form \( K_T(N^*, T, r_d) = C_{K_T} \times T \) is found to be equal to \( 9.9 \times 10^{-4} \), in good agreement with our previous result [58, 61]. Then, those are plotted in Figure 9a as functions of \( T \), suggesting that at a given \( T \) the thermal conductivity \( K_T \) decreases with increasing \( r_d \), due to the donor-size effect.

![Figure 9a](image-url)

**Figure 9a.** Our results of \( K_T(N^*, T, r_d) = C_{K_T} \times T \) are plotted as functions of \( T \), suggesting that at a given \( T \) the thermal conductivity \( K_T \) decreases with increasing \( r_d \), due to the donor-size effect.

Then, from Eqs. (77, 78) for \( L^{(1)} \) and \( L^{(2)} \), and Eq. (D1) of the Appendix D for \( E_{Fm} \), the absolute thermoelectric power \( Q \) can be defined by [58, 61]

\[
Q(N^*, T, r_d) \equiv \frac{1}{T} \left[ \frac{L^{(2)}}{L^{(1)}} + \frac{E_{Fm}}{q} \right].
\]  

(81)

This result (81) is a function commonly used to describe the following thermoelectric coefficients [58, 61], such as: the Thomson coefficient,

\[
T_S(N^*, T, r_d) \equiv T \times \frac{dQ(N^*, T, r_d)}{dT},
\]  

(82)

the Seebeck thermoelectric potential,

\[
S_p(N^*, T, r_d) \equiv \int_0^T Q(N^*, T, r_d) \, dT,
\]  

(84)

and finally the dimensionless figure of merit,

\[
ZT(N^*, T, r_d) \equiv \frac{T \times S_p(N^*, T, r_d) \times \sigma(N^*, T, r_d)}{K_T(N^*, T, r_d)},
\]  

(85)

We now evaluate the above results (81-85) in the following.

In degenerate (d)-Si systems, for \( N = 10^{21} \text{cm}^{-3} \) and in the temperature range from \( T=3 \) to 300 K, our numerical calculation indicates that: (i) the maximal value of absolute relative deviations between \( Q \) determined in Eq. (81) and its approximate form: \(-C_Q \times T\) is found to be equal to \( 6.16 \times 10^{-3} \), and (ii) the maximal value of absolute relative deviations between \( T_S \) determined in Eq. (82) and its approximate form: \(-C_S \times T\) is equal to \( 0.019 \). So, our numerical results of \( Q = -C_Q \times T\) and \( T_S = -C_S \times T\) are plotted in Figures 9b and 9c, as functions of \( T \), respectively, suggesting that at a given \( T \), \( Q \) and \( T_S \) both decrease with increasing \( r_d \), due to the donor-size effect.

![Figure 9b](image-url)

**Figure 9b.** Our results of \( Q = -C_Q \times T \) are plotted as functions of \( T \), suggesting that, at a given \( T \), \( Q \) decreases with increasing \( r_d \), due to the donor-size effect.
Figure 9c. Our results of $T_x = -C_0 \times T$ are plotted as functions of $T$, suggesting that at a given $T$, $T_x$ decreases with increasing $r_x$, due to the donor-size effect.

Finally, in the following Figures 9d, 9e and 9f, our numerical results of Peltier coefficient $P_T$, Seebeck thermoelectric potential $S_B$, and dimensionless figure of merit $ZT$, calculated using Eqs. (83-85), are plotted as functions of $T$, respectively.

Figure 9d. Our results of Peltier coefficient $P_T$ are plotted as functions of $T$, suggesting that at a given $T$, $P_T$ increases with increasing $r_x$, due to the donor-size effect.

Figure 9e. Our results of Seebeck thermoelectric potential $S_B$ are plotted as functions of $T$, suggesting that at a given $T$, $S_B$ decreases with increasing $r_x$, due to the donor-size effect.

Figure 9f. Our results of dimensionless figure of merit $ZT$ are plotted as functions of $T$.

