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Abstract We introduce an analogue of Ramanujan's Master Theorem that greatly generalizes some recent results on the construction of $\frac{1}{\pi}$ series. Implementing this technique through the use of computer algebra systems produces "proof signatures" for closed-form evaluations for new classes of infinite series and definite integrals. Using this integration method, we offer symbolic computations for a variety of new series, including new binomial-harmonic series for $\frac{1}{\pi}$, such as the elegant series

$$\sum_{n=1}^{\infty} \frac{\binom{2n}{n}^2 H_n H_{2n}}{16^n (n+1)} = 4 - \frac{7\pi}{3} + 8\ln(2) + \frac{48\ln^2(2) - 16G - 16\ln(2)}{\pi}$$

introduced in our article, letting G denote Catalan's constant, as well as new closed-form evaluations for ${}_3F_2(1)$ series, as in the equality

$${}_3F_2\left[\left.\begin{matrix} \frac{1}{4}, \frac{5}{4}, \frac{3}{2} \\ 1, \frac{9}{4} \end{matrix}\,\right|\, 1\right] = \frac{\Gamma^2\left(\frac{1}{4}\right)\left(15\sqrt{2} - 5\ln\left(1+\sqrt{2}\right)\right)}{16\pi^{3/2}}$$

introduced in our article.

Keywords Ramanujan's Master Theorem · Infinite series · Closed form · Pi formula · Harmonic number

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1 Introduction

Many of Ramanujan's well-known formulas are based upon the evaluation of moments for Maclaurin-type power series. In particular, Ramanujan's Master Theorem

$$\int_0^\infty x^{s-1}\left(\sum_{i=0}^\infty \frac{(-x)^i}{i!} f(i)\right) dx = \Gamma(s) f(-s) \tag{1}$$

was frequently used by Ramanujan to construct closed-form evaluations of definite integrals and infinite series [1,3]. In the case whereby the series in (1) is hypergeometric, we obtain an evaluation of the following form [17].

$$\int_0^\infty x^{s-1} {}_pF_q\left[\begin{matrix} c_1, c_2, \ldots, c_p \\ d_1, d_2, \ldots, d_q \end{matrix} \,\middle|\, -x\right] dx$$
$$= \Gamma(s) \frac{\Gamma(c_1 - s)\cdots\Gamma(c_p - s)\Gamma(d_1)\cdots\Gamma(d_q)}{\Gamma(c_1)\cdots\Gamma(c_p)\Gamma(d_1 - s)\cdots\Gamma(d_q - s)}$$

There are many analogues and variants of classical versions of Ramanujan's Master Theorem [2,14,18,19], and there are a number of different formulations of this theorem which all provide evaluations for the moments of sufficiently well-behaved functions defined in terms of power series, as in the identity whereby

$$\int_0^\infty x^{-n-1}\left(\sum_{k=0}^\infty (-1)^k a(k) x^k\right) dx = -\frac{\pi}{\sin \pi n} a(n).$$

Given the wide-ranging applications of Ramanujan's Master Theorem, we are inspired to construct new variants and analogues of this result. Instead of dealing specifically with moments of power series, we consider the problem of constructing Ramanujan-like identities for integrals over products consisting of a parametric power of an input function with another input mapping given as a factor in the integrand.

Our present article is inspired in large part by recent research that has dealt with the use of symbolic evaluations for moment-like integrals to construct new closed-form evaluations for infinite series and definite integrals. New results on the construction of Ramanujan-like series for $\frac{1}{\pi}$ are given in [8] that are based on the evaluation of the moments of the elementary function $\frac{\ln(1-x^2)}{\sqrt{1-x^2}}$ on the domain $[0, 1)$, and the integration technique from [8] is heavily utilized in [7, 9]. A somewhat similar approach is given in [12], in which the moments of the elementary expression $\arcsin(x)\ln(x)$ on $(0, 1]$ are used to prove new rational approximations for constants involving $\frac{1}{\pi}$, as in the intriguing equality

$$\frac{4G - 12\ln 2 + 6}{\pi} = \sum_{n=0}^\infty \frac{\binom{2n}{n}^2 H_{2n}}{16^n (2n-1)^2},$$

letting G denote Catalan's constant. In [23], the evaluation of moments of the form

$$\int_0^1 x^{2k-1} \cdot \ln x \cdot {}_3F_2\left[\begin{matrix} \frac{1}{2}, 2, 2 \\ \frac{p+3}{2}, \frac{p+4}{2} \end{matrix} \,\middle|\, x^{2k}\right] dx$$

is explored. Integrals of the form

$$\int_a^b x^n \ln^m (\sin(x)) \, dx$$

are applied in [20] to establish some new results on the log-sine integral function, and a variety of moment formulas are used in [21] to construct new evaluations for series involving the Riemann zeta function. For example, symbolic formulas for moments of the following forms are applied in [21], motivating a full exploration on the application of the class of moments in Lemma 1 below in the construction of new infinite series evaluations.

$$\int_0^{\pi z} x^p \cot(x) \, dx \qquad \int_0^{2\pi z} x^p \cdot \mathrm{Cl}_m(x) \, dx \qquad \int_0^{z} x^p \psi(x) \, dx$$

A formula for integrals involving $\frac{x \ln^{2m-1} x}{1-x}$ is used in [24] in the symbolic computation of some harmonic summations, and a variety of new integral evaluations and series evaluations are proved in [6] by substituting logarithmic functions into Maclaurin series expansions and integrating term-by-term; this illustrates the utility of exploring variants of (1) that make use of non-standard moments such as logarithmic moments, as elaborated below.

The crux of the technique from [6] is based on the observation that logarithmic moments for rational expressions of the form $\frac{1}{1 \pm x^n}$ often may be expressed in a simple way with a zeta-type function and a gamma-type expression as a factor, so that if logarithmic expressions are substituted into the Maclaurin series for trigonometric or hyperbolic functions, if we integrate term-by-term, we often obtain a series involving the Riemann zeta function that Mathematica can evaluate directly. Through the use of non-standard moments such as logarithmic moments, the "signature method" given by Lemma 1 below, which is something of an analogue of Ramanujan's Master Theorem, often leads us to one-line proofs of new integral evaluations, such as

$$\int_0^1 \frac{(x-1)^2}{x(x+1)\ln^2(x)} \, dx = 12\ln(A) - 1 - \frac{4\ln(2)}{3}, \tag{2}$$

letting A denote the Glaisher–Kinkelin constant. We present interesting applications of non-standard moments in the upcoming section through the evaluation of difficult series containing harmonic-type numbers, using Lemma 1, and by using non-standard moments to evaluate new classes of ${}_3F_2(1)$ series.

Letting $e_1(x)$ and $e_2(x)$ be functions on a given domain (a, b) whereby the parameters for this interval are not necessarily finite, the crux of our strategy for generating and classifying Ramanujan-like series and integral formulas involves the evaluation of definite integrals of the form $\int_a^b (e_1(x))^n e_2(x) \, dx$ for

$n \in \mathbb{N}_0$. For the purposes of this article, our strategy may be summarized with the following observation.