9. Concluding Remarks

Using the effective autocorrelation function for potential fluctuations $W_n$, developed in Eq. (B.6) of the Appendix B, expressed in terms of the Heisenberg uncertainty relation given in Eq. (B.5), and an expression for the Gaussian average of $E_k^2$, $\langle E_k^2 \rangle_{KIM}$, obtained in Eq. (20) by the Kane integration method (KIM), we developed the expressions for density of states, optical absorption coefficient, and electrical conductivity, obtained in various degenerate d-Si systems,
being due to the effects of donor-size and heavy doping, as given respectively in Eqs. (36, 49, 72). It should be noted that this average expression was found to be equivalent to that obtained by the Feynman path-integral method. Then, those above results were expressed in terms of \(\mathbb{E}^{\alpha-(1/2)}\), as given in Eq. (24) for \(\mathbb{E} \geq 0\) and \(\alpha \geq 1\), vanished at the band edge: \(\mathbb{E} = 0\), and exhibited their exponential tail behaviors for \(\mathbb{E} \leq 0\), as obtained in Eqs. (28-31), in Tables 4, 7, 14, and in Figures 1, 2a (b), 3a (b, c), 4, 5a (b), 7 and 8a (b). Furthermore, in Figures 1, 4, and 7, some important conclusions were obtained as follows.

(i) First, for a given value of \(-\mathbb{E}\), those exponential tails increased with increasing \(r_{d}\), being due to the donor-size effect.

(ii) Secondly, they vanished at the conduction-band edge \(\mathbb{E} = -0\), as given in Eq. (26), in good accordance with our other results obtained in Eq. (A3) of the Appendix A. Furthermore, those exponential tail-results were also compared with other theoretical ones, being found to be constant, at \(\mathbb{E} = -0\), obtained in the small time approximation [21, 29, 30] and in the full ground-state case and deep-tail approximation [21]. Thus, their results should not be correct, as discussed in Eq. (26).

(iii) Finally, for \(\mathbb{E} \leq 0\), they went to zero as \(\mathbb{E} \to -0\) and \(- \infty\) and presented the maxima, being found to be in good accordance with an asymptotic form for the exponential conduction-band tail, obtained by Halperin and Lax [19], using the minimum counting methods. Hence, the problem posed in the past for those exponential tails [14, 17, 19, 21, 23, 25, 29, 30, 37] should now be solved.

Then, an expression for the average of \(\mathbb{E}^{p}\), at low temperatures and for \(p \geq 3/2\), calculated by the Fermi-Dirac distribution function, was determined in Eq. (34), being used to evaluate, in degenerate d-Si systems, the mobility, conductivity, resistivity, Hall coefficient, Hall factor, Hall mobility, thermal conductivity, diffusion coefficient, absolute thermoelectric power, Thomson coefficient, Peltier coefficient, Seebeck thermoelectric potential, and finally the density of states \((\text{DOS})\), being defined in Eq. (20), we obtained:

\[\langle 2m\mathbb{H}^2/2m\mathbb{H}^2 \times |\mathbb{E}| \rangle = \frac{B}{k} = \frac{B}{k_p n} \sqrt{|\mathbb{E}|}, \quad \text{(A1)}\]

where from the LFL-method [18, 20]: \(B = B_{\text{LFL}} = \pi\) and \(\mathbb{E}_0 = \mathbb{E}_{\text{DOS}(LFL)} = \frac{B_{\text{LFL}}}{k}\), \(k\) being the wave number. Here, \(M_{\text{HDE}}(N, r_d)\) is determined in Eq. (8), and \(|\mathbb{E}| \equiv |\beta\rangle_{\text{FLOP}}\) is determined in Eq. (21). In fact, Eq. (A1) is thus the Heisenberg uncertainty relation, which can be compared with that given in next Eq. (A8) of the Appendix B, as: \(\Delta \beta \equiv \langle \beta | - \mathbb{E} + r \Delta - r^\prime \rangle = \frac{B}{k} = \frac{B}{k_p n} \sqrt{|\mathbb{E}|}, \quad \text{B}_n = 2.7185.\) Here, \(r\) and \(r^\prime\) are the electron positions, according to the first-and-second scatterings at the times \(t\) and \(t^\prime\).

Then, using a transformation given in Eq. (48), as \(a \geq 1\), Eq. (A1) thus becomes

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**Appendices**

**Appendix A: Joint Density of States at 0 K**

Friedberg and Luttinger (FL) [20] studied the behavior of the density of states \((\text{DOS})_n\) or the joint DOS \((\text{DOS})_n \equiv J_n(\mathbb{E})\) for a simple method used for the n-type crystal in the limit of very low positive energies \((\mathbb{E} \to 0_+)\) by reformulating the problem conjectured by Lifshitz (L) [18] as that given in the Brownian motion. Here, they showed that \(J_n(\mathbb{E} \to 0_+) \to 0\), and \(J_n(\mathbb{E})\) vanishes for \(\mathbb{E} \leq 0\), neglecting the exponential conduction-band tail, due to the heavy doping effect, \(\mathbb{E}\) being the total electron energy. Then, those LFL results were reviewed by Mieghem [24]. In Sections III-VI, since \((\text{DOS})_n\) given in Eq. (36) for \(a=1\), is proportional to \((\mathbb{E}^{\frac{1}{2}})_{\text{KIM}}\), being defined in Eq. (20), we obtained: \((\text{DOS})_n \to 0\) when \(\mathbb{E} \to +0\) and \(- 0\), as discussed in Eqs. (23, 26), respectively. In other words, \((\text{DOS})_n \to 0\), when \(|\mathbb{E}| \to 0\) or \((\text{DOS})_n\) vanishes at the conduction-band edge \((\mathbb{E} = 0)\), suggesting thus a generalized (LFL)-method (GLFLM) to be study as follows.

In the very large volume \(V_0 \equiv \left(\frac{4\pi}{3}\right)R_0^3\) of radius \(R_0\), being empty of donors, and for lowest \(|\mathbb{E}| \equiv \frac{\hbar^2 n^2}{2m\mathbb{H}^2} \times |\mathbb{E}| \equiv \frac{\hbar^2 \times k^2}{2m\mathbb{H}^2} R_0\) is thus defined by [18, 20]
The probability of such a large region is proportional to:
\[ \exp(-\frac{N}{\Theta_c} \times \mathcal{V}), \]
where \( \frac{N}{\Theta_c} = \frac{N}{\Theta_c} \mathcal{V} \), such that the reduced \( \mathcal{V} \) can be defined by:
\[ \mathcal{V} = \frac{N}{\Theta_c} \times \mathcal{V} \]
[20]. Here, \( \mathcal{V} \) is determined from the FL-results as [20]: \( \mathcal{V} \equiv 2\lambda_0 \approx \lambda_0 \), since \( c = 0.628 \) and \( \lambda_0 \equiv 1/(4\pi \times \lambda) \) is the skin depth, \( \lambda \) being the scattering length. Moreover, in degenerate d-Si systems, the scattering length \( \mathcal{L} \) could be replaced by the effective screening length, \( \mathcal{L}_s \), where \( \mathcal{L}_s \) is determined in Eq. (4). So, in this GLFLM, for lowest \( |\mathcal{E}| \) (or \( |\mathcal{V}| \)), the kinetic energy of localization by: \( E_{\mathcal{L}_s} \equiv \frac{\hbar^2}{2m_0} \mathcal{L}_s^2 \), one thus obtains:

\[ \ln \mathcal{V} = 0; \mathcal{V} = \frac{4B^3}{9\pi}, \]

where \( \mathcal{L}_s \equiv \frac{\hbar^2}{2m_0} \mathcal{L}_s^2 \) is determined in Eq. (4), being in good accord with our results (23, 26). Furthermore, for \( a=1 \) and \( B = \mathcal{B}_L = \pi \), the first-and-second terms of the last member of Eq. (A3) are found to be identical to the L-and-FL results, respectively.

**Appendix B: Effective Autocorrelation Function**

In degenerate d-Si systems, if denoting the electron positions and the corresponding wave vectors, according to the first-and-second scatterings at the times \( t \) and \( t' \) by \( \mathbf{r} \) and \( \mathbf{k} \), and working with the Fourier transform given in Eq. (18), the effective autocorrelation function for potential fluctuations is then defined by [25]

\[ W_n \equiv \langle V(r)V(r') \rangle = \sum_{\mathbf{k}, \mathbf{k'}} v_i(\mathbf{k})v_j(\mathbf{k'}) \exp \left( i \mathbf{k} \cdot (\mathbf{r} - \mathbf{r'}) \right), \]

where the total potential energy \( V(\mathbf{r}) \) is defined in Eq. (16), and \( f^2 = -1 \). Then, since \( \langle V(\mathbf{r}) \rangle = \langle V(\mathbf{r'}) \rangle = 0 \) as remarked in Eqs. (16, 17), \( W_n \) is non-zero only when \( \mathbf{k} = \mathbf{k}' \) and \( \mathbf{k} = - \mathbf{k}' \), meaning that the electron scattered on each donor twice, and giving:

\[ \Delta k = |\mathbf{k} - \mathbf{k}'| = 2k. \]