Lemma 1 *Letting* $(e_1(x))^n e_2(x)$ *be integrable on* (a, b) *for* $n \in \mathbb{N}_0$*, letting* $I_a^b(e_1, e_2)(n) = I(n) = \int_a^b (e_1(x))^n e_2(x)\, dx$ *for* $n \in \mathbb{N}_0$*, and writing*

$$e_3(x) = \sum_{n=-\infty}^{\infty} x^n f(n), \tag{3}$$

then the equality

$$\int_a^b e_3(e_1(x)) e_2(x)\, dx = \sum_{n=-\infty}^{\infty} I(n) f(n) \tag{4}$$

holds, provided that the series in (3) *and both sides of* (4) *are well-defined.*

Proof This follows immediately by inputting $e_1(x)$ into both sides of (3), multiplying both sides of resultant equation by $e_2(x)$, integrating both sides of the resulting equality over (a, b), and then reversing the order of summation and integration.

Often, if a state-of-the-art computer algebra system (CAS) can evaluate one side of (4), without being able to provide a symbolic evaluation for the other side of the equation, then we obtain a new integral or series formula. The expression $(e_1, e_2, e_3)_{a,b,f}$ (or more compactly, (e_1, e_2, e_3), if the choice of a, b, and f is somehow tacit or obvious) is thus something of a "proof signature" for the evaluation in (4). This approach is reminiscent of, and is partly inspired by, the one-line proofs afforded by the famous Wilf–Zeilberger method from [22]. To illustrate the utility of Lemma 1, we offer an informal one-line proof of the below formulation of the famous Basel problem.

Basel problem: Prove that $\zeta(2) = \frac{\pi^2}{6}$.

Proof: $\left(\sin(x), 1, \left(\sin^{-1}(x)\right)^2\right)$. $\square$

Explicitly, we have that $I(2n) = \int_0^{\pi} (\sin(x))^{2n}\, dx = \frac{\sqrt{\pi}\,\Gamma\left(n+\frac{1}{2}\right)}{\Gamma(n+1)}$ and

$$\left(\sin^{-1}(x)\right)^2 = \frac{1}{2} \sum_{n=1}^{\infty} \frac{(2x)^{2n}}{n^2 \binom{2n}{n}},$$

so that $\frac{\pi^3}{12} = \frac{1}{2} \sum_{n=1}^{\infty} \frac{4^n \frac{\sqrt{\pi}\,\Gamma\left(n+\frac{1}{2}\right)}{\Gamma(n+1)}}{n^2 \binom{2n}{n}}$. Equivalently, $\zeta(2) = \frac{\pi^2}{6}$. This illustrates that the integration method given by Lemma 1 provides us with an efficient and novel way of significantly simplifying proofs of known closed-form formulas.

As noted above, our present article is inspired in part by the integration methods given in [6,8,12]. For a sequence $(s_n)_{n\in\mathbb{N}}$, the fundamental lemma from [8] deals with the integration of infinite series of the form

$$\sum_{n=0}^{\infty}(-1)^n\binom{\frac{1}{2}}{n}s_n\cdot\frac{x^{2n}\ln\left(1-x^2\right)}{\sqrt{1-x^2}} \tag{5}$$

to produce new Ramanujan-like series for $\frac{1}{\pi}$ involving harmonic numbers. For example, we find that the new evaluation

$$\sum_{n=1}^{\infty}\frac{\binom{2n}{n}^2 H_n}{16^n(2n-1)}=\frac{8\ln(2)-4}{\pi} \tag{6}$$

provided in [8] follows immediately from the main lemma from [8] simply by letting $s_n=1$ in (5), and applying a suitable integration operator to (5). So, in other words, the elegant evaluation provided in (6) is equivalent to the proof signature

$$\left(x,\frac{\ln\left(1-x^2\right)}{\sqrt{1-x^2}},\sqrt{1-x^2}\right). \tag{7}$$

This illustrates how Lemma 1 may be regarded as a vast generalization of the integration method from [8], and this leads us to the following question.

Question 1 How can we obtain new classes of Ramanujan-like formulas, with the use of variants of the proof signature in (7)?

The above proof of Lemma 1 is reminiscent of some of Ramanujan's work concerning the application of his Master Theorem. For example, as noted in [3], letting ϕ denote a reasonably well-behaved function, the following corollary of Ramanujan's Master Theorem may be proved by writing $\cos(nx)$ as a Maclaurin series, switching the order of summation and integration, and then applying the Master Theorem, term-by-term.

Corollary 1 *The identity*

$$\int_0^{\infty}\sum_{k=0}^{\infty}\frac{\phi(k)(-x)^k}{k!}\cos(nx)\,dx=\sum_{k=0}^{\infty}\phi(-2k-1)(-n^2)^k$$

holds [3].

Given the interesting Ramanujan-like series results from [8], and given how minor alterations in the corresponding proof signatures for these results provide us with new classes of series that cannot be computed symbolically with current CAS software, this strongly motivates us to continue to explore the application of our Ramanujan-like integration method from Lemma 1.

2 Applications of the signature method

Lemma 1 may be regarded as something of a heuristic tool that motivates the development of new results on closed-form evaluations based on the use of creative combinations of functions in the integrand in (4). We illustrate this idea with a variety of different examples, starting with a variant of the integration methods from [8,9,12] for constructing binomial-harmonic series for $\frac{1}{\pi}$.

2.1 New harmonic summations

The fundamental lemma from Section 1 is very useful in the exploration of variants of the integration technique from [8]. Through a clever application of the symbolic form for the moments of the mapping $\alpha\colon [0,1) \to \mathbb{R}$ whereby $x \mapsto \frac{\ln(1-x^2)}{\sqrt{1-x^2}}$, a very general technique for constructing binomial-harmonic summations for $\frac{1}{\pi}$ is put forth in [8]. It seems natural to consider the symbolic evaluation of moments on variations of the aforementioned transformation α. Since Mathematica is able to directly evaluate the series

$$\sum_{n=1}^{\infty} \frac{\binom{2n}{n}\binom{4n}{2n} H_n}{2^{6n}(n+1)} = \frac{8}{3}\left(2 - \frac{\sqrt{2}\ln(64)}{\pi}\right), \tag{8}$$

in consideration of the recent research from [10,11] concerning the evaluation of series involving binomial products of the form $\binom{2n}{n}\binom{4n}{2n}$, we are inspired to try to generalize (8). From the results in [7,9,8], it would seem that as a natural place to start, consideration of the problem of evaluating series of the form

$$\sum_{n=1}^{\infty} \frac{\binom{2n}{n}\binom{4n}{2n} H_n}{2^{6n}(n+z)}$$

for $z \in \mathbb{Z}_{>0}$ would be suitable. However, as elaborated below, the problem of evaluating

$$\sum_{n=1}^{\infty} \frac{\binom{2n}{n}\binom{4n}{2n} H_n}{2^{6n}(n+2)} \tag{9}$$

turns out to be surprisingly difficult, compared with how Mathematica easily provides a closed-form formula for the series in (8). If we apply a re-indexing argument to express the summation in (8) in terms of (9), we obtain the recalcitrant ${}_pF_q$ series

$${}_4F_3\left[\begin{matrix} \frac{1}{4}, \frac{3}{4}, 1, 1 \\ 2, 2, 3 \end{matrix} \,\middle|\, 1\right]$$

as well as the difficult harmonic series

$$\sum_{n=0}^{\infty} \left(\frac{1}{64}\right)^n \frac{\binom{2n}{n}\binom{4n}{2n} H_n}{(n+1)^2}.$$

Through a direct application of the signature method given by the above lemma, we obtain the following closed-form evaluation for the seemingly recalcitrant series in (9).