Hence, for \( W_n [f^2 - f'^2] \neq 0 \), Eq. (A4) thus becomes:

\[ W_n [f^2 - f'^2] = \mathcal{N} \sum_{\mathbf{k}} |v_i(\mathbf{k})|^2 \exp \left( \mathbf{k} \cdot (\mathbf{r} - \mathbf{r'}) \right), \]

noting that \( v_i(\mathbf{k})v_j(\mathbf{k'}) = v_i(\mathbf{k})v_j(\mathbf{k}) = |v_i(\mathbf{k})|^2 \) for real potential energies. Here, \( v_i(\mathbf{k}) \) is determined in Eq. (18) and \( \mathcal{N} \) is the total number of donors. Further, from the Fourier transform [25], one has:

\[ \exp(-\mathcal{E}_{\mathcal{L}_s} \cdot |f^2 - f'^2|) \equiv \sum_{\mathbf{k}} \frac{1}{k^2 + k_s^2 \mathcal{L}_s^2} \exp \left( \mathbf{k} \cdot (\mathbf{r} - \mathbf{r'}) \right), \]

where \( \Omega \) is the total crystal volume, and taking its partial derivative (i.e., \( \partial/\partial k_s \)), one finally finds:

\[ W_n [f^2 - f'^2] \equiv \mathcal{E}_{\mathcal{L}_s} \exp(-k_s \cdot |f^2 - f'^2|). \]

Here, \( \mathcal{E}_{\mathcal{L}_s} \equiv \mathcal{N} \int_0^\infty v_i(\mathbf{r})^2 \cdot 4\pi r^2 dr = 2\pi \mathfrak{N} q^2 k_s^{-2} \mathcal{E}_s^2 \), where \(-q\) is the electron charge, \( v_i(\mathbf{r}) \) is determined in Eq. (17) and the accurate screening length \( k_s^{-1} \) is determined in Eq. (4). Then, some concluding remarks can be obtained below.

(i) In the small time approximation, \( |f^2 - f'^2| = 0 \), Eq. (A7) is thus reduced to \( \mathcal{E}_{\mathcal{L}_s} \equiv \mathcal{N} \int_0^\infty v_i(\mathbf{r})^2 \cdot 4\pi r^2 dr = 2\pi \mathfrak{N} q^2 k_s^{-2} \mathcal{E}_s^2 \), where \(-q\) is the electron charge, \( v_i(\mathbf{r}) \) is determined in Eq. (17) and the accurate screening length \( k_s^{-1} \) is determined in Eq. (4). Then, some concluding remarks can be obtained below.

(ii) Using Eq. (A5), the Heisenberg uncertainty relation between \( \mathcal{E}_s \) and \( \mathcal{L}_s \) is 2k is given by:

\[ \mathcal{E}_s \times \mathcal{L}_s = \mathcal{H}_n \geq 1/2 \Rightarrow \mathcal{E}_s \times \mathcal{L}_s = \mathcal{H}_n/2, \]

which can be compared with that given in Eq. (A1), as discussed above. Here, the values of empirical Heisenberg parameter \( \mathcal{H}_n \) was proposed in Section V as: \( \mathcal{H}_n = 5.4370 \). Further, if replacing the constant \( B \) given in above Eqs. (A1-A3) by \( B = B_n = \mathcal{H}_n/2 = 2.7165 \), which gives: \( B_{\mathcal{L}_s} = \pi = 3.1416 > B_n \), then, from Eq. (A3), we also obtain:

\[ \ln \mathcal{V} \rightarrow -\infty \]
0, in good accordance with our results (23, 26).

(iii) Finally, using Eq. (A8) and defining the kinetic energy of localization by: \( E_{\text{kin}} \equiv \frac{\hbar^2 k_B^2}{2m^2} \), the effective diffusion coefficient \( D \) with the mobility \( \mu \), obtained for any \( T \) and \( N \), being investigated in our previous papers, with precisions of the order of: \( 2.11 \times 10^{-4} \) and \( 10^{-3} \), respectively, [27, 63] are now summarized in the following.