Theorem 1 *The equality*

$$\sum_{n=1}^{\infty} \left(\frac{1}{64}\right)^n \frac{\binom{2n}{n}\binom{4n}{2n} H_n}{n+2} = \frac{256}{105} + \frac{128\sqrt{2}}{105\pi} - \frac{304\sqrt{2}\ln(2)}{35\pi}$$

holds.

Proof $\left(x^2, \frac{\ln\left(1-x^2\right)}{\sqrt{x}\sqrt[4]{1-x^2}}, \frac{\sqrt[4]{1-x}(x+4)-4}{x^2}\right)$.

Direct applications of Lemma 1 also provide us with the following new results, through the use of moments of the form $\int_0^1 \frac{x^m \ln\left(1-x^2\right)}{\sqrt[4]{1-x^2}}\, dx$.

$$\sum_{n=1}^{\infty} \left(\frac{1}{64}\right)^n \frac{\binom{2n}{n}\binom{4n}{2n} H_n}{n+3} = \frac{16384}{10395} + \frac{4352\sqrt{2}}{3465\pi} - \frac{21136\sqrt{2}\ln(2)}{3465\pi}$$

$$\sum_{n=1}^{\infty} \left(\frac{1}{64}\right)^n \frac{\binom{2n}{n}\binom{4n}{2n} H_n}{n+4} = \frac{262144}{225225} + \frac{795776\sqrt{2}}{675675\pi} - \frac{356656\sqrt{2}\ln(2)}{75075\pi}$$

$$\sum_{n=1}^{\infty} \left(\frac{1}{64}\right)^n \frac{\binom{2n}{n}\binom{4n}{2n} H_n}{n+5} = \frac{67108864}{72747675} + \frac{21632512\sqrt{2}}{19840275\pi} - \frac{94907536\sqrt{2}\ln(2)}{24249225\pi}$$

There are many natural ways of going about generalizing the series evaluations listed above, as suggested in the symbolic form for the below series, which Mathematica 11 cannot evaluate.

Theorem 2 *The evaluation*

$$\sum_{n=0}^{\infty} \left(\frac{1}{27}\right)^n \frac{\binom{2n}{n}\binom{3n}{n} H_n}{3n-4} = \frac{252\sqrt{3}\ln(3) - 279\sqrt{3}}{128\pi}$$

holds.

Proof $\left(x^2, \frac{\ln\left(1-x^2\right)}{\sqrt[3]{x}\sqrt[3]{1-x^2}}, \frac{1}{4}\sqrt[3]{1-x}(3x+1)\right)$.

Using similar signature tuples, we obtain the following new results.

$$\sum_{n=0}^{\infty} \left(\frac{1}{27}\right)^n \frac{\binom{2n}{n}\binom{3n}{n} H_n}{3n-5} = \frac{360\sqrt{3}\ln(3) - 207\sqrt{3}}{500\pi}$$

$$\sum_{n=0}^{\infty} \left(\frac{1}{27}\right)^n \frac{\binom{2n}{n}\binom{3n}{n} H_n}{3n-7} = \frac{28476\sqrt{3}\ln(3) - 33363\sqrt{3}}{21952\pi}$$

$$\sum_{n=0}^{\infty} \left(\frac{1}{27}\right)^n \frac{\binom{2n}{n}\binom{3n}{n} H_n}{3n-8} = \frac{69480\sqrt{3}\ln(3) - 43461\sqrt{3}}{128000\pi}$$

Remark 1 The above results can be generalized using the main lemma from our article together with the identity whereby $\int_0^1 \frac{\ln\left(1-x^k\right)x^\alpha}{(1-x^k)^{1/m}}\,dx$ evaluates as

$$\frac{\Gamma\left(\frac{m-1}{m}\right)\Gamma\left(\frac{\alpha+1}{k}\right)\left(H_{-\frac{1}{m}}-H_{\frac{-k+m+m\alpha}{km}}\right)}{k\Gamma\left(\frac{m-1}{m}+\frac{\alpha+1}{k}\right)}$$

for parameters k, m, and α satisfying the following conditions: $\Re\left(\frac{1}{m}\right) < 1$, $\Re(\alpha+k) > -1$, and $\Re(k) > 0$. For example, if we note that the expression

$$\ln\left(1-x^k\right)\cdot\sum_{i=0}^{\infty}\frac{\binom{\frac{1}{m}}{i}(-x^k)^i}{\left(1-x^k\right)^{1/m}}$$

reduces to $\ln\left(1-x^k\right)$, through the signature method we find that

$$\sum_{i=0}^{\infty}\frac{(-1)^i\Gamma\left(i+\frac{1}{k}\right)H_{i+\frac{1}{k}-\frac{1}{m}}}{\Gamma(1+i)\Gamma\left(1+i+\frac{1}{k}-\frac{1}{m}\right)\Gamma\left(1-i+\frac{1}{m}\right)}$$

may be evaluated as

$$\frac{km\sin\left(\frac{\pi}{m}\right)\left(H_{\frac{1}{k}}+H_{-\frac{1}{m}}\right)}{\pi}.$$

Series involving products of harmonic-type numbers are among the central objects of study in our article, since Lemma 1 seems to naturally given rise to simple ways of evaluating seemingly recalcitrant series of this form, especially when used in conjunction with known generating functions for sequences involving harmonic-like numbers. The study of series involving products of harmonic numbers is a very interesting area [15]. The unexpected [15] nonlinear harmonic series evaluation

$$\sum_{n\in\mathbb{N}}\left(\frac{H_n}{n}\right)^2=\frac{17\pi^4}{360} \tag{10}$$

had been discovered experimentally by Enrico Au-Yeung, and was later proved in [5]. The identity in (10) is generalized in [15], in which an evaluation for summations of the form

$$\sum_{n=1}^{\infty}\frac{H_n}{n}\cdot\frac{H_{n+k}}{n+k}$$

is proved. It appears that there have not previously been any known summations for $\frac{1}{\pi}$ containing products of harmonic-type numbers. This motivates us to apply Lemma 1, together with integration techniques as in [8,9,12], to construct new series of this form. First, our strategy is to exploit the closed-form evaluation for the ordinary generating function for $(\binom{2n}{n}H_n : n\in\mathbb{N})$ given in [13], as suggested in the proof signature for the new evaluation given below. Our use of the moments of $\ln(x)\sqrt{1-x^2}$ with regard to the following proof signature is directly inspired by the integral operator $T_{\ln,\arcsin}$ from [12].

Theorem 3 *The evaluation*

$$\sum_{i=0}^{\infty} \frac{\binom{2i}{i}^2 H_i H'_{2i}}{16^i (i+1)} = \frac{16G - 16\ln^2(2) - 16\ln(2)}{\pi} + 4 + \pi - 8\ln(2)$$

holds.

Proof $\left(x^2, \ln(x)\sqrt{1-x^2}, \frac{\ln\left(\frac{\sqrt{1-x}+1}{2\sqrt{1-x}}\right)}{\sqrt{1-x}}\right)$.

Remark 2 Through the use of the above proof signature, one would need to also evaluate the series

$$\sum_{n=0}^{\infty} \frac{\binom{2n}{n}^2 H_n}{16^n (n+1)^2},$$

which was recently evaluated in closed form in [7]. Also, we need the evaluation of the series

$$\sum_{n=0}^{\infty} \frac{\binom{2n}{n}^2 H_{2n}}{16^n (n+1)^2},$$

given in [9] in the application of the following proof signature to prove the below theorem.

Theorem 4 *The evaluation*

$$\sum_{n=1}^{\infty} \frac{\binom{2n}{n}^2 H_{2n} H'_{2n}}{16^n (n+1)} = \pi - 4\ln(2) + \frac{4 + 8G - 8\ln(2) - 12\ln^2(2)}{\pi}$$

holds.

Proof $\left(x^2, \sqrt{1-x^2}\ln(x), \frac{\ln\left(\frac{\sqrt{1-x}+1}{2(1-x)}\right)}{\sqrt{1-x}}\right)$.

The following result is especially useful.

Theorem 5 *The equality*

$$\sum_{n=1}^{\infty} \frac{\binom{2n}{n}^2 H_n^2}{16^n (n+1)} = \frac{64\ln^2(2) - 32G}{\pi} - \frac{10\pi}{3} + 16\ln(2) \tag{11}$$

holds.

Proof $\left(x^2, \sqrt{1-x^2}\ln\left(1-x^2\right), \frac{\ln\left(\frac{1}{2}\left(\frac{1}{\sqrt{1-x}}+1\right)\right)}{\sqrt{1-x}}\right)$.

Corollary 2 *The equality*

$$\sum_{n=1}^{\infty} \frac{\binom{2n}{n}^2 H_n H_{2n}}{16^n (n+1)} = 4 - \frac{7\pi}{3} + 8\ln(2) + \frac{48\ln^2(2) - 16G - 16\ln(2)}{\pi}$$

holds.

Proof This follows immediately from Theorem 5 and Theorem 3.

Corollary 3 *The equality*

$$\sum_{n=1}^{\infty} \frac{\binom{2n}{n}^2 H_{2n}^2}{16^n (n+1)} = \frac{4 - 8G + 36\ln^2(2) - 24\ln(2)}{\pi} + 4 - \frac{4\pi}{3} + 4\ln(2)$$

holds.

Proof This follows immediately from the preceding corollary together with Theorem 4.

Applying the following proof signature using the main lemma from our present article is quite involed, and requires a number of $\frac{1}{\pi}$ series from [7] and [8], but we omit details for the sake of brevity.

Theorem 6 *The series*

$$\sum_{n=1}^{\infty} \frac{\binom{2n}{n}^2 H_n^2}{16^n (n+2)}$$

is equal to

$$\frac{28}{9} + \frac{64\ln(2)}{9} - \frac{50\pi}{27} + \frac{16 - 128G - 128\ln(2) + 320\ln^2(2)}{9\pi}.$$

Proof $\left(x^2, x^2\sqrt{1-x^2}\ln\left(1-x^2\right), \frac{\ln\left(\frac{\sqrt{1-x}+1}{2\sqrt{1-x}}\right)}{\sqrt{1-x}}\right)$.

We may evaluate series of the form

$$\sum_{n=1}^{\infty} \frac{\binom{2n}{n}^2 H_n^2}{16^n (n+z)}$$

for $z \in \mathbb{Z}_{>0}$ in a recursive fashion, and we may similarly generalize the preceding theorems.

As discussed above, it is interesting how minor modifications to the tuple in (7) can lead to very different evaluations, compared to the rational approximation for $\frac{8\ln(2)-4}{\pi}$ afforded to us by (6). For example, using the signature

$$\left(x, \frac{\ln(1-x)}{\sqrt{1-x}}, \sqrt{1-x^2}\right), \tag{12}$$

we find that the sum

$$\sum_{i=0}^{\infty} \frac{4^i \binom{2i}{i} H_{2i+\frac{1}{2}}}{(2i-1)(4i+1)\binom{4i}{2i}}$$

is equal to

$$\frac{2}{9}\left(7 - 8\sqrt{2} + 3\left(5\sqrt{2}-1\right)\ln(2) + 6\sqrt{2}\ln\left(\sqrt{2}-1\right)\right).$$

This formula is noticeably different compared to the formulas provided by the signature in (7). As suggested in the below theorem, further manipulations of the final entry in the tuple in (12) lead us to interesting new results.

Theorem 7 *The series*

$$\sum_{i=0}^{\infty} \frac{\binom{2i}{i}}{\binom{4i}{2i}} \frac{4^i H_{2i+\frac{1}{2}}}{(2i-1)^2}$$

is equal to

$$-\frac{2}{9}\left(83 - 88\sqrt{2} + 9\left(10\sqrt{2}-3\right)\ln(2) + 36\sqrt{2}\ln\left(\sqrt{2}-1\right)\right).$$

Proof $\left(x, \frac{\ln(1-x)}{\sqrt{1-x}}, -\frac{\sqrt{1-x^2}+3x\sin^{-1}(x)}{\sqrt{1-x}}\right)$.

The above theorem is interesting because modern CAS software cannot compute this sum, and it seems that integral formulas for generalized harmonic numbers also cannot be applied to evaluate the series in Theorem 7. Moreover, it appears that there is not very much mathematical literature on series involving binomial quotients of the following form.

$$\binom{2i}{i}\bigg/\binom{4i}{2i}, \ i \in \mathbb{N}_0 \tag{13}$$

Theorem 7 is vastly generalized through the following new integration result.

Lemma 2 *For a sequence* $(f_n : n \in \mathbb{N})$*, the series*

$$\sum_{n=1}^{\infty} \frac{\binom{2n}{n}}{\binom{4n}{2n}} \cdot \frac{(-4)^n H_{2n+\frac{1}{2}}}{4n+1} \cdot f_n$$

is equal to

$$-2\int_0^1 \sum_{n=0}^{\infty} \binom{-\frac{1}{2}}{n} f_n \left(x^2-1\right)^{2n} \ln(x)\,dx -$$
$$\sqrt{\pi}\ln(2)\sum_{n=1}^{\infty} \frac{2^{2n} f_n \Gamma\left(n+\frac{1}{2}\right)}{\Gamma\left(\frac{1}{2}-n\right)\Gamma\left(2n+\frac{3}{2}\right)} - 2f_0,$$

provided that the above expressions are all well-defined.

Proof This follows immediately from the symbolic evaluation of the antiderivative of $(x^2-1)^{2n}\ln(x)$, for a parameter n.