### Appendix C: Fermi Energy and Generalized Einstein Relation

The Fermi energy \( E_{\text{F}} \), and the generalized Einstein relation \( \frac{D}{\mu} \equiv \frac{2k_B}{\hbar} \left( \frac{dE_{\text{F}}}{dN} \right) \), which relates the diffusion coefficient \( D \) with the mobility \( \mu \), obtained for any \( T \) and \( N \), being investigated in our previous papers, with precisions of the order of: \( 2.11 \times 10^{-4} \) and \( 10^{-3} \), respectively, [27, 63] are now summarized in the following.

**Fermi Energy, \( E_{\text{F}} \)**

We first define the reduced impurity density as: \( u \equiv \frac{N}{N_{\text{CB}}} \), \( N_{\text{CB}} \) being the effective density of conduction-band (CB)-states defined by: \( N_{\text{CB}} = 2g_e \frac{m^2_{\text{BE}}(N,r_d)^9}{2\pi^2 \hbar^2} \left( \text{cm}^{-3} \right) \), where \( m^2_{\text{BE}}(N,r_d) \) is determined in Eq. (4). Then, for any \( T \) and \( N \), our expression for reduced Fermi energy, \( \theta_n \), determined in our previous paper [27], is given by

\[
\theta_n(u) \equiv \frac{E_{\text{F}}(u)}{k_B T} = \frac{G(u) + A u^B F(u)}{1 + A^B}, \quad A = 0.0005372, \quad B = 4.82842262, \quad (A10)
\]

where, in the degenerate case or as \( \theta_n(u \gg 1) \rightarrow \infty \), Eq. (A10) is reduced to

\[
F(u) = au^{2/3} \left( 1 + bu^{-4/3} + cu^{-8/3} \right)^{-2/3}, \quad a = 3 \sqrt{\pi/4}, \quad b = \frac{1}{8} \left( \frac{9}{2} \right)^2, \quad c = 6.3739855 \times 10^{-6}, \quad (A11)
\]

and in the non-degenerate case or for \( \theta_n(u_n \ll 1) \ll 0 \), to

\[
G(u) \approx \log(u) + 2^{-3} x u e^{-du}, \quad d = 2^{3/2} \left( \frac{\log 2}{15} \right) > 0. \quad (A12)
\]

Further, one notes that Eq. (A11) can thus be rewritten as

\[
E_{\text{F}}(u) = E_{\text{F}} \times \left( 1 + bu^{-4/3} + cu^{-8/3} \right)^{-2/3}, \quad (A13)
\]

being the Fermi energy given in the degenerate d-Si systems. For example, in the degenerate P-Si system at \( T=77 \) K and \( N(10^{19} \text{ cm}^{-3}) = 3 \) and 100, the relative deviations between \( E_{\text{F}} \left( N^*, T, r_d \right) \) determined in Eq. (A13) and \( E_{\text{F}} \left( N^*, T, r_d \right) = \theta_n \times k_B T \) in Eq. (A10), defined by: \( 1 - \frac{E_{\text{F}}}{E_{\text{F}}(u)} \), are equal to 0.052 and 4.6 \times 10^{-4}, respectively.

**Generalized Einstein Relation**

The generalized Einstein relation is defined by [51, 62-67]

\[
\frac{D(u)}{\mu} = \frac{N}{\bar{q}} \times \frac{dE_{\text{F}}}{dN} \equiv \frac{k_B}{\hbar^2} T \times \left( u \frac{d\theta_n}{du} \right), \quad (A14)
\]

where \( \mu \) is the mobility and in particular in the degenerate case it is determined in Eq. (67), \( \theta_n(u) \equiv \frac{V(u)}{W(u)}, \quad V(u) = G(u) + A u^B F(u), \quad W(u) = 1 + A^B [27, 63] \) as those given in Eqs. (A10, A11). Then, differentiating this function \( \theta_n(u) \) with respect to \( u \), one thus obtains \( \frac{D(u)}{\mu} \). Therefore, Eq. (A14) becomes [63]

\[
\frac{D(u)}{\mu} = \frac{k_B}{\hbar^2} T \times \left( u \frac{V(u) \times W(u) - V(u) \times W'(u)}{W(u)} \right), \quad (A15)
\]

where \( W'(u) = A B u^{B-1} \) and \( V'(u) = u^{-1} + 2 \frac{3}{\sqrt{2}} e^{-du} (1 - \frac{u}{B}) \).
References