For example, if we simply let

$$f_n = (-1)^n \frac{4n+1}{(2n-3)^2}$$

for elements n in $\mathbb{N}_0$, then we find that the infinite series

$$\sum_{n=1}^{\infty} \frac{\binom{2n}{n}}{\binom{4n}{2n}} \frac{4^n H_{2n+\frac{1}{2}}}{(2n-3)^2} \tag{14}$$

is equal to

$$\frac{4131664\sqrt{2}-3783176}{165375}+\frac{1544\ln(2)-4112\sqrt{2}\ln(2)-2056\sqrt{2}\ln\left(2-\sqrt{2}\right)}{225}.$$

This evaluation is interesting because if a re-indexing argument is applied to obtain the series in Theorem 7, this would require the evaluation of the following very difficult ${}_pF_q$ series derived from hypergeometric sums involving binomial quotients as in (13). Mathematica 11 is not able to evaluate these difficult expressions, emphasizing our previous point on the lack of mathematical resources on hypergeometric series involving quotients of the form in (13).

$${}_4F_3\left[\begin{matrix}\frac{1}{2},1,\frac{7}{4},\frac{5}{2}\\ \frac{9}{4},\frac{11}{4},\frac{11}{4}\end{matrix}\,\middle|\,1\right]$$

$${}_4F_3\left[\begin{matrix}\frac{3}{2},2,\frac{11}{4},\frac{7}{2}\\ \frac{13}{4},\frac{15}{4},\frac{15}{4}\end{matrix}\,\middle|\,1\right]$$

The above evaluation for the series in (14) is also equivalent to the following proof signature:

$$\left(x^2-1,\ln(x),42x^3\sin^{-1}(x)+23\sqrt{1-x^2}x^2+\sqrt{1-x^2}\right),$$

illustrating the versatility of the signature method. Through direct applications of Lemma 2, we obtain the following new results, as well as many extensions of the evaluation from Lemma 1 given by the tuple provided in (12).

$$\sum_{n=1}^{\infty}\frac{\binom{2n}{n}}{\binom{4n}{2n}}\frac{4^n H_{2n+\frac{1}{2}}}{(2n-5)^2}=$$
$$\frac{312851954224\sqrt{2}-279413817016}{11345882625}+$$
$$\frac{2042776\ln(2)-5609648\sqrt{2}\ln(2)-2804824\sqrt{2}\ln\left(2-\sqrt{2}\right)}{297675}$$

$$\sum_{n=1}^{\infty}\frac{\binom{2n}{n}}{\binom{4n}{2n}}\frac{4^n H_{2n+\frac{1}{2}}}{(2n-7)^2}=$$
$$\frac{892093651361104\sqrt{2}-786424570733896}{30466216155375}+$$
$$\frac{4655127896\ln(2)-12916662064\sqrt{2}\ln(2)-6458331032\sqrt{2}\ln\left(2-\sqrt{2}\right)}{676350675}$$

A similar approach leads us to the following evaluation.

Theorem 8 *The evaluation*

$$\sum_{i=0}^{\infty}\frac{\binom{4i}{2i}}{\binom{2i}{i}}\cdot\frac{H_{i+\frac{1}{2}}}{4^i(2i+1)^2}=16-8\sqrt{2}+4\left(3\sqrt{2}-2\right)\ln(2)-8\sqrt{2}\ln\left(1+\sqrt{2}\right)$$

holds.

Proof $\left(x, \frac{\ln\left(1-x^2\right)}{\sqrt{1-x^2}}, \sqrt{x+1}\right)$.

It should be noted that deriving the evaluation presented in Theorem 8 using the above proof signature, together with the master lemma from Section 1, is nontrivial, although the proof signature

$$\left(x, \frac{\ln\left(1-x^2\right)}{\sqrt{1-x^2}}, \sqrt{x+1}\right) \tag{15}$$

provides us with a very succinct summary of how to go about this derivation. The complexity of the following rigorous proof of Theorem 8 compared with the informal proof signature displayed in (15) is illustrative of the power and versatility of the integration method put forth in our article.

Detailed proof of Theorem 8: Through a direct application of Lemma 1 with regard to the proof signature from (15), we find that

$$\pi \ln(2) - \frac{16}{15} {}_3F_2\!\left[\begin{matrix} 1, 1, \frac{3}{2} \\ \frac{7}{4}, \frac{9}{4} \end{matrix} \,\middle|\, 1\right] \tag{16}$$

must be equal to

$$\sum_{n=1}^{\infty} \frac{\binom{\frac{1}{2}}{n}\left(-\left(2^{-1-n} n\pi \Gamma(n)\left(H_{\frac{n}{2}} + \ln(4)\right)\right)\right)}{\Gamma\left(1+\frac{n}{2}\right)^2}. \tag{17}$$

Mathematica is not able to evaluate the ${}_3F_2$ series displayed in (16), even through the use of the `FunctionExpand` command. The evaluation of hypergeometric series with quarter-integer parameters was recently explored in [10, 11], and it turns out that a clever variant of some of the proof techniques from [10,11] can be applied to obtain a simple closed-form evaluation for (16).

One of the key observations from [10] that is used in the closed-form evaluation of a ${}_3F_2$ series with quarter-integer parameters is that Pochhammer products of the form

$$\left(\frac{2n+1}{4}\right)_i \left(\frac{2n+3}{4}\right)_i$$

can be expressed in a very natural way in terms of central binomial coefficients of the form $\binom{4i}{2i}$, so that integral formulas for binomial expressions of this form can be substituted into the appropriate hypergeometric summand in order to express the corresponding series as a definite integral. With regard to the lower parameters of the ${}_3F_2(1)$ series from (16), we have that

$$\left(\frac{7}{4}\right)_i \left(\frac{9}{4}\right)_i = \frac{1}{15} 64^{-i}(4i+1)(4i+3)(4i+5)(2i)!\binom{4i}{2i},$$

which shows us that

$${}_3F_2\!\left[\begin{matrix} 1, 1, \frac{3}{2} \\ \frac{7}{4}, \frac{9}{4} \end{matrix} \,\middle|\, 1\right] = \sum_{i=0}^{\infty} \frac{15\ 16^i(2i+1)}{(4i+1)(4i+3)(4i+5)\binom{4i}{2i}}. \tag{18}$$

Through the use of the beta function, we find that the equality

$$\frac{1}{\binom{4i}{2i}} = \int_0^1 (4i+1)(1-t)^{2i} t^{2i} \, dt \tag{19}$$

holds for $i \in \mathbb{N}_0$. Replacing the factor $\frac{1}{\binom{4i}{2i}}$ in the summand in (18) with the definite integral in (19), and reversing the order of summation and integration, we find that the summations in (18) are equal to the following seemingly recalcitrant integral.

$$\int_0^1 \frac{1}{256\left((-1+t)^2t^2\right)^{5/4}} \Bigg(15\big(-12\sqrt[4]{(-1+t)^2t^2} + \left(3+4\sqrt{(-1+t)^2t^2}\right)\tan^{-1}\left(2\sqrt[4]{(-1+t)^2t^2}\right) + \left(3-4\sqrt{(-1+t)^2t^2}\right)\tanh^{-1}\left(2\sqrt[4]{(-1+t)^2t^2}\right)\big)\, dt \Bigg)$$

Amazingly, *Mathematica* is able to provide a simple closed-form evaluation for the above integral, yielding the following elegant evaluation.

$${}_3F_2\left[\begin{matrix} 1, 1, \frac{3}{2} \\ \frac{7}{4}, \frac{9}{4} \end{matrix} \,\middle|\, 1\right] = -\frac{15}{8}\left(\sqrt{2}\tanh^{-1}\left(\frac{2\sqrt{2}}{3}\right) - 4\right).$$

Using the above evaluation, and by bisecting the series in (17), we obtain the desired result. □

Using the signature method, we can easily derive many natural variants of the above theorem that cannot be evaluated otherwise, as suggested below.

Theorem 9 *The evaluation*

$$\sum_{n=0}^{\infty} \frac{\binom{4n}{2n}}{\binom{2n}{n}} \cdot \frac{H'_{2n}}{4^n(2n+1)(4n-1)} = \frac{\sqrt{2}\ln(2) + 4\ln(2) + 4\ln\left(\sqrt{2}-1\right)}{3}$$

holds.

Proof $\left(x^2-1, \ln\left(1-x^2\right), \sqrt[4]{x+1}\cos\left(\frac{1}{2}\tan^{-1}\left(\sqrt{x}\right)\right)\right)$.

Through a simple application of Lemma 1, we obtain a very general method of evaluating difficult series involving alternating harmonic numbers and inverses of central binomial coefficients, as suggested by the following theorems.

Theorem 10 *Letting ϕ denote the Golden Ratio, we have that the equation*

$$\sum_{n=0}^{\infty} \frac{(-1)^{n+1} H'_{2n}}{(2n+1)\binom{2n}{n}} = \left(\frac{1}{6} - \frac{1}{5\sqrt{5}}\right)\pi^2 - \ln(\phi)\ln\left(2+\sqrt{5}\right)$$

holds.

Proof $\left((1-x)x, \ln(1-x), \frac{1}{x+1}\right)$.

Theorem 11 *The series*

$$\sum_{n=0}^{\infty} \frac{(-1)^n H'_{2n}}{(n+1)\binom{2n}{n}}$$

is equal to

$$\frac{4\zeta(3)}{5} + \frac{\left(\ln\left(\sqrt{5}+2\right)-8\right)\ln^2(\phi) - 3\ln^3(\phi)}{2} + \frac{2}{75}\pi^2\left(3\sqrt{5} + 5\ln\left(\sqrt{5}-2\right)\right).$$

Proof $\left((1-x)x, \ln(1-x), \frac{\ln(x+1)}{x} - \frac{2}{x+1}\right)$.

It is remarkable how such intricate symbolic forms come out of such simple proof signatures, and are numerically equivalent to such simple and elegant harmonic series. The study of the evaluation of series containing harmonic-type numbers and inverses of binomial coefficients of the form $\binom{2n}{n}$ is an interesting subject in its own right.

The new evaluation provided below is inspired by the nonlinear harmonic sums presented in [5,15].

Theorem 12 *The equality*

$$\sum_{i=1}^{\infty} \frac{H_i H_{i+\frac{1}{2}}}{(2i+1)(2i+3)} = \frac{7\zeta(3)}{4} + \frac{\pi^2}{12} + 2\ln^2(2) - 2\ln(2) - \frac{1}{4}\pi^2\ln(2) \tag{20}$$

holds.

Proof $\left(x, \sqrt{1-x}\ln(1-x), \frac{\ln\left(\frac{\sqrt{1-x}+1}{2\sqrt{1-x}}\right)}{\sqrt{1-x}}\right)$.

Without having already known that the above proof signature may be used to prove the above theorem, it would not be obvious at all as to how to evaluate the series in this theorem. We find that partial fraction decomposition cannot be applied in any kind of trivial way to evaluate this series, since if we rewrite the summation in (20) as

$$\sum_{i=1}^{\infty} H_i H_{i+\frac{1}{2}} \left(\frac{1}{2(2i+1)} - \frac{1}{2(2i+3)}\right),$$

we cannot rewrite the above expression as two separate series based on the above decomposition, since neither of these series is convergent. Similarly, we find that current CAS software cannot evaluate the generating function obtained by replacing one of the four main factors in the summand in (20) with x^i. For example, it is not at all clear as to how to evaluate

$$\sum_{i=1}^{\infty} \frac{H_i x^i}{(2i+1)(2i+3)}$$

so as to be able to make use of a suitable integral formula for the sequence $(H_{i+\frac{1}{2}} : i \in \mathbb{N}_0)$.

Using variants of the above proof signature, we obtain many similar results. For example, using Chen's generating function $G(H_n C_n; x)$ for $(H_n C_n : n \in \mathbb{N}_0)$ presented in [13], we find that the following theorem holds.

Theorem 13 *The series*

$$\sum_{n=1}^{\infty} \frac{H_n H_{n+\frac{1}{2}}}{(2n+1)(2n+3)(2n+5)}$$

is equal to

$$\frac{7\zeta(3)}{16} - \frac{1}{3} + \frac{\pi^2}{72} + \frac{\ln^2(2)}{3} - \frac{\ln(2)}{36} - \frac{1}{16}\pi^2 \ln(2).$$

Proof $\left(x, \sqrt{1-x}\ln(1-x), G(H_n C_n; x)\right)$.

To prove an analogue of the above theorem for products of the form $H_{2i} H_{i+\frac{1}{2}}$, we at first need a few preliminary results.

Lemma 3 *The evaluation*

$$\sum_{n=0}^{\infty} \frac{H_{2n}}{(2n-3)(2n-1)^2} = \frac{1}{96}\left(-63\zeta(3) - 20 - 6\pi^2 + 56\ln(2)\right) \tag{21}$$

holds.

Proof $\left(x^2-1, \ln(x), \frac{1}{3}\sqrt{x+1}(2x-1) + \sqrt{x}\sinh^{-1}\left(\sqrt{x}\right)\right)$.

Remark 3 The above evaluation is interesting in its own right, and in a forthcoming article, we explore in full generality the evaluation of series with summands given by products of even-indexed harmonic numbers and rational expressions. A closed-form evaluation for the equivalent series

$$\sum_{n \in \mathbb{N}_0} \frac{H_{2n}}{(2n-1)(2n+1)^2}$$

is given in [12], through a simple application of the integral operator $T_{\ln,\arcsin}$ from [12]. Simple variants of the above proof signature may be used to obtain the following results.

$$\sum_{n=0}^{\infty} \frac{H_{2n}}{(2n-5)(2n-1)^2} = \frac{1}{960}\left(-315\zeta(3) - 86 - 30\pi^2 + 272\ln(2)\right)$$

$$\sum_{n=0}^{\infty} \frac{H_{2n}}{(2n-7)(2n-1)^2} = -\frac{7\zeta(3)}{32} - \frac{\pi^2}{48} + \frac{2103\ln(2) - 601}{11340}$$

$$\sum_{n=0}^{\infty} \frac{H_{2n}}{(2n-9)(2n-1)^2} = -\frac{21\zeta(3)}{128} - \frac{311}{8640} - \frac{\pi^2}{64} + \frac{173\ln(2)}{1260}$$

$$\sum_{n=0}^{\infty} \frac{H_{2n}}{(2n-11)(2n-1)^2} = -\frac{21\zeta(3)}{160} - \frac{39277}{1485000} - \frac{\pi^2}{80} + \frac{37693\ln(2)}{346500}$$

As an alternative way of proving the above lemma, we may apply partial fraction decomposition to the rational component of the summand in (21), obtaining the expresion

$$\frac{1}{2}\sum_{n=0}^{\infty}\frac{H_{2n}}{(2n-3)(2n-1)}-\frac{1}{2}\sum_{n=0}^{\infty}\frac{H_{2n}}{(2n-1)^2}. \tag{22}$$

A proof signature providing a closed-form evaluation for the former sum is

$$\left(x^2, \frac{\ln(1-x)}{x}, \frac{1}{4}\left(3x^{3/2}\tanh^{-1}\left(\sqrt{x}\right)-3x-\sqrt{x}\tanh^{-1}\left(\sqrt{x}\right)\right)\right),$$

whereas an equivalent version of the latter sum from (22) was introduced in [16], and a highly simplified proof of this evaluation was later given in [12].

Lemma 4 *The equality*

$$\sum_{n=0}^{\infty}\frac{H_{2n}}{(2n+1)(2n+3)^2}=\frac{1}{32}\left(-7\zeta(3)+56-3\pi^2-24\ln(2)\right)$$

holds.

Proof This follows from Lemma 3, through the use of a re-indexing argument.

The series Lemma 4 arises through an application of the proof signature given below. With regard to this signature, we are implicitly making use of the generating function for $(H_{2n}\binom{2n}{n} : n \in \mathbb{N}_0)$ from [13].

Theorem 14 *The evaluation*

$$\sum_{n=1}^{\infty}\frac{H_{2n}H_{n+\frac{1}{2}}}{(2n+1)(2n+3)}=\frac{7\zeta(3)}{16}+\frac{5\pi^2}{48}+\ln^2(2)-\frac{3\ln(2)}{2}$$

holds.

Proof $\left(x^2, \ln(x), \frac{\ln\left(\frac{1}{2}\left(\sqrt{1-x}+1\right)\right)-\ln(1-x)}{\sqrt{1-x}}\right)$.

2.2 New results on hypergeometric series

Many new results concerning the evaluation of hypergeometric series involving quarter-integer parameters were recently introduced in [10, 11]. The techniques in [10, 11] for evaluating series of this form mainly rely upon the use of Wallis-type integrals for central binomial coefficients, as well as the use of Fourier–Legendre series. Our Ramanujan-like integration technique from Lemma 1 is very powerful, so it seems natural to try to make use of this lemma to derive new results inspired by [10, 11]. Many of the main results from [11] are given by closed-form evaluations for ratios of ${}_3F_2$ series, as in the interesting identity whereby

$$\frac{\pi}{4} = \frac{{}_3F_2\left[\begin{matrix}-\eta, \frac{1}{2}, 1\\ \frac{3}{2}, 2+\eta\end{matrix}\ \middle|\ -1\right]}{{}_3F_2\left[\begin{matrix}\frac{1}{2}, \frac{1}{2}, 1+\eta\\ 1, 2+\eta\end{matrix}\ \middle|\ 1\right]}$$

for $\eta > -1$ proved in [11]. The following new identity, which is of interest in its own right, does not follow from the ${}_3F_2$ quotients from [11], and may be used to derive very interesting corollaries through the use of a lemma from [10]. As discussed in [10], the symbolic computation of ${}_3F_2(1)$ series is an interesting subject with many applications, and there are no known general formulas for summations of this form. Since ${}_3F_2(1)$ series have important applications in combinatorics, mathematical physics, and countless other areas [10, 26], this strongly motivates the exploration of the master lemma introduced in this article in the evaluation of ${}_3F_2(1)$ series that would otherwise seem to have no possible closed-form expression.

Theorem 15 *The identity*

$$\frac{4\Gamma\left(\frac{5}{2}-a\right)\Gamma(a)}{\pi^{3/2}} = \frac{{}_3F_2\left[\begin{matrix}\frac{3}{2}, \frac{1}{2}-a, \frac{3}{2}-a\\ 1, \frac{5}{2}-a\end{matrix}\ \middle|\ 1\right]}{{}_3F_2\left[\begin{matrix}-\frac{1}{2}, 1-a, a\\ \frac{1}{2}, \frac{3}{2}\end{matrix}\ \middle|\ 1\right]} \tag{23}$$

holds for $a > 0$.

Proof $\left(\sin^2(x), \cos(x), {}_3F_2\left[\begin{matrix}-\frac{1}{2}, 1-a, a\\ \frac{1}{2}, \frac{1}{2}\end{matrix}\ \middle|\ x\right]\right)$.

In [10], it is noted that Pochhamer products given by consecutive quater-integer parameters, i.e., products of the form

$$\left(\frac{2n+1}{4}\right)_i \left(\frac{2n+3}{4}\right)_i \tag{24}$$

are equal to

$$\frac{\binom{4i}{2i}(2i)!}{64^i(2n-1)!!}\prod_{j=1}^{|n|}(4i+\operatorname{sgn}(n)(2j-1))^{\operatorname{sgn}(n)}$$

for $n \in \mathbb{Z}$ and any $i \in \mathbb{N}_0$. As discussed in [10], this often allows us to make use of Wallis' integral formula for central binomial coefficients to explicitly evaluate ${}_pF_q$ series with consecutive quarter-integer parameters as in (24). However, it appears that the problem of evaluating hypergeometric series containing parameters in $(\mathbb{Z}/4)\setminus(\mathbb{Z}/2)$ that do not form Pochhammer products as in (24) can be much more difficult, thus motivating the interesting corollary provided below.

Corollary 4 *The closed-form evaluation*

$$ {}_3F_2\left[\begin{matrix}\frac{1}{4},\frac{5}{4},\frac{3}{2}\\ 1,\frac{9}{4}\end{matrix}\,\middle|\,1\right]=\frac{\Gamma^2\left(\frac{1}{4}\right)\left(15\sqrt{2}-5\ln\left(1+\sqrt{2}\right)\right)}{16\pi^{3/2}} \tag{25}$$

holds.

Proof Letting $a=\frac{1}{4}$ with regard to (23), we find that

$$ {}_3F_2\left[\begin{matrix}-\frac{1}{2},\frac{1}{4},\frac{3}{4}\\ \frac{1}{2},\frac{3}{2}\end{matrix}\,\middle|\,1\right]=\frac{\pi^{3/2}}{4\Gamma\left(\frac{1}{4}\right)\Gamma\left(\frac{9}{4}\right)}\,{}_3F_2\left[\begin{matrix}\frac{1}{4},\frac{5}{4},\frac{3}{2}\\ 1,\frac{9}{4}\end{matrix}\,\middle|\,1\right].$$

We see that the hypergeometric expression in the left-hand side of the above equality contains a Pochhammer product of the form indicated in 24. Applying the lemma from [10], we see that

$$ {}_3F_2\left[\begin{matrix}-\frac{1}{2},\frac{1}{4},\frac{3}{4}\\ \frac{1}{2},\frac{3}{2}\end{matrix}\,\middle|\,1\right]=\sum_{i=0}^{\infty}\left(\frac{1}{16}\right)^i\frac{\binom{4i}{2i}}{(1-2i)(2i+1)}, \tag{26}$$

with Mathematica 11 unable to symbolically evaluate the right-hand side of (26). Through the use of a standard formulation of Wallis' integral identity, we see that

$$ {}_3F_2\left[\begin{matrix}-\frac{1}{2},\frac{1}{4},\frac{3}{4}\\ \frac{1}{2},\frac{3}{2}\end{matrix}\,\middle|\,1\right]=\int_0^{2\pi}\frac{1+\tanh^{-1}\left(\cos^2 t\right)\left(\sec^2 t-\cos^2 t\right)}{4\pi}\,dt,$$

and this provides us with the desired result.

The evaluation provided in the above corollary is especially interesting, since it is not otherwise obvious as to how to evaluate the ${}_3F_2$ series from Corollary 4. We see that the lemma from [10] cannot be applied, since there are three quarter-integer parameters in the left-hand side of (25), and a Pochhammer product of the form in (24) does not appear. The hypergeometric function

$$ {}_2F_1\left[\begin{matrix}\frac{1}{4},\frac{5}{4}\\ \frac{9}{4}\end{matrix}\,\middle|\,x\right]$$

produces a very complicated expression that cannot be simplified, so Wallis-type functions cannot be applied. As in the case with the main ${}_3F_2(1)$ series under consideration from [11], it is easily seen that Watson's theorem on the evaluation of series of the form

$$ {}_3F_2\left[\begin{matrix}a,b,c\\ \frac{1}{2}(a+b+1),2c\end{matrix}\,\middle|\,1\right].$$

cannot be applied directly to prove the above Corollary, and this is also the case for Clausen's product

$$\left({}_2F_1\left[\begin{matrix}a,b\\ a+b+\frac{1}{2}\end{matrix}\,\middle|\,z\right]\right)^2={}_3F_2\left[\begin{matrix}2a,a+b,2b\\ a+b+\frac{1}{2},2a+2b\end{matrix}\,\middle|\,z\right].$$

A similar approach leads us to the following new result.

Corollary 5 *The evaluation*

$$ {}_3F_2\left[\begin{matrix} -\frac{1}{4}, \frac{3}{4}, \frac{3}{2} \\ 1, \frac{7}{4} \end{matrix} \;\middle|\; 1\right] = \frac{\Gamma^2\left(-\frac{1}{4}\right)\left(9\sqrt{2} - 3\ln\left(1+\sqrt{2}\right)\right)}{64\pi^{3/2}} $$

holds.

Proof This may be proved by letting $a = \frac{3}{4}$ in the above theorem, and by mimicking the preceding proof.

Experimentation with the proof signature from Theorem 15 leads us to many new results of a similar flavor, as suggested below. From the variety of results on ${}_3F_2$ quotients from [11], as in the intriguing evaluation

$$ \pi = \frac{4^{m+1}}{\binom{2m}{m}(2m+1)} \cdot \frac{{}_3F_2\left[\begin{matrix} \frac{1}{2} - m, 1, -\eta \\ \frac{3}{2} + m, 2 + \eta \end{matrix} \;\middle|\; -1\right]}{{}_3F_2\left[\begin{matrix} \frac{1}{2} - m, \frac{1}{2}, 1 + \eta \\ 1, 2 + \eta \end{matrix} \;\middle|\; 1\right]} $$

for $m \geq 0$ and $\eta > -1$ introduced and proved in [11], we are inspired to develop new results in the same vein.

Theorem 16 *The identity*

$$ \frac{{}_3F_2\left[\begin{matrix} \frac{5}{2}, \frac{1}{2} - a, \frac{5}{2} - a \\ 2, \frac{7}{2} - a \end{matrix} \;\middle|\; 1\right]}{{}_3F_2\left[\begin{matrix} -\frac{3}{2}, 1 - a, a \\ -\frac{1}{2}, \frac{3}{2} \end{matrix} \;\middle|\; 1\right]} = -\frac{16\Gamma\left(\frac{7}{2} - a\right)\Gamma(a)}{3\pi^{3/2}(2a-3)(2a+1)} $$

holds for $a > 0$.

Proof $\left(\sin^2(x), \cos(x), {}_3F_2\left[\begin{matrix} -\frac{3}{2}, 1 - a, a \\ -\frac{1}{2}, \frac{1}{2} \end{matrix} \;\middle|\; x\right]\right)$.

Corollary 6 *The evaluation*

$$ {}_3F_2\left[\begin{matrix} \frac{1}{4}, \frac{9}{4}, \frac{5}{2} \\ 2, \frac{13}{4} \end{matrix} \;\middle|\; 1\right] = \frac{\Gamma^2\left(\frac{1}{4}\right)\left(61\sqrt{2} - 15\ln\left(1+\sqrt{2}\right)\right)}{64\pi^{3/2}} $$

holds.

Proof This follows from Theorem 16 by letting $a = \frac{1}{4}$ and by mimicking the proof of Corollary 4.

The above corollaries show us how the master lemma from Section 1 can be used to provide us with a very powerful technique for evaluating new ${}_3F_2(1)$ series. Theorem 15 may be greatly generalized as follows.

Theorem 17 *The identity*

$$\frac{{}_3F_2\left[\begin{matrix}1-a,a,b\\ \frac{3}{2},b+1\end{matrix}\,\middle|\,1\right]}{{}_3F_2\left[\begin{matrix}1,\frac{3}{2},\frac{3}{2}-b\\ \frac{5}{2}-a,a+\frac{3}{2}\end{matrix}\,\middle|\,1\right]}=\frac{2b\cos(\pi a)}{(2a-3)(2a-1)(2a+1)}$$

holds for $a>0$ and $b>0$.

Proof $\left(\sin^2(x),\cos(x),{}_3F_2\left[\begin{matrix}1-a,a,b\\ \frac{1}{2},b+1\end{matrix}\,\middle|\,x\right]\right)$.

Corollary 7 *The equation*

$${}_3F_2\left[\begin{matrix}1,\frac{5}{4},\frac{3}{2}\\ \frac{7}{4},\frac{9}{4}\end{matrix}\,\middle|\,1\right]=\frac{15\pi}{4\sqrt{2}}+\frac{15\ln\left(1+\sqrt{2}\right)}{2\sqrt{2}}-\frac{15}{2}$$

holds.

Proof Let $a=b=\frac{1}{4}$ with regard to Theorem 17.

3 Conclusion

Our paper introduces an integration method that is very general but also very powerful, and that seems to "unite" many different kinds of proofs for series/integral evaluations. In a sense, Ramanujan's Master Theorem can be thought of as a special case of Lemma 1. Through the use of the master lemma introduced in this article, simple combinations of input functions lead us to unexpected results as in the evaluations presented in Section 2. It seems likely that this lemma will continue to lead to surprising new results in classical analysis-related fields. The main goal of our article is to show how new infinite series that are seemingly recalcitrant can be easily proved using Lemma 1, and to illustrate how this lemma provides us with a convenient and systematic way of generating and classifying evaluations for series that can be proved using moment identities as applied term-by-term with respect to a given generating function. The results given in Section 2 are new and highly nontrivial, in that the new series in this section cannot be proved using any "obvious" or standard summation techniques. The one-line proofs in this article are instructive in that these proofs show us how 3-tuples of elementary functions can "generate" new Ramanujan-like formulas, and how experimentation with the entries in these tuples can lead to great generalizations and variations of these formulas.

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